THE LEADING IDEAL OF A COMPLETE INTERSECTION OF HEIGHT TWO, PART II

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Abstract. Let $(S, \mathbf{n})$ be a regular local ring and let $I = (f, g)$ be an ideal in S generated by a regular sequence f, g of length two. Let $R = S/I$ and $\mathfrak{m} = \mathfrak{n}/I$. As in [GHK], we examine the leading form ideal I^* of I in the associated graded ring $G = \mathrm{gr}_{\mathbf{n}}(S)$. If $\mathrm{gr}_{\mathfrak{m}}(R)$ is Cohen-Macaulay, we describe precisely the Hilbert series $\mathrm{H}(\mathrm{gr}_{\mathfrak{m}}(R), \lambda)$ in terms of the degrees of homogeneous generators of I^* and of their successive GCD's. If $D = \mathrm{GCD}(f^*, g^*)$ is a prime element of $\mathrm{gr}_{\mathbf{n}}(S)$ that is regular on $\mathrm{gr}_{\mathbf{n}}(S)/(\frac{f^*}{D}, \frac{g^*}{D})$, we prove that I^* is 3-generated and a perfect ideal. If $\mathrm{ht}_{\mathrm{gr}_{\mathfrak{n}}(S)}(f^*, g^*, h^*) = 2$, where $h \in I$ is such that h^* is of minimal degree in $I^* \setminus (f^*, g^*)\,\mathrm{gr}_{\mathbf{n}}(S)$, we prove I^* is 3-generated and a perfect ideal of $\mathrm{gr}_{\mathbf{n}}(S)$, so $\mathrm{gr}_{\mathfrak{m}}(R) = \mathrm{gr}_{\mathbf{n}}(S)/I^*$ is a Cohen-Macaulay ring. We give several examples to illustrate our theorems.

1. Introduction

This paper examines generators of the defining ideal of the tangent cone of a complete intersection of codimension two. We fix the following notation.

Setting 1.1. Let $(S, \mathfrak{n})$ be a regular local ring of dimension $s \geq 2$ and let $I = (f, g)$ be an ideal in S generated by a regular sequence f, g of length two. For simplicity we assume that the residue class field $k = S/\mathfrak{n}$ is infinite. We put $R = S/I$ and $\mathfrak{m} = \mathfrak{n}/I$. Let

$$\mathrm{R}'(\mathfrak{n}) = \sum_{i \in \mathbb{Z}} \mathfrak{n}^i t^i \subseteq S[t, t^{-1}] \quad \text{and} \quad \mathrm{R}'(\mathfrak{m}) = \sum_{i \in \mathbb{Z}} \mathfrak{m}^i t^i \subseteq R[t, t^{-1}]$$

denote the Rees algebras of $\mathfrak{n}$ and $\mathfrak{m}$ respectively, where t is an indeterminate. We put

$$G = \mathrm{gr}_{\mathfrak{n}}(S) = \mathrm{R}'(\mathfrak{n})/t^{-1}\mathrm{R}'(\mathfrak{n}) \quad \text{and} \quad \mathrm{gr}_{\mathfrak{m}}(R) = \mathrm{R}'(\mathfrak{m})/t^{-1}\mathrm{R}'(\mathfrak{m}).$$

For each $0 \neq h \in S$ let $\mathrm{o}(h) = \sup\{i \in \mathbb{Z} \mid h \in \mathfrak{n}^i\}$ and put $h^* = \overline{ht^n}$, where $n = \mathrm{o}(h)$ and $\overline{ht^n}$ denotes the image of ht^n in G. The canonical map $S \to R$ induces the

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epimorphism $\varphi : G \to \mathrm{gr}_{\mathfrak{m}}(R)$ of the associated graded rings. We put

$$I^* = \mathrm{Ker}\ (G \xrightarrow{\varphi} \mathrm{gr}_{\mathfrak{m}}(R)).$$

Then the homogeneous components $\{[I^*]_i\}_{i\in\mathbb{Z}}$ of the leading form ideal I^* of I are given by

$$[I^*]_i = \{\overline{ht^i} \mid h \in I \cap \mathfrak{n}^i\}$$

for each $i \in \mathbb{Z}$. We throughout assume that $a = \mathrm{o}(f) \le b = \mathrm{o}(g)$ and that $f^* \nmid g^*$ in G. The latter part of the condition is equivalent to saying that f^*, g^* form a part of a minimal homogeneous system of generators of I^*.

The original motivation for our work comes from a paper of S. C. Kothari [K]. Kothari answers several questions raised by Abyhankar concerning the local Hilbert function of a pair of plane curves. Let $\ell_S(*)$ denote length over S. In the case where $\dim S = 2$, Kothari proves that $0 \le \dim_k[\mathrm{gr}_{\mathfrak{m}}(R)]_i - \dim_k[\mathrm{gr}_{\mathfrak{m}}(R)]_{i+1} \le 1$ for all $i \ge a$ and that $\ell_S(R) \ge ab$; moreover, one has the equality $\ell_S(R) = ab$ if and only if $f^*,\ g^*$ are coprime in G, that is, $f^*,\ g^*$ form a G-regular sequence.

We have subsequently learned from an informative referee report of other work in this area. Indeed, F. Macaulay in a 1904 paper [M] employs a different method to determine the same necessary condition as Kothari on the Hilbert function of a pair of plane curves. Using his inverse systems, Macaulay establishes the structure of the Hilbert function $H(A)$ of a complete intersection quotient $A = k[[x, y]]/(f, g)$ to be of the form

$$H = (1, 2, \dots, a, t_a, \dots, t_j, 0), \tag{1}$$

where $a \ge t_a \ge t_{a+1} \ge \cdots \ge t_j = 1$ and $|t_i - t_{i+1}| \le 1$ for all i. Thus the Hilbert function H after an initial rising segment breaks up into platforms and regular flights of descending stairs, each step of height one. The structure of $H(A)$ is studied from the point of view of parametrizations by J. Briançon [Br] and by A. Iarrobino [Ia1] and [Ia2]. These authors prove that every sequence satisfying the conditions in Equation 1 is realizable as the Hilbert function $H(A)$ of some Gorenstein Artin algebra of the form $A = k[[x, y]]/(f, g)$.

Let $v(H) = 2 + \#\{\text{platforms}\}$. Iarrobino [Ia1], [Ia2] proves that I^* needs two initial generators f^*, g^* and requires a new generator following each platform, and that $v(H)$ is the minimum possible number of generators of a graded ideal defining a standard algebra with Hilbert function H. In [Ia1, Theorem 2.2.A], Iarrobino characterizes those graded ideals corresponding to I^* for which I is a complete intersection of height two. He proves they are exactly the graded ideals with $v(H)$ generators. The referee has pointed out that our results in Theorem 1.2 and Theorem 1.3 can be deduced from these results of Iarrobino. While acknowledging the priority of these results of Iarrobino, we hope that our different approach is still of some interest.

Theorem 1.2. *Let notation be as in Setting 1.1 and assume that* $\dim S = 2$ *and* $n = \mu_G(I^*)$. *Then* I^* *contains a homogeneous system* $\{\xi_i\}_{1\le i\le n}$ *of generators that satisfy the following three conditions.*

(1) $\xi_1 = f^*$ *and* $\xi_2 = g^*$.
(2) $\deg\xi_i + 2 \le \deg\xi_{i+1}$ *for all* $2 \le i \le n-1$.
(3) $\mathrm{ht}_G(\xi_1, \xi_2, \cdots, \xi_{n-1}) = 1$.

Let $\{\xi_i\}_{1\le i\le n}$ be a homogeneous system of generators of I^* satisfying conditions (1) and (2) in Theorem 1.2. We prove that the ideals

$$\{(\xi_j \mid 1 \le j \le i)G\}_{1\le i\le n}$$

of G are independent of the particular choice of the family $\{\xi_i\}_{1\le i\le n}$ and are uniquely determined by I. Moreover, if $D_i = \mathrm{GCD}(\xi_j \mid 1 \le j \le i)$ and $d_i = \deg D_i$, then one has the strictly descending sequence

$$a = d_1 > d_2 > \cdots > d_{n-1} > d_n = 0$$

and $\dfrac{\xi_{i+1}}{D_{i+1}} \in (\dfrac{\xi_1}{D_i}, \dfrac{\xi_2}{D_i}, \cdots, \dfrac{\xi_i}{D_i})$ for all $1 \le i \le n-1$ (Lemma 3.2). Let $c_i = \deg\,\xi_i$ and let

$$\mathrm{H}(\mathrm{gr}_{\mathfrak{m}}(R), \lambda) = \sum_{i=0}^{\infty} \dim_k [\mathrm{gr}_{\mathfrak{m}}(R)]_i \lambda^i$$

denote the Hilbert series of $\mathrm{gr}_{\mathfrak{m}}(R)$. We explicitly describe $\mathrm{H}(\mathrm{gr}_{\mathfrak{m}}(R), \lambda)$ and the difference $\ell_S(R) - ab$ in terms of c_i and d_i, sharpening results proved by Kothari in [K].

Theorem 1.3. *Let notation be as in Setting 1.1 and assume that* $\dim S = 2$ *and* $n = \mu_G(I^*)$. *The following assertions hold true.*

(1) $\mathrm{H}(\mathrm{gr}_{\mathfrak{m}}(R), \lambda) = \frac{\sum_{i=2}^{n} \lambda^{d_i}(1-\lambda^{d_{i-1}-d_i})(1-\lambda^{c_i-d_i})}{(1-\lambda)^2}$.
(2) $\ell_S(R) = \sum_{i=2}^{n}(d_{i-1} - d_i)(c_i - d_i) = ab + \sum_{i=2}^{n-1} d_i{\cdot}[(c_{i+1} - c_i) - (d_{i-1} - d_i)]$.
(3) $c_{i+1} - c_i > d_{i-1} - d_i > 0$ *for all* $2 \le i \le n-1$.
(4) [K, Corollary1] $\ell_S(R) = ab$ *if and only if* $n = 2$, *i.e.,* f^*, g^* *is a* G*-regular sequence.*

Remark 1.4. In the case where $\dim S = s > 2$, it is still true that $\mathrm{ht}_G(f^*, g^*) > 1$ implies f^*, g^* is a G-regular sequence, and therefore $I^* = (f^*, g^*)G$ also in this case. Thus we assume that $\mathrm{ht}_G(f^*, g^*) = 1$ and put $D_2 = \mathrm{GCD}(f^*, g^*)$ and $d_2 = \deg D_2$. Let $f^* = D_2\xi$ and $g^* = D_2\eta$. Notice that ξ, η is a regular sequence in G. We have $b \geq a > d_2 > 0$, and $\mu_G(I^*) = n \geq 3$. There exists a minimal homogeneous system $\{\xi_1, \xi_2, \dots, \xi_n\}$ of generators of I^* such that $\xi_1 = f^*$ and $\xi_2 = g^*$, and $c_i := \deg \xi_i \leq \deg \xi_{i+1} := c_{i+1}$ for each $i \leq n-1$. However, the ideal I^* may fail to be perfect, and it is possible to have $D_3 := \mathrm{GCD}(\xi_1, \xi_2, \xi_3) = D_2$ as is illustrated in [GHK, Example 1.6]. We prove in [GHK, Theorem 1.2] that I^* is perfect if $n = 3$. We also prove in [GHK] that $\xi_3 = h^*$, where h has the form $h = \alpha f + \beta g \in I$ with $\mathrm{o}(\alpha) = b - d_2$, and $\mathrm{o}(\beta) = a - d_2$, and that $c_3 := \mathrm{o}(h) > a + b - d_2$. Moreover, if $q = \sigma f + \tau g$ is such that $q^* \notin (f^*, g^*)G$ and $(o)(\sigma) = b - d_2$, then $\mathrm{o}(q) = \mathrm{o}(h)$ and $(f^*, g^*, h^*)G = (f^*, g^*, q^*)G$. Thus the ideal $(\xi_1, \xi_2, \xi_3)G$ is independent of the choice of ξ_3. In the case where $n \geq 4$, we also prove that $c_4 \geq c_3 + 2$ [GHK, Proposition 2.4]. However, examples shown to us by Craig Huneke and Lance Bryant show that it is possible to have $c_{i+1} = c_i$ for $i \geq 4$. This resolves a question mentioned in [GHK, Discussion 2.5]).

If $\mathrm{gr}_{\mathfrak{m}}(R)$ is a Cohen-Macaulay ring, we prove in Section 4 by passing to the factor ring of G modulo a suitable linear system of parameters for $\mathrm{gr}_{\mathfrak{m}}(R)$ that it is possible to reduce the problems to the case where $\dim S = 2$ and obtain results corresponding to those proved in Section 3 about the Hilbert series $\mathrm{H}(\mathrm{gr}_{\mathfrak{m}}(R), \lambda)$. In particular, if I^* is perfect, then $c_{i+1} > c_i + 1$ for each i with $2 \leq i \leq n-1$.

With notation as in Setting 1.1, let $\mathrm{e}^0_{\mathfrak{m}}(R)$ denotes the multiplicity of R with respect to $\mathfrak{m}$. Using Theorem 1.2, we prove in Section 4:

Theorem 1.5. *Assume notation as in Setting 1.1 and Remark 1.4, and let $D := D_2$, $d := d_2$ and $c := c_3$. If $\mathrm{ht}_G(f^*, g^*, h^*) = 2$, then the following assertions hold true.*

(1) $I^* = (f^*, g^*, h^*)$.
(2) $\mathrm{gr}_{\mathfrak{m}}(R)$ *is a Cohen-Macaulay ring.*
(3) $\mathrm{H}(\mathrm{gr}_{\mathfrak{m}}(R), \lambda) = \frac{(1-\lambda^c)(1-\lambda^d)+\lambda^d(1-\lambda^{a-d})(1-\lambda^{b-d})}{(1-\lambda)^{\dim S}}$.
(4) $\mathrm{e}^0_{\mathfrak{m}}(R) = ab + d\cdot[(c+d) - (a+b)]$.

