THE LEADING IDEAL OF A COMPLETE INTERSECTION OF HEIGHT TWO

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Abstract. Let $(S, \mathfrak{n})$ be a Noetherian local ring and let $I = (f, g)$ be an ideal in S generated by a regular sequence f, g of length two. Assume that the associated graded ring $\mathrm{gr}_{\mathfrak{n}}(S)$ of S with respect to $\mathfrak{n}$ is a UFD. We examine generators of the leading form ideal I^* of I in $\mathrm{gr}_{\mathfrak{n}}(S)$ and prove that I^* is a perfect ideal of $\mathrm{gr}_{\mathfrak{n}}(S)$, if I^* is 3-generated. Thus, in this case, letting $R = S/I$ and $\mathfrak{m} = \mathfrak{n}/I$, if $\mathrm{gr}_{\mathfrak{n}}(S)$ is Cohen-Macaulay, then $\mathrm{gr}_{\mathfrak{m}}(R) = \mathrm{gr}_{\mathfrak{n}}(S)/I^*$ is Cohen-Macaulay. As an application, we prove that if $(R, \mathfrak{m})$ is a one-dimensional Gorenstein local ring of embedding dimension 3, then $\mathrm{gr}_{\mathfrak{m}}(R)$ is Cohen-Macaulay if the reduction number of $\mathfrak{m}$ is at most 4.

1. Introduction

Setting 1.1. Let $(S, \mathfrak{n})$ be a Noetherian local ring and let $I = (f, g)$ be an ideal in S generated by a regular sequence f, g of length two. Let $R = S/I$ and $\mathfrak{m} = \mathfrak{n}/I$. Let

$$\mathrm{R}'(\mathfrak{n}) = \sum_{i \in \mathbb{Z}} \mathfrak{n}^i t^i \subseteq S[t, t^{-1}] \quad \text{and} \quad \mathrm{R}'(\mathfrak{m}) = \sum_{i \in \mathbb{Z}} \mathfrak{m}^i t^i \subseteq R[t, t^{-1}]$$

denote the extended Rees algebras of $\mathfrak{n}$ and $\mathfrak{m}$ respectively, where t is an indeterminate. Let

$$\mathrm{gr}_{\mathfrak{n}}(S) = \mathrm{R}'(\mathfrak{n})/t^{-1}\mathrm{R}'(\mathfrak{n}) \quad \text{and} \quad \mathrm{gr}_{\mathfrak{m}}(R) = \mathrm{R}'(\mathfrak{m})/t^{-1}\mathrm{R}'(\mathfrak{m}).$$

Then the canonical map $S \to R$ induces the homomorphism $\varphi : \mathrm{gr}_{\mathfrak{n}}(S) \to \mathrm{gr}_{\mathfrak{m}}(R)$ of the associated graded rings. We put

$$I^* = \mathrm{Ker}\ (\mathrm{gr}_{\mathfrak{m}}(S) \stackrel{\varphi}{\to} \mathrm{gr}_{\mathfrak{m}}(R)).$$

Then the ideal I^* is generated by the initial forms of elements of I and $\mathrm{gr}_{\mathfrak{m}}(R) \cong \mathrm{gr}_{\mathfrak{n}}(S)/I^*$. We assume that $G = \mathrm{gr}_{\mathfrak{n}}(S)$ is a UFD. Hence $\mathrm{ht}_G I^* = \mathrm{grade}_G I^* = 2$.

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We are interested in determining generators for I^* and thereby obtaining conditions in order that $\mathrm{gr}_{\mathfrak{m}}(R)$ be Cohen-Macaulay. The goal of the paper is to prove Theorem 1.2, the proof of which is given in Section 2.

Theorem 1.2. *Assume notation as in Setting 1.1, so, in particular,* $\mathrm{gr}_{\mathfrak{n}}(S)$ *is a UFD. If* I^* *is* 3*-generated, then* I^* *is a perfect ideal of* $\mathrm{gr}_{\mathfrak{n}}(S)$*. Therefore if* $\mathrm{gr}_{\mathfrak{n}}(S)$ *is Cohen-Macaulay, then* $\mathrm{gr}_{\mathfrak{m}}(R) = \mathrm{gr}_{\mathfrak{n}}(S)/I^*$ *is Cohen-Macaulay.*

As an immediate corollary to Theorem 1.2, we have

Corollary 1.3. *With notation as in Setting 1.1, if* $(S, \mathfrak{n})$ *is a regular local ring and* I^* *is* 3*-generated, then* $\mathrm{gr}_{\mathfrak{m}}(R)$ *is Cohen-Macaulay.*

In Section 3 we discuss some consequences of Theorem 1.2.

