PROJECTIVELY FULL IDEALS IN NOETHERIAN RINGS (II)

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Abstract

Let R be a Noetherian commutative ring with unit $1 \neq 0$, and let I be a regular proper ideal of R. The set $\mathbf{P}(I)$ of integrally closed ideals projectively equivalent to I is linearly ordered by inclusion and discrete. There is naturally associated to I and to $\mathbf{P}(I)$ a numerical semigroup $S(I)$; we have $S(I) = \mathbb{N}$ if and only if every element of $\mathbf{P}(I)$ is the integral closure of a power of the largest element K of $\mathbf{P}(I)$. If this holds, the ideal K and the set $\mathbf{P}(I)$ are said to be projectively full. A special case of the main result in this paper shows that if R contains the rational number field $\mathbb{Q}$, then there exists a finite free integral extension ring A of R such that $\mathbf{P}(IA)$ is projectively full. If R is an integral domain, then the integral extension A has the property that $\mathbf{P}((IA + z^*)/z^*)$ is projectively full for all minimal prime ideals z^* in A. Therefore in the case where R is an integral domain there exists a finite integral extension domain $B = A/z^*$ of R such that $\mathbf{P}(IB)$ is projectively full.

1 INTRODUCTION.

All rings in this paper are commutative with a unit $1 \neq 0$. Let I be a regular proper ideal of the Noetherian ring R (that is, I contains a regular element of R and $I \neq R$). Recall that an ideal J in R is **projectively equivalent** to I in case $(J^j)_a = (I^i)_a$ for some positive integers i and j (where K_a denotes the integral closure in R of an ideal K of R). The concept of projective equivalence of ideals and the study of ideals projectively equivalent to I was introduced by Samuel in [16] and further developed by Nagata in [8]. Making use of interesting work of Rees in [13], McAdam, Ratliff, and Sally in [7, Corollary 2.4] prove that the set $\mathbf{P}(I)$ of integrally closed ideals projectively equivalent to I is linearly ordered by inclusion and is discrete. They also prove that if I and J are projectively equivalent, then the set Rees I of Rees valuation rings of I is equal to the set Rees J of Rees valuation rings of J and the values of I and J with respect to these Rees valuation rings are proportional [7, Proposition 2.10]. We observe in [1] that the converse also holds and further develop the connections between projectively equivalent ideals and their Rees valuation rings. For this purpose, we define in [1] the ideal I to be **projectively full** if the set $\mathbf{P}(I)$ of integrally closed ideals projectively equivalent to I is precisely the set $\{(I^n)_a\}$ consisting of the integral closures of the powers of I. If there exists a projectively full ideal J that is projectively equivalent to I, we say that $\mathbf{P}(I)$ is **projectively full**. As described in [1], there is naturally associated to I and to the projective equivalence class of I a numerical semigroup $S(I)$. One has $S(I) = \mathbb{N}$, the semigroup of nonnegative integers under addition, if and only if $\mathbf{P}(I)$ is projectively full.

In [7, (3.6)] and in [1, (4.13)] it is noted that $\mathbf{P}(I)$ is projectively full for each nonzero ideal I in a regular local ring of altitude two. On the other hand, in [2] we give an example of an integrally closed local (Noetherian) domain (L, M) of altitude two such that M (and hence $\mathbf{P}(M)$) is not projectively full. We mention in the paragraph just before Proposition 4.3 of [2] that a problem we have not been able to solve is whether, for a given nonzero ideal I of a Noetherian domain R, there always exists a finite integral extension domain A of R such that $\mathbf{P}(IA)$ is projectively full. In [2, Proposition 4.3] we give a "logical candidate" for A and prove for this A that there exists an ideal H of A such that every $J \in \mathbf{P}(I)$ has the property that $(JA)_a$ is the integral closure of a power of H. A special case of Theorem 2.4 in the present paper shows that if I is a regular proper ideal in a Noetherian ring R that contains the rational number field, then there exists a finite integral extension ring A of R such that $\mathbf{P}(IA)$ is projectively full. To obtain in Theorem 2.4 such an extension ring A of R, the additional requirement needed in the construction given in Proposition 4.3 of [2] is that certain subsets of the Rees valuation rings of I are unramified with respect to the extension.

We now give a brief summary of the contents of this paper

In Section 2 we show in Theorem 2.4 that if $I = (b_1, \dots, b_g)R$ and $\{(V_1, N_1), \dots, (V_n, N_n)\}$ is a nonempty subset of Rees I such that: (a) $b_i V_j = IV_j$ $(= {N_j}^{e_j}$, say) for $i = 1, \dots, g$ and $j = 1, \dots, n$; and, (b) the greatest common divisor c of $e_1, \dots, e_n$ is a unit in R; then $A = R[x_1, \dots, x_g]$ $(= R[X_1, \dots, X_g]/(({X_1}^c - b_1, \dots, {X_g}^c - b_g)))$ is a finite free integral extension ring of R such that its ideal $J = (x_1, \dots, x_g)A$ is projectively full and projectively equivalent to I, so $\mathbf{P}(IA)$ is projectively full. Also, if R is an integral domain and if ${z_1}^*, \dots, {z_m}^*$ are the minimal prime ideals in A, then $\mathbf{P}(IB_h)$ is projectively full for $h = 1, \ldots, m$, where $B_h = A/{z_h}^*$. Then in Remark 2.6.1 and Remark 2.6.2 it is shown that I has a basis $b_1, \ldots, b_g$ such that (a) holds if either R is local with an infinite residue field, or $n = 1$. In Remark 2.6.4 it is shown that (b) may be replaced with the weaker assumption that $c \notin (N_1 \cap R) \cup \cdots \cup (N_n \cap R)$. Corollary 2.7 states that if R is a Noetherian ring that contains the field of rational numbers, then for each regular proper ideal I of R there exists a finite free integral extension ring A of R such that $\mathbf{P}(IA)$ is projectively full. If R is an integral domain, there exists a finite integral extension domain $B = A/z^*$ of R such that $\mathbf{P}(IB)$ is projectively full.

In Proposition 3.1 of Section 3 we observe the following: (i) R and A satisfy the Theorem of Transition as formulated by Nagata in [9, Section 19]; (ii) $A/J = R/I$, so there is a one-to-one correspondence between the ideals H in R that contain I and the ideals H' in A that contain J; (iii) A is Cohen-Macaulay if and only if R is Cohen-Macaulay; and (iv) $b_1, \ldots, b_g$ is an R-sequence if and only if $x_1, \ldots, x_g$ is an A-sequence. The relation between the ideals H in $\mathbf{P}(I)$ and the ideals $(HA)_a$ in $\mathbf{P}(IA)$ is considered in Corollary 3.2 and Remark 3.3. The special case of Theorem 2.4 where R is local and I is an open ideal is considered in Corollary 3.4.

In Section 4 we concentrate on the case of Theorem 2.4 where $n = 1$, that is, only one Rees valuation ring (V, N) of I is considered. In this case, (a) of Theorem 2.4 holds by Remark 2.6.2. If the integer c such that $IV = N^c$ is a unit in V, then it is shown in Lemma 4.2.3 and Corollary 4.3 (together with Proposition 4.1.3) that there exists a valuation ring (U, M) extending V and a minimal prime ideal z^* in A such that H is projectively full for all ideals H in all Noetherian rings B such that $A/z^* \subseteq B \subseteq U$ and $JB \subseteq H \subseteq M \cap B$. In particular, if B is such a ring, then there exists a prime ideal P containing JB such that JB, P, JB_P, and PB_P are projectively full.

In Example 5.1.1 of Section 5 it is shown that a regular ideal I of R is projectively full if the associated graded ring $\mathbf{G}(R, I)$ has a minimal divisor p that is its own p-primary component of (0), while in Example 5.2 it is shown that the projectively full ideal J of Theorem 2.4 may have an embedded prime divisor P that is the center of a Rees valuation ring (U, M) such that $JU = M$. Then some cases where J_a is a prime (resp., radical) ideal are considered in Example 5.3 (resp., Example 5.4).

In Example 6.1 of Section 6, we consider the behavior of the projectively full property between R and R^+, where R is a Noetherian domain and R^+ is a Noetherian integral extension domain of R contained in the field of fractions of R. For a nonzero proper ideal I of R, (i) if IR^+ is projectively full, then I is projectively full, but the converse fails, (ii) there exist examples where $\mathbf{P}(I)$ is projectively full and $\mathbf{P}(IR^+)$ fails to be projectively full, and examples where, conversely, $\mathbf{P}(I)$ fails to be projectively full and $\mathbf{P}(IR^+)$ is projectively full. In Example 6.4 we present several examples of Noetherian domains R that are not integrally closed and have the property that $\mathbf{P}(I)$ is projectively full for each nonzero proper ideal I of R. In Example 6.6 we present a family of examples of Noetherian domains R for which there exists an integral extension domain B that differs from the integral extension domain obtained using Theorem 2.4, and has the property that $\mathbf{P}(IB)$ is projectively full for each nonzero proper ideal I of R. In Example 6.8 we present an example of a normal local domain (R, M) of altitude two such that M is projectively full and the associated graded ring $\mathbf{G}(R, M)$ is not reduced. In Remark 6.9, we present an argument of J. Lipman to show that if (R, M) is a normal local domain of altitude two that has a rational singularity, then $\mathbf{P}(I)$ is projectively full for each M-primary ideal I of R.

Our notation is as in [9] and [5]. Thus, for example, elements $b_1, \dots, b_g$ in an ideal I form a **basis** of I if they generate I.

2 FINITE FREE EXTENSION RINGS A OF R IN WHICH P(IA) IS PROJECTIVELY FULL.

Projectively full ideals are introduced in [1, Section 4]. It is observed in [1, (4.13)] that $\mathbf{P}(I)$ is projectively full for every nonzero proper ideal I in a regular local domain of altitude two; see also [7, (3.6)]. In [2] a number of basic properties of a projectively full ideal are developed, and then it is asked if, for a given regular proper ideal I in a Noetherian ring R, there exists a finite integral extension ring A of R such that $\mathbf{P}(IA)$ is projectively full. It follows from Theorem 2.4 that this is frequently the case.

The following two remarks and definition will be useful in the proof of Theorem 2.4.