Let $M = [\mathrm{gr}_{\mathfrak{m}}(R)]_+$ and let $\mathrm{H}^{s-2}_M(\mathrm{gr}_{\mathfrak{m}}(R))$ denote the $s-2$ $\underline{th}$ local cohomology module of $\mathrm{gr}_{\mathfrak{m}}(R)$ with respect to M. Recall that

$$\mathrm{a}(\mathrm{gr}_{\mathfrak{m}}(R)) = \max\{i \in \mathbb{Z} \mid [\mathrm{H}^{s-2}_M(\mathrm{gr}_{\mathfrak{m}}(R))]_i \neq (0)\}$$

is the a-invariant of $\mathrm{gr}_{\mathfrak{m}}(R)$. Using this notation and setting $Q = (X_1, \dots, X_{s-2})G$, where $X_1, \dots, X_s$ are suitably chosen homogeneous elements of degree one in G such that $G = k[X_1, \dots, X_s]$, and using the formula

$$\mathrm{a}(\mathrm{gr}_{\mathfrak{m}}(R)/Q\mathrm{gr}_{\mathfrak{m}}(R)) = \mathrm{a}(\mathrm{gr}_{\mathfrak{m}}(R)) + (s-2)$$

of [GW, Remark (3.1.6)], we establish the following result in Section 4.

Theorem 1.6. *Assume notation as in Setting 1.1 and Remark 1.4. If $\mathrm{gr}_{\mathfrak{m}}(R)$ is a Cohen-Macaulay ring and $n = \mu_G(I^*)$, then the following assertions hold true.*

(1) $\mathrm{H}(\mathrm{gr}_{\mathfrak{m}}(R), \lambda) = \dfrac{\sum_{i=2}^{n} \lambda^{d_i}(1-\lambda^{d_{i-1}-d_i})(1-\lambda^{c_i-d_i})}{(1-\lambda)^s}$.

(2) $\mathrm{e}^0_{\mathfrak{m}}(R) = ab + \sum_{i=2}^{n-1} d_i{\cdot}[(c_{i+1}-c_i)-(d_{i-1}-d_i)]$ *with*

$$c_{i+1} - c_i > d_{i-1} - d_i > 0$$

for all $2 \le i \le n-1$.

(3) $\mathrm{e}^0_{\mathfrak{m}}(R) \le a{\cdot}[c_n + d_{n-1} - a]$, *where the equality holds true if and only if* $n = 2$.

(4) $\mathrm{a}(\mathrm{gr}_{\mathfrak{m}}(R)) = c_n + d_{n-1} - s$.

Sections 5 is devoted to some examples, which illustrate our theorems. Let $H = \langle n_1, n_2, n_3 \rangle$ be a Gorenstein numerical semigroup generated by the three integers n_1, n_2, n_3, where $0 < n_1 < n_2 < n_3$ and $\mathrm{GCD}(n_1, n_2, n_3) = 1$. Let $S = k[[X_1, X_2, X_3]]$ and $T = k[[t]]$ be formal power series rings over a field k. We denote by $\varphi : S \to T$ the k-algebra map defined by $\varphi(X_i) = t^{n_i}$ for $i = 1, 2, 3$. Let $I = \mathrm{Ker}\,\varphi$, $R = k[[t^{n_1}, t^{n_2}, t^{n_3}]]$, $\mathfrak{n} = (X_1, X_2, X_3)S$, and $\mathfrak{m} = (t^{n_1}, t^{n_2}, t^{n_3})R$. Then, as was essentially shown in [H2] and [RV], $\mathrm{gr}_{\mathfrak{m}}(R)$ is a Cohen-Macaulay ring if and only if I^* is 3-generated. We shall recover this result in our context. In Example 5.5, we present a family of examples due to Takahumi Shibuta that demonstrates that for $I = \mathrm{Ker}\,\varphi$ as above, there is no bound on the number of elements needed to generate I^*.

2. Preliminaries

Throughout this section, let notation be as in Setting 1.1, assume that $\dim S = 2$ and let $\mathfrak{n} = (x, y)$.

Lemma 2.1. *Let $h \in S$ with $m = \mathrm{o}(h)$ and assume that $x^* \nmid h^*$. Then $h = \varepsilon y^m + x\varphi$ for some $\varepsilon \in \mathrm{U}(S)$ and $\varphi \in \mathfrak{n}^{m-1}$.*

Proof. Let $\overline{S} = S/(x)$ and denote by $\overline{*}$ the image in $\overline{S}$. Let $\ell = \mathrm{o}(\overline{h})$. Then $\ell \geq m$ and $\overline{h} = \overline{\varepsilon} \cdot \overline{y}^{\ell}$ for some $\varepsilon \in \mathrm{U}(S)$. We write $h = \varepsilon y^{\ell} + x\varphi$ with $\varphi \in S$. Then $\varphi \in \mathfrak{n}^{m-1}$, because $(x) \cap \mathfrak{n}^m = x\,\mathfrak{n}^{m-1}$. Hence $\ell = m$, as $x^* \nmid h^*$. $\square$

Lemma 2.2. *There exist elements x, y, u, and $g_1 \in S$ satisfying the following conditions.*

(1) $\mathfrak{n} = (x, y)$ *and* $x^* \nmid f^*$.
(2) $u \in \mathrm{U}(S)$, $\mathrm{o}(g_1) = b - 1$, *and* $g = uy^{b-a}f + xg_1$

Proof. Let $\mathfrak{n} = (x, y)$. Then, since $k = S/\mathfrak{n}$ is infinite, we have $x^* + cy^* \nmid f^*$ and $x^* + cy^* \nmid g^*$ for some $c \in k$. Let $c \equiv \alpha \mod \mathfrak{n}$ ($\alpha \in S$) and $z = x + \alpha y$. Then $\mathfrak{n} = (z, y)$. Because $z^* \nmid f^*$ and $z^* \nmid g^*$, by Lemma 2.1, we have

$$f = \varepsilon y^a + z\xi \quad \text{and} \quad g = \tau y^b + z\eta$$

for some $\varepsilon, \tau \in \mathrm{U}(S)$, $\xi \in \mathfrak{n}^{a-1}$, and $\eta \in \mathfrak{n}^{b-1}$. Let $g_1 = \eta - uy^{b-a}\xi$ where $u = \tau\varepsilon^{-1}$. Then $g = uy^{b-a}f + zg_1$ and $\mathrm{o}(g_1) = b - 1$, because $g_1 \in \mathfrak{n}^{b-1}$ and $f^* \nmid g^*$. Replacing x with z, we get the required elements x, y, u, and $g_1 \in S$ as claimed. $\square$

In what follows let x, y, u, and $g_1 \in S$ be elements which satisfy conditions (1) and (2) in Lemma 2.2. We put $I_1 = (f, g_1)$, $X = x^*$, and $Y = y^*$.

Proposition 2.3. *The following assertions hold true.*

(1) $I = (f, xg_1)$ *and* $I :_S x = I_1$.
(2) $(f^*, g^*) = (f^*, Xg_1^*)$ *whence* $f^* \nmid {g_1}^*$.
(3) f^*, X *is a* G*-regular sequence.*
(4) $I = \mathfrak{n}$, *if* $b = 1$.
(5) ([K]) *Suppose that* $b > 1$. *Then* I_1 *is a parameter ideal in* S *and* $I^* = (f^*) + XI_1^*$. *Hence* $I^* :_G X = I_1^*$.

Proof. (1) Since $g = uy^{b-a}f + xg_1$, we get $I = (f, xg_1)$, whence $xI_1 \subseteq I$. Let $\varphi \in I :_S x$ and write $x\varphi = \alpha f + \beta(xg_1)$ ($\alpha, \beta \in S$). Then $x(\varphi - \beta g_1) \in (f)$ so that $\varphi - \beta g_1 \in (f)$, because f, x is a regular sequence in S (recall that $x \nmid f$). Hence $\varphi \in (f, g_1) = I_1$ and thus $I :_S x = I_1$.

(2) Recall that $g^* = u^*Y^{b-a}f^* + Xg_1^*$.

(3) This is clear, since $X \nmid f^*$.

(4) We have $a = 1$, since $a \leq b$. Hence $\mathrm{o}(g_1) = 0$ and $\mathrm{o}(f \mod (x)) = 1$ (cf. Proof of Lemma 2.1), so that we have $I = (f, xg_1) = (f, x) = \mathfrak{n}$.

(5) Since $b > 1$, we get $I \subseteq I_1 \subsetneq S$. Hence I_1 is a parameter ideal of S. Let $i \geq a-1$ be an integer. Then, thanks to Proof of [K, Lemma], we see that for every k -basis $W_1, W_2, \cdots, W_r$ of $[I_1^*]_i$, the elements $Y^{i+1-a}f^*, XW_1, XW_2, \cdots, XW_r$ form a k-basis of $[I^*]_{i+1}$. Consequently, $[I^*]_{i+1} \subseteq (f^*) + XI_1^* \subseteq I^*$ (recall that $xI_1 \subseteq I$), whence $I^* = (f^*) + XI_1^*$, because $[I^*]_i = (0)$ for $i \leq a-1$. As f^*, X is a G-regular sequence, we have the equality $I^* :_G X = I_1^*$ similarly as in the proof of assertion (1). $\square$

Corollary 2.4. *Suppose that $b > 1$. Then $\mathrm{H}(G/I^*, \lambda) = \sum_{i=0}^{a-1} \lambda^i + \lambda \cdot \mathrm{H}(G/I_1^*, \lambda)$.*

Proof. Notice that $(X, f^*)/I^* = (X, f^*)/[(f^*) + XI_1^*] \cong (X)/XI_1^* \cong (G/I_1^*)(-1)$, because $(X) \cap (f^*) = (Xf^*)$ and $f^* \in I_1^*$. Then we get the exact sequence

$$0 \to (G/I_1^*)(-1) \to G/I^* \to G/(X, f^*) \to 0$$

of graded G-modules, so that

$$\begin{aligned} \mathrm{H}(G/I^*, \lambda) &= \mathrm{H}(G/(X, f^*), \lambda) + \lambda \cdot \mathrm{H}(G/I_1^*, \lambda) \\ &= \sum_{i=0}^{a-1} \lambda^i + \lambda \cdot \mathrm{H}(G/I_1^*, \lambda) \end{aligned}$$

as claimed. $\square$

The following fact plays a key role in our argument.

Corollary 2.5. *Suppose that $b > 1$. Let $n = \mu_G(I^*)$ and $\ell = \mu_G(I_1^*)$.*

(1) *Suppose that $a < b$. Then $n = \ell$ and, for every homogeneous system $\{\eta_i\}_{1 \leq i \leq n}$ of generators of I_1^* with $\eta_1 = f^*$ and $\eta_2 = g_1^*$, we have $I^* = (f^*, g^*) + (X\eta_i \mid 3 \leq i \leq n)$.*

(2) *Suppose that $a = b$ and $g_1^* \nmid f^*$. Then $n = \ell$ and, for every homogeneous system $\{\eta_i\}_{1 \leq i \leq n}$ of generators of I_1^* with $\eta_1 = g_1^*$ and $\eta_2 = f^*$, we have $I^* = (f^*, g^*) + (X\eta_i \mid 3 \leq i \leq n)$.*

(3) *Suppose that $a = b$ but $g_1^* \mid f^*$. Then $n = \ell + 1$. Choose $f_1 \in S$ so that $\mathrm{o}(f_1) > a$, $I_1 = (g_1, f_1)$, and $g_1^* \nmid f_1^*$. Then, for every homogeneous system $\{\eta_i\}_{1 \leq i \leq n-1}$ of generators of I_1^* with $\eta_1 = g_1^*$ and $\eta_2 = f_1^*$, we have $I^* = (f^*, g^*) + (X\eta_i \mid 2 \leq i \leq n-1)$.*

Proof. (1) By Proposition 2.3 (2) we have $f^* \nmid g_1^*$. Let $\{\eta_i\}_{1 \leq i \leq \ell}$ be a homogeneous system of generators of I_1^* with $\eta_1 = f^*$ and $\eta_2 = g_1^*$. Then, because $I^* = (f^*) + XI_1^*$ and $(f^*, g^*) = (f^*, Xg_1^*)$ (cf. Proposition 2.3, (2) and (5)), we have $I^* = (f^*, X\eta_2) +$

$(X\eta_i \mid 3 \leq i \leq \ell)$. To see that $n = \ell$, we shall check that $f^*, X\eta_2, X\eta_3, \cdots, X\eta_\ell$ is a minimal system of generators of I^*. Since $f^* \notin (X)$, it suffices to show that $X\eta_i \notin (f^*) + (X\eta_2, \cdots, X\eta_{i-1}, X\eta_{i+1}, \cdots, X\eta_\ell)$ for any $2 \leq i \leq \ell$. Assume the contrary and write $X\eta_i = f^*\varphi + \sum_{2 \leq j \leq \ell, j \neq i} X\eta_j\varphi_j$ with $\varphi, \varphi_j \in G$. Then $X[\eta_i - \sum_{2 \leq j \leq \ell, j \neq i} \eta_j\varphi_j] \in (f^*)$. Because f^*, X form a G-regular sequence, we get $\eta_i \in (f^*) + (\eta_2, \cdots, \eta_{i-1}, \eta_{i+1}, \cdots \eta_\ell)$, which is impossible (recall that $f^* = \eta_1, \eta_2, \cdots, \eta_\ell$ is a minimal system of generators of I_1^*). Thus $n = \ell$.

(2) Let $\{\eta_i\}_{1 \leq i \leq \ell}$ be a homogeneous system of generators of I_1^* with $\eta_1 = g_1^*$ and $\eta_2 = f^*$. Then $I^* = (f^*, X\eta_1) + (X\eta_i \mid 3 \leq i \leq \ell)$. For the same reason as in the proof of assertion (1), $f^*, X\eta_1, X\eta_3, \cdots, X\eta_\ell$ is a minimal system of generators of I^* and we get $n = \ell$.

(3) Let $\{\eta_i\}_{1 \leq i \leq \ell}$ be a homogeneous system of generators of I_1^* such that $\eta_1 = g_1^*$ and $\eta_2 = f_1^*$. Then $I^* = (f^*, X\eta_1) + (X\eta_i \mid 2 \leq i \leq \ell)$. We want to show that $f^*, X\eta_1, X\eta_2, \cdots, X\eta_\ell$ is a minimal system of generators of I^*. Let $1 \leq i \leq \ell$ and assume that $X\eta_i \in (f^*) + (X\eta_1, \cdots, X\eta_{i-1}, X\eta_{i+1}, \cdots X\eta_\ell)$. Then $X[\eta_i - \sum_{1 \leq j \leq \ell, j \neq i} \eta_j\varphi_j] \in (f^*)$ for some $\varphi_j \in G$, so that we have $\eta_i \in (f^*) + (\eta_1, \cdots, \eta_{i-1}, \eta_{i+1}, \eta_\ell)$. If $i = 1$, then $\eta_1 = g_1^* \in (f^*) + (\eta_2, \eta_3, \cdots, \eta_\ell)$. Since $\deg f^* = a > \deg g_1^* = a - 1$, this forces $\eta_1 \in (\eta_2, \eta_3, \cdots, \eta_\ell)$, which is impossible. Hence $i > 1$. Then, because $\eta_1 \mid f^*$, we have $\eta_i \in (\eta_1, \cdots, \eta_{i-1}, \eta_{i+1}, \cdots, \eta_\ell)$, which is absurd. Thus $f^*, X\eta_1, X\eta_2, \cdots, X\eta_\ell$ constitute a minimal system of generators of I^* and so $n = \ell + 1$. $\square$

We close this section with the following.