Notation 1.4. Let $G = \mathrm{gr}_{\mathfrak{n}}(S)$. For each $f \in S$ let $o(f) = \sup\{i \in \mathbb{Z} \mid f \in \mathfrak{n}^i\}$, the order of f. We put

$$f^* = \begin{cases} \overline{ft^i} & \text{if } f \neq 0 \text{ and } i = o(f), \\ 0 & \text{if } f = 0 \end{cases}$$

and call it the *initial form* of f, where $\overline{ft^i}$ denotes the image in G of $ft^i \in \mathfrak{n}^i t^i$ in $\mathrm{R}'(\mathfrak{n})$. Then for all $f, g \in S$ we have

$o(fg) = o(f) + o(g), \quad (fg)^* = f^*g^*,$

$o(f+g) \geq \min\{o(f), o(g)\}$, and

$o(f+g) = \min\{o(f), o(g)\}$ if $o(f) \neq o(g)$.

With this notation the following two simple examples illustrate the situation we are considering. In both examples we let $S = k[[x, y, z]]$ be the formal power series ring in the three variables x, y, z over a field k.

Example 1.5. Let $R = k[[w^5, w^6, w^9]]$ be the subring of the formal power series ring $k[[w]]$ and define the homomorphism $\phi : S \to R$ of k-algebras by $\phi(x) = w^5$, $\phi(y) = w^6$, and $\phi(z) = w^9$. Then the ideal $I = \mathrm{Ker}\ \phi$ is generated by $f = z^2 - y^3$ and $g = zy - x^3$, whence R is a complete intersection of dimension one. We have $\mathrm{gr}_{\mathfrak{n}}(S) = k[x^*, y^*, z^*]$, $f^* = z^{*2}$, and $g^* = z^*y^*$. Let $h = yf - zg = zx^3 - y^4$. Then $h^* = z^*x^{*3} - y^{*4}$. Let

$$J = (f^*, g^*, h^*) = (z^{*2}, z^*y^*, z^*x^{*3} - y^{*4}) \subseteq I^*.$$

Then the Hilbert series of the graded ring $\mathrm{gr}_{\mathfrak{n}}(S)/J$ is

$$\frac{1+2t+t^2+t^3}{1-t} = 1+3t+4t^2+5t^3+5t^4+\cdots+5t^n+\cdots$$

and these values are the same as those in the Hilbert series of $\mathrm{gr}_{\mathfrak{m}}(R) = \mathrm{gr}_{\mathfrak{n}}(S)/I^*$, so that $J = I^*$. The reduction number of $\mathfrak{m} = (w^5, w^6, w^9)$ with respect to the principal reduction (w^5) is 3 and the relation type of $\mathrm{gr}_{\mathfrak{m}}(R)$ is 4. The ideal I^* has grade 2 and is generated by the 2×2 minors of the following matrix

$$\begin{bmatrix} y^* & z^* & 0 \\ -x^{*3} & -y^{*3} & z^* \end{bmatrix}.$$

Hence, by the theorem of Hilbert-Burch [BH, Theorem 1.4.17], I^* is a perfect ideal and $\mathrm{gr}_{\mathfrak{n}}(S)/I^* = \mathrm{gr}_{\mathfrak{m}}(R)$ is a Cohen-Macaulay ring.

Example 1.6. Let $R = k[[w^6, w^7, w^{15}]]$ be the subring of the formal power series ring $k[[w]]$ and consider the homomorphism $\phi : S \to R$ of k-algebras defined by $\phi(x) = w^6$, $\phi(y) = w^7$, and $\phi(z) = w^{15}$. Then $I = \mathrm{Ker}\ \phi$ is generated by $f = z^2 - x^5$ and $g = zx - y^3$, whence R is a complete intersection of dimension one. We have $\mathrm{gr}_{\mathfrak{n}}(S) = k[x^*, y^*, z^*]$, $f^* = z^{*2}$, and $g^* = z^*x^*$. Let $h = xf - zg = zy^3 - x^6$. Then $h^* = z^*y^{*3}$ and $(f^*, g^*, h^*) = (z^{*2}, z^*x^*, z^*y^{*3}) \subsetneq I^*$. The inclusion is strict, since $\mathrm{ht}_{\mathrm{gr}_{\mathfrak{n}}(S)} I^* = 2$ and z^* is a common factor of f^*, g^*, and h^*. We have $o(f) = o(g) = 2$ and $o(h) = 4$. Let $h_1 = xh - y^3 g = y^6 - z^7 \in I$. Then $h_1^* = y^{*6}$. We put

