Lectures on Vector Fields and the Unity of Mathematics

by

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1. Introduction

Prediction: A theorem of Marston Morse, [M], which I call the Law of Vector Fields, will come to occupy a position in Mathematics rivaling that of DeMoivre's Formula or the Pythagorian Theorem. This equation, which we call the Law of Vector Fields was discovered in 1929 and has not played a role at all commensurate with our prediction up until now. The purpose of these lectures is to convince you of that fact and to teach you to become proficient in using the theorem.

We describe the equation, which we call the Law of Vector Fields. Let M be a compact manifold with boundary. Let V be a vector field on M with no zeros on the boundary. Then consider the open set of the boundary of M where V is pointing inward. Let $\partial_- V$ denote the vector field defined on this open set on the boundary which is given by projecting V tangent to the boundary. The Euler characteristic of M is denoted by $\chi(M)$, and $\text{Ind}(V)$ denotes the index of the vector field. Then the Law of Vector Fields is

$$\text{Ind}(V) + \text{Ind}(\partial_- V) = \chi(M)$$

In section 2 we investigate the *unity of Mathematics*. This leads us to propose the *Function Principle* which predicts that concepts fashioned out of simple concepts involving functions are broad. This leads us to the prediction that the Law of Vector Fields must be a central result in mathematics.

In section 3 we propose two strategies to get mathematics from the Law of Vector Fields and we report on their successes so far.

In section 4 we indicate how the Law of Vector Fields can be used to define the Index of a vector field and we define the concept of otopy.

In section 5 we discuss vertical vector fields on a fibre bundle. This concept is the proper generalization of otopy.

In section 6 we give an elementary discussion of vector fields in one dimension. This section can be read independently of the rest of the paper and it illustrates with pictures many of the issues arising in the definition of index and proper vector fields.

In section 7 the main equations related to the index are listed for the convenience of the reader and many exercises are given to develop the skill involved in their use.

In section 8 we apply the index to evaluation subgroups to get a relationship between the Euler–Poincare number and the center of a group.

Portions of this exposition are taken verbatim from [G–S], [G7], and [G8].

2. The Unity of Mathematics

We take the following definition of Mathematics:

DEFINITION. *Mathematics is the study of well–defined concepts.*

Now well–defined concepts are creations of the human mind. And most of those creations can be quite arbitrary. There is no limit to the well–defined imagination. So if one accepts the definition that Mathematics is the study of the well–defined, then how can Mathematics have an underlying unity? Yet it is a fact that many savants see a underlying unity in Mathematics, so the key question to consider is:

QUESTION. *Why does Mathematics appear to have an underlying unity?*

If mathematical unity really exists then it is reasonable to hope that there are a few basic principles which explain the occurrence of those phenomena which persuade us to believe that Mathematics is indeed unified; just as the various phenomena of Physics seem to be explained by a few fundamental laws. If we can discover these principles, it would give us great insight into the development of Mathematics and perhaps even insight into Physics.

Now what things produce the appearance of an underlying unity in Mathematics? Mathematics appears to be unified when a concept, such as the Euler characteristic, appears over and over in interesting results; or an idea, such as that of a group, is involved in many different fields and is used in science to predict or make phenomena precise; or an equation, like De Moivre's formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

yields numerous interesting relations among important concepts in several fields in a mechanical way.

Thus underlying unity comes from the ubiquity of certain concepts and objects, such as the numbers π and e and concepts such as groups and rings, and invariants such as the Euler characteristic and eigenvalues, which continually appear in striking relationships and in diverse fields of Mathematics and Physics. We use the word *broad* to describe these concepts.

Compare broad concepts with *deep* concepts. The depth of an idea seems to be a function of time. As our understanding of a field increases, deep concepts become elementary concepts, deep theorems are transformed into definitions and so on. But something broad, like the Euler characteristic, remains broad, or becomes broader as time goes on. The relationships a broad concept has with other concepts are forever.

The Function Principle. *Any concept which arises from a simple construction of functions will appear over and over again throughout Mathematics.*

We assert the principle that function is one of the broadest of all mathematical concepts, and any concept or theorem derived in a natural way from that of functions must itself be broad. We will use this principle to assert that the underlying unity of Mathematics at least partly stems from the breadth of the concept of function. We will show how the breadth of category and functor and equivalence and e and π and de Moivre's formula and groups and rings and Euler Characteristic all follow from this principle. We will subject this principle to the rigorous test of a scientific theory: It must predict new broad concepts. We make such predictions and report on evidence that the predictions are correct.

The concept of a function as a mapping $f\colon X \to Y$ from a source set X to a target set Y did not develop until the twentieth century. The modern concept of a function did not even begin to emerge until the middle ages. The beginnings of Physics should have given a great impetus to the notion of function, since the measurements of the initial conditions of an experiment and the final results gives implicitly a function from the initial states of an experiment to the final outcomes; but historians say that the early physicists and mathematicians never thought this way. Soon thereafter calculus was invented. For many years afterwards functions were thought to be always given by some algebraic expression. Slowly the concept of a function or mapping grew. Cantor's set theory gave the notion a good impulse but the modern notion was adopted only in the Twentieth Century.

The careful definition of function is necessary so that the definition of the composition of two functions can be defined. Thus fog is only defined when the target of g is the source of f. This composition is associative: $(fog)oh = fo(goh)$ and f composed with the identity of either the source or the target is f again. We call a set of functions a *category* if it is closed under compositions and contains the identity functions of all the sources and targets.

Category was first defined by S. Eilenberg and S. MacLane and was employed by Eilenberg and N. Steenrod in the 1940's to give homology theory its functorial character. Category theory became a subject in its own right, its practitioners joyfully noting that almost every branch of Mathematics could be organized as a category. The usual definition of category is merely an abstraction of functions closed under composition. The functions are abstracted into things called morphisms and composition becomes an operation on sets of morphisms satisfying exactly the same properties that functions and composition satisfy. Most mathematicians think of categories as very abstract things and are surprised to find they come from such a homely source as functions closed under composition.

A *functor* is a function whose source and domain are categories and which preserves composition. That is, if F is the functor, then $F(fog) = F(f)oF(g)$. This definition also is abstracted and one says category and functor in the same breath.

Now consider the question: What statements can be made about a function f which would make sense in every possible category? There are basically only four statements since the only functions known to exist in every category are the identity functions. We can say that f is an identity, or that f is a *retraction* by which we mean that there is a function g so that fog is an identity, or that f is a *cross–section* by which we mean that there is a function h so that hof is an identity, or finally that f is an *equivalence* by which we mean that f is both a retraction and a cross–section. In the case of equivalence the function h must equal the function g and it is called the *inverse* of f and it is unique.

Retraction and cross–section induce a partial ordering of the sources and targets of a category, hereafter called the *objects* of the category. Equivalences induce an equivalence relation on the objects and give us the means of making precise the notion that two mathematical structures are the same.

Now consider the self equivalences of some object X in a category of functions. Since X is both the source and the target, composition is always defined for any pair of functions, as are inverses. Thus we have a *group*. The definition of a group in general is just an abstraction, where the functions become undefined elements and composition is the undefined operation which satisfies the group laws of associativity and existence of identity and inverse, these laws being the relations that equivalences satisfy. The notion of functor restricted to a group becomes that of *homomorphism*. The equivalences in the category of groups and homomorphisms are called *isomorphisms*.

The concept of groups arose in the solution of polynomial equations, with the first ideas due to Lagrange in the late eighteenth century, continuing through Abel to Galois. Felix Klein proposed that geometry should be viewed as arising from groups of symmetries in 1875. Poincare proposed that the equations of Physics should be invariant under the correct symmetry groups around 1900. Since then groups have played an increasingly important role in Mathematics and in Physics. The increasing appearance of this broad concept must have fed the feeling of the underlying unity of Mathematics. Now we see how naturally it follows from the Function Principle.