Proposition 2.6. *Let $P = k[X, Y]$ be the polynomial ring in two variables X, Y over a field k. Let J be a graded ideal of P with $\mu_P(J) = n$ and $\sqrt{J} = (X, Y)$. Let $\{\xi_i\}_{1 \leq i \leq n}$ be a homogeneous system of generators of J and set $D_i = \mathrm{GCD}(\xi_1, \xi_2, \cdots, \xi_i)$ for $1 \leq i \leq n$. If $\deg D_i > \deg D_{i+1}$ and $\frac{\xi_{i+1}}{D_{i+1}} \in (\frac{\xi_1}{D_i}, \frac{\xi_2}{D_i}, \cdots, \frac{\xi_i}{D_i})$ for all $1 \leq i \leq n - 1$, then the Hilbert series $\mathrm{H}(P/J, \lambda) = \sum_{i=0}^{\infty} \dim_k [P/J]_i \lambda^i$ of P/J is given by the formula*

$$\mathrm{H}(P/J, \lambda) = \frac{\sum_{i=2}^{n} \lambda^{\deg D_i}(1 - \lambda^{\deg D_{i-1} - \deg D_i})(1 - \lambda^{\deg \xi_i - \deg D_i})}{(1 - \lambda)^2}.$$

In particular,

$$\mathrm{H}(G/(X^{3i}Y^{m-i-1} \mid 0 \leq i \leq m - 1), \lambda) = \frac{\sum_{i=2}^{m} \lambda^{m-i}(1 - \lambda^{3(i-1)})}{1 - \lambda}$$

for all $2 \leq m \in \mathbb{Z}$.

Proof. If $n = 2$, then ξ_1, ξ_2 is a P-regular sequence and we get $\mathrm{H}(P/J, \lambda) = \frac{(1-\lambda^{\deg \xi_1})(1-\lambda^{\deg \xi_2})}{(1-\lambda)^2}$. Suppose that $n > 2$ and that our assertion holds true for $n-1$. Let $D = D_{n-1}$. Then $J \subseteq (D, \xi_n)$ and D, ξ_n form a P-regular sequence (recall that $\mathrm{GCD}(D, \xi_n) = 1$). We write $\xi_i = D\eta_i$ $(1 \leq i \leq n-1)$ and put $K = (\eta_i \mid 1 \leq i \leq n-1)$. Then $\xi_n = \frac{\xi_n}{D_n} \in (\frac{\xi_1}{D}, \frac{\xi_2}{D}, \cdots, \frac{\xi_{n-1}}{D}) = K$ and so $\mu_P(K) = n-1$, since $J = DK + (\xi_n)$. Let $E_i = \mathrm{GCD}(\eta_1, \eta_2, \cdots, \eta_i)$. Then $D_i = DE_i$ so that we have $\deg E_i > \deg E_{i+1}$ and $\frac{\eta_{i+1}}{E_{i+1}} = \frac{\xi_{i+1}}{D_{i+1}} \in (\frac{\xi_1}{D_i}, \frac{\xi_2}{D_i}, \cdots, \frac{\xi_i}{D_i}) = (\frac{\eta_1}{E_i}, \frac{\eta_2}{E_i}, \cdots, \frac{\eta_i}{E_i})$ for all $1 \leq i \leq n-2$. Therefore, thanks to the exact sequence

$$0 \to (P/K)(-\deg D) \to P/J \to P/(D, \xi_n) \to 0$$

of graded P-modules (recall that $(D, \xi_n)/J = (D, \xi_n)/[DK + (\xi_n)] \cong (D)/[DK + (D) \cap (\xi_n)] = (D)/DK \cong (P/K)(-\deg D)$, since $(D) \cap (\xi_n) = (D\xi_n)$ and $\xi_n \in (K)$) and the hypothesis of induction on n, we get

$$\begin{aligned} \mathrm{H}(P/J, \lambda) &= \mathrm{H}(P/(D, \xi_n), \lambda) + \lambda^{\deg D} \cdot \mathrm{H}(P/K, \lambda) \\ &= \frac{(1-\lambda^{\deg D})(1-\lambda^{\deg \xi_n})}{(1-\lambda)^2} \\ &+ \frac{\lambda^{\deg D} \cdot \sum_{i=2}^{n-1} \lambda^{\deg E_i}(1-\lambda^{\deg E_{i-1} - \deg E_i})(1-\lambda^{\deg \eta_i - \deg E_i})}{(1-\lambda)^2} \\ &= \frac{\sum_{i=2}^{n} \lambda^{\deg D_i}(1-\lambda^{\deg D_{i-1} - \deg D_i})(1-\lambda^{\deg \xi_i - \deg D_i})}{(1-\lambda)^2} \end{aligned}$$

as claimed.

For the last assertion, let $\xi_i = X^{3(i-1)} Y^{m-i}$ for $1 \leq i \leq m$. Then $D_i = Y^{m-i}$ and $\dfrac{\xi_i}{D_i} = X^{3(i-1)}$ for all $1 \leq i \leq m$. Hence

$$\begin{aligned} \mathrm{H}(G/(X^{3i} Y^{m-i-1} \mid 0 \leq i \leq m-1), \lambda) &= \frac{\sum_{i=2}^{m} \lambda^{m-i}(1-\lambda)(1-\lambda^{3(i-1)})}{(1-\lambda)^2} \\ &= \frac{\sum_{i=2}^{m} \lambda^{m-i}(1-\lambda^{3(i-1)})}{1-\lambda}. \end{aligned}$$

□

3. Proof of Theorems 1.2 and 1.3

Proof of Theorem 1.2. Assume that Theorem 1.2 fails to hold and choose the ideal I so that $a = \mathrm{o}(I) := \sup\{i \in \mathbb{Z} \mid I \subseteq \mathfrak{n}^i\}$ is as small as possible among the counterexamples. We furthermore choose our ideal I so that $b = \mathrm{o}(g)$ is the smallest among the counterexamples I with $\mathrm{o}(I) = a$. Then $n > 2$, whence $b > 1$ (Proposition 2.3 (4)). Choose elements x, y, u, and $g_1 \in S$ so that conditions (1) and (2) in Lemma 2.2 are satisfied and put $I_1 = (f, g_1)$. We then have the following three cases: (i) $a < b$, (ii) $a = b$ and $g_1^* \nmid f^*$, and (iii) $a = b$ but $g_1^* \mid f^*$.

Suppose that case (i) occurs. Then $\mu_G(I_1^*) = n$ (cf. Corollary 2.5). Since $\mathrm{o}(I_1) = a$ but $\mathrm{o}(g_1) = b - 1$, we may choose a minimal homogeneous system $\{\eta_i\}_{1 \le i \le n}$ of generators of I_1^* so that

(1) $\eta_1 = f^*$ and $\eta_2 = g_1^*$,
(2) $\deg \eta_i + 2 \le \deg \eta_{i+1}$ for all $2 \le i \le n-1$, and
(3) $\mathrm{ht}_G(\eta_1, \eta_2, \cdots, \eta_{n-1}) = 1$.

Then, thanks to Corollary 2.5 (1), we get $I^* = (f^*, g^*) + (X\eta_3, \cdots, X\eta_n)$. Letting $\xi_1 = f^*$, $\xi_2 = g^*$, and $\xi_i = X\eta_i$ $(3 \le i \le n)$, we certainly have conditions (1) and (2) in Theorem 1.2, because $\deg g_1^* = b - 1 \le \deg \eta_3 - 2$. Since $(\xi_1, \xi_2, \cdots, \xi_{n-1}) = (f^*, g^*) + (\xi_3, \cdots, \xi_{n-1}) = (f^*, Xg_1^*) + (X\eta_3, \cdots, X\eta_{n-1}) \subseteq (\eta_1, \eta_2) + (\eta_3, \cdots, \eta_{n-1})$, we get $\mathrm{ht}_G(\xi_1, \xi_2, \cdots, \xi_{n-1}) = 1$. Thus case (i) cannot occur.

Suppose case (ii) occurs. Then $\mu_G(I_1^*) = n$. Since $\mathrm{o}(I_1) = a - 1$, we may choose a minimal homogeneous system $\{\eta_i\}_{1 \le i \le n}$ of generators of I_1^* so that

(1) $\eta_1 = g_1^*$ and $\eta_2 = f^*$,
(2) $\deg \eta_i + 2 \le \deg \eta_{i+1}$ for all $2 \le i \le n-1$, and
(3) $\mathrm{ht}_G(\eta_1, \eta_2, \cdots, \eta_{n-1}) = 1$.

Then $I^* = (f^*, g^*) + (X\eta_3, \cdots, X\eta_n)$ by Corollary 2.5 (2). Let $\xi_1 = f^*$, $\xi_2 = g^*$, and $\xi_i = X\eta_i$ $(3 \le i \le n)$. Then $\deg \xi_2 = b = a$ and $\deg \xi_3 = \deg \eta_3 + 1 \ge \deg \eta_2 + 3 = a + 3$, so that conditions (1) and (2) in Theorem 1.2 are safely satisfied for the family $\{\xi_i\}_{1 \le i \le n}$. Since $(\xi_1, \xi_2, \cdots, \xi_{n-1}) = (f^*, g^*) + (\xi_3, \cdots, \xi_{n-1}) = (f^*, Xg_1^*) + (\xi_3, \cdots, \xi_{n-1}) \subseteq (\eta_1, \eta_2, \cdots, \eta_{n-1})$, we also have condition (3) in Theorem 1.2 to be satisfied. Hence case (ii) cannot occur.

Thus we have case (iii). Hence $\mu_G(I_1^*) = n - 1$. We choose $f_1 \in S$ so that $\mathrm{o}(f_1) = a_1 > a$, $I_1 = (g_1, f_1)$, and $g_1^* \nmid f_1^*$. Because $\mathrm{o}(I_1) = a - 1 < a = \mathrm{o}(I)$, we may choose a minimal homogeneous system $\{\eta_i\}_{1 \le i \le n-1}$ of generators of I_1^* so that

(1) $\eta_1 = g_1^*$ and $\eta_2 = f_1^*$,
(2) $\deg \eta_i + 2 \le \deg \eta_{i+1}$ for all $2 \le i \le n-2$, and
(3) $\mathrm{ht}_G(\eta_1, \eta_2, \cdots, \eta_{n-2}) = 1$.

Then $I^* = (f^*, g^*) + (X\eta_2, X\eta_3, \cdots, X\eta_{n-1})$. Let $\xi_1 = f^*$, $\xi_2 = g^*$, and $\xi_i = X\eta_{i-1}$ for $3 \le i \le n$. Because $\deg \eta_2 = a_1 > a$, we have $\deg \xi_3 \ge a + 2$, so that conditions (1) and (2) in Theorem 1.2 are satisfied for the family $\{\xi_i\}_{1 \le i \le n}$. Since

$(f^*, g^*) + (\xi_3, \cdots, \xi_{n-1}) = (f^*, Xg_1^*) + (\xi_3, \cdots, \xi_{n-1}) \subseteq (g_1^*) + (\eta_2, \cdots, \eta_{n-2}) = (\eta_1, \eta_2, \cdots, \eta_{n-2})$ (recall that $g_1^* \mid f^*$), we also have condition (3). This is absurd and thus Theorem 1.2 holds true.

Discussion 3.1. Let $\xi_1, \xi_2, \cdots, \xi_n$ be a homogeneous system of generators for I^* which satisfies conditions (1) and (2) in Theorem 1.2. Let $c_i = \deg \xi_i$ for $1 \le i \le n$, and let $G_+ = \sum_{i>0} G_i$. We then have $\{c_1, c_2, \cdots, c_n\} = \{i \in \mathbb{Z} \mid [I^*/G_+{\cdot}I^*]_i \neq (0)\}$, whence the *degree* sequence $(c_1, c_2, \cdots, c_n)$ is independent of the choice of $\{\xi_i\}_{1 \le i \le n}$. Because $\xi_1 = f^*$, $\xi_2 = g^*$, and $c_1 = a \le c_2 = b < c_3 < \cdots < c_n$, the ideals $(\xi_1, \xi_2, \cdots, \xi_i)$ $(1 \le i \le n)G$ also do not depend on the choice of $\{\xi_i\}_{1 \le i \le n}$. We put $D_i = \mathrm{GCD}(\xi_1, \xi_2, \cdots, \xi_i)$ and $d_i = \deg D_i$ $(1 \le i \le n)$. (Hence $D_1 = \xi_1$ and $D_n = 1$.) Since the ideal $(\xi_1, \xi_2, \cdots, \xi_{n-1})$ is independent of the choice of $\{\xi_i\}_{1 \le i \le n}$, we have condition (3) in Theorem 1.2 that $\mathrm{ht}_G(\xi_1, \xi_2, \cdots, \xi_{n-1}) = 1$ is always satisfied for every homogeneous system of generators $\{\xi_i\}_{1 \le i \le n}$ of I^* which satisfies conditions (1) and (2) of Theorem 1.2. Similarly, the fact whether $\frac{\xi_{i+1}}{D_{i+1}} \in (\frac{\xi_1}{D_i}, \frac{\xi_2}{D_i}, \cdots, \frac{\xi_i}{D_i})$ or not does not depend on the particular choice of a homogeneous system $\{\xi_i\}_{1 \le i \le n}$ of generators of I^* which satisfies conditions (1) and (2) in Theorem 1.2.

Lemma 3.2. *With notation as in Discussion 3.1, the following assertions hold true.*

(1) $d_i > d_{i+1}$ *and* $\frac{\xi_{i+1}}{D_{i+1}} \in (\frac{\xi_1}{D_i}, \frac{\xi_2}{D_i}, \cdots, \frac{\xi_i}{D_i})$ *for all* $1 \le i \le n-1$.
(2) $c_{i+1} - c_i > d_{i-1} - d_i > 0$ *for all* $2 \le i \le n-1$.
(3) $c_n + d_{n-1} \ge d_{i-1} + c_i - d_i$ *for all* $2 \le i \le n$.
(4) $c_n + d_{n-1} \ge a + b$.