$$J = (z^{*2}, z^*x^*, z^*y^{*3}, y^{*6}) \subseteq I^*.$$

Then the Hilbert series of $\mathrm{gr}_{\mathfrak{n}}(S)/J$ is given by

$$\frac{1+2t+t^2+t^3+t^5}{1-t} = 1+3t+4t^2+5t^3+5t^4+6t^5+\cdots+6t^n+\cdots$$

and these values are the same as those in the Hilbert series of $\mathrm{gr}_{\mathfrak{m}}(R) = \mathrm{gr}_{\mathfrak{n}}(S)/I^*$, so that $J = I^*$. The reduction number of $\mathfrak{m} = (w^6, w^7, w^{15})$ with respect to the principal reduction (w^6) is 5 and the relation type of $\mathrm{gr}_{\mathfrak{m}}(R)$ is 6. The ring $\mathrm{gr}_{\mathfrak{m}}(R)$ is not Cohen-Macaulay. This is implied by the gap in the numerator of the Hilbert series, and can be deduced also from the fact that the ideal I^* has radical (y^*, z^*) and the ideal $I^* : z^*$ is primary with $\sqrt{I^* : z^*} = (x^*, y^*, z^*)$.

2. Proof of Theorem 1.2

The purpose of this section is to prove Theorem 1.2. We assume notation as in Setting 1.1. Let $G = \mathrm{gr}_{\mathfrak{n}}(S)$ and $J = I^*$. We choose $f, g \in S$ so that $I = (f, g)$ with $a = o(f) \leq b = o(g)$. Without loss of generality we may assume that $f^* \notin NJ$ and $g^* \notin NJ + (f^*)$, where $N = G_+$. Hence the elements f^*, g^* form part of a minimal system of homogeneous generators of J. Notice that if $\mathrm{ht}_G(f^*, g^*) = 2$, then the sequence f^*, g^* is G-regular whence $J = (f^*, g^*)$. In what follows we assume that

$$\mathrm{ht}_G(f^*, g^*) = 1.$$

Let $D = \mathrm{GCD}(f^*, g^*)$ and write $f^* = \xi D$, $g^* = \eta D$, where D, ξ, η are homogeneous elements of G with degree $d > 0$, $a - d$, and $b - d$, respectively. Then $\{\xi, \eta\}$ is a G-regular sequence.

We begin with Lemma 2.1 which gives some information about homogeneous elements of J that are not in the ideal (f^*, g^*).

Lemma 2.1. *Let $\alpha, \beta \in S$ and $h = \alpha f + \beta g$. Assume that $h^* \notin (f^*, g^*)$. Then*

(1) $o(\alpha f) = o(\beta g) < o(h)$.
(2) $o(\alpha) + a = o(\beta) + b$, $o(\alpha) \geq b - d$, *and* $o(\beta) \geq a - d$.
(3) $\alpha^* \xi + \beta^* \eta = 0$.

Proof. We have $o(h) \geq \min\{o(\alpha f), o(\beta g)\}$. If $o(\alpha f) < o(\beta g)$, then $o(h) = o(\alpha f)$ and $h^* = \alpha^* f^* \in (f^*)$, which is impossible. We similarly have $o(\alpha f) = o(\beta g)$. Hence $o(h) > o(\alpha f) = o(\beta g)$, because $h^* \notin (f^*, g^*)$. Thus $\alpha^* f^* + \beta^* g^* = (\alpha^* \xi + \beta^* \eta) D = 0$ whence $\alpha^* \xi + \beta^* \eta = 0$. Therefore, since the sequence ξ, η is G-regular, we get $\alpha^* = -\varphi \eta$ and $\beta^* = \varphi \xi$ for some homogeneous element φ of G. Thus $o(\alpha) = \deg \varphi + (b - d)$ and $o(\beta) = \deg \varphi + (a - d)$, so that $o(\alpha) + a = o(\beta) + b$, $o(\alpha) \geq b - d$, and $o(\beta) \geq a - d$, as was claimed. □

The existence of a third generator of the leading ideal J of a certain form is guaranteed by Proposition 2.2.

