LORENTZ TRANSFORMATIONS AND STATISTICAL MECHANICS

Daniel Henry Gottlieb

Abstract.

1. Introduction

In [Gottlieb (1998)] and [Gottlieb (2000)] we launched a study of Lorentz transformations. We find that every Lorentz transformation can be expressed as an exponential e^F where F is a skew symmetric operator with respect to the Minkowski metric $\langle\ ,\ \rangle$ of form $-+++$. We provided F with the notation of electromagnetism. Thus we can describe boosts as pure $\vec{E}$ fields and rotations as pure $\vec{B}$ fields. An important class of Lorentz transformations corresponding to radiation made its appearance. We have yet to see a description of these "radiation" transformations in the Physics literature.

The complexification of the Lorentz Transformations yield very beautiful algebra which is related to Clifford Algebras as well as Lie Algebra theory.

The purpose of this paper is to confront the basic part of Statistical Mechanics with the Lorentz transformation formalism and see how much of Statistical Mechanics can be described in this alternative language.

Using our Lorentzian picture, we identify the central exponentials of Statistical Mechanics, $e^{-\beta\epsilon}$, as the eigenvalues of suitable Lorentz transformations. Since we can have complex eigenvalues, we have a natural scheme which gives Fermions with half integers and Bosons with whole integers and naturally produces the formulas.

$$\langle n_k \rangle = \frac{1}{e^{\beta(\epsilon_k - \mu)} \mp 1}$$

for the mean state occupations numbers $\langle n_k \rangle$ for bosons, with the -1 and fermions with the $+1$. The same qualities play a role in the description of the Maxwell-Boltzman, Bose-Einstein, and Fermi-Dirac statistics.

Finally, we can also give a count of the energy density, in a manner analogous to Rayleigh-Jeans, but not counting by squares of integers because of the wave equation, instead using an argument analogous to the appearance of the sum of squares using Pauli matrices in spin arguments.

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2. Lorentz Transformations

Here we follow [Gottlieb (1998)] and [Gottlieb (2000)] and recall the notation for Lorentz transformations. Let M be Minkowski space with inner product $\langle\ ,\ \rangle$ of the form $-+++$. Let e_0, e_1, e_2, e_3 be an orthonormal basis with e_0 a time-like vector. A linear operator $F : M \to M$ which is skew symmetric with respect to the inner product $\langle\ ,\ \rangle$ has a matrix representation of the form

$$F = \begin{pmatrix} 0 & \vec{E}^T \\ \vec{E} & \times\vec{B} \end{pmatrix}$$

where $\times\vec{B}$ is a 3×3 matrix such that $(\times\vec{B})\vec{v} = \vec{v} \times \vec{B}$, the cross product of $\vec{v}$ with $\vec{B}$. The dual F^* of F is given by

$$F^* = \begin{pmatrix} 0 & -\vec{B}^T \\ -\vec{B} & \times\vec{E} \end{pmatrix}$$

We complexify F by $cF := F - iF^*$. Its matrix representations is

$$F = \begin{pmatrix} 0 & \vec{A}^T \\ \vec{A} & \times(-i\vec{A}) \end{pmatrix} \text{ where } \vec{A} = \vec{E} + i\vec{B}$$

These complexified operators satisfy some remarkable Properties:

a) $cF_1 cF_2 + cF_2 cF_1 = 2\langle\vec{A}_1, \vec{A}_2\rangle I$ where $\langle\ ,\ \rangle$ denotes the complexification of the usual inner product of $\mathbb{R}^3$. Note, $\langle\ ,\ \rangle$ is not the Hermitian form, that is our inner product satisfies $i\langle\vec{v}, \vec{w}\rangle = \langle i\vec{v}, \vec{w}\rangle = \langle\vec{v}, i\vec{w}\rangle$.

b) The same property holds for the complex conjugates $c\overline{F}_1$ and $c\overline{F}_2$ of course. But also, they commute, $c\overline{F}_1 c\overline{F}_2 = c\overline{F}_2 c\overline{F}_1$

c) $cFc\overline{F} = 2T_F$ where T_F is proportional to the stress-energy tensor of electromagnetic fields.

Now $e^F = I + F + \dfrac{F^2}{2!} + \dfrac{F^3}{3!} + \ldots : M \to M$ is a proper Lorentz transformation. It satisfies $e^F = e^{\frac{cF}{2}} e^{\frac{c\overline{F}}{2}}$. The algebraic properties of cF give rise to a simple expression for its exponential:

$$e^{cF} = \cosh(\lambda_{cF})I + \frac{\sinh(\lambda_{cF})}{\lambda_{cF}} cF$$

where λ_{cF} is an eigenvalue for cF. There are only, at most, two values for the eigenvalues of cF, namely λ_{cF} and $-\lambda_{cF}$.

Now every proper Lorentz transformation $L \in SO(3,1)^+$ is equal to the exponential of some skew symmetric linear operator. That is, $L = e^F$ where $F \in \mathfrak{so}(3,1)$. This was proved by Mitsuru Nishikawa in [1983] for $SO(p,1)^+$. I will prove it below (Theorem 4.1), since the proof is novel even though it only works for $p = 3$. In general, it is unusual that a non-compact connected Lie group such as $SO(3,1)^+$ has every element expressed as an exponential. But it is true for $SO(3,1)^+$.

On the other hand, L can be the exponential of more than one F, that is $L = e^F = e^G$ where $F \neq G$. An important example is the fact that the identity is the exponential of an infinite number of skew operators. In fact, $I = e^F$ for $F \neq 0$ if and only if $\lambda_{cF} = 2\pi ni$ where n is an integer. Thus we may say that a Lorentz transformation has "internal states" corresponding to integers. We will identify the odd n with fermions and the even n with bosons for reasons seen below.