If we consider a set of functions S from a fixed object X into a group G we can induce a group structure on S by defining the multiplication of two functions f and g to be $f * g$ where $f * g(a) = f(a) \cdot g(a)$ where a runs through all the elements in X and $\cdot$ is the group multiplication in G. This multiplication can be easily shown to satisfy the laws of group multiplication. The same idea applied to maps into the Real Numbers or the Complex Numbers gives rise to addition and multiplication on functions. These satisfy properties which are abstracted into the concepts of abelian rings . If we consider the set of self homomorphisms of an abelian group and use composition and addition of functions, we get an important example of a non–commutative ring. The natural functors for rings should be ring homomorphisms. In the case of a ring of functions into the Real or Complex numbers we note that a ring homomorphism h fixes the constant maps. If we consider all functions which fix the constants and preserve the addition, we get a category of functions from rings to rings; that is, these functions are closed under composition. We call these functions *linear transformations*. They contain the ring homomorphisms as a subset. Study the equivalences of this category. We obtain the concepts of *vector spaces* and linear transformations after the usual abstraction.

Now we consider a category of homomorphisms of abelian groups. We ask the same question which gave us equivalence and groups, namely: What statements can be made about a homomorphism f which would make sense in every possible category of abelian groups? Now between every possible abelian group there is the trivial homomorphism $0: A \to B$ which carries all of A onto the identity of B. Also we have for every integer N the homomorphism from A to itself which adds every element to itself N times, that is multiplication by N.

Thus for any homomorphism $h: A \to B$ there are three statements we can make which would always make sense. First Noh is the trivial homomorphism 0, second that there is a homomorphism $\tau: B \to A$ so that $ho\tau$ is multiplication by N, or third that τoh is multiplication by N. So we can give to any homomorphism three non–negative integers: The *exponent*, the *cross–section degree*, and the *retraction degree*. The *exponent* is the smallest positive integer such that Noh is the trivial homomorphism 0. If there is no such N then the exponent is zero. Similarly the *cross–section degree* is the smallest positive N such that there is a τ, called a *cross–section transfer*, so that $ho\tau$ is multiplication by N. Finally the *retraction degree* is the smallest positive N such that there is a τ, called a *retraction transfer*, so that τoh is multiplication by N.

In accordance with the Function Principle, we predict that these three numbers will be seen to be broad concepts. Their breadth should be less than the breadth of equivalence, retraction and cross–section because the concepts are valid only for categories of abelian groups and homomorphisms. But exponent, cross–section degree and retraction degree can be pulled back to other categories via any functor from that category to the category of abelian groups. So these integers potentially can play a role in many interesting categories. In fact for the category of topological spaces and continuous maps we can say that any continuous map $f: X \to Y$ has exponent N or cross–section degree N or retraction degree N if the induced homomorphism $f_*: H_*(X) \to H_*(Y)$ on integral homology has exponent N or cross–section degree N or retraction degree N respectively.

As evidence of the breadth of these concepts we point out that for integral homology, cross–section transfers already play an important role for fibre bundles. There are natural transfers associated with many of the important classical invariants such as the Euler characteristic and the index of fixed points and the index of vector fields,[B–G], and the Lefschetz number and coincidence number and most recently the intersection number, [G–O]. And a predicted surprise relationship occurs in the case of cross–section degree for a map between two spaces. In the case that the two spaces are closed oriented manifolds of the same dimension, the cross–section degree is precisely the absolute value of the classical Brouwer degree. The retraction degree also is the Brouwer degree for closed manifolds if we use cohomology as our functor instead of homology, [G2].

The most common activity in Mathematics is solving equations. There is a natural way to frame an equation in terms of functions. In an equation we have an expression on the left set equal to an expression on the right and we want to find the value of the variables for which the two expressions equal. We can think of the expressions as being two function f and g from X to Y and we want to find the elements x of X such that $f(x) = g(x)$. The solutions are called *coincidences*. Coincidence makes sense in any category and so we would expect the elements of any existence or uniqueness theorem about coincidences to be very broad indeed. But we do not predict the existence of such a theorem. Nevertheless in topology there is such a theorem. It is restricted essentially to maps between closed oriented manifolds of the same dimension. It asserts that locally defined coincidence indices add up to a globally defined coincidence number which is given by the action of f and g on the homology of X. In fact it is the alternating sum of traces of the composition of the umkehr map $f_!$, which is defined using Poincare Duality, and g_*, the homomorphism by g. We predict, at least in topology and geometry, more frequent appearances of both the coincidence number and also the local coincidence index and they should relate with other concepts.

If we consider self maps of objects, a special coincidence is the fixed point $f(x) = x$. From the point of view of equations in some algebraic setting, the coincidence problem can be converted into a fixed point problem, so we do not lose any generality in those settings by considering fixed points. In any event the fixed point problem makes sense for any category. Now the relevant theorem in topology is the Lefschetz fixed point theorem. In contrast to the coincidence theorem, the Lefschetz theorem holds essentially for the wider class of compact spaces. Similar to the coincidence theorem, the Lefschetz theorem has locally defined fixed point indices which add up to a globally defined Lefschetz number. This Lefschetz number is the alternating sum of traces of f_*, the homomorphism induced by f on homology. This magnificent theorem is easier to apply than the coincidence theorem and so the Lefschetz number and fixed point index are met more frequently in various situations than the coincidence number and coincidence indices.

In other fields fixed points lead to very broad concepts and theorems. A linear operator gives rise to a map on the one dimensional subspaces. The fixed subspaces are generated by *eigenvectors*. Eigenvectors and their associated eigenvalues play an important role in Mathematics and Physics and are to be found in the most surprising places.

Consider the category of C^∞ functions on the Real Line. The derivative is a function from this category to itself taking any function f into f'. The derivative practically defines the subject of calculus. The fixed points of the derivative are multiples of e^x. Thus we would predict that the number e appears very frequently in calculus and any field where calculus can be employed. Likewise consider the set of analytic functions of the Complex Numbers. Again we have the derivative and its fixed point are the multiples of e^z. Now it is possible to relate the function e^z defined on a complex plane with real valued functions by

$$e^{(a+ib)} = e^a(\cos(b) + i\sin(b)).$$

We call this equation de Moivre's formula. This formula contains an unbelievable amount of information. Just as our concept of space–time separation is supposed to break down near a black hole in Physics, so does our definition–theorem view of Mathematics break down when considering this formula. Is it a theorem or a definition? Is it defined by sin and cos or does it define those two functions?

Up to now the function principle predicted only that some concepts and objects will appear frequently in undisclosed relationships with important concepts throughout Mathematics. However the de Moivre equation gives us methods for discovering the precise forms of some of the relationships it predicts. For example, the natural question "When does e^z restrict to real valued functions?" leads to the "discovery" of π. From this we might predict that π will appear throughout calculus type Mathematics, but not with the frequency of e. Using the formula in a mechanical way we can take complex roots, prove trigonometric identities, etc.