Proposition 2.2. *Assume that the local ring S is $\mathfrak{n}$-adically complete. Then there exist elements α, β of S such that $o(\alpha) = b - d$, $o(\beta) = a - d$, and $(\alpha f + \beta g)^* \notin (f^*, g^*)$.*

Proof. Assume the contrary. Let $f_0, g_0 \in S$ with $o(f_0) = a - d$ and $o(g_0) = b - d$ such that $\xi = f_0^*$ and $\eta = g_0^*$. We are going to construct two sequences $\{f_i\}_{i=0,1,2,\ldots}$ and $\{g_i\}_{i=0,1,2,\ldots}$ of elements in S which satisfy the following conditions: Let $h_i = (-\sum_{k=0}^i g_k)f + (\sum_{k=0}^i f_k)g$ for each $i \geq 0$. Then

(1) $h_i \neq 0$,

(2) $o(h_i) < o(h_{i+1})$,

(3) $o(h_i) - b \leq o(f_{i+1})$ and $o(h_i) - a \leq o(g_{i+1})$

for all $i \geq 0$.

To construct the sequences, firstly we put $h_0 = (-g_0)f + f_0 g$. Then $o(f_0) = a-d$ and $o(g_0) = b-d$. We notice $h_0 \neq 0$, because $b-d = o(g_0) < o(g) = b$ (recall that f, g is a regular sequence). Hence $h_0^* \in (f^*, g^*)$ by our assumption. We write $h_0^* = f^*\varphi + g^*\psi$ with $\varphi \in G_{o(h_0)-a}$ and $\psi \in G_{o(h_0)-b}$. Let $\varphi = \overline{g_1 t^{o(h_0)-a}}$ and $\psi = \overline{(-f_1)t^{o(h_0)-b}}$ with $g_1 \in \mathfrak{n}^{o(h_0)-a}$ and $f_1 \in \mathfrak{n}^{o(h_0)-b}$. Then $h_0 = g_1 f + (-f_1)g + h_1$ for some $h_1 \in \mathfrak{n}^{o(h_0)+1}$; hence

$$h_1 = [-(g_0 + g_1)]f + (f_0 + f_1)g,$$

where $o(f_1) \geq o(h_0) - b$, $o(g_1) \geq o(h_0) - a$, and $o(h_1) > o(h_0)$. Because

$$\begin{array}{rcl} \overline{h_0 t^{a+b-d}} & = & \overline{(-g_0)t^{b-d}} \cdot \overline{ft^a} + \overline{f_0 t^{a-d}} \cdot \overline{gt^b} \\ & = & (-\eta f^*) + \xi g^* \\ & = & (-\eta \cdot \xi D) + \xi(\eta D) \\ & = & 0, \end{array}$$

we get $o(h_0) > a+b-d$, so that $o(f_1) \geq o(h_0) - b > a - d$ and $o(g_1) \geq o(h_0) > b - d$. Thus $o(g_0 + g_1) = o(g_0) = b - d < b$ and $o(f_0 + f_1) = o(f_0) = a - d < a$, whence $h_1 = [-(g_0 + g_1)]f + (f_0 + f_1)g \neq 0$. Repeating this procedure, we get the required sequences $\{f_i\}_{i=0,1,2,\ldots}$ and $\{g_i\}_{i=0,1,2,\ldots}$ of elements in S.

Now let $\alpha = -\sum_{k=0}^\infty g_k$ and $\beta = \sum_{k=0}^\infty f_k$. We then have

$$\begin{array}{rcl} \alpha f + \beta g & = & \sum_{k=0}^\infty [(-g_k)f + f_k g] \\ & = & \lim_{i\to\infty}[(-\sum_{k=0}^i g_k)f + (\sum_{k=0}^i f_k)g] \\ & = & \lim_{i\to\infty} h_i \\ & = & 0, \end{array}$$

whence $\beta \in (f)$, which is impossible because $o(\beta) < a$ (recall that $\beta = f_0 + \sum_{k=1}^\infty f_k$, $o(f_0) = a-d$, and $o(f_k) \geq o(h_0) - b > a - d$ for all $k \geq 1$). Thus $(\alpha f + \beta g)^* \notin (f^*, g^*)$ for some elements α, β of S with $o(\alpha) = b - d$ and $o(\beta) = a - d$. □

Remark 2.3. Let $\alpha, \beta \in S$ with $o(\alpha) = b-d$ and assume that $(\alpha f+\beta g)^* \notin (f^*, g^*)$. Then $\alpha^* = -\bar{u}\eta$ and $\beta^* = \bar{u}\xi$ for some unit u in S. Hence α^*, β^* form a G-regular sequence.