This multiplicity of internal states holds for all proper Lorentz transformations L, except for the "radiative" L. That is those $L = e^N$ where N is a radiative, or what we shall now call a *null* skew symmetric operator. That is, $e^F = e^N$ implies $F = N$ if N is null.

3. Occupation Numbers

We will choose [Tolman (1938)] as our text. It is very carefully written classic exposition of Statistical Mechanics. In that text some equations (141.1-3) give the equations for a key concept of Statistical Mechanics: The mean numbers of elements $\langle n_k \rangle$ in a given energy state k with energy ϵ_k at equilibrium. They read:

$$\langle n_k \rangle = \frac{1}{e^{\alpha+\beta\epsilon_k}} \quad \text{(Maxwell-Boltzmann)}$$
$$\langle n_k \rangle = \frac{1}{e^{\alpha+\beta\epsilon_k} - 1} \quad \text{(Bose-Einstein)}$$
$$\langle n_k \rangle = \frac{1}{e^{\alpha+\beta\epsilon_k} + 1} \quad \text{(Fermi-Dirac)}$$

Here α is a constant depending on the chemical potential of the particles, and $\beta = \dfrac{1}{kT}$ where k is Boltzmann's constant and T is the temperature of the equilibrium.

Let us consider the postulate that the exponentials in these equations should be represented as the eigenvalue of a proper Lorentz transformation. Then a first guess would be that $L = e^F$ where $\lambda_F = \alpha + \beta\epsilon$ and $\lambda_{F^*} = 2\pi n$. Thus

$$\lambda_{cF} = \lambda_F - i\lambda_{F^*} = (\alpha + \beta\epsilon_k) - 2\pi ni$$

Let s be a null eigenvector of L. There are only two of them generically. Let s be the eigenvector corresponding to $\lambda_F = \alpha + \beta\epsilon_k > 0$. Then

$$L^{-1}s = e^{-F}s = \frac{1}{e^{\alpha+\beta\epsilon_k}}s = \langle n_k \rangle s$$

for Maxwell-Boltzmann particles, and

$$(L - I)^{-1}s = (e^F - I)^{-1}s = \frac{1}{e^{\alpha+\beta\epsilon_k} - 1}s = \langle n_k \rangle s$$

for Bose-Einstein particles, and

$$(L + I)^{-1}s = (e^F + I)s = \frac{1}{e^{\alpha+\beta\epsilon_k} + 1}s = \langle n_k \rangle s$$

for Fermi-Dirac particles.

Now to present "Physical arguments" that such a choice is reasonable (aside from the fact that it gives the right answer), we follow the history of the "derivation" the three formulas for $\langle n_k \rangle$.

Boltzmann generalizing Maxwell's distribution of velocities in a gas, counted the number of distributions of n particles in k states. Then he used the method of Lagrangian multipliers to find the distribution of maximum probability. This method produces the constants α and β arising in the exponentials as Langrangian multipliers.

Gibbs realized that identical particles should be counted as indistinguishable elements in order to get the right answer to what is called the Gibbs' Paradox. Then to obtain Bose-Einstein and the Fermi-Dirac occupation numbers, one used Boltzmann's procedures based on two different methods of counting identical particles. These methods of counting are called the Einstein-Bose and the Fermi-Dirac statistics. Though they gave the right answers, they did violence to the idea of particle. Eventually with the discovery of Schrodinger's Equation, it was asserted that what was really being counted were the numbers of different combinations of basic eigen-solutions of Schrodinger's equation whose eigenvalues added up to the energy ϵ_k. For Fermi-Dirac statistics, one counts only anti-symmetric combinations of basic solutions, whereas for Bose-Einstein statistics, one counts only the symmetric combinations.

We can give a count of appropriate Lorentz Transformations which will naturally give the same statistics. Let us observe that the three types of statistics arise from the following methods of counting. Suppose G is a free group with k generators $\alpha_1, \dots, a_k$ and we want to count the number of words of length N which can be formed out of these generators such that we have n_i copies of a_i for each i between $1, \dots, N$. Thus we have $n_1 + \dots + n_k = N$. For example, if we have 2 generators a and b and we want to look at the words of length $N = 4$ made out of $3 = n_1$ a's and $1 = n_2$ b's we get $\{ba^3, aba^2, a^2ba, a^3b\}$. This count will give us Maxwell-Boltzmann Statistics.

For Bose-Einstein Statistics, we let G be a free abelian group on k generators $b_1, \dots, b_k$ and we count the number of words of length N for which there are n_i of the b_i. Thus if a and b commute, the set of words with $N = 4$ and $n_1 = 3$ and $n_2 = 1$ is $\{a^3b\}$.

For Fermi-Dirac Statistics, we consider the elementary abelian 2-group $G \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \dots \oplus \mathbb{Z}_k$ with generators $c_1, \dots, c_k$ and we count the number of words of length n. Since $c_i^2 = 1$ we see that n_i is either 0 or 1.

We take advantage of this group theoretical count to use the fact that the Lorentz transformations form a group. To each particle of energy E_i, we assign a Lorentz transformation e^F where F has eigenvalue E_i. If we insist that one of the two null-eigenvectors of F coincide with a fixed null-vector s, then $e^F = e^{F_1}e^{F_2} \dots, e^{F_k}$ and so the eigenvalue of e^F for s is $e^{E_1 + \dots + E_k}$, so $E = E_1 + \dots + E_k$. This corresponds to the fact that the product of eigenfunctions of basic particle states give rise to the sum of eigenvalues under a suitable Hamiltonian, as in [Tolman (1938), equation 87.7].