Remark 2.1 Let R be a Noetherian ring, let $I = (b_1, \ldots, b_g)R$ be a regular proper ideal of R, let c be a positive integer, let $R_g = R[X_1, \ldots, X_g]$, and let $K = ({X_1}^c - b_1, \ldots, {X_g}^c - b_g)R_g$. In Theorem 2.4 (and throughout this paper) we let $A = R[x_1, \ldots, x_g]$ $(= R_g/K)$ and $J = (x_1, \ldots, x_g)A$, so A is a finite free "root" (integral) extension ring of rank c^g of R. Also, for $i = 1, \ldots, g$ it holds that ${x_i}^c = b_i \in IA$, and $IA \subseteq J^c$, so $(IA)_a = (J^c)_a$, hence $\mathbf{P}(IA) = \mathbf{P}(J)$. Note that for each minimal prime ideal z^* in A it holds that $A/z^* = R_g/P$ (where P is a minimal prime divisor of K) has the form $A/z^* = (R/(z^* \cap R))[\overline{x_1}, \ldots, \overline{x_g}]$, where $\overline{x_i} = x_i + z^*$ for $i = 1, \ldots, g$. Since ${x_i}^c = b_i$ in A, it follows that $A/z^* = (R/(z^* \cap R))[\overline{b_1}^{1/c}, \ldots, \overline{b_g}^{1/c}]$, where $\overline{b_i} = b_i + (z^* \cap R)$ (for $i = 1, \ldots, g$), so A/z^* is generated by c-th roots $\overline{b_1}^{1/c}, \ldots, \overline{b_g}^{1/c}$ of $\overline{b_1}, \ldots, \overline{b_g}$, respectively, in a fixed algebraic closure of the quotient field of $R/(z^* \cap R)$.

Definition 2.2 Let I be a regular proper ideal in a Noetherian ring R. Then **Rees I** denotes the set of Rees valuation rings of I, and if $(V, N) \in$ Rees I, then the **Rees integer** of I with respect to V is the integer e such that $IV = N^e$.

Remark 2.3 Let I be a regular proper ideal in a Noetherian ring R. If the greatest common divisor of the Rees integers of I is equal to one, then I is projectively full, by [1, (4.10)]. (The converse is false, by [7, Example 3.4, page 401].) Therefore if there exists an ideal $K \in \mathbf{P}(I)$ whose Rees integers have greatest common divisor equal to one, then K and $\mathbf{P}(I)$ are projectively full. (If such an ideal K exists, then since the ordered sets of Rees integers of I and K are proportional, necessarily K is the largest ideal in the linearly ordered set $\mathbf{P}(I)$.)

It is clear that assumption (a) in Theorem 2.4 holds if $g = 1$ (that is, if I is a regular principal ideal). Additional comments concerning assumptions (a) and (b) of Theorem 2.4 are given in Remarks 2.6.1 - 2.6.3.

Theorem 2.4 *Let I be a regular proper ideal in a Noetherian ring R, let $b_1, \ldots, b_g$ be a basis of I, let $\{(V_1, N_1), \ldots, (V_n, N_n)\}$ be a nonempty subset of* Rees I, *and for $j = 1, \ldots, n$ let e_j be the Rees integer of I with respect to V_j. Assume:*

(a) $b_i V_j = {N_j}^{e_j}$ for $i = 1, \ldots, g$ and $j = 1, \ldots, n$; and,

(b) the greatest common divisor c of $e_1, \ldots, e_n$ is a unit in R.

Let $A = R[x_1, \ldots, x_g]$ and let $J = (x_1, \ldots, x_g)A$ (see Remark 2.1). Then A is a finite free integral extension ring of R, IA and J are projectively equivalent, and J is projectively full, so $\mathbf{P}(IA)$ is projectively full.

Proof. If $c = 1$, then $A = R$ and I and $\mathbf{P}(I)$ are projectively full (by Remark 2.3), so the conclusion holds in this case. Therefore it may be assumed that $c > 1$.

As noted in Remark 2.1, A is a finite free integral extension ring of R and $(IA)_a = (J^c)_a$, so IA and J are projectively equivalent in A. Therefore it suffices to show that J is projectively full.

For this, let $(U_1, M_1), \ldots, (U_k, M_k)$ be all the Rees valuation rings of J, and for $j = 1, \ldots, k$ let f_j be the Rees integer of J with respect to U_j. Then by Remark 2.3 it suffices to show that the greatest common divisor of $f_1, \ldots, f_k$ is 1.

For this, for $j = 1, \ldots, n$ let $D_j = V_j[{u_{1,j}}^{1/c}, \ldots, {u_{g,j}}^{1/c}]$, where $u_{1,j}, \ldots, u_{g,j}$ are units in V_j determined by $b_1, \ldots, b_g$, and let ${V_j}^* = (D_j)_q$ (where q is a minimal prime divisor of $N_j D_j$). Assume it is known that ${V_j}^*$ is a discrete valuation ring such that $q{V_j}^* = N_j {V_j}^*$ and ${V_j}^* = U_h$ for some $h \in \{1, \ldots, k\}$. Then it follows (after resubscripting $U_1, \ldots, U_k$, if necessary) that, for $j = 1, \ldots, n$, $J^c U_j = IU_j = (IV_j)U_j = {N_j}^{e_j} U_j = {M_j}^{e_j}$, so $JU_j = {M_j}^{c_j}$, where c_j is the positive integer such that $c_j c = e_j$. However, by hypothesis $JU_j = {M_j}^{f_j}$, so it follows first that $f_j = c_j$, and then that the greatest common divisor of $f_1, \ldots, f_k$ is 1 (since $k \geq n$ and the greatest common divisor of $c_1, \ldots, c_n$ is 1). Therefore it remains to show that for $j = 1, \ldots, n$: (i) there exists a prime ideal q in $D_j = V_j[{u_{1,j}}^{1/c}, \ldots, {u_{g,j}}^{1/c}]$ such that ${V_j}^* = (D_j)_q$ is a discrete valuation ring whose maximal ideal is generated by N_j; and, (ii) ${V_j}^*$ is a Rees valuation ring of J.

To see that (i) holds, fix $j \in \{1, \ldots, n\}$. Then by the construction of Rees valuation rings (see [1, (2.9)]) there exists a minimal prime divisor z_j of zero in R such that R/z_j is a subring of V_j. Let c_j be the positive integer defined by $c_j c = e_j$ (where c is the greatest common divisor of $e_1, \ldots, e_n$), let π_j be a generator of N_j, and for $i = 1, \ldots, g$ let $b_{i,j} = b_i + z_j$ (so $b_{i,j} \in R/z_j \subseteq V_j$, and $b_{i,j} = b_i$ if R is an integral domain). Then it follows from assumption (a) that, for $i = 1, \ldots, g$, there exists a unit $u_{i,j} \in V_j$ such that $b_{i,j} = u_{i,j}{\pi_j}^{e_j}$ $= u_{i,j}{\pi_j}^{c_j c}$. Fix c-th roots ${b_{1,j}}^{1/c}, \ldots, {b_{g,j}}^{1/c}, {u_{1,j}}^{1/c}, \ldots, {u_{g,j}}^{1/c}$ of $b_{1,j}, \ldots, b_{g,j}, u_{1,j}, \ldots, u_{g,j}$, respectively, in an algebraic closure of the quotient field of V_j. Then since $b_{i,j} = u_{i,j}{\pi_j}^{c_j c}$, it follows that

(*) $V_j[{u_{i,j}}^{1/c}]$ *and* $V_j[{b_{i,j}}^{1/c}]$ *have the same quotient field for* $i = 1, \ldots, g$.

Let $X_1, \ldots, X_g$ be indeterminates and for $i = 1, \ldots, g$ let $Y_i = \frac{X_i}{{\pi_j}^{c_j}}$. Now the derivative of $f_{i,j}(Y_i) = Y_i^c - u_{i,j}$ (with respect to Y_i) is ${f_{i,j}}'(Y_i) = cY_i^{c-1}$. Also, the roots of $f_{i,j}(Y_i) = 0$ are $\omega^h {u_{i,j}}^{1/c}$ ($h = 1, \ldots, c$, where ω is a primitive c-th root of the unit element $1 \in V_j$), so it follows from [9, (10.17)] that the discriminant $\mathrm{Disc}(f_{i,j}(Y_i))$ of $f_{i,j}(Y_i)$ is $\pm\Pi_{h=1}^{c}\ {f_{i,j}}'(\omega^h {u_{i,j}}^{1/c}) = \pm c^c(\omega^{1+\cdots+c})^{c-1}{u_{i,j}}^{c-1} = \pm c^c {u_{i,j}}^{c-1}$. Therefore, since $u_{i,j}$ is a unit in V_j, and since c is is a unit in V_j (since, by assumption (b), c is a unit in R, so c is a unit in $R/z_j \subseteq V_j$), it follows that $\mathrm{Disc}(f_{i,j}(Y_i)) = \pm c^c {u_{i,j}}^{c-1}$ is a unit in V_j. Therefore it follows from [9, (38.9)] that $V_j[y_i] = V_j[Y_i]/(f_{i,j}(Y_i)V_j[Y_i])$ is integrally closed and that $N_jV_j[y_i] = \pi_jV_j[y_i]$ is a radical ideal (so for each prime divisor P of $N_jV_j[y_i]$, $V_j[y_i]_P$ is a discrete valuation ring whose maximal ideal is $N_jV_j[y_i]_P$). Now ${y_i}^c = u_{i,j}$, so it follows that $V_j[y_i]_P = V_j[{u_{i,j}}^{1/c}]_{P_1}$ for some height one prime ideal P_1 in $V_j[{u_{i,j}}^{1/c}]$ that contains N_j, so $V_{1,j} = V_j[{u_{i,j}}^{1/c}]_{P_1}$ is a discrete valuation ring and $P_1V_{1,j} = N_jV_{1,j}$. (Note that, since $V_j[Y_i]$ is a unique factorization domain, it follows that $V_j[{u_{i,j}}^{1/c}] = V_j[Y_i]/(\mu_i(Y_i)V_j[Y_i])$, where $\mu_i(Y_i)$ is the minimal polynomial of ${u_{i,j}}^{1/c}$ over V_j.)