There is yet another fixed point question to consider: What are the fixed points of the identity map? This question not only makes sense in every category; it is solved in every category! The invariants arising from this question should be even broader than those from the fixed point question. But at first glance they seem to be very uninteresting. However, if we consider the fixed point question for functions which are equivalent to the identity under some suitable equivalence relation in a suitable category we may find very broad interesting things. A suitable situation involves the fixed points of maps homotopic to the identity in the topological category. For essentially compact spaces the *Euler characteristic* (also called the Euler–Poincare number) is an invariant of a space whose nonvanishing results in the existence of a fixed point. This Euler characteristic is the most remarkable of all mathematical invariants. It can be defined in terms simple enough to be understood by a school boy, and yet it appears in many of the star theorems of topology and geometry. A restriction of the concept of the Lefschetz number, its occurrence far exceeds that of its "parent" concept. First mentioned by Descartes, then used by Euler to prove there were only five platonic solids, the Euler characteristic slowly proved its importance. Bonnet showed in the 1840's that the total curvature of a closed surface equaled a constant times the Euler characteristic. Poincare gave it its topological invariance by showing it was the alternating sum of Betti numbers. In the 1920's Lefschetz showed that it determined the existence of fixed points of maps homotopic to the identity, thus explaining, according to the Function Principle, its remarkable history up to then and predicting, according to our principle, the astounding frequency of its subsequent appearances in Mathematics.

The Euler characteristic is equal to the sum of the local fixed point indices of the map homotopic to the identity. We would predict frequent appearances of the local index. Now on a smooth manifold we consider vector fields and regard them as representing infinitesimally close maps to the identity. Then the local fixed point index is the local index of the vector field. Now we have Morse's formula, which we call the Law of Vector Fields:

$$\mathrm{Ind}(V) + \mathrm{Ind}(\partial_- V) = \chi(M).$$

Exercises:

a) Consider the set of functions $f\colon G \to G$ where G is a group. What are the relations between composition $f \circ g$ and multiplication $f * g$? What additional relation is there if the f's are homomorphisms? Abstract these two concepts to get two algebraic structures which generalize rings. Why haven't we heard of these concepts?

b) Consider the function $f(z) = ke^z$ for k a constant. This is a function from $\mathbb{C}$ to $\mathbb{C}$ which is a fixed point of the complex derivative. Show e^x has no fixed point for real x. Show that e^z has two fixed points. What are they? What are the two non-negative k for which $f(z) = ke^z$ has only one fixed point?

c) Let X be a set of self maps of Y. Then the evaluation maps $\hat{\omega}\colon X \times Y \to Y\colon \hat{\omega}(x, y) = x(y)$ and $\omega\colon X \to Y\colon \omega(x) = x(*)$ for fixed base point $*$ should be very broad concepts. The induced maps for homotopy group $\omega_*\colon \pi_*(Y) \to \pi_*(Y)$, and homology $\omega_*\colon H_*(X) \to H_*(Y)$ and cohomology $H^*(Y) \to H^*(X)$ should appear throughout topology. Search through the Korean literature to find out who studies questions related to ω. Search through the literature and find predicted relationships between ω_* and the Euler–Poincare number and the Lefschetz numbers and transfers.

3. The Law of Vector Fields

Just as de Moivre's formula gives us mechanical methods which yields precise relationships among broad concepts, we predict that the Law of Vector Fields will give mechanical methods which will yield precise relationships among broad concepts.

Up to now our predictions had a nebulous quality. They are not specific, they only assert that concepts following from the function principle are broad. They do not tell us how they are broad. And if a predicted broad concept is unknown, it may argue against the function principle, or it may be that in time that the concept will make its importance known.

The Law of Vector Fields is a fair test of the Function Principle. It involves the index and the Euler–Poincare number, the oldest concepts of Algebraic Topology tracing back to Poincare. It was discovered in 1929 by one of the greatest mathematicians of the 20th century, Marston Morse. Morse never used the Law of Vector Fields and it fell into obscurity. It was rediscovered by Charles Pugh [P], and later by the author [G3],[G4].

Given this undistinguished 50 year history and the tremendous advances of Topology during that period, most mathematicians would say that there is little chance that much new could come out of the Law of Vector Fields. And some mathematicians firmly believe this is the case, that it is *impossible* to find much of interest. So if we can show the contrary, that the Law of Vector Fields yields interesting new results in a mechanical way, then we will have powerful evidence for the Function Principle.

We propose two methods of drawing Mathematics from the Law.

The following observation will suggest one method: The Law of Vector Fields is practically an inductive definition of the index of a vector field given the Euler characteristic. The induction is on the dimension of the underlying manifold. We say that the index of an empty vector field is zero and the index of a vector field on a finite set of points is the number of points. Then the knowledge of the index on the boundary gives the index in the one higher dimensional compact manifold. And the index can be defined on open sets by setting it equal to the index of a compact manifold with boundary which contains all the zeros of the vector field.

The method follows:

1. Choose an interesting vector field V and manifold M.
2. Adjust the vector field if need be to eliminate zeros on the boundary.
3. Identify the global and local Ind V.
4. Identify the global and local index $\mathrm{Ind}(\partial_- V)$.
5. Substitute 3 and 4 into the Law of Vector Fields.

We predict that this method will succeed because the Law of Vector Fields is morally the definition of index, so all features of the index must be derivable from that single equation. We measure success in the following descending order: 1. An important famous theorem generalized; 2. A new proof of an important famous theorem; 3. A new, interesting result. We put new proofs before new results because it may not be apparent at this time that the new result will famous or important.

In category 1 we already have the extrinsic Gauss-Bonnet theorem of differential geometry [G6],[G5], the Brouwer fixed point theorem of topology [G6], and Hadwiger's formulas of integral geometry [G6] ,[Had], [Sa]. In category 2 we have the Jordan separation theorem, The Borsuk-Ulam theorem, the Poincare-Hopf index theorem of topology; Rouche's theorem and the Gauss-Lucas theorem in complex variables; the fundamental theorem of algebra and the intermediate value theorem of elementary Mathematics; and the not so famous Gottlieb's theorem of group homology, [G7]. Of course we have more results in category 3, but it is not so easy to describe them with a few words. One snappy new result is the following: Consider any straight line and smooth surface of genus greater than 1 in three dimensional Euclidean space. Then the line must be contained in a plane which is tangent to the surface, ([G6], theorem 15).

We will discuss the Gauss-Bonnet theorem since that yields results in all three categories as well as having the longest history of all the results mentioned. One of the most well-known theorems from ancient times is the theorem that the sum of the angles of a triangle equals 180 degrees. Gauss showed for a triangle whose sides are geodesics on a surface M in three-space that the sum of the angles equals $\pi + \int_M KdM$, where K is the Gaussian curvature of the surface. Bonnet pieced these triangles together to prove that for a closed surface M the total curvature $\int_M KdM$ equals $2\pi\chi(M)$. Hopf proved that $\int_M KdM$, where M is a closed hypersurface in odd dimensional Euclidean space and K is the product of the principal curvatures must equal the degree of the Gauss map $\hat{N} : M^{2n} \to S^{2n}$ times the volume of the unit sphere. Then he proved $2\deg(\hat{N}) = \chi(M^{2n})$. (Morris Hirsch in [Hi] gives credit to Kronicker and Van Dyck for Hopf's result.) For a history of the Gauss-Bonnet theorem see [Gr], pp. 89-72 or [Sp], p. 385.

Given a map f between Riemannian manifolds M to N and a vector field V on N, we can define a vector field on M, denoted f^*V, which is the pullback of V by f. For a map f from M into the real line R the pullback vector field of the unit positive pointing d/dt is just the gradient of f.

Let $f : M \to R^n$ be a smooth map from a compact Riemannian manifold of dimension n to n-dimensional Euclidean space so that f near the boundary ∂M is an immersion. Let V be any vector field on R^n such that V has no zeros on the image of the boundary $f(\partial M)$. Consider the pullback vector field f^*V on M. The local index on $\partial_- V$ turns out to be the local coincidence number of two different Gauss maps. Substituting into the Law of Vector Fields results in a great generalization of the Gauss-Bonnet Theorem, ([G6], theorem 5). A very special case is the following, which still is a generalization of the Gauss-Bonnet Theorem. We quote the special case only in order to avoid introducing notation. The index of the gradient of $xof : M \to R$, where x is the projection of R^n onto the x-axis, is equal to the difference between the Euler Characteristic and the degree of the Gauss map. Thus

$$\operatorname{Ind}(\operatorname{grad}(xof)) = \chi(M) - \deg \hat{N}.$$

This equation leads to an immediate proof of the Gauss-Bonnet Theorem, since for odd dimensional M and any vector field W, the index satisfies $\operatorname{Ind}(-W) = -\operatorname{Ind}(W)$. Thus the left side of the equation reverses sign while the right side of the equation remains the same. Thus $\chi(M)$ equals the degree of the Gauss-map, which is the total curvature over the volume of the standard $n-1$ sphere. Now $2\chi(M) = \chi(\partial M)$, so we get Hopf's version of the Gauss-Bonnet theorem.