Proof. With the same notation as in the proof of Lemma 2.1 we have $0 \neq \varphi \in G_0 = S/\mathfrak{n}$. Letting $\varphi = \bar{u}$ with a unit u in S, we readily get $\alpha^* = -\bar{u}\eta$ and $\beta^* = \bar{u}\xi$. □

Let $n = \mu_G(J)$ and $k = S/\mathfrak{n}$. In Proposition 2.4 (3) we prove the uniqueness of the order of $o(\alpha f + \beta g)$ for the elements α and β in S given by Proposition 2.2 and the uniqueness of the ideal (f^*, g^*, h^*) as well, where $h = \alpha f + \beta g$.

Proposition 2.4. *Let $\alpha, \beta, \sigma, \tau \in S$ with $o(\alpha) = b - d$. Let $h = \alpha f + \beta g$ and $q = \sigma f + \tau g$. Assume that $h^* \notin (f^*, g^*)$. Then the following assertions hold true.*

(1) *Assume that $q^* \notin (f^*, g^*)$. Then $o(q) \geq o(h) + o(\sigma) - (b - d)$.*
(2) *Assume that $q^* \notin (f^*, g^*, h^*)$. Then $o(q) > o(h) + o(\sigma) - (b - d)$.*
(3) *Assume that $q^* \notin (f^*, g^*)$ and $o(\sigma) = b - d$. Then $o(q) = o(h)$ and $(f^*, g^*, q^*) = (f^*, g^*, h^*)$.*
(4) *The elements f^*, g^*, h^* form a part of a minimal system of homogeneous generators of J.*
(5) *Assume that $n \geq 4$ and $I \subseteq \mathfrak{n}^2$. Then writing $J = \bigoplus J_n$, we have $J \supsetneq (J_i \mid 1 \leq i \leq 5)G$.*

Proof. Assume that $q^* \notin (f^*, g^*)$ and let $c = o(\sigma) - (b - d)$. Then $\sigma^*\xi + \tau^*\eta = 0$ by Lemma 2.1. Choose a unit u in S so that $\alpha^* = -\bar{u}\eta$ and $\beta^* = \bar{u}\xi$. Then, since $\sigma^*\xi\bar{u} + \tau^*\eta\bar{u} = 0$, we get $\sigma^*\beta^* = \tau^*\alpha^*$. Hence $\sigma^* = \alpha^*\delta^*$ and $\tau^* = \beta^*\delta^*$ for some $\delta \in S$ with $o(\delta) = c$, because α^*, β^* is a G-regular sequence. Thus $\sigma = \alpha\delta + \sigma_1$ and $\tau = \beta\delta + \tau_1$ for some $\sigma_1, \tau_1 \in S$ with $o(\sigma_1) > o(\sigma)$ and $o(\tau_1) > o(\tau)$;

$$q = h\delta + (\sigma_1 f + \tau_1 g). \tag{1}$$

Now let

$$\Lambda = \left\{ o(\sigma' f + \tau' g) \;\middle|\; \begin{array}{l} \sigma', \tau' \in S \text{ such that} \\ (\sigma' f + \tau' g)^* \notin (f^*, g^*) \text{ and } o(\sigma') \geq b - d + c \end{array} \right\}.$$

Then $o(q) \in \Lambda$. Let $n = \min \Lambda$ and put

$$\Gamma = \left\{ o(\sigma') \;\middle|\; \begin{array}{l} \sigma' \in S \text{ for which there exists } \tau' \in S \text{ such that} \\ (\sigma' f + \tau' g)^* \notin (f^*, g^*),\ o(\sigma') \geq b - d + c, \text{ and } o(\sigma' f + \tau' g) = n \end{array} \right\}.$$