Proof. Assume that Lemma 3.2 is false and choose an ideal I so that $a = \mathrm{o}(I) = \sup\{i \in \mathbb{Z} \mid I \subseteq \mathfrak{n}^i\}$ is as small as possible among the counterexamples. We furthermore choose the ideal I so that $b = \mathrm{o}(g)$ is the smallest among the counterexamples I with $\mathrm{o}(I) = a$. Then $b > 1$, since $n > 2$. Let x, y, u, and $g_1 \in S$ be elements which satisfy conditions (1) and (2) in Lemma 2.2. We put $I_1 = (f, g_1)$. Then we have the following three cases: (i) $a < b$, (ii) $a = b$ and $g_1^* \nmid f^*$, and (iii) $a = b$ but $g_1^* \mid f^*$. For case (i) we have $f^* \nmid g_1^*$ and for case (iii) we have some $f_1 \in S$ with $\mathrm{o}(f_1) = a_1 > a$ such that $I_1 = (g_1, f_1)$ and $g_1^* \nmid f_1^*$. In any case, because the value a or the value b for I_1 is less than that for I, Lemma 3.2 holds true for the ideal I_1. In what follows, we shall establish a contradiction by showing (i),(ii), and (iii) cannot occur.

Suppose that case (i) occurs. Then $\mu_G(I_1^*) = n$. Let $\{\eta_i\}_{1 \le i \le n}$ be a homogeneous system of generators of I_1^* such that $\eta_1 = f^*$, $\eta_2 = g_1^*$, and $\deg \eta_i + 2 \le \deg \eta_{i+1}$

for $2 \le i \le n-1$. Then Lemma 3.2 holds true for the family $\{\eta_i\}_{1 \le i \le n}$ and by Corollary 2.5 we have

$$I^* = (f^*, g^*) + (X\eta_3, \cdots, X\eta_n).$$

Let $\xi_1 = f^*$, $\xi_2 = g^*$, and $\xi_i = X\eta_i$ $(3 \le i \le n)$. Then the homogeneous system $\{\xi_i\}_{1 \le i \le n}$ of generators of I^* satisfies conditions (1) and (2) in Theorem 1.2. We put $c_i' = \deg \eta_i$, $D_i' = \mathrm{GCD}(\eta_1, \eta_2, \cdots, \eta_i)$, and $d_i' = \deg D_i'$. Then, because $(\xi_1, \xi_2) = (f^*, g^*) = (f^*, Xg_1^*) = (\eta_1, X\eta_2)$ and $X \nmid f^*$, we have $D_i = D_i'$ for all $1 \le i \le n$, while $c_1' = a = c_1$ and $c_i' = c_i - 1$ for all $2 \le i \le n$. Consequently, assertions (2), (3), (4), and the former part of assertion (1) in Lemma 3.2 are safely deduced from those on the ideal I_1. Let us check that $\frac{\xi_{i+1}}{D_{i+1}} \in (\frac{\xi_1}{D_i}, \frac{\xi_2}{D_i}, \cdots, \frac{\xi_i}{D_i})$. Since $D_1 = \xi_1$, we may assume $i \ge 2$. First of all, recall that $\frac{\eta_{i+1}}{D_{i+1}} \in (\frac{\eta_1}{D_i}, \frac{\eta_2}{D_i}, \cdots, \frac{\eta_i}{D_i})$ and we have $\frac{X\eta_{i+1}}{D_{i+1}} \in (\frac{\eta_1}{D_i}, \frac{X\eta_2}{D_i}, \cdots, \frac{X\eta_i}{D_i}) = (\frac{\xi_1}{D_i}, \frac{\xi_2}{D_i}, \cdots, \frac{\xi_i}{D_i})$, because $(\eta_1, X\eta_2, \cdots, X\eta_i) = (\xi_1, \xi_2, \cdots, \xi_i)$. Hence $\frac{\xi_{i+1}}{D_{i+1}} \in (\frac{\xi_1}{D_i}, \frac{\xi_2}{D_i}, \cdots, \frac{\xi_i}{D_i})$ as $\xi_{i+1} = X\eta_{i+1}$. Thus case (i) does not occur.

Suppose case (ii). Then $\mu_G(I_1^*) = n$. Let $\{\eta_i\}_{1 \le i \le n}$ be a homogeneous system of generators of I_1^* such that $\eta_1 = g_1^*$, $\eta_2 = f^*$, and $\deg \eta_i + 2 \le \deg \eta_{i+1}$ for all $2 \le i \le n-1$. Then $I^* = (f^*, g^*) + (X\eta_3, \cdots, X\eta_n)$ by Corollary 2.5. Let $\xi_1 = f^*$, $\xi_2 = g^*$, and $\xi_i = X\eta_i$ for $3 \le i \le n$. Then $\{\xi_i\}_{1 \le i \le n}$ is a homogeneous system of generators of I^* which satisfies conditions (1) and (2) in Theorem 1.2. We put $c_i' = \deg \eta_i$, $D_i' = \mathrm{GCD}(\eta_1, \eta_2, \cdots, \eta_i)$, and $d_i' = \deg D_i'$ for each $1 \le i \le n$. Then $c_1' = a - 1 = c_1 - 1$, $c_2' = a = c_2$, and $c_i' = c_i - 1$ for $3 \le i \le n$. Because $X \nmid f^*$ and $(\xi_1, \xi_2) = (f^*, g^*) = (\eta_2, X\eta_1)$, we get $D_1' = \eta_1 = g_1^*$ and $D_i' = D_i$ for $2 \le i \le n$. Hence $d_1' = a - 1 = d_1 - 1$ and $d_i' = d_i$ for all $2 \le i \le n$. Consequently, it is direct to check that assertions (2), (3), (4), and the former part of assertion (1) hold true for the ideal I. Let us show $\frac{\xi_{i+1}}{D_{i+1}} \in (\frac{\xi_1}{D_i}, \frac{\xi_2}{D_i}, \cdots, \frac{\xi_i}{D_i})$ for all $1 \le i \le n-1$. We may assume $i \ge 2$. Because $\frac{\eta_{i+1}}{D_{i+1}} \in (\frac{\eta_1}{D_i}, \frac{\eta_2}{D_i}, \cdots, \frac{\eta_i}{D_i})$, we have $\frac{X\eta_{i+1}}{D_{i+1}} \in (\frac{X\eta_1}{D_i}, \frac{X\eta_2}{D_i}, \cdots, \frac{X\eta_i}{D_i}) \subseteq (\frac{\xi_1}{D_i}) + (\frac{X\eta_j}{D_i} | 1 \le j \le i, j \ne 2) = (\frac{\xi_1}{D_i}, \frac{\xi_2}{D_i}, \cdots, \frac{\xi_i}{D_i})$ (use the fact $(\xi_1, \xi_2) = (\xi_1, X\eta_1)$). Hence $\frac{\xi_{i+1}}{D_{i+1}} \in (\frac{\xi_1}{D_i}, \frac{\xi_2}{D_i}, \cdots, \frac{\xi_i}{D_i})$ as $\xi_{i+1} = X\eta_{i+1}$. Thus case (ii) does not occur.

Now we consider case (iii). We have $\mu_G(I_1^*) = n-1$. Let $f_1 \in S$ such that $\mathrm{o}(f_1) = a_1 > a$, $I_1 = (g_1, f_1)$, and $g_1^* \nmid f_1^*$. Choose a homogeneous system $\{\eta_i\}_{1 \le i \le n-1}$ of generators for I_1^* so that $\eta_1 = g_1^*$, $\eta_2 = f_1^*$, and $\deg \eta_i + 2 \le \deg \eta_{i+1}$ for all $2 \le i \le n-2$. Then $I^* = (f^*, g^*) + (X\eta_2, \cdots, X\eta_{n-1})$. We put $\xi_1 = f^*$, $\xi_2 = g^*$, and $\xi_i = X\eta_{i-1}$ for $3 \le i \le n-1$. Then the homogeneous system $\{\xi_i\}_{1 \le i \le n}$ of generators of I^* satisfies conditions (1) and (2) in Theorem 1.2. Let $D'_i = \mathrm{GCD}(\eta_1, \eta_2, \cdots, \eta_i)$, $d'_i = \deg D'_i$, and $c'_i = \deg \eta_i$ for each $1 \le i \le n-1$. Then $D'_i = D_{i+1}$ for $1 \le i \le n-1$ (recall that $g_1^* \mid f^*$ and $(\xi_1, \xi_2) = (f^*, g^*) = (f^*, Xg_1^*) = (f^*, X\eta_1)$). Hence $c'_1 = a-1 = c_1 - 1$, $c'_i = c_{i+1} - 1$ for $2 \le i \le n-1$, and $d'_i = d_{i+1}$ for $1 \le i \le n-1$. Consequently, assertions (2), (3), (4), and the former part of assertion (1) hold true (use the fact that $c_3 = a_1 + 1 \ge a+2$, $d'_1 = a-1$, and $c_n \ge a+2$). Let us check the latter part of assertion (1). We may assume $i \ge 2$. Then, since $\frac{\eta_i}{D'_i} \in (\frac{\eta_1}{D'_{i-1}}, \frac{\eta_2}{D'_{i-1}}, \cdots, \frac{\eta_{i-1}}{D'_{i-1}})$, we get $\frac{X\eta_i}{D_{i+1}} \in (\frac{f^*}{D_i}, \frac{X\eta_1}{D_i}, \cdots, \frac{X\eta_{i-1}}{D_i})$. Hence $\frac{\xi_{i+1}}{D_{i+1}} \in (\frac{\xi_1}{D_i}, \frac{\xi_2}{D_i}, \cdots, \frac{\xi_i}{D_i})$, because $\xi_{i+1} = X\eta_i$ and $(f^*, X\eta_1) = (\xi_1, \xi_2)$. Thus even case (iii) cannot occur. We conclude that Lemma 3.2 holds true. $\square$

Proof of Theorem 1.3. Items (1) and (3) follow from Proposition 2.6 and Lemma 3.2.

For items (2) and (4), since

$$\mathrm{H}(\mathrm{gr}_{\mathfrak{m}}(R), \lambda) = \sum_{i=2}^{n} \lambda^{d_i} (\sum_{j=0}^{d_{i-1}-d_i-1} \lambda^j)(\sum_{j=0}^{c_i-d_i-1} \lambda^j)$$

and $\ell_S(R) = \dim_k \mathrm{gr}_{\mathfrak{m}}(R)$, we readily get $\ell_S(R) = \sum_{i=2}^{n}(d_{i-1} - d_i)(c_i - d_i) = c_1c_2 + \sum_{i=2}^{n-1} d_i{\cdot}[(c_{i+1} - c_i) - (d_{i-1} - d_i)] = ab + \sum_{i=2}^{n-1} d_i{\cdot}[(c_{i+1} - c_i) - (d_{i-1} - d_i)]$. We have $\ell_S(R) = ab$ if and only if $n = 2$, because $(c_{i+1} - c_i) - (d_{i-1} - d_i) > 0$ for all $2 \le i \le n-1$ by Lemma 3.2 (2). Since $I^* = (f^*, g^*)$ if and only if $n = 2$, we have $\ell_S(R) = ab$ if and only if f^*, g^* form a regular sequence in G.

Corollary 3.3. *Assume notation as in Theorem 1.3, and let* $a(\mathrm{gr}_{\mathfrak{m}}(R)) = \max\{i \in \mathbb{Z} \mid [\mathrm{gr}_{\mathfrak{m}}(R)]_i \neq (0)\}$. *The following assertions hold true.*

(1) $a(\mathrm{gr}_{\mathfrak{m}}(R)) = c_n + d_{n-1} - 2$.
(2) $\ell_S(R) \le a{\cdot}[c_n + d_{n-1} - a]$.
(3) $\ell_S(R) = a{\cdot}[c_n + d_{n-1} - a]$ *if and only if* $n = 2$.

Proof. Since $a(\mathrm{gr}_{\mathfrak{m}}(R)) = \deg \mathrm{H}(\mathrm{gr}_{\mathfrak{m}}(R), \lambda)$, thanks to Theorem 1.3 (1), we have $a(\mathrm{gr}_{\mathfrak{m}}(R)) = \max\{d_i + (d_{i-1} - d_i - 1) + (c_i - d_i - 1) \mid 2 \le i \le n\}$. Hence $a(\mathrm{gr}_{\mathfrak{m}}(R)) = c_n + d_{n-1} - 2$ by Lemma 3.2 (3). Because $d_1 - 1 = a - 1 \ge d_i$ for all $2 \le i \le n-1$

and $c_n + d_{n-1} \geq a + b$ by Lemma 3.2 (1), (4), we get by Theorem 1.3 (2) that

$$\begin{aligned}
\ell_S(R) &\leq ab + (a-1)\cdot\sum_{i=2}^{n-1}[(c_{i+1} - c_i) - (d_{i-1} - d_i)] \\
&= ab + (a-1)\cdot[c_n + d_{n-1} - (a+b)] \\
&= a\cdot[c_n + d_{n-1} - a] - [c_n + d_{n-1} - (a+b)] \\
&\leq a\cdot[c_n + d_{n-1} - a].
\end{aligned}$$

If the equality $\ell_S(R) = a\cdot[c_n + d_{n-1} - a]$ holds true, then $c_n + d_{n-1} - (a+b) = 0$, so that $\ell_S(S/I) = a\cdot[c_n + d_{n-1} - a] = ab$, whence $n = 2$. Since $c_2 = b$ and $d_1 = a$, we certainly have $\ell_S(R) = a(c_n + d_{n-1} - a)$ if $n = 2$. This completes the proof of Corollary 3.3. $\square$

Suppose that $\mathrm{ht}_G(f^*, g^*) = 1$. Let $D = \mathrm{GCD}(f^*, g^*)$ and $d = \deg D$. We write $f^* = D\xi$ and $g^* = D\eta$ with $\xi, \eta \in G$. Then $b \geq a > d > 0$ and by [GHK, Proposition 2.2] we may choose $h = \alpha f + \beta g$ with $\alpha, \beta \in S$ so that $\mathrm{o}(\alpha) = b - d$, $\mathrm{o}(\beta) = a - d$, and $h^* \notin (f^*, g^*)$. We call such an element h^* the third generator of I^*. We put $c = \mathrm{o}(h)$. With this notation we have the following.

