A GRAVITATIONAL LENS NEED NOT PRODUCE AN ODD NUMBER OF IMAGES

by

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Abstract

Given any space-time M without singularities and any event O, there is a natural continuous mapping f of a two dimensional sphere into any space-like slice T not containing O. The set of future null geodesics (or the set of past null geodesics) forms a 2-sphere S^2 and the map f sends a point in S^2 to the point in T which is the intersection of the corresponding geodesic with T. To require that f, which maps a two dimensional space into a three dimensional space, satisfy the condition that any point in the image of f has an odd number of preimages, is to place a very strong condition on f. This is exactly what happens in any case where the odd image theorem holds for a transparent gravitational lens. It is argued here that this condition on f is too restrictive to occur in general; and if it appears to hold in a specific example, then some f should be calculated either analytically or numerically to provide either an illustrative example or counterexample.

I. Introduction

Since 1979 astronomers have been looking for an odd number of images in gravitational lensing events. There have been many discoveries since the first event in 1979. In most cases *only* an even number of discrete images have been found. We assert that the topology and geometry of space-time seem to make it very unlikely that only odd numbers of images exist. Some of the topological arguments for an odd number of images are very persuasive, even though they are based on a Euclidean space-time. Hence the prevalence of even number images should be taken as another vindication of Lorentzian space-time as a model of the universe.

In 1980 C. C. Dyer and R. C. Roeder [1] predicted an odd number of images for a spherical symmetric transparent lens (i. e. Galaxy). In 1981 W. L. Burke [2] claimed that there must be an odd number of images for any bounded transparent lens subject to an assumption that the bending of light rays decreases as the light rays are far from the lens. The argument constructed a vector field on the plane of the lens and showed the index had to be one. So then, assuming the local index of each zero was ± 1, the number of zeros had to be odd to add up to the global index of 1. Each zero corresponds to light rays.

In 1985 R. H. McKenzie [3] wrote down an argument using the degree of a map between two 2 dimensional spheres which asserted that there were an odd number of images. This argument needed no assumptions on the amount of bending and obviously improved Burke's approach. This argument was widely known among astronomers and is very convincing. However it is done in 3-space and not in 4-dimensional space-time. McKenzie notes this and then provides an argument using Morse Theory on 4-dimensional space-time, applying correctly Karen Uhlenbeck's version of Morse Theory for Lorentzian Manifolds, [4]. It is widely believed today that the necessity of an odd number of images has been precisely established and that the contradictory evidence is a result of difficulties of finding the third image, [5], although on page 176 of [6] they state that McKenzie's conditions are physically obscure.

In this paper we translate the degree argument directly into 4-dimensional space-time and we see that an extremely restrictive condition must be true of the space-time in order to obtain the odd images conclusion. The condition is that the pencil of past null-geodesics from the observer must intersect every past space-like slice in a 2-sphere.

We give two examples of 4-dimensional Lorentzian Manifolds for which this condition is false. The first one because of the topology and the second one because of the geometry. Then we argue that the conditions under which McKenzie's Morse Theory argument would apply are even more extremely restrictive.

II. Global lensing in Lorentzian space-time.

We reproduce the topological argument given by McKenzie on page 1592 of [3] which establishes the odd image result for Euclidean space. Then we try to reproduce the argument in Lorentzian space-time.

"There is a relatively simple demonstration of why there are an odd number of images. Although it seems to be well known among astronomers it does not appear to have been published before and so is given here. Consider the situation shown in Fig. 1. A light source is located at S and an observer at O. There is a transparent galaxy G somewhere between S and O. A map f from the small sphere A to the sphere B is defined as follows. The map f maps a point x on A to the point on B where the light ray through O and x intersects B. The number of images of S seen by O is the number of points on A mapped onto S.

FIG. 1. **A galaxy G is located somewhere between a light source S and an observer O. Because of the gravitational field of the galaxy there may be more than one light ray from S to O. f maps the sphere A onto the sphere B. If x is on A then $f(x)$ is defined to be the point on B where the ray through O and x intersects B.**

Suppose $g : M \to N$ is a smooth map between manifolds of the same dimension and that M is compact. If y is a regular value of g then we define

$$\deg(g, y) = \sum_{x \in g^{-1}(y)} sgn\ dg_x,$$

where $sgn\ dg_x = +1(-1)$ if $dg_x : T_x(M) \to T_y(N)$ preserves (reverses) orientations. It turns out that $\deg(g, y)$ is the same for all regular y; it is called the degree of g and denoted $\deg(g)$.

In an actual physical situation it is reasonable to assume that there will be a point y on B such that $f^{-1}(y)$ is a single point, i. e. there is only one ray from O to y. Thus, $\deg(f) = 1$.

Let $n_+(n_-)$ be the number of points x in $f^{-1}(S)$ such that $sgn\ df_x = +1(-1)$. Thus, $n_+(n_-)$ is the number of images of S, seen by O, which have the same (opposite) orientation as the source, and

$$n_+ - n_- = \deg(f, S) = \deg(f) = 1.$$

Thus, if O sees $n = n_+ - n_-$ images of S then $n = 2n$, and so n is odd, and the demonstration is complete."