By repeating much of the previous paragraph (first with $V_{1,j}$ and $u_{h,j}$ (with $h \in \{1, \ldots, g\}$ and $h \neq j$) in place of V_j and $u_{i,j}$ to get $V_{2,j}$, then with $V_{2,j}$ and $u_{m,j}$ (with $m \in \{1, \ldots, g\}$ and $m \neq j, h$) in place of $V_{1,j}$ and $u_{h,j}$ to get $V_{3,j}$, etc.), it follows that, for $j = 1, \ldots, n$, there exists a chain of discrete valuation rings $V_{0,j} = V_j \subseteq V_{1,j} = V_{0,j}[{u_{1,j}}^{1/c}]_{P_1} \subseteq \cdots \subseteq V_{g-1,j}[{u_{g,j}}^{1/c}]_{P_g} = V_{g,j}$ such that $N_jV_{h,j}$ is the maximal ideal of $V_{h,j}$ for $h = 1, \ldots, g$. Let $D_j = V_j[{u_{1,j}}^{1/c}, \ldots, {u_{g,j}}^{1/c}]$, so it follows that $V_{g,j} = (D_j)_q$ for some height one prime ideal q in D_j and that $q(D_j)_q = N_j(D_j)_q$, so (i) holds.

To see that (ii) holds (that is, that $V_{g,j}$ is a Rees valuation ring of $J = (x_1, \ldots, x_g)A$), note that $R/z_j \subseteq V_j$ and by construction (see [1, (2.9)]) there exists a height one prime divisor p of $b_{1,j}{B_j}'$ such that $V_j = ({B_j}')_p$ and $N_j = pV_j$, where ${B_j}'$ is the integral closure of $B_j = (R/z_j)[b_{2,j}/b_{1,j}, \ldots, b_{g,j}/b_{1,j}]$ in its quotient field (here we use assumption (a) (that $IV_j = b_iV_j$ for $j = 1, \ldots, n$ and $i = 1, \ldots, g$)). By integral dependence, there exists a minimal prime ideal ${z_j}^*$ in $A = R[x_1, \ldots, x_g]$ such that ${z_j}^* \cap R = z_j$; then $A/{z_j}^* = (R/z)[\overline{x_1}, \ldots, \overline{x_g}]$ $= (R/z)[{b_{1,j}}^{1/c}, \ldots, {b_{g,j}}^{1/c}]$ (see Remark 2.1). (Note that if R is an integral domain, then each minimal prime ideal z^* in A is a suitable choice for ${z_j}^*$.) Then, since R/z_j and V_j have the same quotient field, it follows from (*) that $A/{z_j}^*$ and $D_j = V_j[{u_{1,j}}^{1/c}, \ldots, {u_{g,j}}^{1/c}]$ have the same quotient field. Also, A is a finite free integral extension ring of R and $b_{i,j}/b_{1,j} \in B_j$ is such that $b_{i,j}/b_{1,j} = (\overline{x_i}/\overline{x_1})^c$ (for $i = 1, \ldots, g$), so it follows that $C_j = (A/{z_j}^*)[\overline{x_2}/\overline{x_{1,j}}, \ldots, \overline{x_{g,j}}/\overline{x_{1,j}}]$ is a finite integral extension domain of B_j. Therefore ${C_j}' = {B_j}'' \subseteq {V_j}''$, where ${C_j}'$ (resp., ${B_j}''$, ${V_j}''$) is the integral closure of C_j (resp., B_j, V_j) in the quotient field of C_j (which is the quotient field of $A/{z_j}^*$ and of D_j). Also, ${u_{i,j}}^{1/c} \in {V_j}''$, since $u_{i,j} \in V_j$, so $D_j = V_j[{u_{1,j}}^{1/c}, \ldots, {u_{g,j}}^{1/c}] \subseteq {V_j}''$, so ${V_j}''$ is an integral extension domain of D_j. Let q be as at the end of the second preceding paragraph, so $V_{g,j} = (D_j)_q$ is a discrete valuation ring. Therefore it follows that $V_{g,j} = ({V_j}'')_{q^*}$, where $q^* = qV_{g,j} \cap {V_j}''$. Since $q^* \cap {B_j}' = (q^* \cap V_j) \cap {B_j}' = N_j \cap {B_j}' = p$ (where p is a height one prime divisor of $b_{1,j}{B_j}'$ (by the start of this paragraph)), and since ${C_j}' = {B_j}'' \subseteq {V_j}''$, it follows that $q_j = q^* \cap {C_j}'$ is a prime ideal in ${C_j}'$ such that $q_j \cap {B_j}' = p$. Therefore q_j is a height one prime divisor of $\overline{x_1}{C_j}' = {b_{1,j}}^{1/c}{C_j}'$, so $({C_j}')_{q_j} = V_{g,j}$ is a Rees valuation ring of J (by [1, (2.9)]), hence (ii) holds. ■

It is clear from the preceding proof that the ring $A = R[x_1, \ldots, x_g]$ and the ideal $J = (x_1, \ldots, x_g)A$ are not canonical, in that they depend on the basis $b_1, \ldots, b_g$ chosen for I. The next two remarks mention several positive things about the extension ring A, the ideal J, and the proof of Theorem 2.4.

Remark 2.5 **(2.5.1)** The proof of Theorem 2.4 shows the following: if V_j is a Rees valuation ring of I, if e_j is the Rees integer of I with respect to V_j, and if c is the greatest common divisor of $e_1, \ldots, e_n$, then $U_j = V_j[{u_{1,j}}^{1/c}, \ldots, {u_{g,j}}^{1/c}]_q$ is a Rees valuation ring of J $= (x_1, \ldots, x_g)R[x_1, \ldots, x_g]$ (for some height one prime ideal q), the Rees integer of J with respect to U_j is $c_j = e_j/c$, and the greatest common divisor of $c_1, \ldots, c_n$ is equal to one. In particular, if $e_1 = \cdots = e_n$ (for example, if $n = 1$), then $e_1 = c$ and $c_1 = \cdots = c_n = 1$.
(2.5.2) If R is a Noetherian domain in Theorem 2.4, then it follows from the last paragraph of the proof of Theorem 2.4 that, for each minimal prime ideal z^* in A, the ideal $(IA +$

$z^*)/z^*$ in A/z^* is such that $\mathbf{P}((IA+z^*)/z^*)$ if projectively full (since the proof shows that $(IA+z^*)/z^*$ has n Rees valuation rings whose Rees integers have greatest common divisor equal to one). Therefore in the case where R is an integral domain there exists a finite integral extension domain $B = A/z^*$ of R such that $\mathbf{P}(IB)$ is projectively full.

Remark 2.6 **(2.6.1)** Concerning assumption (a) of Theorem 2.4 that "$b_1, \dots, b_g$ is a basis of I such that $b_i V_j = IV_j$ for $i = 1, \dots, g$ and $j = 1, \dots, n$", if R is a local ring with maximal ideal M such that R/M is infinite, then there exists such a basis for I for every nonempty subset $\{(V_1, N_1), \dots, (V_n, N_n)\}$ of Rees I.

(2.6.2) Let I be a regular proper ideal in a Noetherian ring R and let $(V, N) \in$ Rees I. Then assumption (a) of Theorem 2.4 holds for I and V: that is, I has a basis (say $b_1, \dots, b_g$) such that $b_i V = IV$ for $i = 1, \dots, g$.

(2.6.3) If R as in Theorem 2.4 contains a field F such that either: char (F) is not a divisor of c; or, char $(F) = 0$; then assumption (b) holds (since the greatest common divisor c of $e_1, \dots, e_g$ is in F). Of course, the larger n is chosen (that is, the more Rees valuation rings of I that are considered), the more likely it is that assumption (b) holds. On the other hand, if H is any ideal that is projectively equivalent to I, then by [7, (2.10)] H and I have the same Rees valuation rings and their corresponding Rees integers are proportional, so by choosing H as the largest ideal in $\mathbf{P}(I)$, the more likely it is that assumption (b) holds (for the greatest common divisor of the Rees integers of H).

(2.6.4) If $c \notin (N_1 \cap R) \cup \cdots \cup (N_n \cap R)$, and if assumption (a) of Theorem 2.4 holds for I, then there exists a finite free integral extension ring A of R and an ideal J in A such that $\mathbf{P}(IA) = \mathbf{P}(J)$ is projectively full.

Proof. For (2.6.1), fix a nonempty subset $\{(V_1, N_1), \dots, (V_n, N_n)\}$ of Rees I, and for $j = 1, \dots, n$ let $H_j = \{x \in I \mid xV_j \subsetneq IV_j\}$. Then it is readily checked that each H_j is an ideal in R that is properly contained in I. Therefore $\overline{H_j} = (H_j + MI)/(MI)$ is a proper subspace (over the field R/M) of $\overline{I} = I/(MI)$. Since R/M is infinite, it follows that $\overline{I}$ has a basis $\overline{b_1}, \dots, \overline{b_g}$ such that no $\overline{b_i}$ is in $\overline{H_1} \cup \cdots \cup \overline{H_n}$. Therefore if $b_1, \dots, b_g$ are preimages in R of $\overline{b_1}, \dots, \overline{b_g}$, then it follows that $b_1, \dots, b_g$ are a basis of I such that $b_i V_j = IV_j$ for $i = 1, \dots, g$ and $j = 1, \dots, n$.

For (2.6.2), let $c_1, \dots, c_g$ be a basis of I, so $IV = c_iV$ for some $i \in \{1, \dots, g\}$. Resubscript the c_i so that $c_hV = IV$ for $h = 1, \dots, f$ and $c_hV \subsetneq IV$ for $h = f+1, \dots, g$. For $h = 1, \dots, f$ let $b_h = c_h$, and for $h = f+1, \dots, g$ let $b_h = b_1 + c_h$. Then it is readily checked that $b_1, \dots, b_g$ is a basis of I such that $b_iV = IV$ for $i = 1, \dots, g$.

For (2.6.4), let $S = R[1/c]$. If $c \notin (N_1 \cap R) \cup \cdots \cup (N_n \cap R)$, and if assumption (a) holds for I, then assumptions (a) and (b) hold for IS. Therefore there exists a finite free integral extension ring $B = S[x_1, \dots, x_g]$ of S such that $J' = (x_1, \dots, x_g)B$ is projectively full (by Theorem 2.4, with S and IS in place of R and I). Let $A = R[x_1, \dots, x_g]$ and $J = (x_1, \dots, x_g)A$, and let $K \in \mathbf{P}(J)$. Then there exist positive integers n, s such that $(K^n)_a = (J^s)_a$, so $((KB)^n)_a = ((K^n)_aB)_a = ((J^s)_aB)_a = ((JB)^s)_a = (J'^s)_a$, hence n divides s, as $JB = J'$ is projectively full. This implies that $K = (J^{s/n})_a$. It follows that $\mathbf{P}(J)$ is projectively full, and $\mathbf{P}(J) = \mathbf{P}(IA)$, by Remark 2.1. ■

Corollary 2.7 *Let R be a Noetherian ring that contains the field $\mathbb{Q}$ of rational numbers. For each regular proper ideal I of R there exists a finite free integral extension ring A of R such that $\mathbf{P}(IA)$ is projectively full. If R is an integral domain, there exists a finite integral extension domain $B = A/z^*$ of R such that $\mathbf{P}(IB)$ is projectively full.*

Proof. Apply Remarks 2.6.2 - 2.6.3, and Remark 2.5.2. ■

In Corollary 2.8, we show that $\mathbf{P}(IA^+)$ is projectively full for certain integral overrings A^+ of the ring A constructed in Theorem 2.4. (A related result is considered in Corollary 4.3 and Remark 4.4 below.)