Note as a by-product we also get $\operatorname{Ind}(\operatorname{grad}(xof)) = 0$ which is a new result thus falling into category 3. Another consequence of the generalized Gauss-Bonnet theorem follows when we assume the map f is an immersion. In this case the gradient of xof has no zeros, so its index is zero so the right hand side in zero and so again $\chi(M) = \deg \hat{N}$. This is Haefliger's theorem [Hae], a category 2 result. Please note in addition that the Law of Vector Fields applied to odd dimensional closed manifolds combined with the category 2 result $\operatorname{Ind}(-W) = -\operatorname{Ind}(W)$, implies that the Euler characteristic of such manifolds is zero, (category 2). So the Gauss-Bonnet theorem and this result have the same proof in some strong sense.

Just as the Gauss-Bonnet theorem followed from pullback vector fields, the Brouwer fixed point theorem is generalized by considering the following vector field. Suppose M is an n-dimensional body in R^n and suppose that $f : M \to R^n$ is a continuous map. Then let the vector field V_f on M be defined by drawing a vector from m to the point $f(m)$ in R^n. Locally the index of $-V_f$ at a zero is a coincidence index of the Gauss map $\hat{N}$ and the "Gauss" map of the vector field V_f given by setting every vector to unit length and translating to the origin. If f maps M into itself with no fixed points on the boundary, then applying the method gives

$$\Lambda_f + \Lambda_{\hat{V}_f, \hat{N}} = \chi(M)$$

This is a category 3 relationship among the main invariants of fixed point theory, coincidence theory and the Euler characteristic. Using the fact that the coincidence number $\Lambda_{\hat{V}_f, \hat{N}}$ is equal to $\deg \hat{N} - \deg \hat{V}_f$ and the fact from the Gauss-Bonnet theorem above that $\deg \hat{N} = \chi(M)$, we see that the Lefschetz number must be equal to $\deg \hat{V}_f$. If we drop the requirement that f maps M into itself, we still have $\operatorname{Ind} V_f = \deg \hat{V}_f$. If the map f satisfies the transversal property, that is the line between m on the boundary of M and $f(m)$ is never tangent to ∂M, than f has a fixed point if $\chi(M)$ is odd (category 1). This last sentence is an enormous generalization of the Brouwer fixed point theorem, yet it remains a small example of what can be proved from applying the Law of Vector Fields to V_f. In fact the Law of Vector Fields applied to V_f is the proper generalization of the Brouwer fixed point theorem.

A second method of producing Mathematics from the Law of Vector Fields involves making precise the statement that the Law defines the index of vector fields, [G–S]. In this method we learn from the Law. The Law teaches us that there is a generalization of homotopy which is very useful. This generalization, which we call *otopy*, not only allows the vector field to change under time, but also its domain of definition changes under time. An otopy is what $\partial_- V$ undergoes when V is undergoing a homotopy. A proper otopy is an otopy which has a compact set of zeros. The proper otopy classes of vector fields on a connected manifold is in one to one correspondence with the integers via the map which takes a vector field to its index. This leads to the fact that homotopy classes of vector fields on a manifold with a connected boundary where no zeros appear on the boundary are in one to one correspondence with the integers. This is not true if the boundary is disconnected.

We find that we do not need to assume that vector fields are continuous. We can define the index for vector fields which have discontinuities and which are not defined everywhere. We need only assume that the set of "defects" is compact and never appears on the boundary or frontier of the sets for which the vector fields are defined. We then can define an index for any compact connected component of defects (subject only to the mild condition that the component is open in the subspace of defects). Thus under an otopy it is as if the defects change shape with time and collide with other defects, and all the while each defect has an integer associated with it. This integer is preserved under collisions. That is the sum of the indices going into a collision equals the sum of the indices coming out of a collision, provided no component "radiates out to infinity", i.e. loses its compactness.

This picture is very suggestive of the way charged particles are supposed to interact. Using the Law of Vector Fields as a guide we have defined an index which satisfies a conservation law under collisions. The main ideas behind the construction involve dimension, continuity, and the concept of pointing inside. We suggest that those ideas might lie behind all the conservation laws of collisions in Physics.

4. The Definition of Index for Discontinuous Vector Field

We assume we are in a smooth manifold N. A vector field is an assignment of tangent vectors to some, not necessarily all, of the points of N. We make no assumptions about continuity. We will call this N the *arena* for our vector fields. We consider the set of defects of a vector field V in N, that is the set D which is the closure of the set of all zeros, discontinuities and undefined points of V. That is we consider a *defect* to be a point of N at which V is either not defined, or is discontinuous, or is the zero vector, or which contains one of those points in every neighborhood.

We are interested in the connected components of the defects and how they change in time. Those connected components of D which are compact we will call topological *particles.* If we can find an open set about a particle which does not intersect any defect not in the particle itself, then we say the particle is *isolated.* If C is an isolated particle we can assign an integer which we call the index of C in V. We denote this by $\mathrm{Ind}(C)$.

The key properties of $\mathrm{Ind}(C)$ are that it is nontrivial, additive over particles, easy to calculate and is conserved under interactions with proper components as V varies under time. For example, let V be the electric vector field generated by one electron in R^3. Then the position of the electron e is the only defect and $\mathrm{Ind}(e) = -1$. Now if V changes under time in such a way that there are only a finite number of particles at each time, all contained in some large fixed sphere, then the sum of the indices of the particles at each time t is equal to -1. Thus the electron vector field can change to the proton vector field only if the set of defects changing under time is unbounded, since the proton has index $+1$ which is different from the index of the electron. In this case we will say that the transformation of the electron to the proton involves " topological radiation".

Vector fields varying under time, and defect components interacting with each other, can be made precise by introducing the concept of *otopy*, which is a generalization of the concept of homotopy. An *otopy* is a vector field on $N \times I$ so that each vector is tangent to a slice $N \times t$. Thus an otopy is a vector field W on $N \times I$ so that $W(n,t)$ is tangent to $N \times t$. We say that V_0 is otopic to V_1 if $V_0(n) = W(n,0)$ and $V_1(n) = W(n,1)$. We say that a set of components C_i of defects on V_0 *transforms* into a set of components of defects D_j of V_1 if there is a connected component T of the defects of W so that $T \cap (N \times 0) = \cup C_i$

and $T \cap (N \times 1) = \cup D_j$. If T is a compact connected component of defects of W, which transforms a set of isolated particles C_i into isolated particles D_j, then we say there is no topological *radiation* and

$$\sum \operatorname{Ind}(C_i) = \sum \operatorname{Ind}(D_j).$$

If T is not compact, we say there is topological radiation.

We define $\operatorname{Ind}(C)$ as follows. Since C is an particle, there is an open set U containing C so that there are no defects in the closure of U except for C. We can define an index for any vector field defined on the closure of an open set so that the set of defects is compact and there is no defect on the frontier of the open set. We say such a vector field is *proper* with *domain* the open set. In the case at hand, V restricted to $\overline{U}$ is proper with domain U. Hence we can define $\operatorname{Ind}(V|U)$. We set $\operatorname{Ind}(C) = \operatorname{Ind}(V|U)$.