Corollary 3.4. *Suppose that* $\mathrm{ht}_G(f^*, g^*) = 1$ *and* $\mathrm{ht}_G(f^*, g^*, h^*) = 2$. *Then the following assertions hold true.*

1. $I^* = (f^*, g^*, h^*)$.
2. $\mathrm{H}(\mathrm{gr}_{\mathfrak{m}}(R), \lambda) = \frac{(1-\lambda^c)(1-\lambda^d)+\lambda^d(1-\lambda^{a-d})(1-\lambda^{b-d})}{(1-\lambda)^2}$.
3. $\mathrm{e}^0_{\mathfrak{m}}(R) = ab + d\cdot[(c+d) - (a+b)]$.

Proof. For each $3 \leq i \leq n$, let $\xi_i = \overline{h_i t^{c_i}}$ where $h_i \in I$ with $\mathrm{o}(h_i) = c_i$. We write $h_i = \alpha_i f + \beta_i g$ with $\alpha_i, \beta_i \in S$. Then $\mathrm{o}(\alpha_i) \geq b-d$ and $\mathrm{o}(h_i) \geq \mathrm{o}(h) + [\mathrm{o}(\alpha_i) - (b-d)] \geq \mathrm{o}(h)$ (cf. [GHK, Proposition 2.4 (1)]). Let $h^* = \sum_{i=1}^n \xi_i \varphi_i$ with $\varphi_i \in G_{c-c_i}$. Then, since $h^* \notin (f^*, g^*) = (\xi_1, \xi_2)$, we have $c - c_i \geq 0$ for some $3 \leq i \leq n$. Hence $c \geq c_i \geq c_3$, so that $c = c_3$, because $c_3 \geq c + [\mathrm{o}(\alpha_3) - (b-d)] \geq c$. We furthermore have $\mathrm{o}(\alpha_3) = b-d$, whence, thanks to [GHK, Proposition 2.4 (3)], we get $(f^*, g^*, \xi_3) = (f^*, g^*, h^*)$. Thus $n = 3$ by Theorem 1.2 (3), because $\mathrm{ht}_G(f^*, g^*, h^*) = 2$ by our assumption, so that $I^* = (\xi_1, \xi_2, \xi_3) = (f^*, g^*, h^*)$ as claimed. Assertions (2) and (3) now readily follow from Theorem 1.3 (1) and (2). $\square$

Remark 3.5. With notation as in Setting 1.1, it follows from Part (1) of Lemma 3.2 that there exists a strictly descending chain

$$(\frac{\xi_1}{D_2}, \frac{\xi_2}{D_2})G \supset (\frac{\xi_1}{D_3}, \frac{\xi_2}{D_3}, \frac{\xi_3}{D_3})G \supset \cdots \supset (\frac{\xi_1}{D_{n-1}}, \frac{\xi_2}{D_{n-1}}, \cdots \frac{\xi_{n-1}}{D_{n-1}})G \supset I^*$$

of height-two ideals of G. In particular, I^* is contained in the ideal $(\frac{\xi_1}{D_2}, \frac{\xi_2}{D_2})G$. This behavior fails to hold in general in the higher dimensional case. The leading ideal of a complete intersection of height two in a three-dimensional regular local ring may fail to have this property as is demonstrated by Example 1.6 of [GHK].

4. Proof of Theorems 1.5 and 1.6

The goal of this section is to prove Theorems 1.5, and 1.6 and deduce several consequences of these theorems. We use the following lemma.

Lemma 4.1. *Assume notation as in Setting 1.1. Let $0 \neq h \in \mathfrak{n}$ and $m = \mathrm{o}(h)$. Let $X_1, X_2, \cdots, X_{s-1} \in G$ be a linear system of parameters for the graded ring $G/(h^*)$ and write $X_i = x_i^*$ with $x_i \in \mathfrak{n}$. Then $x_1, x_2, \cdots, x_{s-1}$ is a part of a regular system of parameters of S and for all $1 \leq \ell \leq s-1$, we have $\mathrm{o}(\overline{h}) = m$, where $\overline{h}$ denotes the image of h in $\overline{S} = S/(x_1, x_2, \cdots, x_\ell)$.*

Proof. Since $X_1, X_2, \cdots, X_{s-1}$ are algebraically independent over k, the elements $x_1, x_2, \cdots, x_{s-1}$ form a part of a regular system of parameters in S. If

$$h \in \mathfrak{n}^{m+1} + (x_1, x_2, \cdots, x_\ell),$$

then since $(x_1, x_2, \cdots, x_\ell) \cap \mathfrak{n}^m = (x_1, x_2, \cdots, x_\ell)\,\mathfrak{n}^{m-1}$, we get

$$h \in \mathfrak{n}^{m+1} + (x_1, x_2, \cdots, x_\ell)\,\mathfrak{n}^{m-1}.$$

Thus $h^* \in (X_1, X_2, \cdots, X_\ell)$, which is impossible, because $X_1, X_2, \cdots, X_\ell, h^*$ forms a regular sequence in G. Hence $\mathrm{o}(\overline{h}) = m$ as claimed. $\square$

Proof of Theorem 1.5. By Corollary 3.4, we may assume that $\dim S = s > 2$. Choose $X_1, X_2, \cdots, X_{s-1} \in G_1$ so that $X_1, X_2, \cdots, X_{s-1}$ is a homogeneous system of parameters for the graded rings $G/(f^*)$, $G/(g^*)$, $G/(h^*)$, $G/(\alpha^*)$, $G/(\beta^*)$, and $G/(D)$ and $X_1, X_2, \cdots, X_{s-2}$ is a homogeneous system of parameters for the graded rings $G/(f^*, g^*, h^*)$, $G/(\xi, \eta)$, and $\mathrm{gr}_{\mathfrak{m}}(R)$. For each i with $1 \leq i \leq s-1$, choose $x_i \in \mathfrak{n}$ such that $x_i^* = X_i$. Then $x_1, x_2, \cdots, x_{s-1}$ form a part of a regular system of parameters for S. Let $\mathfrak{q} = (x_1, x_2, \cdots, x_{s-2})S$. We put $\overline{S} = S/\mathfrak{q}$, $\overline{\mathfrak{n}} = \mathfrak{n}/\mathfrak{q}$, and $\overline{I} = (\overline{f}, \overline{g})$, where overline denotes image in $\overline{S}$. Notice that $\mathfrak{q}R$ is a minimal reduction of $\mathfrak{m}$. Thus $I+\mathfrak{q}$ is a parameter ideal for S and $\overline{I}=(\overline{f},\overline{g})\overline{S}$ is a parameter ideal in the regular local ring $\overline{S}$ of dimension 2. Lemma 4.1 implies that $\mathrm{o}(\overline{f})=a$, $\mathrm{o}(\overline{g})=b$ and $\mathrm{o}(\overline{h})=c$.

Let $Q=(X_1,X_2,\dots,X_{s-2})G$. We prove that the following diagram is commutative:

$$\begin{array}{ccc} S & \longrightarrow & S/\mathfrak{q}:=\overline{S} \\ {\scriptstyle\varphi_1}\big\downarrow & & {\scriptstyle\varphi_2}\big\downarrow \\ G:=\mathrm{gr}_{\mathbf{n}}(S) & \longrightarrow & G/Q:=\widetilde{G}\cong \mathrm{gr}_{\overline{\mathfrak{n}}}(\overline{S}). \end{array}$$

Here φ_1 and φ_2 denote the canonical maps associating an element with its leading form in the associated graded ring, and the identification $\widetilde{G}\cong\mathrm{gr}_{\overline{\mathbf{n}}}(\overline{S})$ is because Q is the leading ideal in G of the ideal $\mathfrak{q}$ of S. We denote with a tilde the image in G/Q of elements and ideals of G. Since $X_1,X_2,\cdots,X_{s-2},\xi,\eta$ is a homogeneous system of parameters in G, $\widetilde{\xi}$, $\widetilde{\eta}$ is a homogeneous system of parameters in G/Q. Thus $\mathrm{GCD}(\widetilde{\xi},\widetilde{\eta})=1$, and $\widetilde{D}=\mathrm{GCD}(\widetilde{f^*},\widetilde{g^*})$. Since $\mathrm{o}(f)=\mathrm{o}(\overline{f})$, we have $\widetilde{f^*}=\overline{f}^*$. Similarly, $\widetilde{g^*}=\overline{g}^*$ and $\widetilde{h^*}=\overline{h}^*$. We have $\widetilde{I^*}\subseteq\overline{I}^*$. Moreover, $\widetilde{I^*}=\overline{I}^*$ if and only if $X_1,\dots,X_{s-2}$ is a regular sequence on G/I^*. Thus $\widetilde{I^*}=\overline{I}^*$ if and only if I^* is a perfect ideal of G.

We furthermore have the following.

Claim 4.2. *The following assertions hold true.*

1. $\overline{f}^*\nmid\overline{g}^*$ *in* $\mathrm{gr}_{\overline{\mathfrak{n}}}(\overline{S})$.
2. $\mathrm{o}(\overline{\alpha})=b-d$, $\mathrm{o}(\overline{\beta})=a-d$, *and* $\mathrm{o}(\overline{h})=c$.
3. $\overline{h}^*\notin(\overline{f}^*,\overline{g}^*)$.

Thus $\overline{h}^*$ *is the third generator of* $\overline{I}^*$ *in* $\mathrm{gr}_{\overline{\mathfrak{n}}}(\overline{S})$.

Proof of Claim 4.2. (1) Suppose that $\overline{f}^*\mid\overline{g}^*$. Then, via the identification $G/Q=\mathrm{gr}_{\overline{\mathfrak{n}}}(\overline{S})$, we have $g^*\in(f^*)+Q$. Let us write $g^*=f^*\varphi+\sum_{i=1}^{s-2}X_i\varphi_i$ with $\varphi,\varphi_i\in G$. Then, since $f^*=D\xi$ and $g^*=D\eta$, we have $D(\eta-\xi\varphi)\in Q$. Hence $\eta-\xi\varphi\in Q$, because $X_1,X_2,\cdots,X_{s-2},D$ is a regular sequence in G. Thus $\eta\in Q+(\xi)$, which is impossible, because $X_1,X_2,\cdots,X_{s-2},\xi,\eta$ is a G-regular sequence. Hence $\overline{f}^*\nmid\overline{g}^*$.

(2) See Lemma 4.1

(3) We have $h^*\in(\xi,\eta)$ ([GHK, Remark 2.3]; recall that $h\in(\alpha,\beta)$). Write $h^*=\xi\varphi+\eta\psi$ with $\varphi,\psi\in G$. Then

$$(f^*,g^*,h^*)=\mathrm{I}_2\left(\begin{array}{c|c|c}\varphi & -\psi & D\\ \hline \xi & \eta & 0\end{array}\right),$$

so that (f^*, g^*, h^*) is a perfect ideal with $\mu_G(f^*, g^*, h^*) = 3$, since $\mathrm{ht}_G(f^*, g^*, h^*) = 2$. Therefore $G/(f^*, g^*, h^*)$ is a Cohen-Macaulay ring, whence $X_1, X_2, \cdots, X_{s-2}$ form a regular sequence in $G/(f^*, g^*, h^*)$. Thus $\overline{h}^* \notin (\overline{f}^*, \overline{g}^*)$, because $\mu_{\mathrm{gr}_{\overline{\mathfrak{n}}}(\overline{S})}(\overline{f}^*, \overline{g}^*, \overline{h}^*) = 3$. □

Therefore $\overline{I}^* = (\overline{f}^*, \overline{g}^*, \overline{h}^*)$ by Corollary 3.4, because $\overline{h}^*$ is the third generator of $\overline{I}^*$ in $\mathrm{gr}_{\overline{\mathfrak{n}}}(\overline{S})$ with $\mathrm{ht}_{\mathrm{gr}_{\overline{\mathfrak{n}}}(\overline{S})}(\overline{f}^*, \overline{g}^*, \overline{h}^*) = 2$. We now look at the estimation $(*)$:

$$\begin{aligned}
\ell_R(R/\mathfrak{q}R) = \ell_{\overline{S}}(\overline{S}/\overline{I}) &= \dim_k \mathrm{gr}_{\overline{\mathfrak{n}}}(\overline{S})/(\overline{f}^*, \overline{g}^*, \overline{h}^*) \\
&= \dim_k G/[Q + (f^*, g^*, h^*)] \\
&\geq \dim_k G/[Q + I^*] \\
&= \dim_k \mathrm{gr}_{\mathfrak{m}}(R)/Q\mathrm{gr}_{\mathfrak{m}}(R) \\
&\geq \mathrm{e}^0{}_{Q\mathrm{gr}_{\mathfrak{m}}(R)}(\mathrm{gr}_{\mathfrak{m}}(R)) \\
&= \mathrm{e}^0_{\mathfrak{m}}(R) \\
&= \ell_R(R/\mathfrak{q}R),
\end{aligned}$$

since $\mathfrak{q}R$ is a minimal reduction of $\mathfrak{m}$. Thus $\mathrm{gr}_{\mathfrak{m}}(R) = G/I^*$ is Cohen-Macaulay, since $\dim_k \mathrm{gr}_{\mathfrak{m}}(R)/Q\mathrm{gr}_{\mathfrak{m}}(R) = \mathrm{e}^0_{Q\mathrm{gr}_{\mathfrak{m}}(R)}(\mathrm{gr}_{\mathfrak{m}}(R))$ (cf. estimation $(*)$), and so the sequence $X_1, X_2, \cdots, X_{s-2}$ is $\mathrm{gr}_{\mathfrak{m}}(R)$-regular. Hence $I^* = (f^*, g^*, h^*)$, because $Q + (f^*, g^*, h^*) = Q + I^*$ and $Q \cap I^* = QI^*$. We furthermore have that

$$\mathrm{H}(\mathrm{gr}_{\mathfrak{m}}(R), \lambda) = \frac{\mathrm{H}(\mathrm{gr}_{\overline{\mathfrak{n}}}(\overline{S})/(\overline{f}^*, \overline{g}^*, \overline{h}^*), \lambda)}{(1-\lambda)^{s-2}},$$

whence

$$\mathrm{H}(\mathrm{gr}_{\mathfrak{m}}(R), \lambda) = \frac{(1-\lambda^c)(1-\lambda^d) + \lambda^d(1-\lambda^{a-d})(1-\lambda^{b-d})}{(1-\lambda)^s}$$

by Corollary 3.4. Thus $\mathrm{e}^0_{\mathfrak{m}}(R) = ab + d\cdot[(c+d) - (a+b)]$ as claimed. This completes the proof of Theorem 1.5. □

Remark 4.3. Without the assumption in Theorem 1.5 that $\mathrm{ht}(f^*, g^*, h^*) = 2$, it is still possible to specialize via $\mathfrak{q}$ and Q to obtain $\widetilde{f^*} = \overline{f}^*$, $\widetilde{g^*} = \overline{g}^*$ and $\widetilde{D} = \mathrm{GCD}(\widetilde{f^*}, \widetilde{g^*})$. However, $\widetilde{h^*} = \overline{h}^*$ may fail to be a minimal generator of $\overline{I}^*$ as we demonstrate in Example 4.4.