Now I conjecture (but I have not proved mathematically) that a generic set of Lorentz transformations e^{F_i} all of whose positive eigenvalue's eigenvector coincide with the null vector s, but whose other eigenvectors are distributed randomly, will look like a free group. These should correspond to Maxwell-Boltzmann statistics.

If our particles are not distinguishable, they cohere in a way, so we will let them be represented by Lorentz transformations both of whose null-eigenvectors agree. This implies that the Lorentz transformation e^{F_i} as well as their exponents F_i, commute. For $\lambda_{F_i} > 0$, the set $\{e^{F_1} \dots , e^{F_k}\}$ should behave like the generators of an abelian free group and thus give rise to Bose-Einstein statistics.

For Fermi-Dirac statistics, we let the particles be represented by e^{F_i} where $\lambda_{cF} := e^{(2ni+1)F_i}$. This represents a rotation around 180°, so $(e^{F_i})^2 = I$. So we have at least a central element in the count of words for $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \dots \oplus \mathbb{Z}_2$.

Boltsmann's method arose from an argument by Maxwell in which particles are moving and colliding at statistical equilibrium. We can use Lorentz transformations to describe some features of a kinetic equilibrium which will make the choice of $e^F - I$ natural.

We consider particles at equilibrium in Minkowski space. They are at equilibrium with respect to an observer u. We let u denote a time-like vector of unit length $< u, u >= -1$. We consider a particle at rest which its excited into motion. Its new four-velocity will be denoted by u'. Then $u' - u$ represent a space-like component of the velocity. We consider more particles at rest. Then the average length of the vectors $||u_i' - u||$ must be a descriptor of the equilibrium process. Now we can choose a skew operator F_i, so that e^{F_i} is a Lorentz transformation, so that $e^{F_i}u = u_i'$. Thus we can describe the $u_i' - u$ as $(e^{F_i} - I)u$. If we restrict ourselves to those F_i, all of which have the null vector s as an eigenvector corresponding to $\lambda_{F_i} > 0$, then we should expect properties of the average $\frac{1}{N}(F_1 + \dots , +F_N)$ to be a descriptor of the equilibrium. An example would be the average of the eigenvalues λ_{cF_i} of the F_i. Thus we should expect an average F so that $(e^F - I)u$ has average length.

We may do the same argument with 4-momenta p_i for appropriate classes of particles. Now $e^{F_i}p_i = p_i'$ has the same length. So we should get $\langle e^{F_i}p_0 - p_0, u\rangle =$ energy difference for i^{th} particle. Then $\langle \sum (e^{F_i} - I)p_0, u\rangle =$ total energy difference Then $\langle \sum \frac{(e^{F_i} - I)}{N} p_0, u\rangle =$ Average energy difference.

Let e^F be Lorentz transformation so that $e^{\lambda_F} = \frac{e^{\lambda_{F_1} + \dots + \lambda_{F_N}}}{N}$ and so that s is a null-eigenvector of F. Then $(e^F - I)s = (e^{\lambda_F} - 1)s = \overline{s}$ where $\overline{s}$ is the average energy times s. So $(e^{\lambda F} - 1)^{-1}\overline{s} = \langle n_k \rangle \overline{s}$.

4. Mathematics

Theorem 4.1. *The exponential map Exp:* $\mathfrak{so}(3,1) \to SO(3,1)^+$ *given by* $F \to e^F$ *is onto. That is, for every proper Lorentz transformation* L*, there exists an* $F \in \mathfrak{so}(3,1)$ *so that* $L = e^F$*.*

To prove the above theorem, we need to consider the complexification $S :=$

$\mathfrak{so}(3,1) \otimes \mathbb{C}$ operating on $\mathbb{R}^{3,1} \otimes \mathbb{C}$. This is isomorphic to $\mathbb{C}^4$ and has an inner product which is of the type $-+++$ on $\mathbb{R}^{3,1}$ and extends to the complex vectors by $\langle i\vec{v}, \vec{w}\rangle = \langle \vec{v}, i\vec{w}\rangle = i\langle \vec{v}, \vec{w}\rangle$. See [Gottlieb (2000), p 2] for more details.

Now let $c : \mathfrak{so}(3,1) \to S$ by $cF = F - iF^*$. The image of c, denoted cS, is a three dimensional complex vector space. The set of operators of the form $aI + bcF$ will be denoted by D. Note that D is a vector space isomorphic to $\mathbb{R}^{3,1} \otimes \mathbb{C}$, and that D is closed under multiplication.

Lemma 4.2. *Let F and $G \in cS$ denote cF and cG. Then* $(aI + \beta F)(\alpha I + G) = (a\alpha + b\beta\langle F, G\rangle)I + (b\alpha F + a\beta G + \dfrac{b\beta}{2}[F, G])$

Now we say that $L \in D$ is a *complex Lorentz transformation* if $\langle Lu, Lv\rangle = \langle u, v\rangle$. Any complex Lorentz transformation L must have the form $L = aI + bF$, where $F \in cS$, such that $a^2 - b^2\lambda_F^2 = 1$.

That is, $L^{-1} = aI - bF$.

Theorem 4.3. *Every complex Lorentz transformation L is an exponential, that is $L = e^F$ for some $F \in cS$, except for $L = -I + N$ where $N \in cS$ is null, that is $N^2 = 0$.*

Proof. Recall [Gottlieb (2000), Theorem 8.5] where $F \in cS$ that

$$e^F = \cosh(\lambda_F)I + \frac{\sinh(\lambda_F)}{\lambda_F}F \tag{**}$$

Now $L = aI + H$ where $H \in cS$ and $a^2 - \lambda_H^2 = 1$. So the first obstruction to showing that L is an exponential is solving the equation $\cosh(\lambda) = a$. We shall show below that such a λ always exists. Next, if $\dfrac{\sinh(\lambda)}{\lambda} \neq 0$, then

$$L = aI + H = \cosh(\lambda)I + \frac{\sinh\lambda}{\lambda}\left(\frac{\lambda}{\sinh\lambda}H\right) = \cosh(\lambda)I + \frac{\sinh\lambda}{\lambda}D = e^D$$

Hence L may not be an exponential if $\dfrac{\sinh(\lambda)}{\lambda} = 0$.

