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Skew Symmetric Bundle Maps on Space-time

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Abstract. We study the "Lie Algebra" of the group of Gauge Transformations of Space-time. We obtain topological invariants arising from this Lie Algebra. Our methods give us fresh mathematical points of view on Lorentz Transformations, orientation conventions, the Doppler shift, Pauli matrices, Electro-Magnetic Duality Rotation, Poynting vectors, and the Energy Momentum Tensor T .

1. Introduction

Let M be a space-time and $T(M)$ its tangent bundle. Thus M is a 4-dimensional manifold with a nondegenerate inner product $\langle\ ,\rangle$ on $T(M)$ of index $-+++$. We study the space of bundle maps $F : T(M) \to T(M)$ which are skew symmetric with respect to the metric, i.e. $\langle Fv, v\rangle = 0$ for all $v \in T_x(M)$ and all $x \in M$.

A skew symmetric F has invariant planes and eigenvector lines in each $T_x(M)$. We give necessary and sufficient conditions as to when these plane systems and line systems form subbundles in Theorem 7.3. Also we determine the space of those F which give the same underlying structure. This is done by introducing the bundle map $T_F = F \circ F - \frac{1}{4}(tr\ F^2)I : T(M) \to T(M)$. Then the space of skew symmetric F which give rise to the same T is homeomorphic to $\mathrm{Map}(M, S^1)$, the space of maps of M into the circle S^1. (See Theorem 7.11.)

We also show that the space of skew symmetric F has a natural complexification. (see Propositions 2.2 and 2.3) This leads to an equivalence between the F and vector fields on the complexified tangent bundle $T(M) \otimes \mathbb{C}$. The complexified study leads to several beautiful relations which link our subject matter to Clifford Algebras and Quaternions. (See Corollaries 4.6 and 4.7 and Theorem 4.8.) We naturally find many points of contact with Physics, especially classical electromagnetism. These considerations frequently govern our choice of notation. The physical motivations and remarks will be explored in the Scholia; and the mathematical motivations and links will be found in the Remarks.

Scholium 1.1. Physical connections.

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Key words and phrases. Electro-magnetism, energy-momentum, vector bundles, Lorentz Transformations, Clifford Algebras, Doppler shift.

a) Each skew symmetric F corresponds to a two-form $\widehat{F}$. The electro-magnetic tensor is a two form. In the classical theory it satisfies Maxwell's equations. The symmetric bundle map T_F corresponds to the energy-momentum tensor of the electro-magnetic field. The homotopy invariants arising from the existence of sub-bundles must give physical information if there is any physical content in Classical Electro-Magnetism. We show that the invariants distinguish the two main cases; a classical free electron and a classical electron in a magnetic field.

b) We give formulas in terms of $\mathbf{E}$ and $\mathbf{B}$ for the eigenvectors of F. Changing observers gives the same eigenvector multiplied by a factor. For "radiative" F, this factor reduces to the Doppler shift. One wonders if the more general shift for non-radiative F has any physical meaning.

c) The space of skew symmetric F has a canonical splitting of space and time. It is mapped isomorphically onto $T(M) \otimes \mathbb{C}$ by a choice of a field of observers. Thus any complex tangent vector field corresponds to a skew-symmetric F. So, for example, if the solutions of the Dirac equation have any physical content, then the homotopy invariants of the corresponding F must have physical import.

Remark 1.2. Mathematical Motivation.

The mathematical point of view of this work stems from the author's study of the space of bundle equivalences in [$\mathbf{G}_1$], [$\mathbf{G}_2$], [$\mathbf{G}_3$]. These bundle equivalences form spaces which later became popular known as groups of gauge transformations. The main result of these papers is that the classifying space of these groups of gauge transformations is the space of maps of the base space into the classifying space of the fibration in question.

This theorem has played an important role, at least in the mathematical part of of Gauge Theory. It entered into the theory via Proposition 2.4 of [**AB**]. But the point of view of these works concerned spaces of connections, instead of spaces of bundle equivalences. The original point of view was furthered in papers by Booth, Heath and Piccinini among others, see for example [**BP**].

In this present work, we study other types of bundle maps. The "Lie Algebras" of "Gauge Transformation Groups" seems to be a natural class to study. The skew-symmetric bundle maps of space-time are the "Lie Algebra" of the group of isometries on $T(M)$, i.e. bundle maps $Q : T(M) \to T(M)$ so that $\langle Qv, Qw \rangle = \langle v, w \rangle$.

Scholium 1.3. Physical Point of View.

Galileo's famous quote that the Laws of Nature are written in the language of geometry should be revised in view of the development of Topology in this century. As topology underlies geometry, one would expect that some Laws of Nature would be expressed in terms of the elementary homotopy invariants of topology. Among these are the degrees of maps and the index of vector fields.

Our method for discovering these laws follows Galileo. To the argument that no one had seen an object travel at a constant velocity forever along a straight line, Galileo replies: Let us assume it is true, derive its mathematical consequences, and see if they relate to what is observed. Thus we begin by studying infinitesimal rigid motions F on space-time M, and observe connections with electromagnetism, etc. The idea of separating the physical from the mathematical arguments via Scholia is borrowed from Newton's Principia.

Remark 1.4. Levels of notation.

We proceed by adding layers of notation to our space-time. We descend one level for every choice we estimate we make. We begin at Level –1 with the inner product and continue by choosing an orientation at Level –2. By Level –10 we have chosen an orthonormal basis for the tangent space of M. We eventually end at Level –16, which are the standard coordinates for Minkowski Space.

This approach permits us to understand that choosing an orientation is like taking a complex conjugate. It also allows a clear view of Lorentz Transformations at the various levels. The major technique of computation in this paper is given by a Level –10 block matrix which allows Level –10 calculations to produce Level -2 statements.

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I have had very productive conversations with Barrett O'Neill, Stephen Parrot, and Solomon Gartenhaus. Gartenhaus gave me a key example which forced me to think more deeply at the beginning of this work. Barrett O'Neill gave me many ideas. The best one is the definition of the complexification map c. Barrett O'Neill's book Semi-Riemannian Manifolds [**O**] exposes space-time in a rigorous mathematical manner. Stephen Parrott's book [**P**] provided great stimulation and guidance.

2. Notation and Preliminaries

A *space-time* M is a smooth 4-dimensional orientable manifold with a Lorentzian metric $\langle\ ,\ \rangle$ defined on the tangent bundle $T(M)$ and a nonzero future pointing time-like vector field. If $x \in M$, then $T_x = T_x(M)$ will denote the 4-dimensional tangent space over x. The space of vectors orthogonal to a vector $u \in T_x$ will be denoted by T_x^u.

A skew symmetric bundle map is a map $F : T(M) \to T(M)$ which covers the identity on the base, is a vector bundle map, and is skew symmetric that is,

$$F(\alpha v_x + \beta w_x) = \alpha F(v_x) + \beta F(w_x) \in T_x \tag{1}$$

$$\text{and } \langle F(v_x), w_x\rangle = \langle v_x, -F(w_x)\rangle. \tag{2}$$

Let ℓ be the vector bundle over M whose fibre ℓ_x is the vector space of skew symmetric linear transformation $F_x : T_x \to T_x$. Then the space of cross-sections $\Gamma(\ell)$ to ℓ corresponds to the space of bundle maps in the usual manner. Let $\Lambda^2(M)$ be the bundle of two forms over M. Thus the fibre $\Lambda^2(M)_x$ are bilinear antisymmetric maps $\widehat{F}_x : T_x \times T_x \to \mathbb{R}$. Any two-form is a cross-section to $\Lambda^2(M)$.

Now ℓ is bundle equivalent to $\Lambda^2(M)$. Let $\rho : \ell \to \Lambda^2(M)$ so that $\rho(F_x) = \widehat{F}_x$ where

$$\widehat{F}_x(v_x, w_x) = \langle v_x, F_x(w_x)\rangle. \tag{3}$$

The non-degeneracy of $\langle\ ,\ \rangle$ implies that ρ is an isomorphism on each fibre, thus ℓ sets up a bijection between two-forms and bundle maps.

Level -1. Lorentz inner product.

Notation plays an important role in Mathematics and Physics. It is a powerful aid to calculation. But notation can blur distinctions and confuse reasoning. For that reason we will introduce notation in Levels. Each improvement of notation is based on more and more choices. The above notation is called Level -1. As we add choices of frame fields and coordinates we descend eventually to Level -16, which is the canonical coordinates of Minkowski Space-time. The number describing the Levels approximates the number of choices made to introduce the notation. We have already made one choice in Level -1 by assuming that $\langle\ ,\ \rangle$ has signature $-+++$, we could have assumed signature $+---$. Level 0 then has innerproduct $\epsilon\langle\ ,\ \rangle$ where ϵ is ± 1. The geometry does not change with the change of ϵ. The geodesics remain the same and skew-symmetric bundle maps remain the same so the choice $-+++$ does not affect our work. But in comparing our results with other authors, be aware that the electro-dynamicists usually choose $+---$. Thus S. Parrott [**8**] chooses $+---$ where as O'Neill [**9**] chooses $-+++$.

Level -2. Orientation.

Since M is orientable, there is a volume form $\Omega \in \Lambda^4(M)$. There are two choices consistent with the metric, $\pm\Omega$. We choose Ω as the orientation. We could have chosen $-\Omega$. Now the Hodge dual is an isomorphism defined on $\Lambda^2(M)$, satisfying $*(*\eta) = -\eta$ for $\eta \in \Lambda^2(M)$. Under $\rho : \ell \to \Lambda^2(M)$ the Hodge dual corresponds to an operator $*$ on $\Gamma(\ell)$. It satisfies

$$(aF)^* = aF^* \quad \textit{and} \quad (F+G)^* = F^* + G^* \quad \textit{and} \quad F^{**} = -F. \tag{4}$$

Let $u \in T_x(M)$ be an *observer.* That is u is a future pointing time-like vector such that $\langle u, u\rangle = -1$. Then we define

$$\mathbf{E}_u = Fu \quad \textit{and} \quad \mathbf{B}_u = -F^*u. \tag{5}$$

Note that $\mathbf{E}_u$ and $\mathbf{B}_u \in T^u$. If we change the orientation, we obtain a new $*'$. This is related to the old $*$ by $F^{*'} = -F^*$. Thus for change of orientation, $\mathbf{E}_u$ remains the same, but $\mathbf{B}_u$ becomes $-\mathbf{B}_u$.

If v and w are space-like vectors in T_x, they span a space-like plane if and only they are linearly independent and

$$v^2w^2 - \langle \mathbf{v}, \mathbf{w}\rangle^2 > 0. \tag{6a}$$

If

$$v^2w^2 - \langle \mathbf{v}, \mathbf{w}\rangle^2 = 0, \tag{6b}$$

they span a light-like plane and if

$$v^2w^2 - \langle \mathbf{v}, \mathbf{w}\rangle^2 < 0 \tag{6c}$$

they span a space-like plane.

Let u be an observer. We define the dot product and cross product on T^u_m.