Corollary 2.8 With the notation and assumptions of Theorem 2.4, let A^+ be a finite integral extension ring of A that is contained in the total quotient ring of A. Then $\mathbf{P}(IA^+)$ is projectively full.

Proof. The Rees valuation rings of IA (and of J) are the Rees valuation rings of IA^+ (and of JA^+), and by integral dependence the Rees integers of IA^+ (resp., JA^+) with respect to these valuation rings are the same as for IA (resp., J). Also, IA^+ and JA^+ are projectively equivalent (since IA and J are projectively equivalent). The conclusion follows from this and Remark 2.3, since the greatest common divisor of these Rees integers of J is equal to one. ■

Corollary 2.9 extends Theorem 2.4 to certain finite collections of regular proper ideals of certain local rings.

Corollary 2.9 *Let (R, M) be a local ring and let $I_1, \dots, I_m$ be regular proper ideals of R. Assume that $\mathbb{Q} \subseteq R$ and that there exist nonempby subsets $\mathbf{C}_i$ of* Rees I_i *such that, for $i \neq j$ in $\{1, \dots, m\}$, there are no containment relations between the centers in R of the valuation rings in $\mathbf{C}_i$ and the centers in R of the valuation rings in $\mathbf{C}_j$. Then there exists a finite free local integral extension ring A of R such that $\mathbf{P}(I_i A)$ is projectively full for $i = 1, \dots, m$.*

Proof. For $i = 1, \dots, m$ let $\mathbf{C}_i = \{(V_{i,1}, N_{i,1}), \dots, (V_{i,n_i}, N_{i,n_i})\}$, and for $h = 1, \dots, n_i$ let $v_{i,h}$ be the valuation of $V_{i,h}$, let $P_{i,h} = N_{i,h} \cap R$ be the center in R of $V_{i,h}$, let $\pi_{i,h} \in V_{i,h}$ such that $N_{i,h} = \pi_{i,h} V_{i,h}$, let $e_{i,h}$ be the Rees integer of I_i with respect to $V_{i,h}$, let c_i be the greatest common divisor of $e_{i,1}, \dots, e_{i,n_i}$, and define $c_{i,h}$ by $c_{i,h} c_i = e_{i,h}$.

Fix $i \in \{1, \dots, m\}$, let $H_{i,(i,h)} = \{x \in I_i \mid v_{i,h}(x) > v_{i,h}(I_i)\}$ (for $h = 1, \dots, n_i$), and let $H_{i,(j,h)} = I_i \cap P_{j,h}$ (for $j \neq i$ in $\{1, \dots, m\}$ and for $h \in \{1, \dots, n_j\}$). Then by the hypothesis concerning the sets $\mathbf{C}_i$ and $\mathbf{C}_j$ it follows that each $H_{i,(j,h)}$ ($j = 1, \dots, m$ and $h \in \{1, \dots, n_j\}$) is a proper subset of I_i, so (since R/M is infinite) there exists a basis $b_{i,1}, \dots, b_{i,g_i}$ of I_i such that no $b_{i,k}$ is in any $H_{i,(j,h)}$. Therefore: (i) for $k = 1, \dots, g_i$ and for $h = 1, \dots, n_i$ it holds that $b_{i,k} V_{i,h} = I_i V_{i,h}$ (so there exist units $u_{k,h} \in V_{i,h}$ such that $b_{i,k} = u_{k,h} {\pi_{i,h}}^{e_{i,h}} = u_{k,h} {\pi_{i,h}}^{c_{i,h} c_i}$, so $(b_{i,k}/\pi_{i,h})^{1/c_i} = {u_{k,h}}^{1/c_i} {\pi_{i,h}}^{c_{i,h}}$); and, (ii) for $k = 1, \dots, g_i$, for $j \neq i \in \{1, \dots, m\}$, and for $h \in \{1, \dots, n_j\}$ it holds that $b_{i,k} V_{j,h} = V_{j,h}$.

Since $\mathbb{Q} \subseteq R$, it follows that assumption (b) of Theorem 2.4 is satisfied for I_1 in place of I, and assumption (a) of Theorem 2.4 is satisfied (for I_1 in place of I) by the preceding paragraph, so let $A_1 = R[x_{1,1}, \dots, x_{g_1,1}]$ ($= R_{g_1}/K_1$, where $R_{g_1} = R[X_{1,1}, \dots, X_{g_1,1}]$ and $K_1 = ({X_{1,1}}^{c_1} - b_{1,1}, \dots, {X_{g_1,1}}^{c_1} - b_{g_1,1}) R_{g_1}$), and let $J_1 = (x_{1,1}, \dots, x_{g_1,1}) A_1$. Then A_1 is a local ring, by Proposition 3.1.5 below, and a finite free integral extension ring of R, by Theorem 2.4. Also, using (i) in the preceding paragraph it follows from Remark 2.5.1 that the greatest common divisor of the Rees integers of J_1 is equal to one, and Theorem 2.4 shows that $\mathbf{P}(I_1 A_1) = \mathbf{P}(J_1)$ is projectively full. Further, by (ii) of the preceding paragraph, each $b_{1,k}$ ($k = 1, \ldots, g_1$) is a unit in each $V_{j,h}$ ($j = 2, \ldots, m$ and $h \in \{1, \ldots, n_j\}$), so by using [9, (38.9)] (as in the proof of Theorem 2.4) it follows that there exists a height one prime ideal $q_{j,h}$ in $V_{j,h}[{u_{1,1}}^{1/c_1}, \ldots, {u_{1,g_1}}^{1/c_1}]$ such that $U_{j,h} = V_{j,h}[{u_{1,1}}^{1/c_1}, \ldots, {u_{1,g_1}}^{1/c_1}]_{q_{j,h}}$ is a Rees valuation ring of I_jA_1 whose maximal ideal is $N_{j,h}U_{j,h} = q_{j,h}U_{j,h}$ (so the Rees integer of I_jA_1 with respect to $U_{j,h}$ is $e_{j,h}$ (so the greatest common divisor of these Rees integers of I_jA_1 is c_j)).

It therefore follows from iterating the preceding paragraph (first with A_1 and I_2A_1 in place of R and I_1, etc.) that the conclusion holds. ■

Before deriving more corollaries of Theorem 2.4, we first observe several properties of the extension ring A.

3 PROPERTIES OF THE FREE EXTENSION RING A.

In this section we record some of the properties of the finite free integral extension ring A of Theorem 2.4. Concerning the Theorem of Transition in Proposition 3.1.1, see [9, Section 19]. Also, for Proposition 3.1.3, recall that the **altitude** of an ideal H is defined to be the maximum of the heights of the minimal prime divisors of H.

Proposition 3.1 *Assume notation as in Theorem 2.4.*

(3.1.1) R and A satisfy the Theorem of Transition.

(3.1.2) For each prime ideal p in R and for each prime ideal P of A such that $P \cap R = p$ it holds that R_p is a subspace of A_P.

(3.1.3) For each ideal H in R it holds that: $\text{ht}(H) = \text{ht}(HA)$; $\text{altitude}(H) = \text{altitude}(HA)$; and $\dim(R/H) = \dim(A/(HA))$.

(3.1.4) $A/J = R/I$.

(3.1.5) There exists a one-to-one correspondence between the ideals H' in A that contain J and the ideals H in R that contain I given by $H = H' \cap R$ and $H' = (J,H)A$ (so if H is prime (resp., primary), then $(J,H)A$ is prime (resp., primary), and if $\cap_{i=1}^k q_i$ is an irredundant primary decomposition of H, then $\cap_{i=1}^k (J, q_i)A$ is an irredundant primary decomposition of $(J,H)A$). In particular: H and $(J,H)A$ have the same number of minimal prime divisors; $\text{ht}((J,H)A) = \text{ht}(H)$; $A/((J,H)A) = R/H$; and A has exactly k maximal ideals containing $(J, H)A$ if H is contained in exactly k maximal ideals of R.

(3.1.6) R is a Cohen-Macaulay ring if and only if A is a Cohen-Macaulay ring.

(3.1.7) $b_1, \ldots, b_g$ is an R-sequence if and only if $x_1, \ldots, x_g$ is an A-sequence.

(3.1.8) If $(V_1, N_1), \ldots, (V_n, N_n)$ are all the Rees valuation rings of I in Theorem 2.4, then $\{c_1, \ldots, c_n\}$ are all the Rees integers of J, where $c_j c = e_j$ for $j = 1, \ldots, n$.

Proof. Since A is a finite free integral extension ring of R, (3.1.1) follows from [9, (19.1)], so (3.1.2) follows from [9, (19.2)(3)], and (3.1.3) follows from [9, (22.9)].

For (3.1.4), as in Remark 2.1 let $R_g = R[X_1, \ldots, X_g]$ and $K = ({X_1}^c - b_1, \ldots, {X_g}^c - b_g)R_g$, so $A = R[x_1, \ldots, x_g] = R_g/K$ and $J = (x_1, \ldots, x_g)A = (X_1, \ldots, X_g, K)/K$. Therefore A/J $= R_g/((X_1, \ldots, X_g, K)R_g) = R_g/((b_1, \ldots, b_g, X_1, \ldots, X_g)R_g) = R/I$.

(3.1.5) follows immediately from (3.1.4) and (3.1.3).

For (3.1.6), apply [5, Theorem 23.3 and Theorem 17.6].

For (3.1.7), since A is a free R-module, it follows that $(H :_R G)A = HA :_A GA$ for all ideals H, G in R, so it follows that $b_1, \ldots, b_g$ are an R-sequence if and only if they are an A-sequence. Since ${x_i}^c = b_i$ for $i = 1, \ldots, g$, it follows that $b_1, \ldots, b_g$ are an A-sequence if and only if $x_1, \ldots, x_g$ are an A-sequence. Therefore $b_1, \ldots, b_g$ are an R-sequence if and only if $x_1, \ldots, x_g$ are an A-sequence.