Now $\dfrac{\sinh\lambda}{\lambda} = 0$ exactly when $\lambda = \pi ni$ for n a non-zero integer. (Note that $\dfrac{\sinh(0)}{0} = 1$). Then

$$a = \cosh(\lambda) = \cosh(\pi ni) = \cos(\pi n) = (-1)^n.$$

If n is even, then $L = I + N = e^N$ where N must be null.

If n is odd, then $a = (-1)^n = -1$, so $L = -I + N$ where N must be null or zero. Now $e^B = -I$ where $B \in cS$ has eigenvalue $(2k+1)\pi i$. But $-I + N = -e^{-N}$ cannot be an exponential, however N must be null. This proves Theorem 4.3 except for the following lemma.

Lemma 4.4.

a) $\cosh(\lambda) = a$ *always has a solution over the complex numbers.*

b) $\sinh(\lambda) = 0$ *if and only if* $\lambda = \pi ni$.

Proof. First we show b). $\sinh(\lambda) = \dfrac{e^{\lambda} - e^{-\lambda}}{2} = 0$

Thus $e^{2\lambda} = 1$, hence $2\lambda = 2\pi ni$ so $\lambda = \pi ni$.

Next we show a). $\cosh(\lambda) = \dfrac{e^{\lambda} + e^{-\lambda}}{2} = a$. Hence $(e^{\lambda})^2 - 2ae^{\lambda} + 1 = 0$

Hence $e^{\lambda} = \dfrac{2a \pm \sqrt{4a^2 - 4}}{2} = a \pm \sqrt{a^2 - 1}$.

Now $e^{\lambda} = b$ has a solution for all b except $b = 0$. But $a \pm \sqrt{a^2 - 1}$ cannot equal zero, hence we have shown there is a solution for each a.

Proof of Theorem 4.1. We show the exponential map is onto $SO(3,1)^{+}$ by showing the products of two exponentials is an exponential. That is $e^F e^G = e^D$ for $F, G, D \in \mathfrak{so}(3,1)$. Now $e^F = e^{\frac{1}{2}cF} e^{\frac{1}{2}\overline{c}F}$ where $\overline{c}F = F + iF^*$. This follows since cF and $\overline{c}F$ commute. Also for this reason, e^{cF} and $e^{\overline{c}G}$ commute. Thus $e^F e^G = e^{\frac{1}{2}cF} e^{\frac{1}{2}cG} e^{\frac{1}{2}\overline{c}F} e^{\frac{1}{2}\overline{c}G}$. Now $e^{\frac{1}{2}cF} e^{\frac{1}{2}cG}$ is a complex Lorentz transformation in D. So either it is an exponential e^{cD}, or it has the form $-I + cN = -e^{cN}$ by Theorem 4.3. Now Theorem 4.3 also holds for $\overline{D} \in \overline{c}S \otimes \mathbb{C}$. Hence we have $e^F e^G = e^{2D}$ or $e^F e^G = (-e^{cN})(-e^{\overline{c}N}) = e^{2N}$.

Corollary 4.5. *The exponential map Exp* : $S \to SO(\mathbb{R}^{3,1} \otimes \mathbb{C})$ *is not onto. If* $N \in \mathfrak{so}(3,1)$ *is null, then* $-e^N$ *is not an exponential even though* $-e^{cN}$ *is an exponential.*

Proof. As explained in [Gottlieb (2000)], we can extend duality F^* to skew symmetric matrices $\begin{pmatrix} 0 & \vec{E} \\ \vec{E} & \times\vec{B} \end{pmatrix}$ where $\vec{E}$ and $\vec{B}$ are complex vectors. Then $cF = F - iF^*$ and $\overline{c}F = F + iF^*$ satisfy the same properties as in the complexification of the real case. Now consider $e^F e^G$ where F, $G \in S$. Then $cF = \frac{1}{2}\,cF + \frac{1}{2}\,\overline{c}F$, so $e^F e^G = e^{\frac{1}{2}cF} e^{\frac{1}{2}\overline{c}F} e^{\frac{1}{2}cG} e^{\frac{1}{2}\overline{c}G}$. Now $cF = cA$ for some $A \in \mathfrak{so}(3,1)$, and $\overline{c}F = \overline{c}A'$ for $A' \in \mathfrak{so}(3,1)$

$$e^F e^G = e^{cA} e^{\overline{c}A'} e^{cB} e^{\overline{c}B'} = (e^{cA} e^{cB})(e^{\overline{c}A'} e^{\overline{c}B'})$$

and so $e^{cA} e^{cB}$ equals either e^{cD} or $-e^{cN}$. But $(-I)e^{cN} = e^{(2n+1)\pi i\overline{c}E} e^{cN} = e^{(2n+i)\pi i\overline{c}E + cN}$ where E has eigenvalue equal to 1. So in both cases $e^{cA} e^{cB}$ is an exponential.