Example 4.4. Let $S = k[[x, y, z]]$ be the formal power series ring in the three variables x, y, z over a field k, and let X, Y, Z denote the leading forms of x, y, z in $G = \mathrm{gr}_{\mathbf{n}}(S) = k[X, Y, Z]$. As in [GHK, Example 1.6], let $I = (f, g)$, where $f = z^2 - x^5$ and $g = zx - y^3$. Thus $R = S/I$ is a complete intersection of dimension one. We have $I^* = (Z^2, ZX, ZY^3, Y^6)G$. We consider several choices for an element $w \in \mathbf{n} \setminus \mathbf{n}^2$ and behavior of the specialization $S \to S/wS = \overline{S}$. Since $\dim G/I^* = 1$ and I^* is not a perfect ideal, one always has the strict inequality $I^*\widetilde{G} \subsetneq (I\overline{S})^*$.

(1) Let $w = x$. Then $\overline{S} = k[[y,z]]$, $\overline{f} = z^2$ and $\overline{g} = -y^3$. We have

$$I^*\widetilde{G} = (Z^2, ZY^3, Y^6)\widetilde{G} \subsetneq (I\overline{S})^* = (Z^2, Y^3)k[Y, Z].$$

The multiplicity of G/I^* is 6 as is the multiplicity of $\mathrm{gr}_{\overline{\mathbf{n}}}(\overline{S})/(I\overline{S})^*$. The Hilbert series for G/I^* is

$$H(G/I^*, \lambda) = \frac{1 + 2\lambda + \lambda^2 + \lambda^3 + \lambda^5}{1 - \lambda},$$

while the Hilbert series for $\mathrm{gr}_{\overline{\mathbf{n}}}(\overline{S})/(I\overline{S})^*$ is

$$H(\mathrm{gr}_{\overline{\mathbf{n}}}(\overline{S})/(I\overline{S})^*, \lambda) = \frac{1 + 2\lambda + 2\lambda^2 + \lambda^3}{1 - \lambda}.$$

The multiplicity of $\widetilde{G}/I^*\widetilde{G}$ is 9, and the Hilbert series for $\widetilde{G}/I^*\widetilde{G}$ is

$$H(\widetilde{G}/I^*\widetilde{G}, \lambda) = \frac{1 + 2\lambda + 2\lambda^2 + 2\lambda^3 + \lambda^4 + \lambda^5}{1 - \lambda}.$$

(2) Let $w = x - y$ and use this to eliminate x. Then $\overline{S} = k[[y,z]]$, $\overline{f} = z^2 - y^5$ and $\overline{g} = zy - y^3$. We have

$$I^*\widetilde{G} = (Z^2, ZY, Y^6)\widetilde{G} \subsetneq (I\overline{S})^* = (Z^2, ZY, Y^5)k[Y, Z].$$

The multiplicity and Hilbert series of G/I^* are as given in part (1). The multiplicity of $\mathrm{gr}_{\overline{\mathbf{n}}}(\overline{S})/(I\overline{S})^*$ is 6, while the multiplicity of $\widetilde{G}/I^*\widetilde{G}$ is 7. The Hilbert series of $\mathrm{gr}_{\overline{\mathbf{n}}}(\overline{S})/(I\overline{S})^*$ is

$$H(\mathrm{gr}_{\overline{\mathbf{n}}}(\overline{S})/(I\overline{S})^*, \lambda) = \frac{1 + 2\lambda + \lambda^2 + \lambda^3 + \lambda^4}{1 - \lambda},$$

while the Hilbert series of $\widetilde{G}/I^*\widetilde{G}$ is

$$H(\widetilde{G}/I^*\widetilde{G}, \lambda) = \frac{1 + 2\lambda + \lambda^2 + \lambda^3 + \lambda^4 + \lambda^5}{1 - \lambda}.$$

Example 4.5. Let $S = k[[x, y, z, u]]$ be the formal power series ring in the four variables x, y, z, u over a field k, and let X, Y, Z, U denote the leading forms of x, y, z, u in $G = \mathrm{gr}_{\mathbf{n}}(S) = k[X, Y, Z, U]$. Let $I = (f, g)S$, where $f = xy$ and $g = xz + u^3$. Thus $R = S/I$ is a complete intersection of dimension two. It can be seen directly, and also is a consequence of Theorem 1.5, that $I^* = (XY, XZ, YU^3)G$. Since I^* is a perfect ideal and $\dim G/I^* = 2$, it is possible to choose $Q = (X_1, X_2)G$, the leading form ideal of $\mathfrak{q} = (x_1, x_2)S$ such that $\widetilde{I^*} = \overline{I}^*$. We illustrate how to successively choose x_1 and x_2.

(1) Let $x_1 = y - u$ and use this to eliminate u. Thus $\overline{S} = k[[x, y, z]]$, $\overline{f} = xy$ and $\overline{g} = xz + y^3$. We have

$$I^*\widetilde{G} = (XY, XZ, Y^4)\widetilde{G} = (I\overline{S})^* = (XY, XZ, Y^4)k[X, Y, Z].$$

We now apply the process again:

(2) Let $x_2 = z - x$ and use this to eliminate z. Thus $\overline{S} = k[[x, y]]$, $\overline{f} = xy$ and $\overline{g} = x^2 + y^3$. We have

$$I^*\widetilde{G} = (XY, X^2, Y^4)\widetilde{G} = (I\overline{S})^* = (XY, X^2, Y^4)k[X, Y].$$

The numerator polynomial of the Hilbert series in each case is $1+2t+t^2+t^3$.

We record the following corollary to Theorem 1.5.

Corollary 4.6. *Assume notation as in Setting 1.1 and Remark 1.4. If $D := D_2$ is a prime element of G that is regular on $G/(\xi, \eta)$, then $\mu(I^*) = 3$ and I^* is perfect.*

Proof. It suffices to show that $\mathrm{GCD}(f^*, g^*, h^*) = 1$. If this fails, then

$$h^* \in (D) \cap (\xi, \eta) = (D\xi, D\eta) = (f^*, g^*),$$

a contradiction to the assumption that h^* is the third generator of I^*. ☐

Example 4.7. Let $S = k[[x, y, z]]$ be the formal power series ring in the three variables x, y, z over a field k. Let $f = xy^i + z^s$ and $g = xz^j$, where $s > i + 1$ and i and j are positive. By Corollary 4.6, $\mu(I^*) = 3$ and I^* is perfect.

We use Lemma 4.1 and Theorem 1.2 to establish in Theorem 4.8 conditions on the degrees of a minimal homogeneous system of generators for I^* in the case where I^* is perfect.

Theorem 4.8. *Assume notation as in Setting 1.1 and Remark 1.4. If $\mathrm{gr}_{\mathfrak{m}}(R)$ is a Cohen-Macaulay ring and $n = \mu_G(I^*)$, then there exist homogeneous elements $\{\xi_i\}_{1 \le i \le n}$ of G such that*

(1) $I^* = (\xi_1, \xi_2, \cdots, \xi_n)$,
(2) $\xi_1 = f^*$ *and* $\xi_2 = g^*$,
(3) $\deg \xi_i + 2 \le \deg \xi_{i+1}$ *for all* $2 \le i \le n - 1$, *and*
(4) $\mathrm{ht}_G(\xi_1, \xi_2, \cdots, \xi_{n-1}) = 1$.

Proof. By Theorem 1.2, we may assume $s > 2$. If $n = 2$, then $I^* = (f^*, g^*)$ and there is nothing to prove. Assume $n > 2$ and let $D = \mathrm{GCD}(f^*, g^*)$. We write $f^* = D\xi$

and $g^* = D\eta$; hence ξ, η is a G-regular sequence. We choose, similarly as in the proof of Theorem 1.5, the elements $X_1, X_2, \cdots, X_{s-1} \in G_1$ so that $\{X_i\}_{1 \le i \le s-1}$ is a homogeneous system of parameters for the rings $G/(f^*), G/(g^*)$, and $G/(D)$ and $\{X_i\}_{1 \le i \le s-2}$ is a homogeneous system of parameters for the rings $G/(\xi, \eta)$ and $\mathrm{gr}_{\mathfrak{m}}(R)$. Let $x_i \in \mathfrak{n}$ with $X_i = x_i^*$. We put $\mathfrak{q} = (x_i \mid 1 \le i \le s-2)$, $\overline{S} = S/\mathfrak{q}$, $\overline{\mathfrak{n}} = \mathfrak{n}/\mathfrak{q}$, and $\overline{I} = (\overline{f}, \overline{g})$, where $\overline{f}$ and $\overline{g}$ respectively denote the images of f and g in $\overline{S}$. Then $\overline{f}^* \nmid \overline{g}^*$ (cf. Proof of Claim 4.2 (1)). The sequence $X_1, X_2, \cdots, X_{s-2}$ is regular in the ring $\mathrm{gr}_{\mathfrak{m}}(R)$, because $\mathrm{gr}_{\mathfrak{m}}(R)$ is Cohen-Macaulay. We identify

$$\mathrm{gr}_{\overline{\mathfrak{n}}}(\overline{S}) = G/Q \quad \text{and} \quad \overline{I}^* = [I^* + Q]/Q,$$

where $Q = (X_i \mid 1 \le i \le s-2)$. Therefore, since $\mu_{\mathrm{gr}_{\overline{\mathfrak{n}}}\overline{S}}(\overline{I}^*) = \mu_G(I^*) = n$, thanks to Theorem 1.2, the ideal $\overline{I}^*$ contains a homogeneous system $\{\eta_i\}_{1 \le i \le n}$ of generators which satisfies the conditions

(1) $\eta_1 = \overline{f}^*$ and $\eta_2 = \overline{g}^*$,
(2) $\deg \eta_i + 2 \le \deg \eta_{i+1}$ for all $2 \le i \le n-1$, and
(3) $\mathrm{ht}_{\mathrm{gr}_{\overline{\mathfrak{n}}}(\overline{S})}(\eta_i \mid 1 \le i \le n-1) = 1$.

Thus, taking $\xi_i \in I^*$ to be a preimage of η_i, we readily get a homogeneous system $\{\xi_i\}_{1 \le i \le n}$ of generators of I^* which satisfies conditions (2) and (3) in Theorem 4.8.

Let us check condition (4) is also satisfied. Assume the contrary and rechoose the system $\{X_i\}_{1 \le i \le s-1}$ so that $\{X_i\}_{1 \le i \le s-2}$ is also a homogeneous system of parameters for the ring $G/(\xi_i \mid 1 \le i \le n-1)$ of dimension $s-2$. Let $\overline{\xi_i}$ denote the image of ξ_i in G/Q. Then $\{\overline{\xi_i}\}_{1 \le i \le n}$ constitutes a minimal homogeneous system of generators of $\overline{I}^* = [I^* + Q]/Q$ with $\deg \overline{\xi_i} \le \deg \overline{\xi_{i+1}}$ for all $2 \le i \le n-1$. Consequently, even though we do not necessarily have $\eta_i = \overline{\xi_i}$ $(1 \le i \le n)$ for the second choice of $\{X_i\}_{1 \le i \le s-1}$, we still have $(\eta_i \mid 1 \le i \le n-1) = (\overline{\xi_i} \mid 1 \le i \le n-1)$, because the ideals $\{(\eta_j \mid 1 \le j \le i)\}_{1 \le i \le n}$ of $\mathrm{gr}_{\overline{\mathfrak{n}}}(\overline{S})$ are independent of the choice of minimal homogeneous systems $\{\eta_i\}_{1 \le i \le n}$ of generators of $\overline{I}^*$ which satisfy the condition that $\eta_1 = \overline{f}^*$, $\eta_2 = \overline{g}^*$, and $\deg \eta_i + 2 \le \deg \eta_{i+1}$ for all $2 \le i \le n-1$. This is however impossible, since $\mathrm{ht}_{\mathrm{gr}_{\overline{\mathfrak{n}}}(\overline{S})}(\eta_i \mid 1 \le i \le n-1) = 1$ while $\dim G/[Q + (\xi_i \mid 1 \le i \le n-1)] = 0$. Thus $\mathrm{ht}_G(\xi_i \mid 1 \le i \le n-1) = 1$ as claimed. $\square$

Proof of Theorem 1.6. Assume that $\mathrm{gr}_{\mathfrak{m}}(R)$ is a Cohen-Macaulay ring and let $\{\xi_i\}_{1 \le i \le n}$ be a homogeneous system of generators of I^* which satisfies conditions (2) and (3) in Theorem 4.8. Let $X_1, X_2, \cdots, X_{s-2} \in G_1$ and write $X_i = x_i^*$ with $x_i \in \mathfrak{n}$. We put $\mathfrak{q} = (x_i \mid 1 \le i \le s-2)$, $\overline{S} = S/\mathfrak{q}$, $\overline{\mathfrak{n}} = \mathfrak{n}/\mathfrak{q}$, and $\overline{I} = (\overline{f}, \overline{g})$, where $\overline{f}$ and $\overline{g}$

respectively denote the images of f and g in $\overline{S}$. We put $Q = (X_i \mid 1 \leq i \leq s-2)$. Then, choosing $\{X_i\}_{1 \leq i \leq s-2}$ to be sufficiently general, we may assume that

(1) $\{X_i\}_{1 \leq i \leq s-2}$ is a homogeneous system of parameters for $\mathrm{gr}_{\mathfrak{m}}(R)$, so that $\overline{S}$ is a regular local ring of dimension 2 with the parameter ideal $\overline{I}$, and

(2) $\widetilde{D_i} = \mathrm{GCD}(\widetilde{\xi}_1, \widetilde{\xi}_2, \cdots, \widetilde{\xi}_i)$ for all $1 \leq i \leq n$,

where $\widetilde{D_i}$ and $\widetilde{\xi}_i$ respectively denote the image of D_i and ξ_i in $G/Q = \mathrm{gr}_{\overline{\mathfrak{n}}}(\overline{S})$. Then the minimal homogeneous system $\{\widetilde{\xi}_i\}_{1 \leq i \leq n}$ of generators of the ideal $\widetilde{I^*} = \overline{I}^*$ in $G/Q = \mathrm{gr}_{\overline{\mathfrak{n}}}(\overline{S})$ satisfies conditions (1) and (2) in Theorem 1.2. We have

$$\mathrm{H}(\mathrm{gr}_{\mathfrak{m}}(R), \lambda) = \frac{\mathrm{H}(\mathrm{gr}_{\overline{\mathfrak{n}}}(\overline{S})/\overline{I}^*, \lambda)}{(1-\lambda)^{s-2}},$$

because $X_1, X_2, \cdots, X_{s-2}$ form a regular sequence in $\mathrm{gr}_{\mathfrak{m}}(R)$. The assertions in Theorem 1.6 readily follow from this. □

Question 4.9. With notation as in Setting 1.1 and Remark 1.4, if I^* is perfect, does it follow that $I^* \subseteq (\xi_1/D_2, \xi_2/D_2)G$?