For (3.1.8), let z^* be a minimal prime ideal in A and let $z = z^* \cap R$, so z is a minimal prime ideal in R (since A is a finite free integral extension ring of R). Let an overbar denote residue class modulo z^* and let F be the quotient field of $\overline{R}$, so the quotient field of $\overline{A}$ is $E = F[\overline{b_1}^{1/c}, \ldots, \overline{b_g}^{1/c}]$. Let $\overline{\omega}$ be a primitive c-th root of the unit element 1 in F. Then it is clear that $F[\overline{\omega}]$ is a Galois extension field of F, so it follows that $F[\overline{\omega}, \overline{b_1}^{1/c}, \ldots, \overline{b_g}^{1/c}]$ is a Galois extension field of both F and E. Therefore the Rees valuation rings of $JA[\omega] =$ $(x_1, \ldots, x_g)A[\omega]$ (and of $\overline{J} = (\overline{x_1}, \ldots, \overline{x_g})\overline{A} = (\overline{b_1}^{1/c}, \ldots, \overline{b_g}^{1/c})\overline{A}$ (see Remark 2.1)) that lie over a given Rees valuation ring (say, V_j) of $I = (b_1, \ldots, b_g)R$ (and $\overline{I} = (\overline{b_1}, \ldots, \overline{b_g})\overline{R}$) are conjugate, so these Rees integers of J are all equal to $c_j = e_j/c$, by the fourth paragraph of the proof of Theorem 2.4. Thus if $(V_1, N_1), \ldots, (V_n, N_n)$ are all the Rees valuation rings of I, then the Rees integers of J are $\{c_1, \ldots, c_n\}$. ■

Since $IA \subseteq J^c$ (by Remark 2.1), since $J^c \subseteq J^{c-1} \subseteq \cdots \subseteq J$, and since $J \cap R = I$ (by Proposition 3.1.4), it follows that if $J^i = (J^i)_a$ for some $i \in \{1, \ldots, c\}$, then $I = I_a$.

We close this section with two more corollaries of Theorem 2.4. For the first of these, the integer d in Corollary 3.2.2 is the integer d shown to exist in [7, (2.8) and (2.9)] (and denoted $d(I)$ in [1, Section 4] and in [2]). It is a common divisor of the Rees integers of I, and it is the smallest positive integer k such that, for all ideals $G \in \mathbf{P}(I)$, $(G^k)_a = (I^i)_a$ for some positive integer i.

Corollary 3.2 *With the notation and assumptions of Theorem 2.4, assume that H is an ideal in R that is projectively equivalent to I. Then:*

(3.2.1) *If h, i are positive integers such that $(H^h)_a = (I^i)_a$, then $(HA)_a = (J^{ci/h})_a$ and ci/h is a positive integer.*

(3.2.2) *If $e_1, \ldots, e_n$ are all the Rees integers of I in Theorem 2.4, then there exists a positive integer k such that $(H^d)_a = (I^k)_a$, so $(HA)_a = (J^{kd^*})_a$, where d^* is the positive integer c/d.*

Proof. For (3.2.1), if H is projectively equivalent to I, then by definition there exist positive integers h, i such that $(H^h)_a = (I^i)_a$, and then it follows that $(H^h A)_a = (I^i A)_a$. By Theorem 2.4, $(I^i A)_a = (J^{ci})_a$, so $(HA)_a = J_{ci/h}$ $(= \{x \in A \mid \overline{v}_J(x) \geq ci/h\}$; see [7, (2.3)]). Also, HA is projectively equivalent to IA, and IA is projectively equivalent to J, so HA is projectively equivalent to J. However, J is projectively full, by Theorem 2.4, so $(HA)_a = (J^k)_a$ for some positive integer k. It follows that $J_{ci/h} = (HA)_a = (J^k)_a = J_k$ (by [7, (2.3)]), so $ci/h = k$.

For (3.2.2), as noted preceding this corollary, there exists a smallest common divisor d of the Rees integers $e_1, \ldots, e_n$ of I such that for all ideals G that are projectively equivalent to I it holds that $(G^d)_a = (I^k)_a$ for some positive integer k. Let k be the integer such that $(H^d)_a = (I^k)_a$, and let c be the greatest common divisor of $e_1, \ldots, e_n$. Then $c = dd^*$ for some positive integer d^*, so it follows that $(H^c)_a = (H^{dd^*})_a = (I^{kd^*})_a$, so $(HA)_a = (J^{kd^*})_a$ by (3.2.1). ■

Remark 3.3 It is shown in [7, Corollary 2.4] that $\mathbf{P}(I)$ is linearly ordered and discrete, so there exist positive integers $c_1 < c_2 < \cdots$ such that $\mathbf{P}(I) = \{(I^{c_i/d})_a \mid i$ is a positive integer$\}$, where d is as in Corollary 3.2.2. Let $d^* = c/d$ as in Corollary 3.2.2, so $\mathbf{P}(I) = \{(I^{c_i d^*/c})_a \mid i$ is a positive integer$\}$. With this in mind, it follows from Corollary 3.2.2 that $(\mathbf{P}(I))A = \{(J^{c_i d^*})_a \mid i$ is a positive integer$\} \subseteq \mathbf{P}(IA)$ (and $\mathbf{P}(IA) = \{(J^i)_a \mid i$ is a positive integer$\}$, by Theorem 2.4).

Corollary 3.4 *With the notation and assumptions of Theorem 2.4, assume that R is a local ring with maximal ideal M. Then:*

(3.4.1) *If I is an open ideal in R, then there exists a finite free local integral extension ring A of R such that $\mathbf{P}(IA)$ is projectively full.*

(3.4.2) *If $I = M$ in (3.4.1), then $A = R[x_1, \dots, x_g]$ is a finite free local integral extension ring of R whose maximal ideal $N = (x_1, \dots, x_g)A$ is projectively full.*

(3.4.3) *Assume that $b_1, \dots, b_f$ $(f \leq g)$ in (3.4.2) are such that $X = (b_1, \dots, b_f)R$ is a reduction of M, let $A_0 = R[x_1, \dots, x_f]$, and let $C = (x_1, \dots, x_f)A_0$. Then C is a reduction of the maximal ideal $(x_1, \dots, x_f, M)A_0 = (x_1, \dots, x_f, b_{f+1}, \dots, b_g)A_0$ of A_0, and C is projectively full.*

Proof. For (3.4.1), if R is local, then $I = (b_1, \dots, b_g)R \subseteq M$, so $A = R[x_1, \dots, x_g]$ is a local ring with maximal ideal $(M, x_1, \dots, x_g)A$, by Proposition 3.1.5, so the conclusion follows from Theorem 2.4.

(3.4.2) follows from (3.4.1), since if $I = M$, then $MA = (b_1, \dots, b_g)A \subseteq (x_1, \dots, x_g)A$, so it follows that $A/((x_1, \dots, x_g)A) = R/M$, hence $N = (x_1, \dots, x_g)A$.

For (3.4.3), X and M $(= I)$ have the same Rees valuation rings and Rees integers, since X is a reduction of M, so C is projectively full by Theorem 2.4. Also, it is clear that $C \subseteq (C, M)A_0$ and that $(C, M)A_0 = (x_1, \dots, x_f, b_{f+1}, \dots, b_g)A_0$ is the maximal ideal in A_0. Further, $(MA_0)_a = (XA_0)_a$ (since $X_a = M_a = M$ in R) $= (C^c)_a \subseteq C_a$, so $MA_0 \subseteq C_a$. Therefore $(C, M)A_0 \subseteq C_a$, so $C_a = (C, M)A_0$ (since $(C, M)A_0$ is the maximal ideal in A_0), hence C is a reduction of $(C, M)A_0$. ■

4 IDEALS WITH A REES INTEGER EQUAL TO ONE.

The last part of Remark 2.5.1 shows that if the number of Rees valuation rings considered in Theorem 2.4 is one, then the ideal J of Theorem 2.4 has a Rees valuation ring U such that the Rees integer of J with respect to U is equal to one. In this section we consider some consequences of this.

We begin with the following proposition.

Proposition 4.1 *Let I be a regular proper ideal in a Noetherian ring R, let $\mathbf{R} = R[u, tI]$, where t is an indeterminate and $u = 1/t$, and let $\mathbf{R}'$ be the integral closure of $\mathbf{R}$ in its total quotient ring. Then:*

(4.1.1) *I has a Rees integer equal to one if and only if $u\mathbf{R}'$ has a primary component that is prime.*

(4.1.2) *Every Rees integer of I is equal to one if and only if $u\mathbf{R}'$ is a radical ideal.*

(4.1.3) *If there exists an ideal K in $\mathbf{P}(I)$ such that some Rees integer of K is equal to one, then K and $\mathbf{P}(I)$ are projectively full.*

Proof. For (4.1.1), it follows from [2, (2.3)] that the Rees valuation rings V of I correspond to the rings $\mathbf{R}'_p$, where p is a (height one) prime divisor of $u\mathbf{R}'$, and the Rees integer e of I with respect to V is given by $u\mathbf{R}'_p = p^e\mathbf{R}'_p$. Therefore it follows that I has a Rees integer equal to one if and only if $u\mathbf{R}'$ has a (height one) prime divisor p such that $u\mathbf{R}'_p = p\mathbf{R}'_p$. The conclusion readily follows from this.

(4.1.2) follows immediately from (4.1.1).

For (4.1.3), if some Rees integer of K is equal to one, then the greatest common divisor of the Rees integers of K is equal to one, so K and $\mathbf{P}(I)$ are projectively full, by [1, (4.10)]. ■

Concerning Proposition 4.1.1, some properties of a regular ideal I in a Noetherian ring R such that $u\mathbf{R}$ (rather than $u\mathbf{R}'$) has a primary component that is prime are noted in Examples 5.1.1 and 5.1.2 below.