Now $-e^{cN}$ is an exponential since $-e^{cN} = e^{\pi i\overline{c}E} e^{cN} = e^{\pi i\overline{c}E + cN}$ where E has eigenvalue $\lambda_{cE} = 1$. On the other hand $-e^N$, where N is null and real, cannot be an exponential, since if $-e^N = e^F$, then s, the unique eigenvector for e^N, applied to this equation gives $-s = e^F s = e^{\lambda_F} s$, so $\lambda_F = (2n+1)\pi i$ for some n. Thus F has another eigenvector, which contradicts $-e^N$ having only one. □

5. Lorentz transformations sharing eigenvectors

Suppose that we have a set of complex Lorentz transformations $L_i \in cS$ so that there is a null vector s which is an eigenvector for all the L_i. Let N be a null operator in cS. Then each F_i can be expressed as $F_i = \lambda_i E + \alpha_i N$ where E and N share the eigenvector s and has $\lambda_E = 1$, and $L_i = e^{F_i}$; or else $L_i = -e^{\alpha_i N}$.

Now $F_i N = \lambda_i N$. So $e^{F_i} N = e^{\lambda_i} N$ and $N e^{F_i} = e^{-\lambda_i} N$. This follows since N and F_i share an eigenvector, so $N F_i = -F_i N$.

Lemma 5.1. *If F, $N \in cS$ and $N^2 = 0$ and $FN = \lambda N$, then $e^F e^N = e^F + e^\lambda N = e^H$ where $H = F + \dfrac{\lambda_F e^{\lambda_F}}{\sinh \lambda_F} N$. The last equality holds only if $\lambda_F \neq n\pi i$ for non zero integers n.*

Proof.

$$e^F e^N = e^F(I+N) = e^F + e^F N = e^F + e^\lambda N = \cosh\lambda I + \frac{\sinh\lambda}{\lambda} F + e^\lambda N$$
$$= \cosh\lambda I + \frac{\sinh\lambda}{\lambda} H,$$

provided $\dfrac{\sinh\lambda}{\lambda} \neq 0$.

Theorem 5.2. *Let $L_1 L_2 \dots L_k$ be a product of complex Lorentz transformations $L_i \in cS$ all sharing a common eigenvector s. Assume that none of the L_i have eigenvalues of the form $\lambda_i = n\pi i$ for $n \neq 0$, and that the sum of the eigenvalues $\sum_1^k \lambda_i$ is not equal to $n\pi i$ for nonzero n. So each $L_i = e^{\lambda_i E + \alpha_i N}$ or else $-e^{\alpha_i N}$. Then $L_1 \dots L_k = \pm \left(e^{(\sum_1^k \lambda_i)E} + \left(\sum_{i=1}^k e^{\lambda_1 + \dots + \lambda_{i-1} - \lambda_{i+1} + \dots + \lambda_k} \alpha_i \dfrac{\sinh \lambda_i}{\lambda_i} \right) N \right)$.*

Proof.

$$L_1 \dots L_k = \pm \prod_1^k e^{\lambda_i E + \alpha_i N} = \pm \prod_1^k (e^{\lambda_i E} + \alpha_i \frac{\sinh\lambda_i}{\lambda_i} N)$$
$$= \pm \left(e^{(\sum_1^k \lambda_i)E} + \sum_{j=1}^k e^{(\sum_1^{j-1} \lambda_i)} \alpha_j \frac{\sinh\lambda_j}{\lambda_j} N e^{(\sum_{j+1}^k \lambda_i)E} + 0 + 0 + \dots + 0 \right)$$
$$= \pm \left(e^{(\sum_1^k \lambda_i)E} + \left(\sum_{j=1}^k e^{(\sum_1^{j-1} \lambda_i - \sum_{j+1}^k \lambda_i)} \frac{\sinh\lambda_j}{\lambda_j} \alpha_j \right) N \right]$$
$$= \pm e^H$$

where

$$H := \frac{\sum_1^k \lambda_i}{\sinh(\sum_1^k \lambda_i)} \left(\left(\sum_1^k \lambda_i \right) E + \sum_{j=1}^k \left(e^{\sum_1^{j-1} \lambda_j - \sum_{j+1}^k \lambda_j} \right) \frac{\sinh\lambda_j}{\lambda_j} \alpha_j N \right)$$

assuming $\sum_1^k \lambda_i \neq n\pi i$ for $n \neq 0$.

Corollary 5.3. *Suppose the L_i commute. Then $L_i = e^{\lambda_i(E+N)}$, or else $L_i = \pm e^{\alpha_i N}$ for all i. Then $\prod_1^k L_i = e^{aE} + bN$ where $a = \sum_i^k \lambda_i$ and $b = \sum_1^k \exp\left[\sum_1^{j-1} \lambda_i + \sum_{j+1}^k (-\lambda_i)\right] \frac{\sinh \lambda_j}{\lambda_j}$. Hence $\prod L_i = e^{(\sum_1^k \lambda_i)(E+N)}$.*

Let $F \in cS$ and let suppose N^+ and N^- are null operators in cS so that $FN^+ = \lambda_F N^+$ and $FN^- = -\lambda_F N^-$. Then $F + aN^+$ has the same eigenvalue as F with N^+ as the same "eigenvector". Here $a \in \mathbb{C}$. The question we pose is: What is the "eigenvector" N' corresponding to $-\lambda_F$?

Theorem 5.4.

a) Suppose $(F + aN^+)N' = -\lambda N'$. Then $a = \dfrac{-\langle F, N'\rangle}{\langle N^+, N'\rangle}$.

b) The N' for $F + aN^+$ is given by any complex multiple of

$$N' = \frac{-a}{2\lambda^2} F + \frac{1}{2} N^- - \frac{a^2}{2\lambda^2} N^+$$

We are choosing N^- so that $\langle N^+, N^-\rangle = 2$.

Proof of a). Expand $(F + aN^+)N' = -\lambda N'$ and use the equation $FG = \langle F, G\rangle I + \frac{1}{2}[F, G]$ found in [Gottlieb (2000), Lemma 5.5] to arrive at the equation

$$(\langle F, N'\rangle + a\langle N^+, N'\rangle)I + \frac{1}{2}[F + aN_+, N'] = -\lambda N'.$$

The coefficient of I must be zero since skew operators have zero trace. The unique a so that the coefficient is zero is the required result.