Definition 2.1. We define the dot product and the cross product in the space T^u_m. Let $\mathbf{v}$ and $\mathbf{w} \in T^u_m$. Define the dot product by

$$\mathbf{v}\cdot\mathbf{w} = \mathbf{v}\cdot_u \mathbf{w} = \langle \mathbf{v}, \mathbf{w}\rangle \tag{7}$$

Then $v^2 = \mathbf{v}\cdot\mathbf{v}$ and $\mathbf{v}\cdot\mathbf{w} = vw\cos\theta$ where θ is defined to be the angle between $\mathbf{v}$ and $\mathbf{w}$. Define the cross product in T^u_m by $\mathbf{v}\times\mathbf{w} = \mathbf{v}\times_u\mathbf{w} =$ the unique vector orthogonal to $\mathbf{v}$ and $\mathbf{w}$ in T^u_m of length $|vw\sin\theta|$ so that $\Omega(\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{v}\times\mathbf{w}) \geq 0$.

This cross product satisfies the usual relations:

$$\begin{aligned}
&\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v} \\
&\mathbf{v} \times (\alpha\mathbf{w} + \beta\mathbf{x}) = \alpha(\mathbf{v} \times \mathbf{w}) + \beta(\mathbf{v} \times \mathbf{x}) \\
&\mathbf{v} \times \mathbf{w} = \mathbf{0} \ \text{ if and only if } \alpha\mathbf{v} = \beta\mathbf{w} \\
&(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{w} \cdot \mathbf{u})\mathbf{v} - (\mathbf{w} \cdot \mathbf{v})\mathbf{u} \\
&\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = -\mathbf{v} \cdot (\mathbf{u} \times \mathbf{w}) \\
&\mathbf{v} \cdot (\mathbf{v} \times \mathbf{w}) = 0
\end{aligned} \tag{8}$$

We use F^* to impose a complex structure on ℓ. Define

$$e^{i\theta}F = \cos\theta F + \sin\theta F^*. \tag{9}$$

Proposition 2.2. *The action $e^{i\theta}$ on $\Gamma(\ell)_x$ induces a complex structure.*

Proof. Any complex number $z = ae^{i\theta}$, so $z \cdot F = e^{i\theta}(aF)$. We check that $e^{i\theta'}(e^{i\theta}F) = e^{i(\theta+\theta')}F$ and $e^{i\theta} \cdot (F + F') = e^{i\theta} \cdot F + e^{i\theta} \cdot F'$. □

Consider $T(M) \otimes \mathbb{C}$. We define the innerproduct $\langle\ ,\ \rangle_{\mathbb{C}}$ on $T(M) \otimes \mathbb{C}$ by

$$\langle iu, v\rangle_{\mathbb{C}} = \langle u, iv\rangle_{\mathbb{C}} = i\langle u, v\rangle \quad when \quad u, v \in T_x(M). \tag{10}$$

If $\mathbf{a}$ and $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ are in T_x^u, we define

$$(\mathbf{a} + i\mathbf{b}) \times (\mathbf{c} + i\mathbf{d}) = (\mathbf{a} \times \mathbf{c} - \mathbf{b} \times \mathbf{d}) + i(\mathbf{b} \times \mathbf{c} + \mathbf{a} \times \mathbf{d}). \tag{11}$$

Let $\ell_{\mathbb{C}}$ be the bundle of linear maps $\mathbb{F} : T_x \otimes \mathbb{C} \to T_x \otimes \mathbb{C}$ skew symmetric with respect to $\langle\ ,\ \rangle_{\mathbb{C}}$. Let $F_x \in \ell_x$ act on $T_x \otimes \mathbb{C}$ by

$$F(\mathbf{a} + i\mathbf{b}) = F(\mathbf{a}) + iF(\mathbf{b}). \tag{12}$$

Define $c : \ell \to \ell_{\mathbb{C}}$ and $\overline{c} : \ell \to \ell_{\mathbb{C}}$ by

$$cF = F - iF^* \quad \text{and} \quad \overline{c}F = F + iF^*. \tag{13}$$

Note that changing the orientation means replacing F^* by $F^{*'} := -F^*$. Hence the complex structure is changed so that c becomes $\overline{c} = c'$.

Proposition 2.3. *c is a complex bundle map.*

Proof. cF_x is skew symmetric on $T_x \otimes \mathbb{C}$. Also c commutes with addition and multiplication. It is complex because

$$c(e^{i\theta} \cdot F) = e^{i\theta}(cF). \tag{14}$$

This follows because

$$\begin{aligned}
e^{i\theta}(cF) &= (\cos\theta + i\sin\theta)(F - iF^*) \\
&= \cos\theta F + \sin\theta F^* + i(\sin\theta F - \cos\theta F^*) \\
&= e^{i\theta} \cdot F - i(e^{i\theta} \cdot F^*) = e^{i\theta} \cdot F - i(e^{i\theta}F)^* \\
&= c(e^{i\theta}F).
\end{aligned}$$

□

We will show presently that c is injective.

Scholium 2.4. Maxwell's equations and Lorentz' Law.

a) We chose $\rho : \ell \to \Lambda^2$ to be given by (3), $\widehat{F}(v,w) = \langle v, F(w)\rangle$, in order to agree with the standard index conventions of tensor analysis. Parrott's otherwise careful book makes the opposite choice, $\widehat{F}(v,w) = \langle F(v), w\rangle$, and is thus inconsistent with his index conventions. This has little import for his book, since he deals mostly with forms, but it could cause confusion if one is using skew symmetric operators.

b) Electro-magnetic tensors are two-forms. Classically they satisfy Maxwell's equations:

$$d\widehat{F} = 0, \quad d * \widehat{F} = J. \tag{15}$$

We can write Maxwell's equations in terms of skew symmetric bundle maps as follows.

$$\text{div } F = j \ , \quad \text{div } F^* = 0 \tag{16}$$

where j is a one form. We may reduce this to one equation by extending div to the complex case by $\text{div}(iF) = i \ \text{div}(F)$. Then F satisfies Maxwell's equation if and only if $\text{div}(cF)$ is real.

c) *The Lorentz Law:* Suppose a particle with charge q is moving in an electromagnetic field $\widehat{F}$ with 4–velocity u. Then its acceleration is $a = qFu$ where $\rho(F) = \widehat{F}$. This is the reason we chose the symbol $\mathbf{E}$ to equal Fu. The charge is motionless with respect to the u observer, hence its acceleration is given by the electric field $\mathbf{E}$ as seen by that observer. Also $\mathbf{B} = -F^*u$ corresponds to the magnetic field, as will be seen shortly.

Level −9. Orthonormal Bases. We may choose orthonormal vector fields e_0, e_1, e_2, e_3, so

$$\langle e_0, e_0\rangle = -1 \quad \text{and} \quad \langle e_i, e_j\rangle = \delta_{ij}. \tag{17}$$

Already this notation restricts the topology of the M. It must be parallelizable for such a basis to exist. Fortunately we can find local regions which admits these orthogonal frame fields. Now $F(e_i) = \sum F_{ij} e_j$. So $\langle F(e_i), e_j\rangle = F_{ij}\langle e_j, e_j\rangle$. Hence F is skew symmetric if and only if $F_{ji}\langle e_i, e_i\rangle = -F_{ij}\langle e_j, e_j\rangle$. So we can represent F by a matrix of the form

$$F = \left(\begin{array}{c|ccc} 0 & E_1 & E_2 & E_3 \\ \hline E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{array}\right) \quad \begin{array}{l} \text{where } F_{0i} = F_{i0} = E_i \\ \text{and } F_{ij} = -F_{ji} = B_k. \end{array} \tag{18}$$

We find it convenient to partition this matrix into blocks. So

$$F = \left(\begin{array}{c|c} 0 & \mathbf{E}^T \\ \hline \mathbf{E} & \times\mathbf{B} \end{array}\right). \tag{19}$$

where $E = \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix}$. Here the notation $\times\mathbf{B}$ means

$$(\times\mathbf{B}) \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = (v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3) \times (B_1 e_1 + B_2 e_2 + B_3 e_3),$$

or

$$(\times\mathbf{B})\mathbf{v} = \mathbf{v} \times \mathbf{B} \tag{20}$$

for short. This assumes that $e_1 \times e_2 = e_3$. If $e_1 \times e_2 = -e_3$, then $(\times \mathbf{B})\mathbf{v} = \mathbf{B} \times \mathbf{v}$.

Level -10. Oriented Orthonormal Bases. Same as in Level –9, but here we require $e_1 \times e_2 = e_3$.

Now in Level –9 we have

$$F = \begin{pmatrix} 0 & \mathbf{E}^T \\ \mathbf{E} & \times \mathbf{B} \end{pmatrix} \quad \text{and} \quad F^* = \begin{pmatrix} 0 & -\mathbf{B}^T \\ -\mathbf{B} & \times \mathbf{E} \end{pmatrix}. \tag{21}$$

Then

$$cF = \begin{pmatrix} 0 & \mathbf{A}^T \\ \mathbf{A} & \times(-i\mathbf{A}) \end{pmatrix} \quad \text{where} \quad \mathbf{A} = \mathbf{E} + i\mathbf{B} \tag{22}$$

Note that any matrix of the form $\begin{pmatrix} 0 & \mathbf{E}^T \\ \mathbf{E} & \times \mathbf{B} \end{pmatrix}$ represents a skew symmetric linear map.

Scholium 2.5. Lorentz transformation at level -2.
Let u and u' be observers. Then

$$u' = \frac{1}{\sqrt{1-w^2}}\,(u + \mathbf{w}) \tag{23}$$

where $\mathbf{w}$ is space-like in T^u_x. We call $\mathbf{w}$ the velocity of u' relative to u. There is a symmetric formula

$$u = \frac{1}{\sqrt{1-w'^2}}\,(u' + \mathbf{w}')$$

But note that $\mathbf{w}'$ does not lie in the same subspace as $\mathbf{w}$. However $w = w'$ and $\mathbf{w}$ and $\mathbf{w}'$ both lie in the u, u' plane. Now if a particle moves along u' as seen by u, then

$$\mathbf{a} = qFu' = \frac{q}{\sqrt{1-w^2}} \begin{pmatrix} 0 & \mathbf{E}^T \\ \mathbf{E} & \times \mathbf{B} \end{pmatrix} \begin{pmatrix} 1 \\ \mathbf{w} \end{pmatrix} = q\left[(\mathbf{E}\cdot\mathbf{w})u + \mathbf{E} + \mathbf{w}\times\mathbf{B}\right]/\sqrt{1-w^2}. \tag{24}$$

This is a more familiar form of the Lorentz Law.

The block matrix of Level –10 gives a very effective way of discovering facts about F. Most of the time we will use Level –2 proofs or Level –10 proofs. But what are definitely superior are Level –2 statements.

Now from the block matrices of Level –10 we quickly find several facts.

Proposition 2.6. *a)* $\dim_{\mathbb{R}} \ell_x = 6$, *so* $\dim_{\mathbb{C}} \ell_x = 3$.

b) For a given observer field u, there is an F for every pair of vector fields $\mathbf{E}$ *and* $\mathbf{B}$ *in T^u.*

c) The map $c : \ell \to \ell_{\mathbb{C}}$ is injective, since the map $\phi_u : \ell \to T^u \otimes \mathbb{C}$ is a vector bundle equivalence where

$$\phi_u(F) = cFu = Fu - iF^*u = \mathbf{E} + i\mathbf{B} \tag{25}$$

3. Key Relations

Using the notation of Level –10 we obtain the following facts by straight forward calculation.