5. Examples with $\mu_G(I^*) = 3$ and with given $\mu_G(I^*)$

Let $0 < n_1 < n_2 < n_3$ be integers such that $\mathrm{GCD}(n_1, n_2, n_3) = 1$ and let $S = k[[X_1, X_2, X_3]]$ and $T = k[[t]]$ be the formal power series rings over a field k. We denote by $\varphi : S \to T$ the k-algebra map defined by $\varphi(X_i) = t^{n_i}$ for $i = 1, 2, 3$. Let

$$I = \mathrm{Ker}\varphi,\ R = k[[t^{n_1}, t^{n_2}, t^{n_3}]],\ \mathfrak{n} = (X_1, X_2, X_3)S, \text{ and } \mathfrak{m} = (t^{n_1}, t^{n_2}, t^{n_3})R.$$

We then have the following, which is essentially due to J. Herzog [H2] (see p.191–192) and L. Robbiano and G. Valla [RV]. Let us include a brief proof in our context for the sake of completeness.

Theorem 5.1. *Suppose that $\mu_S(I) = 2$, namely, R is a Gorenstein ring. Then $\mathrm{gr}_{\mathfrak{m}}(R)$ is a Cohen-Macaulay ring if and only if the leading form ideal I^* of I is 3-generated.*

Proof. See [GHK, Theorem 1.2] for the proof of the *if* part. Suppose now that $\mathrm{gr}_{\mathfrak{m}}(R)$ is a Cohen-Macaulay ring. Let $G = \mathrm{gr}_{\mathfrak{n}}(S)$, which we shall identify with the polynomial ring $k[X_1, X_2, X_3]$ over k. We will show that $\mu_G(I^*) \leq 3$. Since $\mu_S(I) = 2$, as for the system of generators of I we distinguish the following four cases ([H1]):

(1) $I = (X_1^{c_1} - X_2^{c_2}, X_1^{c_1} - X_3^{c_3})$,

(2) $I = (X_2^{c_2} - X_3^{c_3}, X_1^{c_1} - X_2^{s_{12}} X_3^{s_{13}})$ $(s_{12} > 0, s_{13} > 0)$,
(3) $I = (X_1^{c_1} - X_3^{c_3}, X_2^{c_2} - X_1^{s_{21}} X_3^{s_{23}})$ $(s_{21} > 0, s_{23} > 0)$, and
(4) $I = (X_1^{c_1} - X_2^{c_2}, X_3^{c_3} - X_1^{s_{31}} X_2^{s_{32}})$ $(s_{31} > 0, s_{32} > 0)$

where $c_i = \min\{0 < c \in \mathbb{Z} \mid 0 \neq X_i^c - X_1^{\alpha_1} X_2^{\alpha_2} X_3^{\alpha_3} \in I$ for some $0 \leq \alpha_1, \alpha_2, \alpha_3 \in \mathbb{Z}\}$. For cases (1), (3), and (4), the ideal $I+(X_1)$ is generated by monomials in X_1, X_2, X_3 and so, thanks to [H2, Theorem 1], we have $\mu_G(I^*) = \mu_S(I) = 2$, once $\mathrm{gr}_{\mathfrak{m}}(R)$ is a Cohen-Macaulay ring. We are now concentrated in case (2), where

$$I = (X_2^{c_2} - X_3^{c_3}, X_1^{c_1} - X_2^{s_{12}} X_3^{s_{13}})$$

for some integers $s_{12} > 0$ and $s_{13} > 0$. Then $c_1 = (n_2, n_3)$, $n_2 = c_1 c_3$, and $n_3 = c_1 c_2$ ([H1]); hence $c_3 < c_2$. We write $s_{13} = c_3 q + s'_{13}$ with integers q, s'_{13} such that $0 \leq q, 0 \leq s'_{13} < c_3$ and put $s'_{12} = c_2 q + s_{12}$. Then

$$s'_{13} = 0 \quad \text{or} \quad c_1 + c_3 - s'_{13} \geq c_2 + s'_{12},$$

because $\mathrm{gr}_{\mathfrak{m}}(R)$ is a Cohen-Macaulay ring (see [H2, p.192]). Let $f = X_2^{c_2} - X_3^{c_3}$ and $g = X_1^{c_1} - X_2^{s'_{12}} X_3^{s'_{13}}$. Then $I = (f, g)$, since $g \equiv X_1^{c_1} - X_2^{s_{12}} X_3^{s_{13}} \mod (f)$. If $s'_{13} = 0$, then $g^* = X_1^{c_1}$ if $c_1 < s'_{12}$, $g^* = X_1^{c_1} - X_2^{s'_{12}}$ if $c_1 = s'_{12}$, and $g^* = -X_2^{s'_{12}}$ if $c_1 > s'_{12}$. Since $f^* = -X_3^{c_3}$ (recall that $c_3 < c_2$), in any case the forms f^*, g^* constitute a regular sequence in G, so that we have $I^* = (f^*, g^*)$.

Assume that $s'_{13} > 0$. Then $g^* = -X_2^{s'_{12}} X_3^{s'_{13}}$, since $c_1 - (s'_{12} + s'_{13}) \geq c_2 - c_3 > 0$. We put $h := X_2^{s'_{12}} f + X_3^{c_3 - s'_{13}} g = X_1^{c_1} X_3^{c_3 - s'_{13}} - X_2^{c_2 + s'_{12}}$. Let $J = (f^*, g^*, h^*) \subseteq I^*$. Then

$$J = (X_2^{c_2 + s'_{12}}, X_2^{s'_{12}} X_3^{s'_{13}}, X_3^{c_3}) = \mathrm{I}_2 \begin{pmatrix} 0 & X_2^{c_2} & X_3^{s'_{13}} \\ X_2^{s'_{12}} & X_3^{c_3 - s'_{13}} & 0 \end{pmatrix}$$

$$(\text{resp. } J = (X_1^{c_1} X_3^{c_3 - s'_{13}} - X_2^{c_2 + s'_{12}}, X_2^{s'_{12}} X_3^{s'_{13}}, X_3^{c_3}) = \mathrm{I}_2 \begin{pmatrix} X_1^{c_1} & X_2^{c_2} & X_3^{s'_{13}} \\ X_2^{s'_{12}} & X_3^{c_3 - s'_{13}} & 0 \end{pmatrix})$$

if $c_1 + c_3 - s'_{13} > c_2 + s'_{12}$ (resp. $c_1 + c_3 - s'_{13} = c_2 + s'_{12}$). We now want to show $I^* = J$. For this purpose we firstly look at the exact sequence

$$0 \to I^*/J \to G/J \to \mathrm{gr}_{\mathfrak{m}}(R) \to 0.$$

Then, since $a = t^{n_1}$ is a minimal reduction of the ideal $\mathfrak{m}$, the element $X_1 \in G$ acts on the Cohen-Macaulay ring $\mathrm{gr}_{\mathfrak{m}}(R)$ as a non-zerodivisor, whence we have the exact sequence

$$0 \to (I^*/J)/X_1(I^*/J) \to G/[(X_1) + J] \xrightarrow{\varepsilon} \mathrm{gr}_{\mathfrak{m}/(a)}(R/(a)) \to 0.$$

Therefore, to show $I^* = J$, by Nakayama's lemma it is enough to check that ε is an isomorphism, or equivalently, to check that

$$\dim_k G/[(X_1) + J] \leq \dim_k \mathrm{gr}_{\mathfrak{m}/(a)}(R/(a)).$$

We have

$$\dim_k \mathrm{gr}_{\mathfrak{m}/(a)}(R/(a)) = \ell_R(R/(a)) = \mathrm{e}^0_{\mathfrak{m}}(R) = n_1$$

and $n_1 = c_3 s'_{12} + c_2 s'_{13}$ (recall that $n_1 c_1 = n_2 s'_{12} + n_3 s'_{13}$, $n_2 = c_1 c_3$, and $n_3 = c_1 c_2$). On the other hand, since

$$G/[(X_1) + J] \cong k[X_2, X_3]/(X_2^{c_2+s'_{12}}, X_2^{s'_{12}} X_3^{s'_{13}}, X_3^{c_3}),$$

we readily get $\dim_k G/[(X_1) + J] \leq c_3(c_2 + s'_{12}) - c_2(c_3 - s'_{13}) = c_3 s'_{12} + c_2 s'_{13} = n_1$. Hence $I^* = J$ so that we have $\mu_G(I^*) = 3$ as claimed. $\square$

Corollary 5.2 (to the proof)**.** *Assume that $\mu_S(I) = 2$. Then $\mu_G(I^*) = 3$ if and only if there exist integers $\alpha, \beta \in \mathbb{Z}$ such that $0 < \alpha, 0 < \beta < c_3, c_1 + c_3 \geq c_2 + (\alpha + \beta)$, and $I = (X_2^{c_2} - X_3^{c_3}, X_1^{c_1} - X_2^{\alpha} X_3^{\beta})$. When this is the case, we have $c_1 = \mathrm{GCD}(n_2, n_3), n_2 = c_1 c_3, n_3 = c_1 c_2, n_1 = c_3 \alpha + c_2 \beta$, and the leading form ideal I^* of I is given by*

$$I^* = \mathrm{I}_2 \begin{pmatrix} 0 & X_2^{c_2} & X_3^{\beta} \\ X_2^{\alpha} & X_3^{c_3-\beta} & 0 \end{pmatrix} \ (\textit{resp. } I^* = \mathrm{I}_2 \begin{pmatrix} X_1^{c_1} & X_2^{c_2} & X_3^{\beta} \\ X_2^{\alpha} & X_3^{c_3-\beta} & 0 \end{pmatrix})$$

if $c_1 + c_3 > c_2 + (\alpha + \beta)$ (resp. $c_1 + c_3 = c_2 + (\alpha + \beta)$).

Remark 5.3. This result classifies Gorenstein numerical semigroups $H = \langle n_1, n_2, n_3 \rangle$ generated by 3 integers $n_i's$ with $0 < n_1 < n_2 < n_3$ and $\mathrm{GCD}(n_1, n_2, n_3) = 1$, for which the associated graded rings $\mathrm{gr}_{\mathfrak{m}}(R)$ ($R = k[[t^{n_1}, t^{n_2}, t^{n_3}]], k$ a field) are non-Gorenstein Cohen-Macaulay rings. In fact, firstly we choose integers c_2, c_3 so that $2 \leq c_3 < c_2$ and $\mathrm{GCD}(c_2, c_3) = 1$. Let α, β be integers such that $0 < \alpha, 0 < \beta < c_3$ and put $n_1 = c_3 \alpha + c_2 \beta$. We choose an integer c_1 so that $c_1 > \frac{n_1}{c_3}$, $\mathrm{GCD}(n_1, c_1) = 1$, and $c_1 + c_3 \geq c_2 + (\alpha + \beta)$. Lastly let $n_2 = c_1 c_3$ and $n_3 = c_1 c_2$. Then for $H = \langle n_1, n_2, n_3 \rangle$ we easily get the equality

$$I = (X_2^{c_2} - X_3^{c_3}, X_1^{c_1} - X_2^{\alpha} X_3^{\beta})$$

and $c_i = \min\{0 < c \in \mathbb{Z} \mid 0 \neq X_i^c - X_1^{\alpha_1} X_2^{\alpha_2} X_3^{\alpha_3} \in I$ for some $0 \leq \alpha_1, \alpha_2, \alpha_3 \in \mathbb{Z}\}$ as well for each $i = 1, 2, 3$. Hence by Corollary 5.2 the ring $\mathrm{gr}_{\mathfrak{m}}(R)$ is a non-Gorenstein Cohen-Macaulay ring. Let $f = X_2^{c_2} - X_3^{c_3}$, $g = X_1^{c_1} - X_2^{\alpha} X_3^{\beta}$, and $h = X_1^{c_1} X_3^{c_3-\beta} - X_2^{c_2+\alpha}$ ($= X_2^{\alpha} f + X_3^{c_3-\beta} g$). Then, since $c_2 > c_3$ and $c_1 - (\alpha + \beta) \geq c_2 - c_3 > 0$, we have $f^* = -X_3^{c_3}$ and $g^* = -X_2^{\alpha} X_3^{\beta}$ whence $\mathrm{GCD}(f^*, g^*) = X_3^{\beta}$, while $h^* = -X_2^{c_2+\alpha}$ (resp. $h^* = X_1^{c_1}X_3^{c_3-\beta} - X_2^{c_2+\alpha}$) if $c_1 + c_3 > c_2 + (\alpha+\beta)$ (resp. $c_1 + c_3 = c_2 + (\alpha+\beta)$), which is the third generator of I^*. Hence

$$\mathrm{H}(\mathrm{gr}_{\mathfrak{m}}(R), \lambda) = \frac{(1-\lambda^{c_2+\alpha})(1-\lambda^{\beta}) + \lambda^{\beta}(1-\lambda^{c_3-\beta})(1-\lambda^{\alpha})}{(1-\lambda)^3}$$

and $\mathrm{e}^0_{\mathfrak{m}}(R) = c_3(\alpha+\beta) + \beta[\{(c_2+\alpha)+\beta\} - \{c_3 + (\alpha+\beta)\}] = c_3\alpha + c_2\beta = n_1$ by Theorem 1.5.

Let us note more concrete examples.

Example 5.4. (1) Let $q \geq 0$ be an integer and put $n_1 = 6q+5$, $n_2 = 2(3q+4)$, and $n_3 = 3(3q+4)$. Then, letting $c_2 = 3, c_3 = 2, \alpha = 3q+1$, $\beta = 1$, and $c_1 = 3q+4$, by Corollary 5.2 and Remark 5.3 we get $I^* = I_2\begin{pmatrix} 0 & X_2^3 & X_3 \\ X_2^{3q+1} & X_3 & 0 \end{pmatrix}$. If we take $q = 0$, then $n_1 = 5, n_2 = 8, n_3 = 12$.