Proof of b). Let u be an observer. Then Fu completely determines $F \in cS$. So choose a basis in $\mathbb{R}^{3,1} \otimes \mathbb{C}$ so that $Fu = (0, 0, \lambda)$, $N^+u = (1, i, 0)$ and $N^-u = (1, -i, 0)$. Let $N'u = (r, s, t)$. We can choose N' so that $\langle N^+, N'\rangle = 1$. Then $\langle F, N'\rangle = -a$. Also $\langle N', N'\rangle = 0$. From these three equations we obtain three equations

$$r + si = 1$$
$$\lambda t = -a$$
$$r^2 + s^2 + t^2 = 0.$$

The solution is $N'u = \left(\dfrac{\lambda^2 - a^2}{2\lambda^2}, -\dfrac{(\lambda^2 + a^2)i}{2\lambda^2}, -\dfrac{a}{\lambda}\right)$. We can write this as

$$\begin{aligned} N'u &= -\frac{a}{2\lambda}(0, 0, 1) + \frac{1}{2}(1, -i, 0) - \frac{a^2}{2\lambda^2}(1, i, 0) \\ &= -\frac{a}{2\lambda^2} Fu + \frac{1}{2} N^- u - \frac{a^2}{2\lambda^2} N^+ u. \end{aligned}$$

This gives us the general equation since it holds for all vectors if it holds for u. □

A simple method to calculate the real eigenvector s' is to calculate the complex 3 vector $N'u =: \vec{E} + i\vec{B}$. Then $s' = E^2 u + \vec{E} \times \vec{B}$.

We see from this process that $F+aN^+$ has a different eigenvectors s' corresponding to $-\lambda_F$ and they are in $1-1$ correspondence with the points in $\mathbb{C}$, assuming F is not null.

Another way to calculate the other eigenvector is to note that $F+N^+ = e^{-\frac{N^+}{2}}Fe^{\frac{N^+}{2}}$. Now $e^{-\frac{N^+}{2}}N^-$ is an eigen-operator for $F+N^+$ associated with $-\lambda_F$. It follows that a real eigenvector can be obtained by $s^- = e^{-N}s^-$ where the N denotes the real null operator of N^+ .

Now there is a cononical equation for an eigenvector s corresponding to the λ_F eigenvalue of $F \in \mathfrak{so}(3,1)$. For a given observer u, F is determined by $\vec{E}$ and $\vec{B}$. Then the canonical eigenvector s is given by the following equation, [Gottlieb (1998), Corollary 6.8].

$$\begin{aligned} s &:= \frac{1}{2}(\lambda_{cF}I + cF)(\overline{\lambda_{cF}I - cF}) \\ &= \left(\lambda_T + \frac{E^2+B^2}{2}\right)u + \vec{E}\times\vec{B} + \lambda_F\vec{E} - \lambda_{F^*}\vec{B}. \end{aligned}$$

Theorem 5.5. *Let $F_0 + N = F$, where F_0, N, and $F \in \mathfrak{so}(3,1)$ and suppose N is null whose eigenvector, s_1, lies along the eigenvector, s_0, of F_0 corresponding to the eigenvalue λ_{cF_0}. Then $s = s_0 + s_1$, where s is the canonical eigenvector of F, if we assume that F_0 has linearly dependent $\vec{E}$ and $\vec{B}$ for observer u.*

Proof. Now in u's rest space, F_0 is composed out of $\vec{E}_0$ and $\vec{B}_0$ where $\lambda_{F_0}\vec{E}_0 - \lambda_{F*}\vec{B}_0 =: \vec{k}$, and $\vec{E}_0\times\vec{B}_0 = 0$. N_1 is determined by $\vec{E}_1$ and $\vec{B}_1$ where $E_1 = B_1$, $\vec{E}_1\cdot\vec{B}_1 = 0$ and $\vec{E}_1\cdot\vec{k} = \vec{B}_1\cdot\vec{k} = 0$. Then F is determined by $\vec{E} = \vec{E}_0 + \vec{E}_1$ and $\vec{B} = \vec{B}_0 + \vec{B}_1$. Under these conditions,

$$\text{(a)} \qquad s = \left(\lambda_T + \frac{E^2+B^2}{2}\right)u + \vec{E}\times\vec{B} + \lambda_F\vec{E} - \lambda_{F*}\vec{B}$$

$$\text{(b)} \qquad s_0 = \left(\lambda_T + \frac{E_0^2+B_0^2}{2}\right)u + \lambda_F\vec{E}_0 - \lambda_{F*}\vec{B}_0$$

$$\text{(c)} \qquad s_1 = E_1^2 u + \vec{E}_1\times\vec{E}_2.$$

Now we eliminate $\vec{E}$ and $\vec{B}$ in equation (a), use the fact that

$$\frac{E^2+B^2}{2} = \frac{E_0^2+B_0^2}{2} + \frac{E_1^2+B_1^2}{2} = \lambda_T + E_1^2$$

and then that $\vec{E}\times\vec{B} = \vec{E}_1\times\vec{B}_1 + \vec{E}_0\times\vec{B}_1 + \vec{E}_1\times\vec{B}_0$ and equations b) and c) to obtain $s = s_0 + s_1 +$ space-like vectors. But the space-like vectors must add up to a light-light vector which implies that they must add up to $\vec{0}$. This implies that $s_0 + s_1 = s$.

6. Bosons and Fermions

Now recall that the mean occupation number $\langle n_k \rangle$ of states of a particle at equilibrium at a temperature T is given by

$$\langle n_k \rangle = \frac{1}{e^{\beta(\epsilon_k - \mu)} \mp 1}$$

depending on whether the particle is a boson or a fermion. Here ϵ_k is the energy of the kth state, $\beta = \frac{1}{kT}$ and μ is the chemical potential.