Then $\Gamma \neq \emptyset$ and $\gamma < n - a$ for all $\gamma \in \Gamma$ (cf. Lemma 2.1 (1)). Let $\gamma = \max \Gamma$ and choose $\sigma', \tau' \in S$ so that $(\sigma' f + \tau' g)^* \notin (f^*, g^*)$, $\gamma = o(\sigma') \geq b - d + c$, and $o(\sigma' f + \tau' g) = n$. Let $q' = \sigma' f + \tau' g$. Then, because $q'^* \notin (f^*, g^*)$, similarly as in equation (1) we have

$$q' = h\delta' + (\sigma_2 f + \tau_2 g)$$

for some $\delta', \sigma_2, \tau_2 \in S$ with $o(\delta') = o(\sigma') - (b-d)$, $o(\sigma_2) > o(\sigma')$, and $o(\tau_2) > o(\tau')$. Let $q'' = \sigma_2 f + \tau_2 g$ and assume that $o(q') < o(h\delta')$. We then have

$$n = o(q') = o(q'') \qquad \text{and} \qquad q'^* = q''^*,$$

whence $q''^* \notin (f^*, g^*)$. On the other hand, because $o(\sigma_2) > o(\sigma') \geq b - d + c$, we get $o(\sigma_2) \in \Gamma$, which is impossible (recall that $o(\sigma') = \max \Gamma$). Thus $o(q') \geq o(h\delta')$ and so

$$\begin{aligned} o(q) &\geq n = o(q') \geq o(h) + o(\delta') \\ &= o(h) + o(\sigma') - (b-d) \\ &\geq o(h) + [(b-d) + c] - (b-d) \\ &= o(h) + c, \end{aligned}$$

as was claimed. This proves assertion (1).

Now assume that $q^* \notin (f^*, g^*, h^*)$. Then $o(q) \geq o(h) + c$ by assertion (1), where $c = o(\sigma) - (b-d)$. Assume $o(q) = o(h) + c$ and write $q = h\delta + (\sigma_1 f + \tau_1 g)$ for some $\delta, \sigma_1, \tau_1 \in S$ with $o(\delta) = c$, $o(\sigma_1) > o(\sigma)$, and $o(\tau_1) > o(\tau)$ (cf. equation (1)). We put $q_1 = \sigma_1 f + \tau_1 g$. Then, because $o(q) = o(h\delta) \geq \min\{o(h\delta), o(q_1)\}$, $o(q_1) \geq o(h\delta)$. If $o(q_1) > o(h\delta)$, then we have $q^* = (h\delta)^* = h^*\delta^* \in (f^*, g^*, h^*)$, which is impossible. Hence $o(q_1) = o(h\delta) = o(q)$ so that $q^* = h^*\delta^* + q_1^* \notin (f^*, g^*, h^*)$. Consequently $q_1^* \notin (f^*, g^*)$ and so we get by assertion (1) that

$$\begin{aligned} o(h) + c &= o(h\delta) = o(q_1) \\ &\geq o(h) + o(\sigma_1) - (b-d) \\ &\geq o(h) + [o(\sigma) + 1] - (b-d) \\ &= o(h) + c + 1, \end{aligned}$$

which is absurd. Hence $o(q) > o(h) + c$. This proves assertion (2).

To show assertion (3), thanks to assertion (2), it is enough to check the equality $o(q) = o(h)$. The inequality $o(q) \geq o(h)$ follows from assertion (1), whence $o(h) = o(q)$ by symmetry.

We now prove assertions (4) and (5). Let $V = J/NJ$ and choose homogeneous elements $\delta_1, \delta_2, ..., \delta_n$ of J so that their images $\bar{\delta_1}, \bar{\delta_2}, ..., \bar{\delta_n}$ in V form a k-basis of V. We may assume $\delta_1 = f^*$, $\delta_2 = g^*$. Hence $J = (f^*, g^*, \delta_3, ..., \delta_n)$. For each $3 \leq i \leq n$ let $\delta_i = q_i^*$ with $q_i \in I$ and write $q_i = \sigma_i f + \tau_i g$ for some $\sigma_i, \tau_i \in S$. Then $o(\sigma_i) \geq b - d$ by Lemma 2.1. We have $o(q_i) = o(h)$ and $(f^*, g^*, q_i^*) = (f^*, g^*, h^*)$ (resp. $o(q_i) > o(h)$), if $o(\sigma_i) = b - d$ (resp. if $o(\sigma_i) > b - d$) by assertion (3) (resp. assertion (1)). Hence $o(q_i) \geq o(h)$. We may assume $o(q_3) \leq o(q_4) \leq ... \leq o(q_n)$. Then, because $h^* \in (f^*, g^*, \delta_3, \delta_4, ..., \delta_n)$ but $h^* \notin (f^*, g^*)$, we get $\deg h^* = o(h) \geq \deg \delta_3 = o(q_3)$ so that $o(q_3) = o(h)$, whence $(f^*, g^*, \delta_3) = (f^*, g^*, h^*)$ by assertion (3). Thus assertion (4) follows. Suppose that $n \geq 4$. Then $\delta_4 = q_4^* \notin (f^*, g^*, \delta_3) = (f^*, g^*, h^*)$. Therefore $o(\sigma_4) > b - d$. Hence by assertion (2) we have

$$\begin{aligned} \deg \delta_4 &= o(q_4) \\ &\geq o(h) + [o(\sigma_4) - (b-d) + 1] \geq o(h) + 2 \\ &\geq (a + b - d) + 3 \qquad \text{(by Lemma 2.1)} \\ &\geq b + 4 \geq a + 4. \end{aligned}$$