Next we define $\operatorname{Ind}(V)$ with domain U to be equal to the index of $V|M$ where $M \subset U$ is a smooth compact manifold with boundary containing the defects of V in its interior. We can find such an M since the defects are a compact set in U.

We call a vector field V defined on a compact manifold M *proper with respect to the boundary* if there are no defects on the boundary. Consider the open set of the boundary where V points inside. We denote that set by $\partial_- M$. We define the vector field $\partial_- V$ with domain $\partial_- M$ in the arena ∂M by letting $\partial_- V$ be the end product of first restricting V to the boundary and then projecting each vector so that it is tangent to ∂M which results in a vector field ∂V tangent to ∂M, and then finally restricting ∂V to $\partial_- M$ to get $\partial_- V$. Then we define $\operatorname{Ind}(V)$ by the equation

$$\operatorname{Ind}(V) = \chi(M) - \operatorname{Ind}(\partial_- V) \tag{*}$$

where $\chi(M)$ denotes the Euler-Poincare number of M. We know that $\partial_- V$ is a proper vector field with domain $\partial_- M$ since the set of defects is compact unless there is a defect at the the frontier of $\partial_- M$. If there were such a defect, it would be a zero of V tangent to ∂M and hence a zero of V on the boundary, so V would not have been proper.

Now $\partial_- V$ is a proper vector field with domain the open set $\partial_- M$ which is one dimension lower than M. Then $\operatorname{Ind}(\partial_- V)$ is defined in turn by finding a compact manifold containing the defects of $\partial_- V$ and using equation (*). We continue this process until either $\partial_- M$ is a zero dimensional manifold where every point is a defect and so $\operatorname{Ind}(\partial_- V)$ is simply the number of points, or where $\partial_- M$ empty in which case $\operatorname{Ind}(\partial_- V) = 0$.

To summarize, we define the index of a proper vector field V with domain U assuming that the index for vector fields is already defined for compact manifolds with boundary. Then the index of V is defined to be the index of V restricted to a compact smooth manifold with boundary of codimension zero containing all the defects of V in U. In [G–S] it is shown that this definition is well-defined, that is it does not depend on the chosen manifold with boundary, by showing that a vector field with no defects defined on a compact manifold with boundary has index zero.

5. Vertical Vector Fields

Let $F \to E \xrightarrow{p} B$ be a fibre bundle whose fibre F is a smooth manifold and whose structure group is the group of diffeomorphisms of F. Then we have a vector bundle α over B of vectors tangent to the fibres. That is $\alpha|F =$ tangent bundle of F. A vertical vector field V on E is an assignment to a point e of E a vector in α at e. V might be empty or it might be defined on part of E. A more precise way to express this is that $V : S \to \alpha$ is a cross-section from some subset of S into α.

If $B = I$, the unit interval, then V is called an otopy. If V is an otopy which is continuous and defined over all of E, then V is called a homotopy.

A vertical vector field V is *proper* if $D \cap p^{-1}(C)$ is compact for all compact subsets C of B where D is the set of defects of V.

A vertical vector field V is *proper with respect to an open set* $U \subset E$ if $(D \cap U) \cap p^{-1}(C)$ is compact for all compact C in B and if V can be extended continuously over the frontier $\overline{U} - U$ so that there are no zeros on $\overline{U} - U$.

If $F \to E \xrightarrow{p} B$ is a fibre bundle so that F has boundary $\dot{F}$, we say a vertical vector field V is *proper with respect to the boundary* if $D \cap \dot{E} = \emptyset$ where $\dot{E} \subset E$ is the set of points in E on the boundary of some fibre.

Note that V proper with respect to the boundary implies that V is proper with respect to the open set $E - \dot{E}$.

The above definitions restrict to the concepts of proper homotopy and proper otopy.

In each of those three cases we can only define the index for certain nice situations: The component of defects must be compact and isolated from the other components; the boundary ∂M of the compact manifold M must contain no defects; the open set U has no defects on its frontier and the set of defects is compact in U. We use the term *proper vector field with respect to the boundary* in the compact manifold case and *proper vector field with respect to* U in the open set case to mean vector fields satisfying the appropriate conditions so that the appropriate index can be defined.

Now in each of these situations there is a notion of *proper otopy* under which the three forms of the index are preserved. These naturally generalize to the concepts of Proper Vertical Vector Fields.

If W is a vertical vector field, and if V is the restriction of W to a fixed fibre, then we say the defects of V *interact* via W if they are contained in a connected set of defects of W. An important class of questions is the following. If $F \to E \xrightarrow{p} B$ is a fibre bundle and V is a vector field on a fibre F, is it possible to extend V to a vertical vector field W so that certain defects of V do not interact, or so that the defects of W satisfy some condition such as they are compact?

We can phrase the question, "Are vector fields V_0 and V_1 homotopic?" in terms of the extension question. Let V_1 and V_2 be vector fields on a manifold F. Let $\pi\colon S^1 \times$

$F \xrightarrow{p\times 1} S^1 \times F \xrightarrow{Pr} S^1$ be the composition where $p: S^1 \to S^1$ is the double covering. So $\pi: S^1 \times F \to S^1$ is a fibre bundle with fibre $F \cup F$, two disjoint copies of F. Consider the vector field on $F \cup F$ given by V_0 on one copy of F and V_1 on the second copy of F. Then any homotopy V_t gives rise to a vertical vector W which extends $V_0 \cup V_1$, and conversely any vertical vector field gives rise to a homotopy from V_0 to V_1.

The extension of vector fields with nonzero indices puts strong conditions on the homology of the fibre bundle as the following results from [B–G] show:

Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a smooth fibre bundle with F a compact manifold with boundary ∂F and B a finite complex. Let V be a proper vertical vector field defined on an open set of E. We assume that V has no zeros on $\dot{E}$. We will call such vector fields *vertical* vector fields.

In [B–G], we defined an S-map $\tau_V : B^+ \to E^+$ associated with V. This transfer τ_V has the usual properties:

a) If V is homotopic to a vertical vector field V' by a homotopy of vertical vector fields, so in particular no zeros appear on $\dot{E}$, then $\tau_{V'}$ is homotopic to τ_V.

b) $\tau_V^*(p^*\alpha \cup \beta) = \alpha \cup \tau_V^*(\beta)$ for cohomology theories h^* with cup products.

c) For ordinary homology or cohomology, $p_* \circ \tau_{V_*}$ and $\tau_V^* \circ p^*$ is multiplication by the index of V restricted to a fibre F, denoted $\mathrm{Ind}(V|F)$.

Also in [B–G] the following theorem is shown. Given fibre bundle $F \xrightarrow{i} E \xrightarrow{p} B$ where V is a vertical vector field, that

$$0 = \mathrm{Ind}(V|F)\omega_* : \{X, \Omega B\} \to \{X, F\}$$

is trivial. Here we assume that X is a finite complex, $\Omega B \xrightarrow{\omega} F$ is the transgression map induced by the fibre bundle, $\{X, Y\}$ denotes the group of stable homotopy classes from X to Y.

It follows that

$$0 = \mathrm{Ind}(V|F)\omega_* : H_*(\Omega B) \to H_*(F)$$

Vertical vector fields can be constructed from equivariant vector fields on a manifold with a group action. Thus if V is a G-vector field on a manifold M, then the trace of the action (G, M), denoted by $\mathrm{tr}(G, M)$ and defined in [G2], must divide $\mathrm{Ind}\, V$. In symbols,

$$\mathrm{tr}(G, M) | \,\mathrm{Ind}(V).$$

6. One Dimensional Examples of Vector Fields

An 0–dimensional manifold consists of a discrete set of points. A vector field on this manifold consists of zero vectors. To each zero we assign the index 1.