Our first approximation to describing this was to postulate that there was a Lorentz transformation e^F so that $\lambda_F = \beta(\epsilon_k - \mu)$. However, there are different F's which give rise to this Lorentz transformation. So we can consider F with eigenvalues $\lambda_{cF} = \beta(\epsilon_k - \mu) + 2\pi s i$ where s is a half integer $\frac{n}{2}$. Then

$$\frac{1}{e^{\lambda_{cF}} - 1} = (-1)^{2s} \langle n_k \rangle.$$

We then should correspond the integers s to bosons and the half integers s to fermions in order to get the correct formulas.

We can associate a cF to an equilibrium as easily as F. The $(e^{cF} - I)$ will give us a real vector which describes a mean change of direction, and a complex vector which describes some sort of mean rotation or spinning. Then $\lambda_{cF} = \beta(\epsilon_k - \mu) + 2\pi s i$. Hence

$$(e^{cF} - I)^{-1} cN = (-1)^{2s} \langle n_k \rangle cN$$

where cN is the null eigen-operator for cF, that is $cFcN = \lambda_{cF} cN$.

Now suppose e^{cF} represents some mean description of a system in equilibrium at temperature T. We expect λ_{cF} to remain constant in time. But e^{cF} distinguishes different directions, so we would expect e^{cF} to vary in time so that $e^{\lambda_{cF}}$ remains constant.

There is a beautiful formula, due to Helgason, which gives the differential of the exponential map. See [Gottlieb (2000), §4] or [Helgason (1978), p.105, Theorem 7.1].

Theorem 6.1. (Helgason). *Let $\mathfrak{g}$ be the Lie Algebra for a Lie group $\mathbf{G}$. Then $\nabla_G e^F := \frac{d}{dt}(e^{F+tG})|_{t=0} = e^F \cdot (g[adF](G))$ where $ad\ F \colon \mathfrak{g} \to \mathfrak{g}$ sends $X \longmapsto [F, X]$ and $g[\xi]$ is the power series of the function $g(\xi) = \dfrac{e^{-\xi} - 1}{-\xi}$.*

Now cF has two eigen-operators, cN^+ and cN^- so that $cFcN^+ = \lambda_{cF} cN^+$ and $cFcN^- = -\lambda_{cF} cN^-$. Hence we get

$$\nabla_{cN} e^{cF} = e^{\lambda_{cF}} \left(\frac{e^{-2\lambda_{cF}} - 1}{-2\lambda_{cF}} \right) cN = \frac{e^{\lambda_{cF}} - e^{-\lambda_{cF}}}{2\lambda_{cF}} = \frac{\sinh(\lambda_{cF})}{\lambda_{cF}} cN.$$

This also holds for cN^- and so it is true for any linear combination of cN and cN^-. This subspace is characterized by cG such that $\langle cG, cF \rangle = 0$. This $G \longmapsto \nabla_G e^F$ is completely characterized by the following theorem.

Theorem 6.2. $\nabla_{cG} e^{cF} = \dfrac{\sinh \lambda_{cF}}{\lambda_{cF}} cG$ *if* $\langle cF, cG\rangle = 0$, *and* $\nabla_{cF} e^{cF} = e^F \circ F$.

Now if $e^F \in SO(3,1)^+$ represents the process in equilibrium, we saw that $\lambda_{cF} = \beta(\epsilon_k - \mu) + 2n\pi i$ was a reasonable choice of eigenvalue. Since $e^F = e^{\frac{1}{2}cF} e^{\frac{1}{2}\overline{c}F}$, the choice for a complexified cF representing the process should have eigenvalue $\lambda_{cF} = \frac{1}{2}\beta(\epsilon_k - \mu) + n\pi i$. Hence

$$(*) \qquad \begin{aligned} \frac{\sinh \lambda_{cF}}{\lambda_{cF}} &= e^{-\lambda_{cF}} \left[\frac{e^{+2\lambda_{cF}} - 1}{+2\lambda_{cF}} \right] = \frac{e^{-\frac{1}{2}\beta(\epsilon_k - \mu) - n\pi i} \left(\frac{1}{\langle n_k \rangle} \right)}{\beta(\epsilon_k - \mu) + 2n\pi i} \\ &= \frac{1}{\langle n_k \rangle} \cdot \frac{(\beta(\epsilon_k - \mu) - 2n\pi i)}{\beta^2(\epsilon_k - \mu)^2 + 4n^2\pi^2} \cdot (-1)^n e^{-\frac{1}{2}\beta(\epsilon_k - \mu)}. \end{aligned}$$

Scholium 6.3 Let cF represent an equilibrium process then

a) $\nabla_G e^{cF} = \dfrac{(-1)^n}{\langle n_k \rangle} e^{-\frac{1}{2}\beta(\epsilon_k - \mu)} \dfrac{(\beta(\epsilon_k - \mu) - 2n\pi i)}{\beta^2(\epsilon_k - \mu)^2 + 4n^2\pi^2} cG$ if $\langle cG, cF\rangle = 0$.

b) $e^{cF} = \cosh(\dfrac{1}{2}\beta(\epsilon_k - \mu) + n\pi i) I + \dfrac{(-1)^n}{\langle n_k \rangle} e^{-\frac{1}{2}\beta(\epsilon_k - \mu)} \dfrac{(\beta(\epsilon_k - \mu) - 2n\pi i)}{\beta^2(\epsilon_k - \mu)^2 + 4n^2\pi^2} cF$.

Proof. We prove a) by substituting (*) into the previous theorem. We prove b) by substituting (*) into equation (**) in proof of Theorem 4.3.