Consequently, $\deg \delta_4 = o(q_4) \geq 6$, if $I \subseteq \mathfrak{n}^2$. Hence $J \supsetneq (J_i \mid 1 \leq i \leq 5)G$, which completes the proof of Proposition 2.4. $\square$

We are now ready to prove Theorem 1.2 .

Proof of Theorem 1.2. We may assume that S is complete and $\mathrm{ht}_G(f^*, g^*) = 1$. Hence $\mu_G(J) = 3$. Choose $\alpha, \beta \in S$ so that $o(\alpha) = b - d$ and $(\alpha f + \beta g)^* \notin (f^*, g^*)$. Let $h = \alpha f + \beta g$. Then $J = (f^*, g^*, h^*)$ by Proposition 2.4 (4). We furthermore have $h^* \in (\alpha^*, \beta^*)$, because α^*, β^* is a G-regular sequence (cf. Remark 2.3) and $h \in (\alpha, \beta)$. Let $h^* = \alpha^* \varphi + \beta^* \psi$ with $\varphi, \psi \in G$. Then, since $\alpha^* = -\bar{u}\eta$ and $\beta^* = \bar{u}\xi$ for some unit u in S, we see

$$J = I_2 \begin{pmatrix} \bar{u}\varphi & \bar{u}\psi & D \\ \xi & \eta & 0 \end{pmatrix}$$

where $D \in G$ is the element such that $f^* = \xi D$ and $g^* = \eta D$. Thus J is a perfect ideal of G, because $\mathrm{grade}_G J = 2$. $\square$

Discussion 2.5. Assume notation as in Setting 1.1 and also assume that $I \subset \mathfrak{n}^2$. Let $\mu(I^*)$ denote the minimal number of generators of I^*. If $\mu(I^*) = 3$, then $I^* = (f^*, g^*, h_0^*)G$, where $h_0 = \alpha f + \beta g$ and $o(\alpha) = b - d$. We have

$$2 \leq \deg f^* \leq \deg g^* < \deg g^* + 2 \leq \deg h_0^*,$$

so $\deg h_0^* \geq 4$. If $\mu(I^*) \geq 4$, then there exist homogeneous generators for I^* so that

$$I^* = (f^*, g^*, h_0^*, h_1^*, \ldots, h_r^*)G,$$

where we have $r = \mu(I^*) - 3$, and

$$2 \leq \deg f^* \leq \deg g^* < \deg g^* + 2 \leq \deg h_0^* < \deg h_0^* + 2 \leq \deg h_1^* \leq \cdots \leq \deg h_r^*.$$

The inequality $\deg h_1^* \geq \deg h_0^* + 2$ is by Proposition 2.4 (2). In particular, if $\mu(I^*) \geq 4$, then the relation type of $\mathrm{gr}_{\mathfrak{m}}(R)$ is greater than or equal to 6.

It would be interesting to know whether $\deg h_i^* + 2 \leq \deg h_{i+1}^*$ holds for all i with $0 \leq i < r$, or, if this fails to hold in general, whether $\deg h_i^* + 1 \leq \deg h_{i+1}^*$. An interesting result of Kothari [K] shows that if S is a 2-dimensional regular local ring containing a coefficient field, then $\deg h_i^* + 1 \leq \deg h_{i+1}^*$ for all i with $1 \leq i < r$.

3. Applications of the theorem

Let us give some consequences of Theorem 1.2. We begin with the following.