LEMMA 3.1. *Let the commutator be denoted by* $[x, y] = xy - yx$.
a) $(\times\mathbf{B})(\times\mathbf{A}) = \mathbf{A}\mathbf{B}^T - (\mathbf{A} \cdot \mathbf{B})I$
where I *is the* 3×3 *identity matrix and all vectors are column vectors.*
b) $[(\times\mathbf{B}), (\times\mathbf{B}')] = \times(\mathbf{B}' \times \mathbf{B})$.
c) $\mathbf{E}'\mathbf{E}^T - \mathbf{E}\mathbf{E}'^T = \times(\mathbf{E}' \times \mathbf{E})$.
d) $\times(\mathbf{B} + \mathbf{B}') = \times\mathbf{B} + \times\mathbf{B}'$.
e) $(\times\mathbf{B})^T = \times(-\mathbf{B}) = -(\times\mathbf{B})$
f) $\mathbf{v}^T(\times\mathbf{B}) = (\mathbf{B} \times \mathbf{v})^T$

PROOF OF F).

$$\begin{aligned} \mathbf{v}^T(\times\mathbf{B}) &= [(\times\mathbf{B})^T\mathbf{v}]^T = [\times(-\mathbf{B})\mathbf{v}]^T \\ &= [\mathbf{v} \times (-\mathbf{B})]^T = [\mathbf{B} \times \mathbf{v}]^T. \end{aligned}$$

□

A key result is the following

THEOREM 3.2.

$$FF^* = F^*F = -(\mathbf{E} \cdot \mathbf{B})I.$$

PROOF. Use (21) and multiply out using Lemma 3.1a.

COROLLARY 3.3. $\langle Fv, F^*v\rangle = (\mathbf{E}\cdot\mathbf{B})\langle v, v\rangle$ *for any* $v \in T(M)$. *Hence* $\mathbf{E}_u\cdot\mathbf{B}_u = \mathbf{E}_{u'} \cdot \mathbf{B}_{u'}$ *for any two observers.*

PROOF.

$$\begin{aligned} \langle Fv, F^*v\rangle &= -\langle F^*Fv, v\rangle = -\langle -(\mathbf{E} \cdot \mathbf{B})v, v\rangle \\ &= \mathbf{E} \cdot \mathbf{B}\langle v, v\rangle. \end{aligned}$$

Thus $\mathbf{E}_{u'} \cdot (-\mathbf{B}_{u'}) = \mathbf{E} \cdot \mathbf{B}(-1)$. □

COROLLARY 3.4. $-\mathbf{E} \cdot \mathbf{B} = \lambda_F\lambda_{F^*}$ *where* λ_F *is the eigenvalue for an eigenvector* s *of* F *and* λ_{F^*} *is the eigenvalue of* s *for* F^*.

PROOF. Since F and F^* commute, they have a common eigenvector s. Then

$$\lambda_{F^*}\lambda_F s = F^*Fs = -(\mathbf{E} \cdot \mathbf{B})s.$$

□

COROLLARY 3.5. $F^2 - F^{*2} = (E^2 - B^2)I$.

PROOF. Apply Theorem 3.2 to $(F + F^*)(F + F^*)^*$. So

$$\begin{aligned} (F + F^*)(F + F^*)^* &= -\langle (F + F^*)u, -(F + F^*)^*u\rangle I \\ -(F^2 - F^{*2}) &= -(\mathbf{E} - \mathbf{B}) \cdot (\mathbf{B} + \mathbf{E})I \\ F^2 - F^{*2} &= (E^2 - B^2)I. \end{aligned}$$

The second equation follows from (4) and the definition of $\mathbf{E}$ and $\mathbf{B}$. □

COROLLARY 3.6. $E_u^2 - B_u^2 = E_{u'}^2 - B_{u'}^2$.

COROLLARY 3.7. $\lambda_F^2 - \lambda_{F^*}^2 = E^2 - B^2$.

DEFINITION 3.8. Let $T_F = \frac{1}{2}(F^2 + F^{*2})$. Thus T_F is a bundle map which is symmetric with respect to $\langle\ ,\ \rangle$.

PROPOSITION 3.9. $T_F = F^2 - \frac{(E^2-B^2)}{2} I$.

PROOF. Use Corollary 3.5.

PROPOSITION 3.10.

$$T_F = \left[\begin{array}{c|c} \frac{E^2+B^2}{2} & -(\mathbf{E}\times\mathbf{B})^T \\ \hline \mathbf{E}\times\mathbf{B} & \mathbf{E}\mathbf{E}^T + \mathbf{B}\mathbf{B}^T - \frac{E^2+B^2}{2} I \end{array}\right].$$

PROOF. Compare [**P**], p.117, equation (28). Use equations Lemma 3.1a and Proposition 3.9. □

COROLLARY 3.11. Trace $(T_F) = 0$.

COROLLARY 3.12. Trace $(F^2) = 2(E^2 - B^2)$, *hence*

$$T_F = F^2 - \frac{1}{4}\operatorname{tr}(F^2) I.$$

PROOF. Use Corollary 3.11 and Proposition 3.9.

SCHOLIUM 3.13. ENERGY-MOMENTUM TENSOR.

a) Physically T_F is a multiple of the energy-momentum tensor. See [**P**], p.116, equation (20).

b) The Poynting 4-vector as seen by observer u is

$$Tu = \frac{E^2 + B^2}{2}\, u + \mathbf{E}\times\mathbf{B}. \tag{26}$$

Thus $\frac{E^2+B^2}{2}$ is interpreted as the energy of the electromagnetic field F, and $\mathbf{E}\times\mathbf{B}$ is interpreted as the 3-momentum per unit volume of the field F.

4. The Complex Structure and Commutators

Using the commutator relations Lemma 3.1b and c and matrix multiplication, we obtain the following key result for commutators $[F_1, F_2] = F_1F_2 - F_2F_1$.

THEOREM 4.1.

$$[F_1, F_2] = \left[\begin{array}{c|c} 0 & (-\mathbf{E}_1\times\mathbf{B}_2 - \mathbf{B}_1\times\mathbf{E}_2)^T \\ \hline (-\mathbf{E}_1\times\mathbf{B}_2 - \mathbf{B}_1\times\mathbf{E}_2) & \times(\mathbf{E}_1\times\mathbf{E}_2 - \mathbf{B}_1\times\mathbf{B}_2) \end{array}\right]$$

In other words

$$[F', F]u = -\mathbf{E}'\times\mathbf{B} - \mathbf{B}'\times\mathbf{E}$$
$$-[F', F]^* u = \mathbf{E}'\times\mathbf{E} - \mathbf{B}'\times\mathbf{B}.$$

We remark that this result also holds for complex F_1 and F_2 since the argument is just formal.

Corollary 4.2.

$$[F_1, F_2]^* = [F_1, F_2^*] = [F_1^*, F_2]. \tag{27}$$

Hence

$$e^{i(\theta+\phi)} \cdot [F_1, F_2] = [e^{i\theta} \cdot F_1 \quad , \quad e^{i\phi} \cdot F_2]. \tag{28}$$

Proof. (27) follows from (Theorem 4.1) and (28) follows from (27).

Hence the complexification of $\Gamma(\ell)$ commutes with the Lie algebra structure of $\Gamma(\ell)$.

Theorem 4.3.

$$[cF, cG] = 2c([F, G]).$$

Proof.

$$\begin{aligned}
[cF, cG] &= (F - iF^*)(G - iG^*) - (G - iG^*)(F - iF^*) \\
&= FG - F^*G^* - i(FG^* + F^*G) - GF + G^*F - i(-GF^* - G^*F) \\
&= [F, G] + [G^*, F^*] - i([F, G^*] + [F^*, G]) \\
&= [F, G] + [G, F]^{**} - i([F, G] + [F, G])^* \\
&= 2([F, G] - i[F, G]^*) = 2c[F, G].
\end{aligned}$$

where the last equality comes from the definition of c, and the previous two equalities come from (27) and (4). □

Corollary 4.4.

$$(c[F_1, F_2])u = i(\mathbf{E}_1 + i\mathbf{B}_1) \times (\mathbf{E}_2 + i\mathbf{B}_2)$$

for observer u.

Proof. $c[F_1, F_2] = \frac{1}{2}[cF_1, cF_2]$ by Theorem 4.3. Now $cF_1 = \begin{pmatrix} 0 & \mathbf{A}_1^T \\ \mathbf{A}_1 & \times(-i\mathbf{A}_1) \end{pmatrix}$ where $A_1 = \mathbf{E}_1 + i\mathbf{B}_1$ and similarly for cF_2. Now by Theorem 4.1 for complex F, we have

$$\begin{aligned}
[cF_1, cF_2]u &= -\mathbf{A}_1 \times (-i\mathbf{A}_2) - (-i(\mathbf{A}_1) \times \mathbf{A}_2) \\
&= 2i\mathbf{A}_1 \times \mathbf{A}_2 = 2i(\mathbf{E}_1 + i\mathbf{B}_1) \times (\mathbf{E}_2 + i\mathbf{B}_2).
\end{aligned}$$

□

Theorem 4.5. *Let* $\mathbb{F} = \begin{pmatrix} 0 & \mathbf{A}^T \\ \mathbf{A} & \times\mathbf{C} \end{pmatrix}$ *where* $\mathbf{A}$ *and* $\mathbf{C}$ *are complex 3-vectors. Then* $\mathbb{F}^2 = kI$ *if and only if* $k = \mathbf{A} \cdot \mathbf{A}$ *and* $\mathbf{C} = \pm i\mathbf{A}$.

Proof. Assume $\mathbb{F}^2 = kI$. Then $\mathbb{F}^2 w = kw$ for any vector w. For $w = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ we get $\mathbf{A} \cdot \mathbf{A} = k$ and $\mathbf{A} \times \mathbf{C} = \mathbf{0}$. Thus $\mathbf{C} = s\mathbf{A}$ for some $s \in \mathbb{C}$. Hence $\mathbb{F} = \begin{pmatrix} 0 & \mathbf{A}^T \\ \mathbf{A} & \times(s\mathbf{A}) \end{pmatrix}$ and $\mathbb{F}^2 = (\mathbf{A} \cdot \mathbf{A})I$. Apply $\begin{pmatrix} 0 \\ \mathbf{v} \end{pmatrix}$ to this last equation. We obtain

$$(\mathbf{v} \cdot \mathbf{A})\mathbf{A} + (\mathbf{v} \times s\mathbf{A}) \times (s\mathbf{A}) = (\mathbf{A} \cdot \mathbf{A})\mathbf{v}.$$

Using the third and fourth equations of (8) and rearranging terms we get

$$(1+s^2)(\mathbf{A}\cdot\mathbf{v})\mathbf{A} = (1+s^2)(\mathbf{A}\cdot\mathbf{A})\mathbf{v}$$

for arbitrary $\mathbf{v}$. Thus $1+s^2=0$. Hence $s=\pm i$.