Now we consder this argument in Lorentzian space-time, M. We consider every past directed null geodesics eminating from the observer O. There is one geodesic for each point in the celestial sphere A. (More precisely, take a "unit" sphere in the past null cone of O in the tangent space to O. Then there is a unique past directed null line for each point of the sphere and the exponential map maps this line onto a past directed geodesic.) Now let T be a space-like slice containing the source S. Then it is natural to assume that each geodesic intersects T exactly once. (If this assumption does not hold it makes the odd image "theorem" even more dubious.) So we can define a map $f : S^2 \to T$. Now what corresponds to the sphere B? It must be the image $f(S^2)$ of S^2 in the three dimensional manifold T. It seems unlikely that $f(S^2)$ would be a sphere if f is not injective. Only if $f(S^2)$ were a toplogical sphere would we be entitled to use the degree argument, otherwise it is invalid.

This would strike any differential topologist or geometer as obvious. It may be possible to construct such an M, but these M's would be quite special.

III. Two Examples.

We give two examples of space-times which do not have the property that pencils of null geodesics intersect space-like slices in 2-spheres. Many more examples can be constructed using Barrett O'Neill's book [7], Corollary 57 on page 89 and warped products on pages 207–209.

a) Let $M = S^1 \times S^1 \times S^1 \times \mathbb{R}$. The universal covering space is $\widetilde{M} = \mathbb{R}^4$. Let $\widetilde{M}$ be Minkowski space, so it has the Minkowski metric. It induces the same metric on M. The geodesics of $\widetilde{M}$ are straight lines and their images are the geodesics of M. Pencils of null geodesics do not intersect space-like slices in spheres in this M.

b) Let $M = \mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$. Let the second $\mathbb{R}^2$ have the Minkowski metric. We will put a Riemannian metric on first $\mathbb{R}^2$ and then we take the product metric. We note that a geodesic of the first $\mathbb{R}^2$ factor coupled with a time-like line in the second factor is a null geodesic in M, (i. e. if $\alpha : \mathbb{R} \to \mathbb{R}^2$ is a geodesic of the first factor and $\beta : \mathbb{R} \to \mathbb{R}^2$ is a time-like geodesic of the second factor with the same speed as α, then $\alpha \times \beta : \mathbb{R} \to \mathbb{R}^2 \times \mathbb{R}^2$ is a null geodesic of M). So if we produce an $\mathbb{R}^2$ so that the exponential map of geodesics eminating from a point x carries some circle in the tangent plane at x into a set in $\mathbb{R}^2$ which is not a circle, then the pencil of null geodesics intersecting a space-like slice in M is not a two sphere.

One can visualize a metric on $\mathbb{R}^2$ by embedding it as a surface T in Euclidean 3-space. The geodesics are characterized as those paths in T whose acceleration is orthogonal to the surface T. Now it is easy to construct examples with the desired property.

One that works is the following. Take an arc of a circle whose length exceeds a half circle. Extend the ends of this arc by the tangent lines at the ends of the arc. The lines intersect in a Point A. Now take a small interval perpendicular to the plane in which the curve just constructed, γ, lies. Move this interval along γ so that it is perpendicular to the plane over the arc and so that it lies in the plane along most of the two extended lines including their intersection A. The interval should be twisted in moving from the ends of the arc so that the interval sweeps out a smooth surface with two boundary components. Then extend this "old fashioned men's collar" to a surface T in $\mathbb{R}^3$.

Let O be the midpoint of the circular arc on T. Then the geodesics of fixed length greater than OA on T near γ clearly do not end in a circle.

FIG. 2

We can adjust this example so that the nonflat part of $T \times \mathbb{R}$ is bounded in any space-line slice $T \times \mathbb{R} \times s \subset \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} = M$. The technique for the adjustment is the warped product construction, which can be found in [7].

IV. Morse Theory.

R. H. McKensie in [3] "proves" the odd image result by applying Uhlenbeck's version of Morse theory of Lorentzian manifolds [4]. The relevant theorems are Theorem 4 (which he calls the local theorem) and Theorem 5 (the global theorem). The global theorem is less relevant to the study of gravitational lensing then the local theorem according to McKensie. This is the case both for practical considerations of how observations are made, and because the hypotheses of the global theorem do not hold in realistic space-time models.

The statement of the local theorem is difficult to understand since McKensie does not make clear how the points q and r and set B are related to the history of the source T and the observer p. The most reasonable interpretation is that $\Omega(T,p)^c$ is a deformation retract of $\Omega(T,p)$ which he assumes is contractible. This is a wordy way of assuming that $\Omega(T,p)^c$ is contractible. Now as c varies, $\Omega(T,p)^c$ will not be contractible in general since every time c passes through a critical value of T, the topology of $\Omega(T,p)^c$ is altered by attaching a cell (which corresponds to a new geodesic from T to p). But it is impossible to attach only one cell to a contractible space and still have it be contractible. Thus for "most" c the hypothesis is not true unless there are pairs of geodesics from T to p for each critical value for c.

V. Discussion.

The fact that there is no odd image theorem shows that even for very mild lensing the Euclidean approximation is fundamentally wrong. As a general rule, every Euclidean argument should be looked at in Lorentzian space-time to see if it can be reproduced there in principle. If the argument cannot be reproduced, then its consequences are a test for the Lorentzian model of the universe.

The argument does not really depend on mapping the entire two sphere into a spacelike slice. If a portion of the light cone two-sphere is mapped into a space-like slice, it would give rise, with the most usual choices, to a mapping f of a plane into three space. The requirement that the mapping have only an odd number of points in each coincidence is still a strong condition. For convincing special cases where it appears the odd images theorem should hold, the mapping f should be calculated, either analytically or by ray tracing with a computer. Then if the odd image property is true for that f, we may be able to infer properties for such mappings in odd imaging cases.

References

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FIG. 2