Corollary 3.1. *Let $(R, \mathfrak{m})$ be a d-dimensional Gorenstein local ring. Assume that $\mathfrak{m}$ is minimally generated by $d + 2$ elements. Then $\mathrm{gr}_{\mathfrak{m}}(R)$ is a Cohen-Macaulay ring, if the relation type of $\mathrm{gr}_{\mathfrak{m}}(R)$ is less than or equal to 5.*

Proof. We may assume that $(R, \mathfrak{m})$ is complete. Hence, thanks to the structure theorem of Cohen ([BH, Theorem A.21]), we get $R = S/I$, where I is an ideal of a $(d+2)$-dimensional regular local ring $(S, \mathfrak{n})$. Because R is a Gorenstein ring and $\dim R = d$, the ideal I is generated by a regular sequence f, g of length 2. Let $J = \mathrm{Ker}\ (\mathrm{gr}_{\mathfrak{n}}(S) \xrightarrow{\varphi} \mathrm{gr}_{\mathfrak{m}}(R))$, where $\varphi : \mathrm{gr}_n(S) \to \mathrm{gr}_m(R)$ denotes the canonical map. We may assume that $\mu_{\mathrm{gr}_n(S)}(J) \geq 3$. Then by Proposition 2.4 (5) the ideal J is 3-generated, because the relation type of $\mathrm{gr}_{\mathfrak{m}}(R)$ is at most 5, whence by Theorem 1.2, $\mathrm{gr}_{\mathfrak{m}}(R)$ is a Cohen-Macaulay ring since the polynomial ring $\mathrm{gr}_{\mathfrak{n}}(S)$ is a UFD. □

Corollary 3.2. *Let* $(R, \mathfrak{m})$ *be a one-dimensional Gorenstein local ring and assume that* $\mathfrak{m}$ *is minimally generated by* 3 *elements. If the reduction number of* $\mathfrak{m}$ *is less than or equal to* 4*, then* $\mathrm{gr}_{\mathfrak{m}}(R)$ *is a Cohen-Macaulay ring.*

Proof. The result of Huckaba [H, Theorem 2.3] shows that in our setting the relation type of $\mathrm{gr}_{\mathfrak{m}}(R)$ is at most one more than the reduction number of $\mathfrak{m}$. Hence by Corollary 3.1 the ring $\mathrm{gr}_{\mathfrak{m}}(R)$ is Cohen-Macaulay. □

The example studied in Example 1.6 shows that Corollary 3.2 may fail if the reduction number of $\mathfrak{m}$ is 5. The following example is explored by Sally [S, Example 2.2] and shows that Corollary 3.1 may fail if we assume that R is a Cohen-Macaulay (rather than Gorenstein) ring.

Example 3.3. Let $S = k[[x, y, z]]$ be the formal power series ring with three variables x, y, z over a field k . Let $R = k[[w^4, w^5, w^{11}]]$ be the subring of the formal power series ring $k[[w]]$ and consider the homomorphism $\phi : S \to R$ of k-algebras defined by $\phi(x) = w^4$, $\phi(y) = w^5$, and $\phi(z) = w^{11}$. Then $I = \mathrm{Ker}\ \phi$ is generated by $xz - y^3$, $yz - x^4$, and $z^2 - x^3y^2$. We have $\mathrm{gr}_{\mathfrak{n}}(S) = k[x^*, y^*, z^*]$,

$$I^* = (z^{*2}, z^*y^*, z^*x^*, y^{*4}),$$

and the ring $\mathrm{gr}_{\mathfrak{m}}(R) = \mathrm{gr}_{\mathfrak{n}}(S)/I^*$ is not Cohen-Macaulay. The relation type of $\mathrm{gr}_{\mathfrak{m}}(R)$ is 4 and the reduction number of $\mathfrak{m}$ is 3.

Corollary 3.4. *Let* $(R, \mathfrak{m})$ *be a one-dimensional Gorenstein local ring and assume that* $\mathfrak{m}$ *is minimally generated by* 3 *elements. If the reduction number* r *of* $\mathfrak{m}$ *is less than or equal to* 4*, then* $\mathrm{gr}_{\mathfrak{m}}(R)$ *is a Gorenstein ring if and only if* $J^r : \mathfrak{m}^r = \mathfrak{m}^r$*, where* J *is a reduction of* $\mathfrak{m}$.

Proof. By Corollary 3.2, $\mathrm{gr}_{\mathfrak{m}}(R)$ is Cohen-Macaulay. Therefore all the powers of $\mathfrak{m}$ are closed in the sense of Ratliff-Rush. Hence $\mathrm{gr}_{\mathfrak{m}}(R)$ is a Gorenstein ring if and only if $J^r : \mathfrak{m}^r = \mathfrak{m}^r$ (cf. [HKU, Corollary 4.8]). □

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