For the converse, first suppose $s=-i$, so $\mathbb{F} = \begin{pmatrix} 0 & \mathbf{A}^T \\ \mathbf{A} & \times(-i\mathbf{A}) \end{pmatrix}$. Then $\mathbb{F} = F - iF^* = cF$ where $cFu = \mathbf{E}+i\mathbf{B} = \mathbf{A}$. Then

$$\begin{aligned} \mathbb{F}^2 &= (cF)^2 = (F-iF^*)^2 = F^2 - F^{*2} - i(FF^* + F^*F) \\ &= (E^2-B^2)I - 2i(-\mathbf{E}\cdot\mathbf{B})I \\ &= (E^2-B^2+2\mathbf{E}\cdot\mathbf{B})i)I \\ &= (\mathbf{E}+i\mathbf{B})\cdot(\mathbf{E}+i\mathbf{B})I = (\mathbf{A}\cdot\mathbf{A})I. \end{aligned}$$

Similarly if $s=i$, $\mathbb{F}^2 = (\mathbf{A}\cdot\mathbf{A})I$ implies $\mathbb{F}^2 = (\mathbf{A}\cdot\mathbf{A})I$. □

COROLLARY 4.6.

$$\begin{aligned} (cF)^2 &= (\mathbf{A}\cdot\mathbf{A})I = \lambda_{cF}^2 I \\ (\overline{c}F)^2 &= \lambda_{\overline{c}F}^2 I. \end{aligned}$$

PROOF. The inner equality of the first line was proved above. Apply this equation to an eigenvector s to get the last equality of the first line. The second equality follows from complex conjugation.

COROLLARY 4.7. $cF_1cF_2 + cF_2cF_1 = 2(\mathbf{A}_1\cdot\mathbf{A}_2)I$.

PROOF.

$$\begin{aligned} cF_1cF_2 + cF_2cF_1 &= (cF_1+cF_2)^2 - cF_1^2 - cF_2^2 \\ &= [(\mathbf{A}_1+\mathbf{A}_2)\cdot(\mathbf{A}_1+\mathbf{A}_2) - (\mathbf{A}_1\cdot\mathbf{A}_1 - \mathbf{A}_2\cdot\mathbf{A}_2]I = 2(\mathbf{A}_1\cdot\mathbf{A}_2)I. \end{aligned}$$

□

THEOREM 4.8. $cF_1\overline{cF}_2 = \overline{cF}_2cF_1$.

PROOF. We must show that $[cF_1, \overline{cF}_2] = 0$. Apply theorem 4.1 where $\mathbf{E}_j = \mathbf{A}_j$ and $\mathbf{B}_j = (-1)^j i\mathbf{A}_j$ for $j=1,2$. Then all cross products must be zero in Theorem 4.1 and we obtain the desired result. □

REMARK 4.9. CLIFFORD ALGEBRAS.

According to the first proposition of [**LM**], Corollary 4.6 is a clue that $c : \ell_x \to \mathbb{C}(4)$ involves representations of Clifford modules. Here $\mathbb{C}(4)$ is the space of linear maps on $T_x\otimes\mathbb{C}$, a 16 dimensional space which is a complex Clifford Algebra. The image of c in $\mathbb{C}(4)$ generates the Quaternions tensored with $\mathbb{C}$. The complex conjugate $\overline{c}$ generates another complex representation of the Quaternions in $\mathbb{C}(4)$. The two representations commute, and they generate all of $\mathbb{C}(4)$ under composition. This probably has something to do with the fact that $so(4)\simeq so(3)\times so(3)$? But it might be that this particular representation by means of $F-iF^*$ is new.

Scholium 4.10. Pauli Matrices.

The Pauli matrices of physics play an important role in quantum mechanics. The relations among their products are their key features, the actual form of the matrices is not important. Thus we have $\boldsymbol{\sigma}_x$, $\boldsymbol{\sigma}_y$, $\boldsymbol{\sigma}_z$ so that $\{\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_j\} = 2\delta_{ij} I$ and $[\boldsymbol{\sigma}_x, \boldsymbol{\sigma}_y] = 2i\boldsymbol{\sigma}_z$, [**F, III, 11-4**]. We get the same relations using cF as follows. Let E_x, E_y, E_z be the F with zero $\mathbf{B}$ field and with unit $\mathbf{E}$ fields pointing along the x, y, z axes, respectively, of Minkowski space. So for example $E_x = \begin{pmatrix} 0 & \mathbf{e}_x \\ \mathbf{e}_x & 0 \end{pmatrix}$. Denote $\boldsymbol{\sigma}_x = cE_x$, $\boldsymbol{\sigma}_y = cE_y$ and $\boldsymbol{\sigma}_z = cE_z$. Then $\boldsymbol{\sigma}_x$, $\boldsymbol{\sigma}_y$, $\boldsymbol{\sigma}_z$ satisfy the Pauli matrix relations. In addition, $\overline{\boldsymbol{\sigma}}_x$, $\overline{\boldsymbol{\sigma}}_y$, $\overline{\boldsymbol{\sigma}}_z$ commute with the $\boldsymbol{\sigma}$'s and satisfy the Pauli relations among themselves except that $\overline{\boldsymbol{\sigma}}_x \overline{\boldsymbol{\sigma}}_y = -i\overline{\boldsymbol{\sigma}}_z$. Also $\boldsymbol{\sigma}_x$, $\boldsymbol{\sigma}_y$, $\overline{\boldsymbol{\sigma}}_x$, $\overline{\boldsymbol{\sigma}}_y$ generate the Clifford algebra $\mathbb{C}(4)$. This can be shown by brute force.

5. Eigenvectors

Recall our notation in which $\bar{c}F = \overline{cF}$ and $\lambda_{\bar{c}F} = \overline{\lambda_{cF}}$.

Proposition 5.1. *$cF \circ \overline{cF} = 2T_F$. Hence $\lambda_{cF}\overline{\lambda_{cF}} = 2\lambda_T$.*

Proof. $cF \circ \bar{c}F = (F - iF^*)(F + iF^*) = F^2 + F^{*2}$ since $FF^* = F^*F$. Now apply the definition of T_F, (Definition 3.8).

Corollary 5.2.

$$T_{e^{i\theta}\cdot F} = T_F.$$

Proof.

$$\begin{aligned} T_{e^{i\theta}\cdot F} &= \frac{1}{2} c(e^{i\theta} \cdot F) \circ \overline{c(e^{i\theta} \cdot F)} \\ &= \frac{1}{2}(e^{i\theta} cF) \circ e^{-i\theta}\overline{cF} = \frac{1}{2} cF\bar{c}F = T_F. \end{aligned}$$

□

Corollary 5.3. *$T^2 = \lambda_T^2 I$ where λ_T is an eigenvalue of T.*

Proof. $T^2 = \frac{1}{4}((cF)(\overline{cF}))^2 = \frac{1}{4}(cF)^2(\overline{cF})^2 = \frac{1}{4}\,\lambda_{cF}^2 \overline{\lambda}_{cF}^2 I$ by Theorem 4.8. So $T^2 = \lambda_T^2 I$.

Theorem 5.4. *Let $F \in \Gamma(\ell)$ and let λ_F be an eigenvalue of F and λ_T be an eigenvalue of T_F.*

a) $\lambda_T = \sqrt{(\frac{E^2 - B^2}{2})^2 + (\mathbf{E} \cdot \mathbf{B})^2}$

b) $\lambda_F = \pm\sqrt{\lambda_T + \frac{(E^2 - B^2)}{2}}, \quad \lambda_{F^*} = \pm\sqrt{\lambda_T - \frac{(E^2 - B^2)}{2}}.$

c) $\lambda^4 - (E^2 - B^2)\lambda^2 - (\mathbf{E} \cdot \mathbf{B})^2$ *, or equivalently,*
$\lambda^4 - (\lambda_F^2 - \lambda_{F^*}^2)\lambda^2 - (\lambda_F \lambda_{F^*})^2$*, is the characteristic polynomial of F.*

Proof. Corollaries 3.4 and 3.7 gives the equations $\lambda_F \lambda_{F^*} = -\mathbf{E} \cdot \mathbf{B}$ (Corollary 3.4) and $\lambda_F^2 - \lambda_{F^*}^2 = E^2 - B^2$ (Corollary 3.7). Eliminating λ_{F^*} from (Corollary 3.4) and (Corollary 3.7) gives $\lambda_F^4 - (E^2 - B^2)\lambda_F^2 - (\mathbf{E} \cdot \mathbf{B})^2 = 0$. Solving gives b). Then a) follows from $\lambda_T = \lambda_F^2 - \frac{(E^2 - B^2)}{2}$ which follows from Corollary 3.7. To be absolutely certain that c) is the characteristic polynomial, one must calculate $\det(F - \lambda I)$ for F represented as a matrix in (18). □

PROPOSITION 5.5. *If s is an eigenvector of $F \in \ell_x$, then $\lambda_F\langle s,s\rangle = 0$. So if $\lambda_F \neq 0$, then s is a null vector. Both λ_F and λ_{F^*} are zero if and only if $\lambda_T = 0$. In that case s is a multiple of $\frac{E^2+B^2}{2}\, u + \mathbf{E}\times\mathbf{B}$, which is null.*

PROOF. $\lambda_F\langle s,s\rangle = \langle \lambda_F s,s\rangle = \langle Fs,s\rangle = -\langle s,F_s\rangle = -\lambda_F\langle s,s\rangle$. The same argument holds for the complex cF, so $\lambda_{cF}\langle s,s\rangle = 0$. Since $\lambda_T = \frac{1}{2}\,\lambda_{cF}\overline{\lambda}_{cF}$, we get the second sentence. Now $\lambda_T = 0$ if and only if $E = B$ and $\mathbf{E}\cdot\mathbf{B} = 0$. Under those conditions, use (19) to show that $\frac{E^2+B^2}{2}\, u + \mathbf{E}\times\mathbf{B}$ is an eigenvector and is a null vector. □

SCHOLIUM 5.6. THE NULL AND NON NULL CASES.

The null and non-null cases are when $\lambda_T = 0$ and $\lambda_T \neq 0$ respectively. If $\lambda_T = 0$ then $E = B$ and $\mathbf{E}\cdot\mathbf{B} = 0$. This is called the null case mathematically. Physicists identify an electro-magnetic field with $E = B$ and $\mathbf{E}\cdot\mathbf{B} = 0$ as the *radiative* or *wave-like* case. In the null case $\lambda_F = \lambda_{F^*} = \lambda_T = \lambda_{cF} = 0$. The characteristic polynomial is λ^4, $T = F^2$, F^2u is the eigenvector of F^2.