Now consider the real line $\mathbb{R}$. This is a 1–dimensional manifold. We will look at vector fields on $\mathbb{R}$ carefully even though the index is fairly easy. There are many issues which can be explained clearly in one dimension.

Let V be a continuous vector field defined on all of $\mathbb{R}$. We would draw a picture as follows.

Figure 0

Now this is a cluttered picture, so let us adopt the convention that a vector pointing upward represents a vector of the same length pointing to the right and a vector pointing downwards represents a vector of the same length pointing to the left. Thus if V is the vector field so that $V(x)$ is a one dimensional vector field $x^2 - 1$ based at the point x on $\mathbb{R}$, then the pictorial representation described above is

Figure 1

Now notice that there is a zero of V at $x = -1$ and $x = 1$. We assign the index -1 to the zero at $x = -1$ and the index $+1$ to the zero at $x = 1$. We say the index of V is $-1 + 1 = 0$.

The rule for assigning indices is the following. Find a small interval about the zero containing no other zeros. If both vectors on the boundary of the interval are pointing inside, then assign the index -1, if both vectors are pointing outside, assign the value 1, and if one vector is pointing inside and one is pointing outside assign the index 0.

Figure 2

Exercise: Relate this definition to the definition in section 4.

Note that the definition of index based on pointing inside or outside is independent of orientation. Now the reader will be tempted to define the index by looking at the graph and noticing that it is -1 at a decreasing zero and $+1$ at an increasing zero and 0 at a tangent zero. But suppose we change our conventions, quite literally changing the orientation of $\mathbb{R}$, by representing left pointing vectors by upward pointing vector and right pointing by downward pointing. In this case, figure 1 becomes

Figure 3

Now the official definition still gives the zero at $x = -1$ the index -1 and the zero at $x = 1$ the index $+1$, whereas the tempted reader's definition changes the sign of the indices.

There is a lesson to be learned here. The definition which leads to the more easily computable concept is not necessarily the best definition. The fact that the official definition is invariant under choices of orientation will more than make up for the fact that it is harder for humans to compute with. There is another thing we should mention here. Why couldn't we have chosen the index to be $+1$ when the vectors point inside? Keep this question in mind when you look at the properties of the index in section 7.

Now consider what happens if V changes in time. This is called a *homotopy* and is frequently denoted V_t. Consider the homotopy $V_t(x) = x^2 + t$ where t runs from -1 to 1. Then $V_{-1}(x)$ is our example, and $V_0(x) = x^2$ has only one zero at the origin of index 0, and $V_1(x) = x^2 + 1$ has no zeros whatsoever.

Figure 4

As t runs from -1 to 1, the zeros at $x = -1$ and $x = 1$ move toward each and finally collide in a zero of index zero at the origin at $t = 0$. The zero then disappears for $0 < t$. It is as if an electron of charge -1 and a positron of charge $+1$ collide and annihilate each other.

We can represent this movement of zeros by another type of diagram

Figure 5

The vertical lines represent the vector fields V_{-1}, V_0, V_1 on $\mathbb{R}$. If we graph the paths of the zeros we get

Figure 6

We say the zeros at -1 and 1 *interact* under the homotopy V_t since the set of zeros in $[-1,1] \times \mathbb{R}$ is connected.

Now consider the projection $[-1,1] \times \mathbb{R} \xrightarrow{p} [-1,1]$. Here p is a trivial example of a fibre bundle with fibre $\mathbb{R}$. The homotopy V_t can be thought of as a vector field on the total space $E = [-1,1] \times \mathbb{R}$ which is tangent to the fibres $F = \mathbb{R}$. Such a vector field will be called a *vertical vector field* on E. Vertical vector fields are a generalization of homotopy.

Consider again the example $V(x) = x^2 - 1$ on $\mathbb{R}$. If $I = [a,b]$ is a closed interval, we define the $\mathrm{Ind}_I V$ on $[a,b]$ by the same rules we used to define the index of a zero. Namely: $\mathrm{Ind}_I V = -1$ if the vectors at a and b are pointing outward, $\mathrm{Ind}_I V = -1$ if the vectors are pointing inward and $\mathrm{Ind}_I V = 0$ if one vector is pointing outward and one vector is pointing inward. If there is a zero at either a or b, then $\mathrm{Ind}_I V$ is not defined.

Exercise: What is $\mathrm{Ind}_I V$ when $I = [-2,2]?,\ [-2,0]?,\ [-2,-1]?$

From the exercise you may note that $\mathrm{Ind}_I V$ is the sum of the indices of the zeros contained in I. This is true in general. This sum equation forces the conservation of indices under a homotopy. For example, notice that $\mathrm{Ind}_I V_t = 0$ for all $t \in [-1,1]$. Hence the sum of the indices of the zeros of V_t inside $[-2,2]$ must equal zero. But $\mathrm{Ind}_I V_t = -1$ when $I = [-2,0]$ until $t = 0$ when it is undefined, and then for $t > 0$ we see that $\mathrm{Ind}_I V_t = 0$. So the change of the index occurred when a zero migrated across the boundary of I.

We say a homotopy V_t on a closed interval is *proper* if $\mathrm{Ind}_I V_t$ is always defined for each t. That is, no zeros appear on the boundary of I. Thus $\mathrm{Ind}_I V_t$ is constant during a proper homotopy.

Exercise: Show that if $\mathrm{Ind}_I V_0 = \mathrm{Ind}_I V_1 \neq 0$, then V_0 is properly homotopic, with respect to the boundary, to V_1. Show this is not the case for $\mathrm{Ind}_I V_0 = \mathrm{Ind}_I V_1 = 0$.

Now suppose we have a vector field on $\mathbb{R}$ whose zeros are not isolated. An example is given in Figure 7.

Figure 7

The indices of the connected components are written under the component. The rule is the following. Find a closed interval I around a connected component of the zeros so that there are no other zeros in I and no zeros on the boundary of I. Then define the index of the zero component to be $\mathrm{Ind}_I V$.

Everything above applies to these connected sets of zeros, except for two difficulties. There are two possible ways for $\mathrm{Ind}(C)$, where C is a connected set of zeros, not to be defined. First and by far the most important, $\mathrm{Ind}(C)$ is not defined if C is not compact. For then it is impossible to find a compact interval I which contains C. The second way is if C is not isolated. That is, it is impossible to find a closed interval I which contains C and no other zero.

The following is an example of such a C. Let $V(x) = x \sin \frac{1}{x}$ for $x \neq 0$ and $V(0) = 0$.

Figure 8

Then $\mathrm{Ind}(0)$ is not defined. Prove that $\mathrm{Ind}_I V = 0$ if $I = [-2, 2]$. So although $\mathrm{Ind}(0)$ is not defined and there are an infinite number of zeros in I, any small change of V which results in a finite number of zeros will have the sum of their indices equal to 0.

Now the reader might think that the index of a zero is an intersection number. In fact, this is a fruitful way to look at Index for continuous vector fields everywhere defined. But we can make the same definitions for vector fields which are not defined everywhere, or are discontinuous. We say that a *defect* of a vector field V is a point where the vector field has a zero, or is not defined, or is discontinuous, or is a limit point of those kinds of points. Thus the set of defects D is a closed set.

A one–dimensional example with the indices written under the defects is shown in the next figure.

Figure 9

The indices of a compact connected component C of D are defined as before. Find a closed interval I containing C so that there are no defects on the boundary and no other defects of V inside I. Then the index $\text{Ind}(C) = \text{Ind}_I V$.

Consider the vector field, for example, $V(x) = \ln(x)$. This is represented in the following figure.