(2) Similarly, let $q \geq 0$ be an integer and put $n_1 = 6q+5$, $n_2 = 2(3q+3)$, and $n_3 = 3(3q+3)$. Then, letting $c_2 = 3, c_3 = 2, \alpha = 3q+1$, $\beta = 1$, and $c_1 = 3q+3$, by Corollary 5.2 and Remark 5.3 we get $I^* = I_2\begin{pmatrix} X_1^{3q+3} & X_2^3 & X_3 \\ X_2^{3q+1} & X_3 & 0 \end{pmatrix}$. If we take $q = 0$, then $n_1 = 5, n_2 = 6, n_3 = 9$, which is [GHK, Example 1.5].

We close this section with an example due to Takahumi Shibuta (Kyusyu University). His example shows that, unless $\mathrm{gr}_{\mathfrak{m}}(R)$ is Cohen-Macaulay, we do not necessarily have the descending sequence

$$a = d_1 > d_2 > \cdots > d_{n-1} > d_n = 0$$

of degrees of GCD's of $\xi_i's$ even for a minimal homogeneous system $\{\xi_i\}_{1 \leq i \leq n}$ of generators of I^* which satisfies the conditions in Theorem 4.8.

Example 5.5. Let $2 \leq m \in \mathbb{Z}$ and put $n_1 = 3m, n_2 = 3m+1$, and $n_3 = 6m+3$. Then $I = (X_1^{2m+1} - X_3^m, X_2^3 - X_1X_3)$ in S and $I^* = (X_1X_3) + (X_2^{3i}X_3^{m-i} \mid 0 \leq i \leq m)$ in $G = k[X_1, X_2, X_3]$ with $\mu_G(I^*) = m+2$. Letting $\xi_1 = X_1X_3$ and $\xi_i = X_2^{3(i-2)}X_3^{m-i+2}$ for $2 \leq i \leq m+2$, we see that the minimal homogeneous system $\{\xi_i\}_{1 \leq i \leq m+2}$ of generators of I^* satisfies the conditions in Theorem 4.8, while $\mathrm{GCD}(\xi_1, \xi_2, \cdots, \xi_i) = X_3$ for $2 \leq i \leq m+1$. We have

$$\mathrm{H}(\mathrm{gr}_{\mathfrak{m}}(R), \lambda) = \frac{\sum_{i=2}^{m} \lambda^{m-i+1} - \sum_{i=2}^{m} \lambda^{m+2i-2} + \sum_{i=0}^{3m-1} \lambda^i}{1-\lambda}.$$

Proof. It is routine to check that $I = (X_1^{2m+1} - X_3^m, X_2^3 - X_1X_3)S$. Hence we have $X_1X_3, X_3^m \in I^*$. Let $h_i = X_1^{2m+i+1} - X_2^{3i}X_3^{m-i}$ for $1 \leq i \leq m$. We put $J = (X_1X_3) + (X_2^{3i}X_3^{m-i} \mid 0 \leq i \leq m)$ in G. Then $h_i \in I$ for all $1 \leq i \leq m$, whence $J \subseteq I^*$. Let $K = (X_1) + (X_2^{3i}X_3^{m-i-1} \mid 0 \le i \le m-1)$ in G. Then $\sqrt{K} = G_+ = (X_1, X_2, X_3)$, $J :_G X_3 = K$, and $(X_3) + J = (X_2^{3m}, X_3)$. Consequently, $\mathrm{H}_N^0(G/J) = (\overline{X_3})$, where $\overline{X_3}$ is the image of X_3 in G/J and $\mathrm{H}_N^0(G/J)$ denotes the $0^{\underline{th}}$ local cohomology module of G/J with respect to $N = G_+$. Hence

$$(0) :_{G/J} N = \sum_{i=1}^{m-1} k\overline{X_2^{3i-1}X_3^{m-i}},$$

because $(\overline{X_3}) \cong [G/K](-1)$ and the k-vector space $(0) :_{G/K} N$ is spanned by the images of $\{X_2^{3i-1}X_3^{m-i-1}\}_{1 \le i \le m-1}$.

Let $\overline{\theta} : G/[J + (X_3)] \to \mathrm{gr}_{\mathfrak{m}}(R)/\mathrm{H}_N^0(\mathrm{gr}_{\mathfrak{m}}(R))$ be the epimorphism induced from the canonical epimorphism $G \to \mathrm{gr}_{\mathfrak{m}}(R)$. Recall that X_1 is a parameter for the ring $\mathrm{gr}_{\mathfrak{m}}(R)$, since t^{3m} is a minimal reduction of $\mathfrak{m}$, so that X_1 is a non-zerodivisor in the Cohen-Macaulay ring $\overline{\mathrm{gr}_{\mathfrak{m}}(R)} = \mathrm{gr}_{\mathfrak{m}}(R)/\mathrm{H}_N^0(\mathrm{gr}_{\mathfrak{m}}(R))$. Hence $\overline{\theta}$ is an isomorphism, because $\dim_k G/[J + (X_1, X_3)] = 3m$ and

$$\begin{aligned}
\dim_k \mathrm{gr}_{\mathfrak{m}}(R)/[\mathrm{H}_N^0(\mathrm{gr}_{\mathfrak{m}}(R)) + X_1\mathrm{gr}_{\mathfrak{m}}(R)] &= \mathrm{e}^0_{X_1\mathrm{gr}_{\mathfrak{m}}(R)}(\overline{\mathrm{gr}_{\mathfrak{m}}(R)}) \\
&= \mathrm{e}^0_{X_1\mathrm{gr}_{\mathfrak{m}}(R)}(\mathrm{gr}_{\mathfrak{m}}(R)) \\
&= \mathrm{e}^0_{\mathfrak{m}}(R) \\
&= 3m.
\end{aligned}$$

Therefore, the kernel of the epimorphism $\theta : G/J \to \mathrm{gr}_{\mathfrak{m}}(R)$ induced from the canonical epimorphism $G \to \mathrm{gr}_{\mathfrak{m}}(R)$ is contained in $(\overline{X_3}) = \mathrm{H}_N^0(G/J)$ and so, to see that θ is an isomorphism, it suffices to show that θ is injective on the socle

$$(0) :_{G/J} N = \sum_{i=1}^{m-1} k\overline{X_2^{3i-1}X_3^{m-i}}$$

of G/J, that is, it is enough to show $\theta(\overline{X_2^{3i-1}X_3^{m-i}}) \neq 0$ in $\mathrm{gr}_{\mathfrak{m}}(R)$ for any $1 \le i \le m-1$, because the degrees of $\overline{X_2^{3i-1}X_3^{m-i}}$ are distinct.

Let $x = t^{n_1}, y = t^{n_2}$, and $z = t^{n_3}$. We put $U = k[x, y, z]$ in R. Hence U is a graded ring with $\deg x = n_1, \deg y = n_2$, and $\deg z = n_3$. Let $M = U_+ = (x, y, z)U$. We denote by U_i the homogeneous component of U of degree i. In what follows we will show that $y^{3i-1}z^{m-i} \notin \mathfrak{m}^{m+2i}$ for any $1 \le i \le m-1$. Assume that $y^{3i-1}z^{m-i} \in \mathfrak{m}^{m+2i}$, or equivalently, assume that $y^{3i-1}z^{m-i} \in M^{m+2i}$ for some $1 \le i \le m-1$. Then we have the following.

Claim 5.6. $y^{3i-1}z^{m-i} \in M^{m+2i+\ell}$ *for all* $0 \le \ell \le m-i$.

Proof of Claim 5.6. When $\ell = 0$, we have nothing to prove. Assume that $0 \le \ell < m - i$ and that our assertion holds true for ℓ. We put $\delta = (3i-1)n_2 + (m-i)n_3 = 6m^2 + 3mi - 1$. Then $t^\delta = y^{3i-1}z^{m-i} \in M^{m+2i+\ell} = \sum_{\alpha=0}^{m+2i+\ell}(x,y)^{m+2i+\ell-\alpha}{\cdot}z^\alpha$ in U. Take $0 \le \alpha \in \mathbb{Z}$ and assume that $m - i - \ell \le \alpha \le m + 2i + \ell$. Then

$$(x,y)^{m+2i+\ell-\alpha}{\cdot}z^\alpha = (x^\beta y^\gamma z^\alpha \mid 0 \le \beta, \gamma \in \mathbb{Z} \text{ such that } \beta + \gamma = m + 2i + \ell - \alpha).$$

We now choose $0 \le \beta, \gamma \in \mathbb{Z}$ so that $\beta+\gamma = m+2i+\ell-\alpha$ and put $\eta = \beta n_1+\gamma n_2+\alpha n_3$. Then

$$\begin{aligned}
\eta &\ge (\beta+\gamma)n_1 + \alpha n_3 \\
&= 3(m+2i+\ell-\alpha)m + \alpha(6m+3) \\
&= 3m^2 + 6mi + 3m\ell + 3m\alpha + 3\alpha \\
&\ge 6m^2 + 3mi + 3(m-i-\ell) \quad (\text{since } \alpha \ge m-i-\ell) \\
&\ge 6m^2 + 3mi \\
&> \delta = 6m^2 + 3mi - 1.
\end{aligned}$$

Consequently, $t^\delta \in \sum_{\alpha=0}^{m-i-\ell-1}(x,y)^{m+2i+\ell-\alpha}{\cdot}z^\alpha$. We write $t^\delta = \sum_{\alpha=0}^{m-i-\ell-1}\varphi_\alpha z^\alpha$ with $\varphi_\alpha \in (x,y)^{m+2i+\ell-\alpha}$ such that $\varphi_\alpha \in U_{\delta-\alpha n_3}$. Let us furthermore write $\varphi_\alpha = \sum_{\beta=0}^{m+2i+\ell-\alpha} w_{\alpha,\beta}{\cdot}x^\beta y^{m+2i+\ell-\alpha-\beta}$ with $w_{\alpha,\beta} \in U_{\delta-\alpha n_3-(\beta n_1+(m+2i+\ell-\alpha-\beta)n_2)}$. Then, if $0 \le \alpha \le m-i-\ell-1$, choosing $0 \le \beta, \gamma \in \mathbb{Z}$ with $\beta+\gamma = m+2i+\ell-\alpha$, we have

$$\begin{aligned}
\beta n_1 + \gamma n_2 + \alpha n_3 &= 3m\beta + (3m+1)\gamma + \alpha n_3 \\
&\le (3m+1)(\beta+\gamma) + \alpha(6m+3) \\
&= 3m^2 + m + 6mi + 2i + 3\ell m + \ell + 3m\alpha + 2\alpha \\
&\le 6m^2 + 3mi - \ell - 2 \quad (\text{since } \alpha \le m-i-\ell-1) \\
&< \delta = 6m^2 + 3mi - 1,
\end{aligned}$$

whence $\beta n_1 + \gamma n_2 < \delta - \alpha n_3$. Consequently, $w_{\alpha,\beta} \in M$ for each α and β, so that $\varphi_\alpha \in M^{m+2i+\ell-\alpha+1}$ for all $0 \le \alpha \le m-i-\ell-1$, whence $t^\delta \in M^{m+2i+\ell+1}$ as claimed. $\square$

Therefore $t^\delta \in M^{2m+i}$, which is however impossible, because

$$\begin{aligned}
\beta n_1 + \gamma n_2 + \tau n_3 \ge (\beta+\gamma+\tau)n_1 &= (2m+i){\cdot}3m \\
&= 6m^2 + 3mi > \delta
\end{aligned}$$

for all $0 \le \beta, \gamma, \tau \in \mathbb{Z}$ with $\beta + \gamma + \tau = 2m + i$. Thus the epimorphism $\theta : G/J \to \mathrm{gr}_{\mathfrak{m}}(R)$ is injective on the socle of G/J, so that θ is an isomorphism. Hence $I^* = J$.

Because $\mathrm{H}_N^0(G/I^*) = (\overline{X_3}) \cong (G/K)(-1)$, thanks to the exact sequence

$$0 \to (G/K)(-1) \to G/I^* \to G/(X_2^{3m}, X_3) \to 0$$

of graded G-modules, we have

$$\mathrm{H}(G/I^*, \lambda) = \lambda{\cdot}\mathrm{H}(G/K, \lambda) + \frac{1-\lambda^{3m}}{(1-\lambda)^2}.$$

Therefore

$$\begin{aligned}\mathrm{H}(\mathrm{gr}_{\mathfrak{m}}(R), \lambda) &= \frac{\sum_{i=2}^{m} \lambda^{m-i+1}(1-\lambda^{3(i-1)})}{1-\lambda} + \frac{1-\lambda^{3m}}{(1-\lambda)^2} \\ &= \frac{\sum_{i=2}^{m} \lambda^{m-i+1} - \sum_{i=2}^{m} \lambda^{m+2i-2} + \sum_{i=0}^{3m-1} \lambda^{i}}{1-\lambda}\end{aligned}$$

by Proposition 2.6, since $G/K = k[X_2, X_3]/(X_2^{3i}X_3^{m-i-1} \mid 0 \le i \le m-1)$.

$\square$

References

[Br] J. Briançon, *Description de $Hilb^n C\{x,y\}$*, Invent. Math. **41** (1977), 45-89.

[GHK] S. Goto, W. Heinzer, and M.-K. Kim, *The leading ideal of a complete intersection of height two*, J. Algebra. **298** (2006), 238-247.

[GW] S. Goto and K.-i. Watanabe, *On graded rings, I*, J. Math. Soc. Japan, **30** (1978), 179-213.

[H1] J. Herzog, *Generators and relations of abelian semigroups and semigroup rings*, manuscripta math., **3** (1970), 175–193.

[H2] J. Herzog, *When is a regular sequence super regular?*, Nagoya Math. J., **83** (1981), 183–195.

[Ia1] A. Iarrobino, *Tangent cone of a Gorenstein singularity*, in Proceedings of the conference in Algebraic Geometry, Berlin 1985, H. Kurke and M. Roczen, eds., Teubnertexte zur Math. **92**, Teubner, Leipzig, 1986, 163-176.

[Ia2] A. Iarrobino, *Associated Graded Algebra of a Gorenstein Artin Algebra*, Memoirs Amer. Math. Soc., **107**, num. 514, 1994.

[K] S. C. Kothari, *The local Hilbert function of a pair of plane curves*, Proc. Amer. Math. Soc., **72** (1978), 439-442.

[M] F. S. Macaulay, *On a method of dealing with the intersection of plane curves*, Trans. Amer. Math. Soc., **5** (1904), 385-410.

[RV] L. Robbiano and G. Valla, *On the equations defining tangent cones*, Math. Proc. Camb. Phil. Soc., **88** (1980), 281-297.

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