PROOF. $F^2(F^2u) = F^4u = T^2u = 0$. So $s = F^2u = Tu = \frac{E^2+B^2}{2}\, u + \mathbf{E}\times\mathbf{B}$. (The Poynting 4-vector). Now s is null, i.e. $\langle s,s\rangle = 0$. Since $\langle s,s\rangle = \langle Ts,Ts\rangle = \langle T^2s,s\rangle = \langle 0,s\rangle = 0$. So image $(T^2) =$ span s. Then dim ker $T = 3$. □

Now consider the non-null case. Then $\lambda_T \neq 0$. Hence $\lambda_{cF} \neq 0$, so one of λ_F or λ_{F^*} is not zero. Hence there are two real null eigenvectors of cF, s for λ_{cF} and s_- for $-\lambda_{cF}$. Both s and s_- are linearly independent. Since $T = \frac{1}{2}\, cF\overline{cF}$, s and s_- are both eigenvectors of T with eigenvalue $\lambda_T > 0$.

Let Π_+ be the space of eigenvectors of T in $T_x(M)$ corresponding to λ_T and let Π_- be the space of eigenvectors corresponding to $-\lambda_T$. Then $\Pi_+ =$ image (Φ_+) and $\Pi_- =$ image (Φ_-) where $\Phi_+ = \lambda_T I + T$ and $\Phi_- = -\lambda_T I + T$. Now $\Phi_\pm$ are symmetric with respect to $\langle\ ,\ \rangle$. Note that $\Phi_\pm^2 = \pm 2\lambda_T\Phi_\pm$ and $\Phi_+\Phi_- = \Phi_-\Phi_+ = 0$, all because of the fact that $T^2 = \lambda_T^2 I$. From this we obtain:

PROPOSITION 5.7. *Let F be non null.*

a) Π_+ *is orthogonal to* Π_-.
b) Π_+ *is time-like and* Π_- *is space like.*
c) $\dim\Pi_+ = \dim\Pi_- = 2$.
d) $F(\Pi_\pm) \subset (\Pi_\pm)$, *i.e.* $\Pi_\pm$ *are invariant subspaces of* F.

PROOF. The following two lemmas prove a), b) and c). And d) follows since for $v \in \Pi_\pm$ we have $\pm\lambda_T F(v) = F(\pm\lambda_T v) = F(T(v)) = T(F(v))$. So $F(v) \in \Pi_\pm$.□

LEMMA 5.8. *Suppose $Q : T_x \to T_x$ is symmetric with respect to $\langle\ ,\ \rangle$. If Q has a time-like eigenvector, then Q has an orthonormal frame of eigenvectors.*

PROOF. Let u be a time-like eigenvector of Q. We may assume that $\langle u,u\rangle = -1$. Consider T_x^u, the space of vectors orthogonal to u. Then $Q : T_x^u \to T_x^u$ since $\langle u,Qv\rangle = \langle Qu,v\rangle = \lambda_Q\langle u,v\rangle = 0$ if $v \in T_x^u$. Hence $Q : T^u \to T^u$. But T^u is space-like and $\langle\ ,\rangle$ on T^u is positive definite and Q is symmetric. Hence there is an orthonormal set of eigenvectors on T^u by a famous theorem. Call them e_1, e_2, e_3. Then u, e_1, e_2, e_3 is the desired frame. □

LEMMA 5.9. *Let $Q : T_x \to T_x$ be a linear map which is*
a) symmetric with respect to $\langle\ ,\ \rangle$, i.e. $\langle Qv, w\rangle = \langle v, Qw\rangle$.
b) $Q^2 = \lambda^2 I$.
c) Trace $(Q) = 0$.
d) $\langle u, Qu\rangle < 0$ for some future timelike u.
Then if $\lambda = 0$, there is a null eigenvector s so that image (Q) = span s. If $\lambda \neq 0$, then the set of all eigenvectors corresponding to $\pm\lambda$ form two 2 dimensional subspaces $\Pi_\pm$, and Π_+ is orthogonal to Π_-, and Π_+ is time-like and Π_- is space like.

PROOF. Suppose $\lambda = 0$. Then $\langle Qv, Qv\rangle = \langle Q^2 v, v\rangle = 0$ for all $v \in T_x$. So the image of Q consists of null-vectors. Since $\langle u, Qu\rangle < 0$ for some time-like u, we see that $Qu \neq 0$ and that $Q(Qu) = 0$. So Qu is the desired s.

Suppose $\lambda \neq 0$. Let $\langle u, Qu\rangle < 0$ for observer u. Consider $\lambda u + Qu$. Then $\langle \lambda u + Qu, \lambda u + Qu\rangle = -2\lambda^2 + 2\lambda\langle u, Qu\rangle < 0$. So $\lambda u + Qu$ is time like. But $Q(\lambda u + Qu) = \lambda Qu + Q^2 u = \lambda(\lambda u + Qu)$. So $\lambda u + Qu$ is a time-like eigenvector. Thus by Lemma 5.8, there is an orthonormal eigenvector frame. Since trace $(Q) = 0$, two of the vectors of the frame correspond to λ and generate a time-like plane Π_+ and the orthogonal two generate Π_- and are space-like. □

COROLLARY 5.10. *If $Q : T_x \to T_x$ is as in the theorem above, there is an antisymmetric $F : T_x \to T_x$ so that $T_F = Q$.*

PROOF. If $\lambda \neq 0$, Π_+ intersects the light cone in two null-subspaces generated by, say, s_+ and s_- respectively. Let $\lambda_F = \sqrt{2\lambda}$. Define $Fs_+ = \lambda_F s_+$ and $Fs_- = -\lambda_F s_-$. Let $F(v) = 0$ for all v in Π_- so we are defining $\lambda_{F^*} = 0$. Then there is a unique linear map which satisfies these conditions and $Q = F^2 - \lambda_F^2 I$. Note F is antisymmetric on Π_- since it is trivial and on Π_+ since

$$\langle \alpha s_+ + \beta s_-, F(\alpha s_+ + \beta s_-)\rangle = \langle \alpha s_+ + \beta s_-, \alpha\lambda_F s_+ - \beta\lambda_F s_-\rangle = 0,$$

since s_+ and s_- are null.

If $\lambda = 0$, choose observer u and let $s = Qu$. Choose $\mathbf{E}$ and $\mathbf{B} \in T^u$ so that $s, \mathbf{E}, \mathbf{B}$ are in the kernel of Q and are mutually orthogonal and of sufficient length so that $s = E^2 u + \mathbf{E} \times \mathbf{E}$ where $B = E$. Then let $Fu = \mathbf{E}$, $F(\mathbf{B}) = 0$, $F(s) = 0$ and $F(\mathbf{E}) = s$. Then F is determined and $F^2 = Q$. □

REMARK 5.11.
The question is, given Q over TM, does there exist an F so that $Q = T_F$?

6. Complex Eigenvectors

Let $\phi_+ = \lambda_{cF} I + cF$ and $\phi_- = -\lambda_{cF} I + cF$. Let $\overline{\phi}_+ = \overline{\lambda}_{cF} I + c\overline{F}$ and $\overline{\phi}_- = -\overline{\lambda}_{cF} I + c\overline{F}$. Since $cF^2 = \lambda_{cF}^2 I$ and $cFc\overline{F} = c\overline{F}cF$, we obtain the following facts.

THEOREM 6.1. *Let $cF : T_x \otimes \mathbb{C} \to T_x \otimes \mathbb{C}$ and $cF \neq 0$.*
a) The image of $(\phi_\pm)$ equals the $\pm\lambda_{cF}$ eigenspace of cF and the image of $(\overline{\phi}_\pm)$ equals the $\pm\overline{\lambda}_{cF}$ eigenspace of $c\overline{F}$.
b) The kernel of $(\phi_\pm)$ equals the $\mp\lambda_{cF}$ eigenspace of cF, and the kernel of $(\overline{\phi}_\pm)$ equals the $\mp\overline{\lambda}_{cF}$ eigenspace of $c\overline{F}$.

c) The eigenspaces of cF and $\overline{cF}$ consist of null vectors.
d) The eigenspaces of cF and $\overline{cF}$ have dimension 2.

PROOF.
We easily see that

$$cF\phi_{\pm} = \pm\lambda_{cF}\ \phi_{\pm}, \quad \overline{cF}\ \overline{\phi}_{\pm} = \pm\overline{\lambda_{cF}}\ \overline{\phi_{\pm}} \tag{29}$$

$$\phi_{\pm}\phi_{\mp} = 0,\ \overline{\phi_{\pm}}\ \overline{\phi_{\mp}} = 0 \tag{30}$$

$$\langle\phi_{\pm}v, w\rangle = \langle v, \phi_{\mp}w\rangle \tag{31}$$

a) follows from (29)
b) follows from (30) and a)
c) follows from a) and (31).
d) For an observer u, the vectors $\phi_+\overline{\phi}_+u$ and $\phi_+\overline{\phi}_-u$ are eigenvectors of cF by (29).

Now $\phi_+\overline{\phi}_+u$ is an eigenvector of cF corresponding to λ_{cF} as well as an eigenvector of $\overline{cF}$ corresponding to $\overline{\lambda}_{cF}$. On the other hand $\phi_+\overline{\phi}_-u$ is an eigenvector of cF corresponding to λ_{cF} and also an eigenvector of $\overline{cF}$ corresponding to $-\overline{\lambda}_{cF}$. If $\phi_+\overline{\phi}_+u$ is linear dependent on $\phi_+\overline{\phi}_-u$, then $-\overline{\lambda}_{cF} = \overline{\lambda}_{cF}$, hence $\lambda_{cF} = 0$, hence F is null. Thus if F is nonnull, $\phi_+\overline{\phi}_+u$ and $\phi_+\overline{\phi}_-u$ are linearly independent eigenvectors. If F is null, then $F^2u = E^2u + \mathbf{E}\times\mathbf{B}$ and $cFu = \mathbf{E} + i\mathbf{B}$ are linearly independent eigenvectors of the eigenspace. Hence $\dim(\mathrm{image}(\phi_+)) \geq 2$ and similarly $\dim(\ker(\phi_+)) \geq 2$. Therefore d) is proved. □

LEMMA 6.2. *Suppose a and b are real vectors in T_x. Then $a + ib \in T_x \otimes \mathbb{C}$ is null if and only if either a and b are linear dependent null vectors, or a and b are both space–like and have the same length and are orthogonal.*

PROOF. Let $v = a + ib$. Now $\langle v, v\rangle = 0$ if and only if $\langle a, a\rangle = \langle b, b\rangle$ and $\langle a, b\rangle = 0$. If a or b is null, so is the other. Since they are orthogonal null vectors, they must be linearly dependent.

On the other hand, if one of a or b is space–like, so is the other and they have equal lengths and are orthogonal. Neither a or b can be time–like, since if one were, they both would be. But no two time–like vectors are orthogonal. □

LEMMA 6.3. *Let $\mathbf{a}$ and $\mathbf{b}$ be space–like in T_x. Then $\mathbf{a}$ and $\mathbf{b}$ span a space–like plane if and only if $a^2b^2 - \langle\mathbf{a}, \mathbf{b}\rangle^2 > 0$. Thus if $\mathbf{a}$ and $\mathbf{b}$ are orthogonal and space–like, they span a space–like plane.*

PROOF. $\langle\alpha\mathbf{a} + \beta\mathbf{b},\ \alpha\mathbf{b} + \beta\mathbf{b}\rangle$ is greater than zero if and only if the determinant of

$$\begin{pmatrix} a^2 & \langle a, b\rangle \\ \langle a, b\rangle & b^2 \end{pmatrix}$$

is greater than zero.