Now we take the differential of e^F for $F \in \mathfrak{so}(3,1)$ and N null with a common eigenvalue.

Theorem 6.4. *If* N, $F \in \mathfrak{so}(3,1)$ *Where* N *and* F *both share an eigenvector and* N *is null, then* $e^{-F} \nabla_N e^F = \dfrac{e^{-\lambda_{cF}} - 1}{-\lambda_{cF}} \cdot N$ *where the dot is electromagnetic duality.*

Proof.

$$\begin{aligned} [F, N] &= \left[\frac{1}{2} cF + \frac{1}{2} cF, \frac{1}{2} cN + \frac{1}{2} cN \right] = \frac{1}{4} \left([cF, cN], [\overline{c}F, \overline{c}N] \right) \\ &= \frac{2}{4} \mathrm{Re}[cF, cN] = \mathrm{Re} \left(\frac{1}{2} (2\lambda_{cF} cN) \right) = \lambda_{cF} \cdot N. \end{aligned}$$

Now $e^{-F} \circ \nabla_N e^F = g([F, N]) = \dfrac{e^{-\lambda_{cF}} - 1}{-\lambda_{cF}} \cdot N$.

Scholium 6.5. *Let* F *represent an equilibrium at temperature* T *with null* N *sharing eigenvector and* $\lambda_{cF} = \beta(\epsilon_k - \mu) + 2\pi n i$. *Then* $e^{-F} \circ \nabla_N e^F = \dfrac{e^{-\lambda_{cf}}}{\langle n_k \rangle \lambda_{cF}} \cdot N$.

7. The Rayleigh-Jeans Factor

The main use of the mean occupations numbers is in the establishment of useful differential formulas. These comprise the *Planck radiation law*, [Tolman (1938), (93.6)]

$$du = \frac{8\pi h v^3}{c^3} \frac{1}{e^{hv/kT} - 1} dv.$$

The number of particles in a Bose-Einsten gas [Tolman (1938), 93.11]

$$dn = \frac{4\pi v g}{h^3} m\sqrt{2m} \frac{\epsilon^{\frac{1}{2}}\, d\epsilon}{e^{\alpha+\beta\epsilon} - 1}$$

The number of particles in a Fermi-Dirac gas [Tolman (1938),94.6]

$$dn = \frac{4\pi v g}{h^3} m\sqrt{2m} \frac{\epsilon^{\frac{1}{2}}\, d\epsilon}{e^{\alpha+\beta\epsilon} + 1}$$

The coefficients $\dfrac{4\pi v g}{h^3} m\sqrt{2m\epsilon}$ of the last two equations are derived in [Tolman (1938), §7]. They arise from solutions of the stationary Schrodinger's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = -\frac{8\pi^2 m}{h^2} E.$$

The appropriate solutions are characterized by triples of non-negative integers (n_1, n_2, n_3) which lie in a small spherical shell in the first octant. This gives the number of eigen-solutions as $G = \dfrac{4\pi v}{h^3} m\sqrt{2mE}\Delta E$, which is [Tolman (1938), (71.16)].

We may produce the same lattice of integers by using skew symmetric operators. Let E_x, E_y, E_z be skew symmetric real operators such that $E_x u = \vec{e}_x$, the unit vector in the x direction. Similarly for E_y and E_z. Let cE_x, cE_y, cE_z be their complexifications. Then $cE_x^2 = cE_y^2 = cE_z^2 = I$ and they all anticommute with each other, i.e. $cE_x cE_y = -cE_y cE_x$.

Now consider $cE = n_1 cE_x + n_2 cE_y + n_3 cE_z$. Then $\lambda_{cE}^2 I = cE^2 = (n_1^2 + n_2^2 + n_3^2)I$. Now λ_{cE} is real, so $\lambda_{cE}^2 = |\lambda_{cE}\overline{\lambda}_{cE}| = 2\lambda_T$. We can identify λ_T with energy since $T = \frac{1}{2} cE\overline{c}E$ corresponds to the stress energy tensor and so $2EV$ (where V is volume) should be the energy. This necessitates letting $\dfrac{h}{2m}\, cE_x$, $\dfrac{h}{2m} cE_y$, $\dfrac{h}{2m} cE_z$ be our basic set of skew operators. Counting the number of allowable triples (n_1, n_2, n_3) as in the usual argument gives us the same formula as in the standard argument.

If we are in a situation where the opposite eigenvector s-corresponding to $\lambda_{cE_x} = -\dfrac{h}{2m}$ is distinguished, as by an opposite spin, then we want to count $\dfrac{h}{2m} cE_x$ twice and so we multiply by two the factor to get the number of eigenstates G corresponding to the energy range E to $E + \Delta E$. That is

$$G = \frac{8\pi V}{h^3} m\sqrt{2m\, E}\Delta E$$

for spin.

Multiplying the mean occupation numbers by the number of particles gives us the formulas above .

For the black body radiation formula, we have

$$du = 2 \cdot \frac{4\pi V^2}{c^3}\, d\nu \cdot kT \cdot \frac{\frac{h\nu}{hT}}{e^{h\nu/kT} - 1}.$$

Now the factor $\dfrac{8\pi\nu^2}{c^2}kT\,d\nu$ gives the Rayeigh-Jeans Law. So

$$\frac{e^{h\nu/kT}-1}{\frac{h\nu}{kT}} = \frac{8\pi\nu^2}{c^3}kT\frac{d\nu}{du}.$$

Now let cN^- be the eigen-operator corresponding to $cFcN^- = -\lambda_{cF} = -\frac{h\nu}{kT}$. Then

$$e^{-cF}\nabla_{cN^-}e^{cF} = \frac{8\pi V^2}{c^3}kT\frac{d\nu}{du}cN^-.$$

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