Lemma 4.2 *Let I be a regular proper ideal in a Noetherian ring R and let $e_1, \ldots, e_n$ be all the Rees integers of I. Then:*

(4.2.1) *$e_j = 1$ for some $j \in \{1, \ldots, n\}$ if and only if there exists a minimal prime ideal z in R such that some Rees integer of $(I + z)/z$ is equal to one. If these hold, then I, $\mathbf{P}(I)$, $(I + z)/z$, and $\mathbf{P}((I + z)/z)$ are projectively full.*

(4.2.2) *$e_j = 1$ for some $j \in \{1, \ldots, n\}$ if and only if there exists a multiplicatively closed subset S in R such that some Rees integer of IR_S is equal to one. If these hold, then I, $\mathbf{P}(I)$, $IR_{S'}$, and $\mathbf{P}(IR_{S'})$ are projectively full for all multiplicatively closed subsets S' of R such that $P \cap S' = \emptyset$ (where $P = N \cap R$ with (V, N) a Rees valuation ring of I such that $IV = N$).*

(4.2.3) *Assume that (V, N) is a Rees valuation ring of I such that the Rees integer of I with respect to V is equal to one. Let B be a Noetherian domain such that $R/z \subseteq B \subseteq V$ for some minimal prime ideal z in R ($z = (0)$, if R is an integral domain), and let K be an ideal in B such that $IB \subseteq K \subseteq N \cap B$. Then V is a Rees valuation ring of K such that the Rees integer of K with respect to V is equal to one, so K is projectively full. In particular, IB is projectively full,*

Proof. For (4.2.1), the construction of Rees valuation rings in [1, (2.9)] shows that, for each minimal prime ideal z in R, each Rees valuation ring of $(I + z)/z$ is a Rees valuation ring (V, N) of I such that the Rees integer of $(I+z)/z$ with respect to V is the Rees integer of I with respect to V. The same construction shows that, for each Rees valuation ring (V, N) of I, there exists a minimal prime ideal z in R such that V is a Rees valuation ring of $(I + z)/z$ and the Rees integer of $(I + z)/z$ with respect to V is the Rees integer of I with respect to V. The conclusion clearly follows from this and Proposition 4.1.3.

The proof of (4.2.2) is similar, so it will be omitted.

For (4.2.3), by hypothesis there exists $b \in I$ such that $bV = IV = N$. Therefore $b \in K \subseteq N = bV$, so $D = B[K/b] \subseteq V$. Also, $C = R[I/b] \subseteq D$, and $N \cap C'$ is a height one prime divisor of bC' (by [1, (2.9)]). Therefore it follows that $N \cap D'$ is a height one prime divisor of bD', so V is a Rees valuation ring of K (by [1, (2.9)]). Since $N = bV \subseteq KV \subseteq N$, it follows that the Rees integer of K with respect to V is equal to one. The remaining conclusions follow from this and Proposition 4.1.3. ■

Example 5.2 below concerns a special case of Lemma 4.2.3

We remark that the hypothesis "e is a unit in V" in Corollary 4.3 holds if either: (i) e is not a multiple of char (V_j/N_j); or, (ii) char $(V_j/N_j) = 0$.

Corollary 4.3 *Let I be a proper nonzero ideal in a Noetherian ring R and assume that R*

has a Rees valuation ring (V, N) such that the Rees integer e of I with respect to V is a unit of V. Then there exists a finite free integral extension ring $A = R[x_1, \ldots, x_g]$ of R and an ideal $J = (x_1, \ldots, x_g)A$ in A such that J has a Rees integer equal to one. Therefore there exists a minimal prime ideal z^* in A such that if B is a Noetherian domain between A/z^* and its integral closure $(A/z^*)'$, then there exists a prime ideal P containing JB such that each of P, JB, PB_P, and JB_P has a Rees integer equal to one.

Proof. Remark 2.6.2 shows that assumption (a) of Theorem 2.4 holds for I with respect to V, and Remark 2.6.4 shows that $\mathbf{P}(IA) = \mathbf{P}(J)$ is projectively full. It follows from Remark 2.5.1 that J has a Rees valuation ring (U, M) such that the Rees integer of J with respect to U is equal to one. The final statement follows from this and Lemma 4.2.3. ■

In Corollary 4.3, P need not be a minimal prime divisor of JB; see Example 5.2 below.

Remark 4.4 It follows immediately from the last part of Corollary 4.3 (and Proposition 3.1.8) that if R is a Noetherian domain, if $\mathrm{Rad}(I)$ is a prime ideal, and if there exists only one prime ideal in the integral closure R' of R that lies over $\mathrm{Rad}(I)$, then PB_P has a Rees integer equal to one for each prime ideal P in B that lies over $(J, \mathrm{Rad}(I))A$.

5 EXAMPLES OF IDEALS WITH SOME REES INTEGER EQUAL TO ONE.

In Proposition 4.1.3 it was noted that if I is a regular proper ideal in a Noetherian ring R such that some Rees integer of I is equal to one, then I is projectively full. In this section we give some examples of such ideals.

Concerning the conclusion of Example 5.1.2, recall that an ideal I is **normal** in case each power I^n of I is integrally closed.

Example 5.1 Let I be a regular ideal in a Noetherian ring R and let $\mathbf{G}(R, I) = \Sigma_{i=0}{}^{\infty}\, I^i/I^{i+1}$ denote its associated graded ring.

(5.1.1) If $\mathbf{G}(R, I)$ has a minimal prime ideal p such that p is its own p-primary component of (0), then I has a Rees integer equal to one.

(5.1.2) If $\mathbf{G}(R, I)$ is reduced, then I is a radical ideal and a normal ideal, and each Rees integer of I is equal to one.

Proof. Let $\mathbf{R} = R[u, tI]$, where t is an indeterminate and $u = 1/t$. It is shown in [14] that: $\mathbf{G}(R, I) = \mathbf{R}/(u\mathbf{R})$; u is a regular element in $\mathbf{R}$; and, $u^n\mathbf{R} \cap R = I^n$ for all positive integers n.

For the proof of (5.1.1), observe that $u\mathbf{R}_p = p\mathbf{R}_p$ implies that $\mathbf{R}_p$ is a discrete valuation ring. It follows that $p' = p\mathbf{R}_p \cap \mathbf{R}'$ is the p'-primary component of $u\mathbf{R}'$, so one of the Rees integers of I is equal to one by Proposition 4.1.1.

For the proof of (5.1.2), if $\mathbf{G}(R, I)$ is a radical ideal, then $u\mathbf{R}$ is a radical ideal. Therefore it follows from [9, (33.11)] that $u\mathbf{R}'$ is a radical ideal, so each Rees integer of I is equal to one by Proposition 4.1.2. Also, $I = u\mathbf{R} \cap R$ is a radical ideal. Further, $u\mathbf{R}_q = q\mathbf{R}_q$ for each (minimal) prime divisor q of $u\mathbf{R}$, so each $\mathbf{R}_q$ is a discrete valuation ring. It follows that, for all positive integers n, $u^n\mathbf{R} = \cap\{u^n\mathbf{R}_q \cap \mathbf{R} \mid q \in \text{Ass}(\mathbf{R}/(u\mathbf{R}))\}$ (by [9, (12.6)]) and that each $u^n\mathbf{R}_q \cap \mathbf{R}$ is integrally closed, so $u^n\mathbf{R} = (u^n\mathbf{R})_a$, by [11, Lemma 4]. Therefore $I^n = u^n\mathbf{R} \cap R = (u^n\mathbf{R})_a \cap R = {I^n}_a$ (by [12, Lemma 2.5]) for all positive integers n, so it follows that I is a normal ideal. ■

Several specific examples of ideals I as in Example 5.1.2 are given in Example 6.6. We delay giving these examples till the next section, since they are also examples of a Noetherian domain R with a proper finite integral extension domain A such that $\mathbf{P}(IA)$ is projectively full for all nonzero ideals I of R, and since they are also closely related to Examples 6.1.4 and 6.1.5.

Example 5.2 Let I be a regular ideal in a Noetherian ring R such that the center q in R of some Rees valuation ring (V, N) of I is not a minimal prime divisor of I and the Rees integer e of I with respect to V is a unit of V. Let $b_1, \dots, b_g$ be a basis of I such that $b_iV = N^e$ for $i = 1, \dots, g$ (see Remark 2.6.2), let $A = R[x_1, \dots, x_g]$, let $J = (x_1, \dots, x_g)A$ be as in Corollary 4.3, and let (U, M) be the extension of V to a Rees valuation ring of J as in the proof of Theorem 2.4. Then $(J, q)A = M \cap A$, $(J, q)A$ properly contains a minimal prime divisor of J, and *every* ideal H between J and $(J, q)A$ has Rees integer equal to one with respect to U.

Proof. It follows from the hypothesis concerning q and Proposition 3.1.5 that $(J, q)A$ is a prime ideal that properly contains a minimal prime divisor of J. Therefore the conclusion follows immediately from Corollary 4.3 and Lemma 4.2.3. ■

Example 5.3 Let I be a nonzero ideal in a Noetherian domain R such that I has a unique Rees valuation ring (V, N) and the Rees integer e of I with respect to V is a unit of V. Let $b_1, \dots, b_g$ be a basis of I such that $b_iV = N^e$ for $i = 1, \dots, g$ (see Remark 2.6.2) and let $A = R[x_1, \dots, x_g]$ and $J = (x_1, \dots, x_g)A$ be as in Corollary 4.3. Then J_a is a prime ideal. Also, each prime ideal in each Noetherian ring A^+ between A and its integral closure A' that lies over J_a has a Rees integer that is equal to one.

Proof. The hypothesis implies that $\text{Rad}(I)$ is a prime ideal and that there exists a unique prime ideal in R' that lies over $\text{Rad}(I)$. Therefore the last statement follows from Corollary 4.3.

Also, $J_a = \cap\{JU_i \cap A \mid U_i$ is a Rees valuation ring of $J\}$, by [15, Theorem 4.12, page 61] (or by [12, (2.5)] together with [2, (2.3)]), and each such U_i is an extension of V, so the maximal ideal M_i of U_i lies over the maximal ideal N of V (so $M_i \cap R = \text{Rad}(I)$), and $JU_i = M_i$ (since the Rees integer of J with respect to U_i is equal to one (by Proposition 3.1.8)), so $JU_i \cap A = M_i \cap A$. Further, there exists a one-to-one correspondence between the minimal prime divisors of I and the minimal prime divisors of J, by Proposition 3.1.5, so it follows that J_a has a unique minimal prime divisor and that J_a is a prime ideal. ■

Example 5.4 generalizes Example 5.3.

Example 5.4 Let R, I, $(V_1, N_1), \dots, (V_n, N_n)$, $e_1, \dots, e_n$, A, and J be as in Theorem 2.4, and let $p_1, \dots, p_h$ be the distinct prime ideals in $\{N_j \cap R \mid j = 1, \dots, n\}$ (subscripted so that $p_j = N_j \cap R$). Assume that $e_1 = \cdots = c_h =$ (say) e is not in N_j for $j = 1, \dots, n$ and that $p_1, \dots, p_h$ are minimal prime divisors of I. Then J_a has h primary components that are prime ideals and each of them has a Rees integer equal to one. In particular, if $p_1, \dots, p_h$ are all the minimal prime divisors of I, then J_a is a radical ideal that is the intersection of h (and no fewer) minimal prime divisors.