LEMMA 6.4. *Any null subspace of $T_x \otimes \mathbb{C}$ has a degenerate inner product. That is any two vectors in a subspace of null vectors are orthogonal.*

PROOF. Suppose s and s' are null vectors in a null subspace V. Then $s + s'$ is in V. Hence $\langle s + s',\ s + s'\rangle = 0$. Expanding the left side yields $\langle s, s'\rangle = 0$.

Remark 6.5. Null planes.

Suppose $\mathbf{E}+i\mathbf{B} \in T^u_x \otimes \mathbb{C}$ is a null vector in the rest space of an observer u. Then $s = E^2 u + \mathbf{E} \times_u \mathbf{B}$ is a real null vector. Now s and $\mathbf{E}+i\mathbf{B}$ span a null plane V, which is the image of cF where F is a null skew symmetric operator with $Fu = \mathbf{E}$ and $F^* u = -\mathbf{B}$.

Also $s_- = E^2 u - \mathbf{E} \times \mathbf{B}$ is a real null vector. Again s_- and $\mathbf{E}+i\mathbf{B}$ span a null plane V', which is the image of $\overline{c}G$ for a null G so that $Gu = \mathbf{E}$ and $G^* u = \mathbf{B}$.

Thus we have two kinds of null planes, those which are the images of null cF and those which are the images of null $\overline{cF}$.

We can think of these null planes from a geometric point of view. Suppose $\mathbf{E}+i\mathbf{B}$ is a space–like null vector. Then $\mathbf{E}$ and $\mathbf{B}$ span a space–like plane $\Pi_s \subset T_x$, by Lemma 6.3. Let Π_t be the time–like plane orthogonal to Π_s. Then Π_t intersects the light cone in two one–dimensional null lines. One of these real null lines and $\mathbf{E}+i\mathbf{B}$ spans a null plane and the other line and $\mathbf{E}+i\mathbf{B}$ spans the "conjugate" null plane containing $\mathbf{E}+i\mathbf{B}$.

Thus given a space–like null vector v, there are exactly two null planes containing v. We say these two planes are *$*$–conjugate with respect to v*. If V is a null plane and contains a light–like null vector v, then we say that $\overline{V}$ *is $*$–conjugate* to V with respect to v. The planes which are the image of a null cF are called *$*$–consistent* null planes and those which are the image of a null $\overline{c}F$ are called *$*$–inconsistent*.

Lemma 6.6. *In $T_x \otimes \mathbb{C}$*

a) Every null plane contains a real null vector

b) The eigenspaces of cF are $*$–consistent planes. The eigenspaces of $\overline{cF}$ are $*$–inconsistent planes.

c) The intersection of a $*$–consistent and a $*$–inconsistent plane is one dimensional.

Proof. a) Choose an appropriate basis and use analysis to obtain the conditions for a null–plane.

b) By continuity and connectivity of $\ell_x \oplus \mathbb{C}$.

c) Let V be the null plane spanned by $\mathbf{A} = \mathbf{E}+i\mathbf{B}$ and $s = E^2 u + \mathbf{E} \times \mathbf{B}$ for u an observer orthogonal to $\mathbf{E}$ and $\mathbf{B}$. Then V is both the image and kernel of a null cF such that $cFu = \mathbf{A}$, since $cF^2 = 0$. Let $\overline{W}$ be a $*$–inconsistent null plane. It is the image of some null $\overline{c}G$. Now $\overline{W} \neq V$ since $\overline{W}$ is $*$–inconsistent, so $cF(\overline{W}) \neq 0$. Since cF and $\overline{c}G$ commute by Theorem 4.8, we see that $\overline{W} \cap V \neq 0$. So $W \cap V$ is one dimensional. $\square$

Theorem 6.7. *Let F and G be skew symmetric bundle maps. Let $\phi = \lambda_{cF} I + cF$ and $\overline{\gamma} = \overline{\lambda}_{cG} I + c\overline{G}$. Note that the choice of which of the two eigenvectors $\pm\lambda_{cF}$ is not reflected in the notation.*

a) $\phi\overline{\gamma} = \overline{\gamma}\phi$.

b) The image of $(\phi\overline{\gamma})$ is one dimensional and is generated by a null vector which is an eigenvector of both cF and $c\overline{G}$ with associated eigenvalues λ_{cF} and $\overline{\lambda}_{cG}$ respectively.

c) The image of $(\phi\overline{\phi})$ is generated by a real null vector s which is an eigenvector of cF corresponding to λ_{cF}.

Proof.

b) From (a), the image of $\phi\overline{\gamma}$ is the one dimensional sub space (image (ϕ)) $\cap$ (image $(\overline{\gamma})$).

c) Let $\gamma = \phi$ and apply (b).

COROLLARY 6.8. *The eigenvector s for a skew symmetric bundle map satisfies the following equation in terms of $\mathbf{E}_u$ and $\mathbf{B}_u$,*

$$s = 2(\lambda_T u + \frac{E^2 + B^2}{2} u + \mathbf{E} \times \mathbf{B} + \lambda_F \mathbf{E} - \lambda_{F^*} \mathbf{B}).$$

PROOF. Recall $s = \phi\overline{\phi}u$ where $\phi\overline{\gamma} = \overline{\gamma}\phi$. Expand that equation and use equations (5), (13), and (26) and Proposition 5.1. $\square$

Define $\Psi : \ell \oplus \mathbb{C} \oplus T(M) \to T(M) \otimes \mathbb{C}$ by $\Psi(F, \alpha, v) = (\alpha I + cF)v$. Let $\Psi_v : \ell \oplus \mathbb{C} \to T(M) \otimes \mathbb{C}$ be defined by

$$\Psi_v(F, \alpha) = \Psi(F, \alpha, v).$$

THEOREM 6.9. *$\Psi_v : \ell \oplus \mathbb{C} \to T(M) \otimes \mathbb{C}$ is a bundle equivalence if v is a non null vector field.*

PROOF. Both bundles are 4 dimensional and Ψ_v is a bundle map, so we only need to show that Ψ_v has zero kernel. So assume $(\alpha I + cF)v = 0$. Then v is an eigenvector of cF, hence by Theorem 6.1c we have, in contradiction to the hypothesis, that v is a null vector.

7. Eigenbundles

Given a skew symmetric $F \in \Gamma(\ell)$, we define a map $\psi_F : M \to \mathbb{C}$ by setting

$$\psi_F(m) = \lambda^2_{cF_m} = (E^2 - B^2) + 2i(\mathbf{E} \cdot \mathbf{B}) \tag{32}$$

evaluated at m.

We define a sequence of open submanifolds $M \supset M_0 \supset M_1$ based on the given F.

$$M_0 = \{m \in M \mid F_m \text{ is defined and not identically zero}\} \tag{33}$$

$$M_1 = \{m \in M \mid F_m \text{ is not null}\} \tag{34}$$

Since F_m is null if and only if $\lambda_{cF} = 0$, we see that

$$\psi_F^{-1}(\mathbb{C} - 0) = M_1. \tag{35}$$

DEFINITION 7.1. We define the degree of F, denoted deg F, to be the degree of $\psi_F : M_1 \to \mathbb{C} - 0$. We define the degree of $\psi : M_1 \to \mathbb{C} - 0$ to be the integer which corresponds to the generator of the subgroup (image (ψ)) $\subset H_1(\mathbb{C} - 0) \cong \mathbb{Z}$.

REMARK 7.2. The degree of ψ in Definition 7.1 is related to the usual Brouwer degree of Algebraic Topology. This can be seen in [$\mathbf{G}_4$]. Note, the definition of deg ψ yields a non–negative integer, in contrast to the usual Brouwer degree.

THEOREM 7.3. *The following are equivalent:*

a) $\deg \psi$ *is even.*
b) There is a line bundle of eigenvectors of F over M_1.
c) The invariant plane bundle Π_+ is an orientable 2 plane bundle over M_1.
d) There is a nonzero vector field of null eigenvectors of F over M_0.

PROOF. Consider $\widetilde{M_1}$, the set of pairs (m, α) where $m \in M_1$, and $\alpha \in \mathbb{C} - 0$ is equal to either one of the two eigenvalues $\pm\lambda_{cF_m}$. Then $\widetilde{M_1}$ is a double covering space of M_1. If $\widetilde{M_1}$ is not connected, then it is possible to choose one α at each m in a continuous way over M_1. The choice of the eigenvector corresponding to $\alpha(m)$ gives the line bundle of eigenvectors. Conversely, a line bundle of eigenvectors over M_1 will select a continuous choice of corresponding eigenvalues, so $\widetilde{M_1}$ will be disconnected. Now we have a commutative diagram

$$\begin{tikzcd} \widetilde{M} \arrow[r, "\widetilde{\psi}"] \arrow[d, "p"] & \mathbb{C} - 0 \arrow[d, "\mathrm{sq}"] \\ M \arrow[r, "\psi"'] & \mathbb{C} - 0 \end{tikzcd}$$

where $p(m, \alpha) = m$ and $\widetilde{\psi}(m, \alpha) = \alpha$ and $\mathrm{sq}(z) = z^2$. If $\widetilde{M}$ is not connected, there is a cross–section s to p. Then deg $\psi =$ deg $(\mathrm{sq} \circ \widetilde{\psi} \circ s)$ is even since the degree of sq is 2. This proves that (a) and (b) are equivalent.

For (c), the plane bundle Π_+ of time–like invariant planes of F is also the eigenbundle of T_F corresponding to $\lambda_T > 0$. Now we can always choose a nonzero time–like vector field u over M. Then $\Phi_+(u) = \lambda_T u + Tu$ is a non zero vector field of eigenvectors of T. Hence there is a trivial line sub–bundle ε in Π_+. Hence $\Pi_+ = \varepsilon \oplus \nu$, where ν is the orthogonal line bundle. If Π_+ were orientable, then ν would be trivial and we could use the direction in ν to choose at each m one of the two null eigenvector subspaces in $(\Pi_+)_m$. Hence we would get a line bundle η of eigenvectors of F. Conversely, if the eigenbundle η existed, then $\Pi_+ = \varepsilon \oplus \eta$. Since η is a trivial line bundle (because M is time–oriented) this implies that Π_+ is orientable.

Now d) is equivalent to b) because we assumed that M was time orientable. Thus the line bundle over M_0 must be trivial and hence gives a nonzero vector field over M_1. This vector field obviously extends continuously over M_0. In fact, the equation of Corollary 6.8 gives the vector field, the possible ambiguity of the choice of eigenvectors being eleminated by the fact that the bundle in b) is trivial. The converse, d) implies b), is obvious. $\square$

SCHOLIUM 7.4. THE PHASE OF ψ.

We may write $\psi_F(m) = \lambda^2_{cF_m} = 2\lambda_{T_m} e^{i\alpha}$ for some angle α, which we will call the *phase* of ψ. Suppose we have two paths in space–time from A to B which do not pass over radiation. If we measure the difference of the phase after having traveled from point A to point B along the two paths, we will find that they differ by a multiple n of 2π. If n is not zero, then the two paths linked wave-like regions. If n is even, then the continuous extension of the same eigenvector at A along the two paths result in the same eigenvector at B.