Figure 10

Then $V(x)$ is not defined for $x \leq 0$. Also there is a zero of index 1 at $x = 1$. So there are two connected components of the defect set D. The component $(-\infty, 1]$ has undefined index since it is not compact and the zero at $\{1\}$ has index 1.

Consider the vector homotopy $V_t(x) = \ln x + \ln(t - x)$. Then for any t the components of D_t are $(-\infty, 0] \cup [t, \infty) \cup \left\{\frac{t-\sqrt{t^2-4}}{2}\right\} \cup \left\{\frac{t+\sqrt{t^2-4}}{2}\right\}$. The index is undefined for the first two components, it is $+1$ for $\left\{\frac{t-\sqrt{t^2-4}}{2}\right\}$ and -1 for $\left\{\frac{t+\sqrt{t^2-4}}{2}\right\}$. At $t = 2$ the two zeros coalesce to a zero at $x = 1$ of index 0. For $t < 2$ there are only the two rays. They coalesce at $t = 0$.

The following picture, drawn with the same conventions as in Figure 6, shows how the zeros interact.

Figure 11

Thus as t runs from 0 to infinity we see that the undefined component which is all of $\mathbb{R}$ at $t = 0$ splits into two undefined components. At $t = 2$, a zero at $x = 1$ appears with index zero and then splits into a zero of index -1, which travels in the positive direction, and a zero of index $+1$ which travels in the negative direction. In this homotopy at $t = 4$ there are 4–path components. The two zeros interact and the two undefined components interact, but the zeros do not interact with undefined components.

Note that the homotopy is not a homotopy since the vector field is not defined everywhere. We will call it an *otopy*. Similarly as before, if we regard Figure 11 as a fibre bundle, then the otopy is a vertical vector field which is not necessarily defined everywhere on the fibre.

Note the vertical vector field described in Figure 11 is not proper since the set of defects is not compact. Let U be the open set formed by the wedge $\{(t, x)|0 < x < t\}$. Then V with domain U is not a proper vertical vector field since V cannot extend continuously to the point, of the wedge at $t = 0$. If $U' = U - (0, \frac{1}{10}] \times \mathbb{R}$, then V with domain U' is a proper vertical vector field with respect to U' .

Consider the open set $U'' = U - (0, 2] \times \mathbb{R}$. Now V with respect to U'' is not proper since the defects inside U'' are not compact over $[2, 3]$. Note that for the subfibre bundle $(2, \infty) \times \mathbb{R} \to (2, \infty)$ that V with respect to U'' is proper. For the fibre bundle $[2, \infty) \times \mathbb{R} \to [2, \infty)$, we see that V with respect to U'' is not proper, but V with respect to $U''' = U - (0, 2] \times \mathbb{R}$ is proper.

A useful example to keep in mind of a vertical vector field which is not proper is the following. Let $E = I \times \mathbb{R}$ and $B = I$ with $p : I \times \mathbb{R} \to I$ the projection, so $F = \mathbb{R}$. Let V be the constant vector field of unit length pointing in the positive direction on $\mathbb{R}$. Let $V(t, s) = t \times V$ if $0 \leq t \leq \frac{1}{2}$ and $V(t, s) = t \times (-V)$ if $\frac{1}{2} < t \leq 1$. Then the set of defects D is $\{(\frac{1}{2}, s)|$ all $s \in \mathbb{R}\}$. Note that V has no defects when restricted to the fibre over $t = \frac{1}{2}$.

figure 12

Now let us consider the non-trivial 1–dimensional bundle $I \to M \to S^1$ where M is the Mobius Band. As an exercise, show that there is a proper vertical vector field with respect to the boundary which restricts to the fibre with index $+1$ and index -1, but there is none which restricts to index 0. On the other hand, show that there is a vertical vector field proper with respect to some open set which does restrict to a fibre with index 0.

7. Properties of Index

The Law of Vector Fields is the following: Let M be a compact smooth manifold and let V be a vector field on M so that $V(m) \neq \vec{0}$ for all m on the boundary ∂M of M. Then ∂M contains an open set $\partial_- M$ which consists of all $m \in \partial M$ so that $V(m)$ points inside. We define a vector field, denoted $\partial_- V$ on $\partial_- M$, so that for every $m \in \partial_- M$ we have $\partial_- V(m) =$ Projection of $V(m)$ tangent to $\partial_- M$. Under these conditions we have

$$\text{Ind } V + \text{Ind } \partial_- V = \chi(M) \tag{1}$$

where Ind V is the index of the vector field and $\chi(M)$ is the Euler characteristic of M. ([M], [G_3-G_6], [P]).

The Law of Vector Fields can be used to define the index of vector fields, so the whole of index theory follows from (1). The definition of index is not difficult, but proving it is well-defined is a little involved, [G–S]. The definition proceeds as follows:

a) The index of an empty vector field is zero.

b) If M is a finite set of points and V is defined on all of the M (the vectors are necessarily zero), then $\text{Ind}(V) =$ number of points in M.

c) If V is a *proper* vector field on a compact M, by which we mean V has no defects on ∂M, then we set

$$\text{Ind } V = \chi(M) - \text{Ind}(\partial_- V).$$

d) If V is defined on the closure of an open subset U of a smooth manifold M so that the set of defects D is compact and $D \subset U$, then we say V is a *proper vector* field. The index Ind V is defined to be $\text{Ind}(V|M)$ where M is any compact manifold such that $D \subset M \subset U$.

e) If C is a connected component of D and C is compact and open in D define $\text{Ind}_C(V)$ to be the index of V restricted to an open set containing C and no other defects of V.

A key idea in proving this definition is well-defined is a generalization of the concept of homotopy which we call otopy. An *otopy* is what $\partial_- V$ undergoes when V undergoes a homotopy. The formal definition is as follows: An *otopy* is a vector field V defined on the closure of an open set $T \subset M \times I$ so that $V(m,t)$ is tangent to the slice $M \times t$. The otopy is *proper* if the set of zeros Z of V is compact and contained in T. The restriction of V to $M \times 0$ and $M \times 1$ are said to be properly otopic vector fields. Proper otopy is an equivalence relation.

The following properties hold for the index:

(2) Let M be a connected manifold. The proper otopy classes of proper vector fields on M are in one to one correspondence via the index to the integers. If M is a compact manifold with a connected boundary, then a vector field V is properly homotopic to W if and only if $\text{Ind } V = \text{Ind } W$.

(3) $\text{Ind}(V|A \cup B) = \text{Ind}(V|A) + \text{Ind}(V|B) - \text{Ind}(V|A \cap B)$

(4) $\text{Ind}(V \times W) = \text{Ind}(V) \cdot \text{Ind}(W)$

(5) $\text{Ind}(-V) = (-1)^{\dim M}(V)$

(6) If V has no zeros, then $\text{Ind}(V) = 0$

(7) $\text{Ind}(V) = \sum_C \text{Ind}_C(V)$ for all compact connected components C, assuming Z is the union of a finite number of compact connected components.

For certain vector fields the index is equal to classical invariants. Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$. Let M be a compact n submanifold. Define V^f by $V^f(m) = f(m)$. If $f : \partial M \to \mathbb{R}^n - \vec{0}$, then

(8) $\text{Ind } V^f = \deg f$.

Suppose $f : U \to \mathbb{R}^n$ where U is an open set of $\mathbb{R}^n$. Let $V_f(m) = \vec{m} - \overrightarrow{f(m)}$. Then

(9) $\text{Ind } V_f =$ fixed point index of f on U.

Suppose $f : M \to N$ is a smooth map between two Riemannian manifolds. Let V be a vector field on N. Let f^*V be the pullback of V on M. We define the pullback by

$$\langle f^*V(m), \vec{v}_m \rangle = \langle V(f(m)), f_*(\vec{v}_m) \rangle.$$

Note that for $f : M \to \mathbb{R}$ and $V = \frac{d}{dt}$, we have $f^*V =$ gradient f.