Proof. It follows from the fourth paragraph of the proof of Theorem 2.4 that the Rees integer of J with respect to each of its Rees valuation rings $(U_1, M_1), \dots, (U_h, M_h)$ (with U_j the extension of V_j constructed in the proof of Theorem 2.4) is equal to one. Therefore JU_j

$= M_j$, so it follows as in the proof of Example 5.3 that $M_j \cap A = (J, p_j)A$, that $J_a A_{(J,p_j)A}$ $= (J, p_j)A_{(J,p_j)A}$ for $j = 1, \ldots, h$, and that each $(J, p_j)A$ has a Rees integer equal to one. Also, there exists a one-to-one correspondence between the minimal prime divisors p of I and the minimal prime divisors P of J (given by $P = (J, p)A$), by Proposition 3.1.5. The conclusions clearly follow from this. ■

6 EXAMPLES OF PROJECTIVELY FULL IDEALS.

In [2, Section 4] we give a number of examples of projectively full ideals. In this section we give some additional examples.

Example 6.1 Let R be a Noetherian domain, let R' be the integral closure of R in its quotient field, and let $R^+ \subseteq R'$ be a Noetherian integral extension domain of R. Let I be a nonzero proper ideal of R.

(6.1.1) We have Rees $I =$ Rees IR^+. Also, for each $V \in$ Rees I, the Rees integer of I with respect to V is equal to the Rees integer of IR^+ with respect to V. Thus the gcd of the Rees integers of I is equal to the gcd of the Rees integers of IR^+.

(6.1.2) If IR^+ is projectively full in R^+, then I is projectively full in R.

(6.1.3) It is possible for I to be projectively full, while IR^+ is not projectively full.

(6.1.4) It is possible for $\mathbf{P}(I)$ to be projectively full in R, while $\mathbf{P}(IR^+)$ is not projectively full in R^+.

(6.1.5) It is possible for $\mathbf{P}(IR^+)$ to be projectively full, while $\mathbf{P}(I)$ is not projectively full.

Proof. To establish (6.1.1), since R^+ is contained in the quotient field of R, we have Rees $I =$ Rees IR^+ and $R^+ \subseteq V$ for each $V \in$ Rees I. Also, $IV = (IR^+)V$, so the Rees integer of I with respect to each V_i is the same as the Rees integer of IR^+ with respect to V_i. The last statement in (6.1.1) is clear from this.

(6.1.2) is proved in [2, (3.2)(1)].

For (6.1.3), we use [7, Example 3.4]. Let X and Y be indeterminates over a field E, let $R^+ = E[X, Y]$ and let $R = E[X^2, XY, Y]$ (so $R^+ = R'$). Then $I = X^2R$ is projectively full, but X^2R^+ is not projectively full.

For (6.1.4), let $R = E[X^2, XY, Y]$ as in the proof of (6.1.3), and let $R^+ = R[X^3] = E[X^2, X^3, XY, Y]$. Since $I = X^2R$ is projectively full, $\mathbf{P}(I)$ is projectively full. However, $(IR^+)_a = (X^2, X^3)R^+ := J$ is such that $\mathbf{P}(J)$ is not projectively full in $R^+ = E[X^2, X^3, XY, Y]$. For if $H := (X^3, X^4)R^+$, then $J^3 = H^2 = (X^6, X^7)R^+$ (so J and H are projectively equivalent), and J and H are not the integral closure of powers of any ideal of R^+.

For (6.1.5), let X be an indeterminate over a field E, let $R = E[[X^2, X^3]]$, and let $I = (X^2, X^3)R$ be the maximal ideal of R. Let $R^+ = E[[X]]$ (so $R^+ = R'$). Then R^+ is a DVR, so $\mathbf{P}(IR^+)$ is projectively full. Let $J = (X^3, X^4)R$. Then $J^2 = I^3 = (X^6, X^7)R$, so it follows that $\mathbf{P}(I)$ is not projectively full. ■

Question 6.2 Does there exist an example of a Noetherian domain R for which Example 6.1.4 holds with R^+ taken to be the integral closure R' of R?

In Example 6.4 we present several examples where R is a Noetherian domain that is not integrally closed and $\mathbf{P}(I)$ is projectively full for all nonzero proper ideals I of R. The following lemma will be used in explaining why these examples hold.

Lemma 6.3 *Let (R, M) be a local domain and let R^+ be a Noetherian integral extension domain of R. Assume that M is the Jacobson radical of R^+ and that $\mathbf{P}(IR^+)$ is projectively full for all nonzero proper ideals I of R. Then $\mathbf{P}(I)$ is projectively full for all nonzero proper ideals I of R.*

Proof. The hypothesis that R and R^+ have the same Jacobson radical implies that $H \subseteq M \subset R$ for each ideal H in R^+ that is projectively equivalent to IR^+. The conclusion readily follows from this. ■

In the three examples in Example 6.4, the Noetherian integral extension domain R^+ of Lemma 6.3 is chosen to be the integral closure R' of R.

Example 6.4 For the following rings R, $\mathbf{P}(IR)$ is projectively full for all nonzero proper ideals I of R.

(6.4.1) Let E be a finite algebraic extension field of a field F, let X be an indeterminate, and let $R' = E[[X]]$ and $R = F + XR'$.

(6.4.2) Let $F \subset E$ be as in (6.4.1), let X, Y be indeterminates, and let $R' = E[[X, Y]]$ and $R = F + (X, Y)R'$.

(6.4.3) Let $R \subset R'$ be as in [9, Example 2, pp. 203-205] in the case where $m = 0$ and $r = 2$.

Proof. For (6.4.1), since $E[[X]]$ is a discrete valuation ring, it follows from Lemma 6.3 that $\mathbf{P}(IR)$ is projectively full for all nonzero proper ideals I of R.

For (6.4.2), since R' is a regular local ring of altitude two, it follows from Lemma 6.3 and either [7, (3.6)] or [1, (4.13)] that $\mathbf{P}(IR)$ is projectively full for all nonzero proper ideals I of R.

For (6.4.3), it is shown in [9] that: $\dim(R) = 2$; the integral closure R' of R is a unique factorization regular domain with exactly two maximal ideals $M = xR'$ and N; R'_M is a discrete valuation ring and R'_N is a regular local domain of altitude two; $M \cap N$ is the maximal ideal of R; and, $R' = R + eR$ for all elements $e \in R' - R$. Using these it can be shown that, for each nonzero ideal I in R, $IR' = x^i q$ $(= IR'^{\prime}_M \cap IR'_N)$ for some positive integer i and for some ideal q in R' such that $q \subseteq N$ and $q \nsubseteq M$. Since R'_N is a regular local domain of altitude two, it follows that $q_a = {Q^m}_a$ for some positive integer m, where Q is the largest element in the projectively full projective equivalence class $\mathbf{P}(q)$ (see either [7, (3.6)] or [1, (4.13)]). Then, since projectively equivalent ideals H, K have the same Rees valuation rings and proportional Rees integers (by [7, Proposition 2.10] and [1]), it follows that $\mathbf{P}(IR')$ is projectively full with largest ideal $x^{i/c}(Q^{m/c})_a$, where c is the greatest common divisor of i and m. The conclusion follows from this and Lemma 6.3. ■

Remark 6.5 If the Noetherian domain R has a finite integral extension domain R^+ that is a regular local domain of altitude two, then [7, (3.6)] or [1, (4.13)] implies that $\mathbf{P}(IR^+)$ is projectively full for every nonzero proper ideal I of R. We present in Example 6.6 specific examples of such rings R.

Example 6.6 Let F be a field, let X, Y be indeterminates, let n be a positive integer, let $R_n = F[[\{X^{n-i}Y^i\}_{i=0}^n]]$, and let $M_n = (\{X^{n-i}Y^i\}_{i=0}^n)R_n$. Then $R_1 = F[[X, Y]]$ is a finite integral extension domain of R_n and a regular local domain of altitude two. Therefore

$\mathbf{P}(IR_1)$ is projectively full for each nonzero proper ideal I in R_n. Also, M_n is a projectively full normal ideal that has only one Rees valuation ring V_n and its Rees integer with respect to V_n is equal to one.

Proof. That $\mathbf{P}(IR_1)$ is projectively full is immediate from Remark 6.5.

For the last statement, note first that $R_n[M_n/X^n] = R_n[Y/X]$ (since $\frac{X^{n-i}Y^i}{X^n} = \frac{Y^i}{X^i}$ for $i = 1, \ldots, n$). For each positive integer j let $C_j = R_j[M_j/X^j]$ and let $C_j{}'$ be the integral closure of C_j. Then, in particular, $C_1 = R_1[Y/X]$, and it is well known that $C_1 = C_1{}'$ and that XC_1 is a prime ideal such that $(C_1)_{XC_1}$ is the ord valuation ring of M_1 (and the only Rees valuation ring of M_1). Also, $C_n[X]$ (resp., $C_n{}'[X]$) is a free integral extension domain of C_n (resp., $C_n{}'$), and $Y = X(Y/X) \in C_n[X]$ (so $R_1 \subset C_n[X]$), so it follows that $C_1 = C_n[X] = C_n{}'[X]$ is a free integral extension domain of C_n and of $C_n{}'$. Therefore, since $C_n \subseteq C_n{}'$, it follows that $C_n = C_n{}'$. Also, X^nC_1 is XC_1-primary, so it follows that X^nC_n is primary for $p_n = XC_1 \cap C_n$. Since the Rees valuation rings of M_n are the rings $(C_n{}')_{p_i}$, where the p_i are the (height one) prime divisors of $X^nC_n{}'$ $(= X^nC_n)$, it follows that $V_n = (C_n)_{p_n}$ is the only Rees valuation ring of M_n.

To see that M_n is a normal projectively full ideal and that the Rees integer of M_n with respect to V_n is equal to one, it suffices (by Example 5.1.2) to show that X^nC_n is a prime ideal.