COROLLARY 7.5. *Let F be a skew symmetric bundle map. There is a plane sub–eigen–bundle η of $T(M_0) \otimes \mathbb{C}$ if and only if* $\deg \psi_F$ *is even.*

PROOF. If deg ψ_F is even, then we can choose continuously one λ_{cF_m} out of the two possible. Thus $\phi = \lambda_{cF} I + cF$ is a well–defined bundle map since there is no ambiguity with λ_{cF}. Now over M_0, the image of (ϕ_m) is always a two plane by Theorem 6.1d. The unambiguous choice of λ_F gives a bundle map ϕ whose image is a plane bundle η. Conversely, if η is a plane eigenbundle, it selects the eigenvalue λ_{cF_m} at each m which correspond to the plane η_m.

SCHOLIUM 7.6. A NEW ELECTRO-MAGNETIC INVARIANT.
First note that from Corollary 6.8, that for any observer the vectors of the form

$$\mathbf{E} \times \mathbf{B} + \lambda_F \mathbf{E} - \lambda_{F^*} \mathbf{B} \tag{36}$$

can never be zero as long as F is defined and not identically equal to zero. Now if $\deg \psi_F$ is even, then (36) gives rise to a vector field which is nonzero over M_0. The index of that vector field on any closed compact space-like manifold whose boundry is contained in M_0 must be an invariant of the field, independent of the observer field. The index is defined in [$\mathbf{G}_5$], for example.

COROLLARY 7.7. *Let* $F \in \Gamma(\ell)$ *and suppose that* $\mathbf{E} \cdot \mathbf{B} = 0$ *for all* $m \in M_0$. *Then* $\deg \psi_F = 0$ *and* $\ker(F)$ *is a plane sub–bundle of* $T(M_0)$.

COROLLARY 7.8. *If* $0 \in \mathbb{C}$ *is a regular value of* ψ_F, *then* $\deg \psi_F = 1$.

PROOF. If 0 is a regular value of ψ_F we can find a small circle about 0 which lifts to M_1. Thus $1 \in \mathbb{Z} \cong H_1(\mathbb{C} - 0;\ \mathbb{Z})$ is the image of ψ_*.

SCHOLIUM 7.9. ELECTRONS.

a) A classical free electron at rest in Minkowski space M can be represented by an F such that $\mathbf{E}(\mathbf{r}, t) = -\dfrac{\mathbf{r}}{r^3}$ and $\mathbf{B}(\mathbf{r}, t) = 0$. Thus $M_0 = M_1 = M-$ (the time axis). The deg of the free electron is zero by Corollary 7.7.
b) A classical electron at rest in a constant magnetic field will be represented by an F such that $\mathbf{E}(\mathbf{r}, t) = -\dfrac{\mathbf{r}}{r^3}$ and $\mathbf{B} = \mathbf{e}_x$. Then $M_0 = M-$ (the time axis) and $M_1 = M_0 - S$ where S in each space slice is a circle of radius 1 in the yz plane centered on the electron. The deg of the election in a constant magnetic field is 1 by Corollary 7.8.
c) An assembly of point charges at rest in a constant magnetic field will have odd degree if the number of charged points is odd. This follows from the considerations of Scholium 7.6: If the degree were even, then (36) gives a nonzero space-like vector field everywhere except at the charged points. The vector field near a charged point points outward if the point has positive charge and inward if the point has negative charge, as can be seen by the equations of Theorem 5.4 where E is much larger than B. Thus the index of each singularity contributes a positive or negative 1 to the index of the vector field, the sign of the 1 depending upon the sign of the charge times the sign of the eigenvalue λ_F. Since the number of charges is odd, the index of the vector field cannot be zero by the summation equation (8) of [$\mathbf{G}_5$]. Hence the index is not zero by hypothesis. On the other hand the index of the vector field must be zero (by the existance of defects, property (8) of [$\mathbf{G}_5$]), since far away from the points it looks like the constant B field (seen by using theorem 5.4 for B much larger than E). Thus the vector field cannot exist, so the degree must be odd.

SCHOLIUM 7.10. ELECTRO–MAGNETIC DUALITY ROTATION.
Equation (9) is called the Electro–Magnetic Duality Rotation by Physicists. We noted that $T_{e^{i\theta}F} = T_F$ in Corollary 5.2. Thus for any map $\varphi : M_0 \to S^1$, the

skew symmetric bundle map $\varphi \cdot F$ defined by $(\varphi \cdot F)_m = \varphi(m)F_m$ gives rise to the same T as does F.

On the other hand, suppose $F' = \varphi F$. Then $\psi_{F'} = \varphi^2 \psi_F$. So $\psi'_* = (2\varphi_* + \psi_*)$ on the first homology groups. Thus deg $\psi' =$ deg $\psi + 2k$ for some k. So deg (φF) has the same parity as deg (F).

THEOREM 7.11. *The space of skew symmetric operators over M_0 which gives rise to the same T is homeomorphic to the space of maps $\varphi : M_0 \to S^1$. The path components of map (M_0, S^1) correspond to the elements of $H^1(M_0; \mathbb{Z})$.*

PROOF. We only need show that given F'_m and F_m with the same T, there is a θ such that $F'_m = e^{i\theta} F_m$. Now F'_m and F_m must have the same invariant planes and the same eigenvectors. Also $\lambda_F^2 + \lambda_{F^*}^2 = 2\lambda_T$ and the same holds for F'. So we can rotate F until $\lambda_F = \lambda_{F'}$ and $\lambda_{F^*} = \lambda_{F'^*}$. So F and F' agree on Π_+. Similarly they agree on Π_-. For null F and F', the $\mathbf{E}$ and $\mathbf{B}$ must have the same length and same $\mathbf{E} \times \mathbf{B}$, so one can rotate into the other.

SCHOLIUM 7.12. ELECTRON "STATES" FOR THE SAME ENERGY MOMENTUM.

a) For the free electron F of 7.10, $H^1(M^0; \mathbb{Z}) \cong 0$. So all the "states" $e^{i\theta}F$ are homotopic to one another. All of them have eigenvector bundles.
b) For the electron in a constant magnetic field, F, there are infinitely many homotopy classes of "states" giving rise to the same energy momentum T_F. Since $H^1(M_1; \mathbb{Z}) \cong \mathbb{Z}$, these states correspond to the integers. Each state has odd degree. Thus there is no eigenvector line bundle over M_1 for any state.

8. Lorentz Transformations

Lorentz Transformations play an important role in Physics. They are an artifact of Level -16, the standard coordinates of Minkowski space. As we move up through the levels of notation they seem to dwindle in importance. That is because one of their main functions, relating different choices of systems of notation, is eliminated as the choices are eliminated. What remains are two things, changes of observers in Level -2 as mentioned in Scholium 2.5, and the Gauge group of bundle isometries of Level 0. At these levels we obtain a fresh perception of the Lorentz Transformation.

LEVEL -16. MINKOWSKI SPACE–TIME.

At Level -10 we have coordinates for the tangent space, but not for the manifold. A choice of 4 functions coordinatizes M. We need to tie in our bases of the tangent bundle with the gradients of the coordinate functions. We can use the gradients as a basis, but usually they will not be orthonormal. Or, we can use the Gram Schmidt process on them to get a more complicated orthonormal basis. The best thing would be to find coordinates whose gradients are orthonormal. That is what is done for Minkowski space.

So let $M = \mathbb{R}^4$. Put coordinates t, x, y, z on M with orthonormal gradients $\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$. We could choose another such coordinate system t', x', y', z'. The formulas relating them are called the Lorentz transformation. See [F], page I - 15 - 3.

SCHOLIUM 8.1. LORENTZ TRANSFORMATIONS OF ELECTRO-MAGNETISM.

Feynman in [F],(Vol. II, Table 26.4), carries out the Lorentz transformation in Level -16, and then tries to express the results in notation at Level -2. Calling $\mathbf{E}'$ and $\mathbf{B}'$ the transformed version of the original $\mathbf{E}$ and $\mathbf{B}$, he relates them by the formula

$$\begin{aligned} \mathbf{E}'_{\|} &= \mathbf{E}_{\|} & \mathbf{B}'_{\|} &= \mathbf{B}_{\|} \\ \mathbf{E}'_{\perp} &= \frac{(\mathbf{E}+\mathbf{w}\times\mathbf{B})_{\perp}}{\sqrt{1-w^2}} & \mathbf{B}'_{\perp} &= \frac{(\mathbf{B}-\mathbf{w}\times\mathbf{E})_{\perp}}{\sqrt{1-w^2}}. \end{aligned} \tag{37}$$

Here $\mathbf{E}'_{\|}$ means the component of $\mathbf{E}'$ parallel to the relative velocity $\mathbf{w}$ of the two coordinate frames and $E'_{\perp}$ means the component of $\mathbf{E}'$ orthogonal to $\mathbf{w}$. This formula is both correct and meaningless.

Let us give a Level -2 derivation of (37). Let $\mathbf{E} = Fu$ and $\mathbf{B} = -F^*u$. Let $u' = \dfrac{1}{\sqrt{1-w^2}}(u+\mathbf{w})$. Then $\mathbf{E}' = Fu'$ and $\mathbf{B}' = -F^*u'$. Substituting (21) and (23) into these formulas results in

$$\mathbf{E}' = \frac{\mathbf{E}\cdot\mathbf{w}}{\sqrt{1-w^2}}\left(u+\frac{\mathbf{w}}{w^2}\right) + \frac{1}{\sqrt{1-w^2}}\left(\mathbf{E}-\frac{\mathbf{E}\cdot\mathbf{w}}{w^2}\mathbf{w}+\mathbf{w}\times_u\mathbf{B}\right) \tag{38}$$

$$\mathbf{B}' = \frac{\mathbf{B}\cdot\mathbf{w}}{\sqrt{1-w^2}}\left(u+\frac{\mathbf{w}}{w^2}\right) + \frac{1}{\sqrt{1-w^2}}\left(\mathbf{B}-\frac{\mathbf{B}\cdot\mathbf{w}}{w^2}\mathbf{w}+\mathbf{w}\times_u\mathbf{E}\right) \tag{39}$$

Each of the four terms on the right hand sides of the above equations are orthogonal to u' and hence lie in $T^{u'}$. The last terms in each equation are orthogonal to $\mathbf{w}$ and lies in a plane orthogonal to the u, u' plane. These are $\mathbf{E}'_{\perp}$ and $\mathbf{B}'_{\perp}$ respectively. The first terms in each equation are the parallel components

$$\mathbf{E}'_{\|} = \frac{(\mathbf{E}\cdot\mathbf{w})}{\sqrt{1-w^2}}\left(u+\frac{\mathbf{w}}{w^2}\right) \text{ and } \mathbf{B}'_{\|} = \frac{(\mathbf{B}\cdot\mathbf{w})}{\sqrt{1-w^2}}\left(u+\frac{\mathbf{w}}{w^2}\right) \tag{40}$$

But $\mathbf{E}_{\|} = \dfrac{\mathbf{E}\cdot\mathbf{w}}{w}\mathbf{w}$ and $\mathbf{B}_{\|} = \dfrac{\mathbf{B}\cdot\mathbf{w}}{w}\mathbf{w}$. So $\mathbf{E}'_{\|} \neq \mathbf{E}_{\|}$ and $\mathbf{B}'_{\|} \neq \mathbf{B}_{\|}$ contrary to the assertion in (37). However they are both in the $u, \mathbf{w}$ plane and $E'_{\|} = E_{\|}$ and $B'_{\|} = B_{\|}$.