Now suppose that $f : M^n \to \mathbb{R}^n$ where M^n is compact and V is a vector field on $\mathbb{R}^n$ so that f has no singular points near ∂M and V has no zeros on $f(\partial M)$. Then if $n > 1$

(10) $\text{Ind } f^*V = \sum w_i v_i + (\chi(M) - \deg \hat{N})$

where $\hat{N} : \partial M \to S^{n-1}$ is the Gauss map defined by the immersion of ∂M if $\mathbb{R}^n$ under f, and $v_i = \mathrm{I}nd_{c_i}(V)$ for the i^{th} zero of V and w_i is the winding number of the i^{th} zero with respect to $f : \partial M \to \mathbb{R}^n$. The winding number is calculated by sending a ray out from the i^{th} zero and noting where it hits the immersed $n-1$ manifold ∂M. At each point of intersection the ray is either passing inward or outward relative to the outward pointing normal N. Add up these point assigning $+1$ if the ray is going from inside to outside, and -1 if the ray goes from outside to inside. This is the generalized Gauss- Bonnet theorem.

Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a smooth fibre bundle with F a compact manifold with boundary ∂F and B a finite complex. Let V be a proper vertical vector field defined on an open set of E. We assume that V has no zeros on $\dot{E}$. Then there is a transfer so that on ordinary homology we have

$$p_* \circ \tau_{V_*} = \mathrm{I}nd(V|F) \tag{11}$$

Exercise: The Euler–Poincare number.

Using the property that the Euler–Poincare number is an invariant of homotopy type and the above properties of the index, prove the following useful properties of the Euler–Poincare number.

1. $\chi(M) = 1$ where M is contractible.
2. $\chi(M_1 \cup M_2) = \chi(M_1) + \chi(M_2) - \chi(M_1 \cap M_2)$ where M_1 and M_2 and $M_1 \cap M_2$ are submanifolds of $M_1 \cup M_2$.
3. $\chi(M_1 \times M_2) = \chi(M_1) \times \chi(M_2)$
4. $\chi(M) = 0$ when M is a closed odd dimensional manifold.
5. $\chi(\partial M) = 2\chi(M)$ if M is even dimensional.

Exercises: Calculating the Index

Answer the three questions below for each problem and the given N, V, M, U.

Let V be a vector field on N.

a) What are the defects of V and what are their indices?
b) What is $\mathrm{Ind}_U V$ for the open set U?
c) What is $\mathrm{Ind}_M V$ for the compact manifold M?

1. Let $V = \nabla f$, $N = \mathbb{R}^3$ and $f\colon \mathbb{R}^3 \to \mathbb{R}$ where $f(x, y, z) = x^2 + 2y^2 + z^2$. Suppose $U = \mathbb{R}^3$ and $M = f^{-1}([0, 1])$
2. Let $V(m) = m \sin \frac{1}{m}$ if $m \neq 0$ and 0 if $m = 0$. Then V is a vector field on $N = \mathbb{R}^1$, $U = (-1, 1)$, $M = [-2\pi, 2\pi]$.
3. Let $N = \mathbb{R}^2$ and let V be the velocity vector field given by rotating $\mathbb{R}^2$ around origin. $U = \mathbb{R}^3$, $M = D^2$.
4. Let $N = \mathbb{R}^3$, let V be the velocity vector field given by rotating about the z–axis.

$U = \mathbb{R}^3,\ M = D^3.$

5. ∂V from problem 4 tangent to unit sphere $S^2 = N$.

   $U = S^2,\ M =$ Northern hemisphere.

6. $N = \mathbb{R}^3,\ V(m) = (-x^3,\ \sin y,\ z).$

   $U = \mathbb{R}^3,\ M = D^3.$

7. $N = \mathbb{R}^3,\ V =$ electrostatic vector field of two electrons, i.e. $V(m) = \frac{\tilde{r}_1 - \tilde{m}}{\|\tilde{r}_1 - \tilde{m}\|^3} + \frac{\tilde{r}_2 - \tilde{m}}{\|\tilde{r}_2 - \tilde{m}\|^3}.$

   $U = \mathbb{R}^3,\ M =$ Ball of radius $\frac{\|\tilde{r}_1 - \tilde{r}_2\|}{2}$ with center at $\tilde{r}_1$.

8. Same as problem 7 except $V(m) = \frac{\tilde{r}_1 - \tilde{m}}{\|\tilde{r}_1 - \tilde{m}\|^3} - \frac{\tilde{r}_2 - \tilde{m}}{\|\tilde{r}_2 - \tilde{m}\|^3}.$

9. The Evaluation Subgroup

THEOREM. *Let π be a group with a finite free resolution. Then if the Euler-Poincare number of the group is non-zero, the center of the group must be trivial.*

This theorem, named Gottlieb's Theorem by Stallings in [St], has been considerably generalized, first by Rossett and then by Cheeger and Gromov. Define $G(X) =$ Image ω_* where ω_* is induced by the evaluation map on the fundamental group.

The key lemma in the original proof is: If X is a compact CW-complex so that $\chi(X) \neq 0$, then $G(X)$ is trivial. Then if X is a $K(\pi, 1)$ we know that $G(X) =$ center of π. Thus we must show that if $F : X \times S^1 \to X$ so that $F|X \times *$ is the identity and if $\chi(X) \neq 0$, then $F|x \times S^1$ is homotopically trivial. The original proof used Nielsen-Wecken fixed point classes [G1]. These were transformed by Stallings [St] into an algebraic setting which is of importance algebraically. The present proof is more elementary and does not need Nielsen-Wecken fixed point theory.

Suppose M is a regular neighborhood of an embedding of X in some $\mathbb{R}^n$. Then we have a map $F : M \times I \to M$ so that $F(m, 0) = m$ and $F(m, 1) = m$ for all $m \in M$. We may adjust F so that $F(m, t) \neq m$ for all $m \in \partial M$ and all t, and so that F_0 and F_1 are the identity outside a small collar neighborhood of the boundary. Define the vector field T so that $T(m, t) = (\overrightarrow{m - F(m, t)}, t)$. Let Z be the zeros of T on $M \times I$. Now Z is compact and $Z \cap (\partial M \times I)$ is empty. Let U be the open set of $M \times I$ so that $\|T(m, t)\| < \epsilon$ where ϵ is so small that two paths $\alpha(t)$ and $\beta(t)$ on M are homotopic if the distance between $\alpha(t)$ and $\beta(t)$ is always less than ϵ. Such an ϵ exists since M is compact. Let W be a path component of U containing $M \times 0$. Then $T|W$ is a proper otopy from T_0 to T_1. Now Ind $T_0 =$ Ind T_1 and Ind $T_0 = \chi(M)$ and Ind $T_1 = \chi(M)$. So the path connected W contains $M \times 0$ and $M \times 1$.

We may find a path $\gamma : I \to U$ so that $\gamma(0) = * \times 0$ and $\gamma(1) = * \times 1$. We can homotopy $F : M \times S^1 \to M$ to a G such that $G(\gamma(t), t) = \gamma(t)$. Then $\gamma \sim \alpha \cdot \gamma$ where $\alpha(t) = G(*, t)$. Hence $\alpha \sim 0$, which was to be shown.

Exercises: Prove the following results using properties of the index. See [G7] for solutions.

1. Intermediate Value Theorem
2. Fundamental Theorem of Algebra
3. Jordan Curve Theorem
4. Any straight line is contained in a plane tangent at some point to a given surface of genus greater than 1.
5. Show that every vertical vector field proper with respect to some open set on the Hopf fibration $S^3 \xrightarrow{i} S^7 \xrightarrow{p} S^4$ has index equal to zero when restricted to a fibre.

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