For this, since X^n, Y^n is a system of parameters in R_n, it is well known that $P = M_nR_n[Y^n/X^n]$ is a prime ideal and that the P-residue class T of Y^n/X^n is transcendental over $F = R_n/M_n$ (so $R_n[Y^n/X^n]/(M_nR_n[Y^n/X^n]) = F[T]$ is a polynomial ring over F). Also, $X^nC_n = M_nC_n$ (since $X^{n-i}Y^i = X^n(Y^i/X^i) \in X^nC_n$ for $i = 0, 1, \ldots, n$), so $C_n/(X^nC_n) = F[\overline{Y/X}]$. Further, $C_n = R_n[Y/X]$ is a finite integral extension ring of $R_n[Y^n/X^n]$, so $P = M_nR_n[Y^n/X^n] = M_nC_n \cap R_n[Y^n/X^n] = X^nC_n \cap R_n[Y^n/X^n]$. It follows that $F[\overline{Y/X}] = C_n/(X^nC_n)$ is a finite integral extension ring of the polynomial ring $R_n[Y^n/X^n]/(M_nR_n[Y^n/X^n]) = F[T]$, hence X^nC_n is a prime ideal. ■

Remark 6.7 If one applies the construction in Theorem 2.4 to the ring R_n of Example 6.6 and the set $\{X^{n-i}Y^i\}_{i=0}^n$ of generators of the ideal $M_n = (\{X^{n-i}Y^i\}_{i=0}^n)R_n$ of R_n, one obtains a finite free integral extension ring A_n of R_n. By Remark 2.1 there exists a minimal prime ideal z^* in A_n such that $A_n/z^* = R_n[(X^n)^{1/n}, (X^{n-1}Y)^{1/n}, \ldots, (Y^n)^{1/n}]$ is a proper finite integral extension domain of $R_1 = F[[X, Y]]$. However, if instead of applying the construction in Theorem 2.4 to the ideal M_n, we instead apply it to the generators X^n, Y^n of the reduction $(X^n, Y^n)R_n$ of M_n, then the free integral extension ring $A_n = R_n[T_1, T_2]/({T_1}^n - X^n, {T_2}^n - Y^n)$ of Theorem 2.4 has a minimal prime ideal z^* such that $A_n/z^* = R_1 = F[[X, Y]]$.

In Example 6.8 we present an example of a normal local domain (R, M) of altitude two such that M is projectively full and the associated graded ring $\mathbf{G}(R, M)$ is not reduced.

Example 6.8 Let F be an algebraically closed field with char $F = 0$, and let R_0 be a regular local domain of altitude two with maximal ideal $M_0 = (x, y)R_0$ and coefficient field F, e.g., $R_0 = F[x, y]_{(x,y)}$, or $R_0 = F[[x, y]]$, where x and y are indeterminates over F. Let $R = R_0[z]$, where $z^2 = x^3 + y^j$, where $j \geq 3$. It is readily checked that R is a normal local domain of altitude two with maximal ideal $M = (x, y, z)R$, and that $\mathbf{G}(R, M)$ is not reduced. We prove that M is projectively full.

Proof. The unique Rees valuation ring of M_0 is $V_0 = R_0[y/x]_{xR_0[y/x]}$. Notice that $I = (x, y)R$ is a reduction of M since z is integral over I. It follows that every Rees valuation ring of M is an extension of V_0. Let V be a Rees valuation ring of M and let v denote the normalized valuation with value group $\mathbb{Z}$ corresponding to V. Then $v(x) = v(y)$ and the image of y/x in the residue field of V is transcendental over F. Since $z^2 = x^3 + y^j$ and $j \geq 3$, we have

$$2v(z) = v(z^2) = v(x^3 + y^j) = 3v(x).$$

It follows that $v(x) = 2$ and $v(z) = 3$. Therefore V is ramified over V_0. This implies that V is the unique extension of V_0 and thus the unique Rees valuation ring of M.

For each positive integer n, let $I_n = \{r \in R \,|\, v(r) \geq n\}$. Thus $I_2 = M$. Since V is the unique Rees valuation ring of M, we have $I_{2n} = (M^n)_a$ for each $n \in \mathbb{N}$. To show M is projectively full, we prove that V is not the unique Rees valuation ring of I_{2n+1} for each $n \in \mathbb{N}$. Consider the inclusions

$$M^2 \subseteq I_4 \subset (z, x^2, xy, y^2)R := J \subseteq I_3 \subset M.$$

Since $\lambda(M/M^2) = 3$ and since the images of x and y in M/M^2 are F-linearly independent, $J = I_3$ and $M^2 = I_4 = (M^2)_a$. Since $x^3 = z^2 - y^j$ and $j \geq 3$, $L = (z, y^2)R$ is a reduction of $I_3 = (z, x^2, xy, y^2)R$. Indeed, $(x^2)^3 \in L^3$ and $(xy)^3 \in L^3$ implies x^2 and xy are integral over L. It follows that V is not a Rees valuation of I_3, for $zV \neq y^2V$. Consider $M^3 \subset I_3M \subseteq I_5 \subset I_4 = M^2$. Since the images of x^2, xy, y^2, xz, yz in M^2/M^3 are an F-basis, it follows that $I_3M = I_5$ and $M^3 = (M^3)_a = I_6$. Proceeding by induction, we assume $M^{n+1} = (M^{n+1})_a = I_{2n+2}$, and consider

$$M^{n+2} \subset I_3M^n \subseteq I_{2n+3} \subset M^{n+1} = I_{2n+2}.$$

Since the images in M^{n+1}/M^{n+2} of $\{x^ay^b \,|\, a+b = n+1\} \cup \{zx^ay^b \,|\, a+b = n\}$ is an F-basis, $\lambda(M^{n+1}/M^{n+2}) = 2n+3$, and the inequalities $\lambda(M^{n+1}/I_{2n+3}) \geq n+2$ and $\lambda(I_3M^n/M^{n+2}) \geq n+1$ imply $I_3M^n = I_{2n+3}$ and $M^{2n+2} = (M^{2n+2})_a$. Therefore the ideal I_{2n+3} has a Rees valuation ring different from V, and thus is not projectively equivalent to M. We conclude that M is projectively full. We have also shown that M is a normal ideal. ■

Remark 6.9 In [4], Joseph Lipman extends Zariski's theory of complete ideals of a regular local domain of altitude two to a situation where R is a normal local domain of altitude two that has a rational singularity. Lipman proves that R satisfies unique factorization of complete ideals if and only if the completion of R is a UFD. For R having this property, it follows that $\mathbf{P}(I)$ is projectively full each nonzero proper ideal I. An example to which this applies is $R = F[[x, y, z]]$, where F is a field and $z^2 + y^3 + x^5 = 0$. In [3, Corollary 3.11], Hartmut Göhner proves that if (R, M) is a normal local domain of altitude two that has a rational singularity, then the set of complete asymptotically irreducible ideals associated to a prime R-divisor v consists of the powers of an ideal A_v which is uniquely determined by v. In our terminology, this says that if I is a nonzero proper ideal of R having only one Rees valuation ring, then $\mathbf{P}(I)$ is projectively full. Göhner's proof involves choosing a desingularization $f : X \to \text{Spec } R$ such that v is centered on a component E_1 of the closed fiber on X. Let $E_2, \ldots, E_n$ be the other components of the closed fiber on X. Let E_X denote the group of divisors having the form $\sum_{i=1}^n n_iE_i$, where $n_i \in \mathbb{Z}$. Define

$$E_X^+ = \{D \in E_X \,|\, D \neq 0 \text{ and } (D \cdot E_i) \leq 0 \text{ for all } 1 \leq i \leq n\}$$

and

$$E_X^{\#} = \{D \in E_X \mid D \neq 0 \text{ and } O(-D) \text{ is generated by its sections over } X\}.$$

Lipman shows in [4] that $E_X^{\#} \subseteq E_X^+$ and that equality holds if R has a rational singularity. Also, if $D = \sum_i n_i E_i \in E_X^+$, then negative-definiteness of the intersection matrix $(E_i \cdot E_j)$ implies $n_i \geq 0$ for all i. For if $D \in E_X^+$ and $D = A - B$, where A and B are effective, then $(A - B \cdot B) \leq 0$ and $(A \cdot B) \geq 0$ imply $(B \cdot B) \geq 0$, so $B = 0$. Let $v = v_1, v_2, \ldots, v_n$ denote the discrete valuations corresponding to $E_1, \ldots, E_n$. Associated with $D = \sum_i n_i E_i \in E_X^{\#}$ one defines the complete M-primary ideal $I_D = \{r \in R \mid v_i(r) \geq n_i \text{ for } 1 \leq i \leq n\}$. This sets up a one-to-one correspondence between elements of $E_X^{\#}$ and complete M-primary ideals that generate invertible O_X-ideals. Lipman suggested to us the following proof that $\mathbf{P}(I)$ is projectively full for each complete M-primary ideal I if R has a rational singularity. Fix a desingularization $f : X \to \operatorname{Spec} R$ such that I generates an invertible O_X-ideal and let $D = \sum_i n_i E_i \in E_X^{\#}$ be the divisor associated to I. Let $g = \gcd\{n_i\}$. Since $E^+ = E^{\#}$, $(1/g)D \in E^{\#}$. The ideals $J \in \mathbf{P}(I)$ correspond to divisors in $E^{\#}$ that are integral multiples of $(1/g)D$. Thus if K is the complete M-primary ideal associated to $(1/g)D$, then each $J \in \mathbf{P}(I)$ is the integral closure of a power of K, so $\mathbf{P}(I)$ is projectively full. Since the rings $R_n = F[[\{X^{n-i}Y^i\}_{i=0}^n]]$ as in Example 6.6 are normal local domains of altitude two that have rational singularities, it follows that $\mathbf{P}(I)$ is projectively full for each nonzero proper ideal I of R_n.

ACKNOWLEDGMENT OF PRIORITY:

(i) In [1, Remark 4.2(d)] we noted that it was shown in [7, (2.9)] that if I is a regular ideal in a Noetherian ring R, then there exists a positive integer d such that, for all ideals J in R that are projectively equivalent to I, $(J^d)_a = (I^n)_a$ for some positive integer n. This result was also proved in [6, (1.4)].

(ii) In [1, Proposition 3.3] we showed that Rees $I \cup$ Rees $J =$ Rees IJ if $\dim(R) \leq 2$, and we noted just prior to [1, Proposition 3.3] that for the case that R is a pseudo-geometric normal Noetherian domain, this result appears in [3, Lemma 2.1]. The equality Rees $I \cup$ Rees $J =$ Rees IJ was first proved for an equicharacteristic integrally closed analytically irreducible local domain of dimension two in [10, Theorem 3.17].

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