PROOF.

$$\begin{aligned} \mathbf{E}'_{\|}\cdot\mathbf{E}'_{\|} &= \frac{(\mathbf{E}\cdot\mathbf{w})^2}{1-w^2}\langle\left(u+\frac{\mathbf{w}}{w^2}\right),\ \left(u+\frac{\mathbf{w}}{w^2}\right)\rangle \\ &= \frac{(\mathbf{E}\cdot\mathbf{w})^2}{1-w^2}\left(-1+\frac{w^2}{w^4}\right) = \\ &= \left(\mathbf{E}\cdot\frac{\mathbf{w}}{w}\right)^2 = E_{\|}^2 \end{aligned}$$

Similarly for $B'_{\|} = B_{\|}$.

SCHOLIUM 8.2. THE DOPPLER SHIFT.

Let s_u be an eigenvector of F corresponding to λ_F as seen by an observer u. Suppose

$$u' = \frac{1}{\sqrt{1-w^2}}(u+\mathbf{w}) \tag{41}$$

is another observer. Then u' sees a different eigenvector $s_{u'}$. But $s_{u'}$ must be a multiple of s_u since they are eigenvectors. So the question is, what is the multiple in terms of $\mathbf{E}, \mathbf{B}$ and $\mathbf{w}$? The answer is:

$$s_{u'} = \frac{1}{\sqrt{1-w^2}} \left[1 + \frac{-(\mathbf{E}\times\mathbf{B})\cdot\mathbf{w} + \lambda_F \mathbf{E}\cdot\mathbf{w} - \lambda_{F^*}\mathbf{B}\cdot\mathbf{w}}{\lambda_T + \dfrac{E^2+B^2}{2}} \right] s_u. \tag{42}$$

PROOF. Define

$$\varphi(v) = \frac{\langle v, s_- \rangle}{\langle u, s_- \rangle} s_u \tag{43}$$

where s_- is an eigenvector corresponding to $-\lambda_F$. Then φ is a linear map whose image is the span of s_u and whose kernel is the space of vectors orthogonal to s_-. Now $\varphi(u) = s_u$.

Now $\Phi := (\lambda_{cF} I + cF) \circ (\overline{\lambda_{cF}} I + c\overline{F})$ has the same properties and let $\Phi(u) := s_u$. Then $\Phi = \varphi$. Let $s_- = \Phi_-(u) = (-\lambda_{cF} I + cF) \circ (\overline{-\lambda_{cF}} I + c\overline{F})u$. Now

$$s_u = 2\left(\lambda_T u + \frac{E^2+B^2}{2} u + \mathbf{E}\times\mathbf{B} + \lambda_F \mathbf{E} - \lambda_{F^*}\mathbf{B}\right) \tag{44}$$

and

$$s_- = 2\left(\lambda_T u + \frac{E^2+B^2}{2} u + \mathbf{E}\times\mathbf{B} - \lambda_F \mathbf{E} + \lambda_{F^*}\mathbf{B}\right) \tag{45}$$

from Corollary 6.8 and s_- is the same with the signs changed on λ_F and λ_{F^*}.

Now $s_{u'} = \varphi(u') = \dfrac{\langle u', s_- \rangle}{\langle u, s_- \rangle} s_u$. Substituting (41) into this equation yields

$$s_{u'} = \frac{1}{\sqrt{1-w^2}} \left(1 + \frac{\langle \mathbf{w}, s_- \rangle}{\langle u, s_- \rangle} \right) s_u. \tag{46}$$

Now

$$\langle u, s_- \rangle = -2 \left(\lambda_T + \frac{E^2+B^2}{2} \right) \tag{47}$$

using (45). Then using (45) to calculate $\langle \mathbf{w}, s_- \rangle$ and substituting this into (46) we obtain (42). $\square$

Now (42) holds for all $F \in \Gamma(\ell)$. If we restrict to null F we should see (42) reduce to a simpler form. In the null case $\lambda_F = \lambda_{F^*} = 0$ and $E = B$. So equation (42) reduces to

$$s_{u'} = \frac{1}{\sqrt{1-w^2}} \left(1 - \mathbf{w} \cdot \frac{(\mathbf{E}\times\mathbf{B})}{E^2} \right) s_u. \tag{48}$$

Now $\mathbf{w} \cdot \dfrac{(\mathbf{E}\times\mathbf{B})}{E^2}$ is the component along the $\mathbf{E}\times\mathbf{B}$ direction. If we assume that $\mathbf{w} = \mathbf{w}_r$, that is $\mathbf{w}$ is pointing in the radial direction, then

$$s_{u'} = \sqrt{\frac{1-w_r}{1+w_r}} s_u. \tag{49}$$

Here $\sqrt{\dfrac{1-w_r}{1+w_r}}$ is the Doppler shift ratio. This suggests that null F propagate along null geodesics by parallel translation.

SCHOLIUM 8.3. ELIMINATING $\mathbf{E} \times \mathbf{B}$.

In the non–null case there is a Lorentz transformation so that $\mathbf{E}' \times \mathbf{B}' = 0$. We may see this clearly using Level -2 methods. Suppose u' is an eigenvector of T_F. Then

$$T_F u' = \lambda_T u' = \frac{E_{u'}^2 + B_{u'}^2}{2} u' + \mathbf{E}_{u'} \times \mathbf{B}_{u'}. \tag{50}$$

The second equality shows that $\mathbf{E}_{u'} \times \mathbf{B}_{u'} = 0$ and $\dfrac{E_{u'}^2 + B_{u'}^2}{2} = \lambda_T$. Now we can always find an eigenvector u' by setting $u' = (\lambda_T I + T)u/k$. Here $k = \sqrt{2\lambda_T{}^2 + 2\lambda_T(E^2 + B^2)/2}$ where the k is the factor which makes u' an observer. Thus the relative velocity is

$$\mathbf{w} = (\mathbf{E}_u \times \mathbf{B}_u) \Big/ \left(\lambda_T + \frac{E_u^2 + B_u^2}{2}\right). \tag{51}$$

At Level -10, the Lorentz transformations become equations relating the choice of orthonormal bases e_0, e_1, e_2, e_3 and e_0', e_1', e_2', e_3'. In the block matrices formalism, the Lorentz transformation becomes an invertible matrix Λ so that

$$\begin{pmatrix} 0 & \mathbf{E}' \\ \mathbf{E}' & \times\mathbf{B}' \end{pmatrix} = \Lambda^{-1} \begin{pmatrix} 0 & \mathbf{E} \\ \mathbf{E} & \times\mathbf{B} \end{pmatrix} \Lambda. \tag{52}$$

Although we used many Level -10 arguments in this paper, our statements were usually Level -2. The only choices necessary were of different observers. The algebraic component of the Lorentz Transformations Λ becomes the bundle isometries of $T(M)$, that is the group of Gauge Transformations. These can be thought of at Level 0.

REMARK 8.4. THE EXPONENTIAL MAP e^F.

The exponential map maps the "Lie Algebra" $\Gamma(\ell)$ onto the group of bundle isometries $\mathcal{G}$ of $T(M)$. This exponential map is a diffeomorphism near the identity. It has a beautiful representation using the e^F notation.

$$e^F := I + F + \frac{1}{2!}F^2 + \frac{1}{3!}F^3 + \dots \tag{53}$$

where F^n means F composed with itself n–times. For F a bundle map, e^F satisfies several properties.

a) e^F is a well–defined bundle map
b) $(e^F)^{-1} = e^{-F}$ if F is skew symmetric
c) $\langle e^F v, w\rangle = \langle v, e^{-F} w\rangle$ if F is skew symmetric
d) $e^{F+F'} = e^F \circ e^{F'}$ if $FF' = F'F$
e) $\left.\dfrac{d}{dt} e^{tF}\right|_{t=0} = F$
f) Every isometry Q can be written as $Q = e^F$ for a skew symmetric F, at least locally.
g) If s is an eigenvector of F corresponding to λ_F, then s is an eigenvector of e^F corresponding to e^{λ_F} .
h) If F is skew symmetric and null, then since $F^3 = 0$ we have $e^F = I + F + \frac{1}{2}F^2$.

Now these properties also hold for $\mathbb{F} \in \Gamma(\ell \otimes \mathbb{C})$ and $e^{\mathbb{F}}$. So the fact that $(cF)^2 = \lambda_{cF}^2 I$ gives us the following striking result.

THEOREM 8.5. $e^{cF} = \cosh(\lambda_{cF})I + \dfrac{\sinh(\lambda_{cF})}{\lambda_{cF}}(cF)$
where $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$
and $\sinh(x) = \dfrac{e^x - e^{-x}}{2}$.

COROLLARY 8.6.

$$e^F = \left(\cosh\left(\frac{\lambda}{2}\right) I + \frac{\sinh\left(\frac{\lambda}{2}\right)}{\lambda} cF\right) \circ \overline{\left(\cosh\left(\frac{\lambda}{2}\right) I + \frac{\sinh\left(\frac{\lambda}{2}\right)}{\lambda} cF\right)}$$

where $\lambda = \lambda_{cF}$.

PROOF. $e^F = e^{(cF+\overline{c}F)/2} = e^{cF/2}e^{\overline{c}F/2}$, this last by Remark 8.4d. Then apply Theorem 8.5.

We leave it as an exercise to the reader to expand Corollary 8.6 and obtain an equation involving only real quantities.

COROLLARY 8.7.

$$e^{-F}(cG)e^F = c(e^{-F}Ge^F) = e^{-cF/2}(cG)e^{cF/2}.$$

PROOF. First we note the following result.

$$(e^{-F}Ge^F)^* = e^{-F}G^*e^F. \tag{54}$$

This follows from the fact that $(e^{-F}Ge^F)^{-1} = e^{-F}G^{-1}e^F$ and that $G^{-1} = \dfrac{G^*}{-(\vec{E}\cdot\vec{B})}$ when $\vec{E}\cdot\vec{B} \neq 0$. The case for $\vec{E}\cdot\vec{B} = 0$ follows by continuity.

Now
$$\begin{aligned} c(e^{-F}Ge^F) &= e^{-F}Ge^F - i(e^{-F}Ge^F)^* \\ &= e^{-F}Ge^F - ie^{-F}G^*e^F \\ &= e^{-F}(cG)e^F \\ &= e^{-cF/2}(cG)e^{cF/2}. \end{aligned}$$

The last equality follows from subsituting $e^F = e^{(cF+\overline{c}F)/2} = e^{cF/2}e^{\overline{c}F/2}$ and the fact that $\overline{cF}$, and hence $e^{\overline{c}F/2}$, commutes with cG. $\square$

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