Catenary local rings with geometrically normal formal fibers

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Abstract. We discuss relations between the catenary property and geometrically normal formal fibers. We present for each integer $n \geq 2$ an example of a catenary Noetherian local integral domain of dimension n which has geometrically regular formal fibers and is not universally catenary. These examples are obtained by means of a construction developed in our previous articles which uses power series rings, homomorphic images and intersections.

1 Introduction

We are happy to dedicate this paper to Shreeram S. Abhyankar in celebration of his seventieth birthday. In his mathematical work Ram has opened up many avenues. In the present paper we are pursuing one of these related to power series and completions. [1]

A Noetherian ring R is said to be *catenary* if, for every pair of comparable prime ideals $P \subseteq Q$ of R, every saturated chain of prime ideals from P to Q has the same length [A, page 11]. The ring R is *universally catenary* if every finitely generated R-algebra is catenary. A Noetherian local ring $(R, \mathbf{m})$ with $\mathbf{m}$-adic completion $\widehat{R}$ has *geometrically normal* (respectively *geometrically regular) formal fibers* if for each prime P of R and for each finite algebraic extension k' of the field $k(P) := R_P/PR_P$, the ring $\widehat{R} \otimes_R k(P) \otimes_{k(P)} k'$ is normal (respectively regular).

In this paper we investigate the catenary property in Noetherian local rings having geometrically normal formal fibers. In Example 4.2 we apply a technique from our earlier papers to construct, for each integer $n \geq 2$, an example of a catenary Noetherian local integral domain of dimension n with geometrically regular formal fibers which is not universally catenary.

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Let $(R, \mathbf{m})$ be a Noetherian local ring. We denote the Henselization of R by R^h. We say $(R, \mathbf{m})$ is *formally equidimensional*, or in other terminology *quasi-unmixed*, provided all the minimal primes of the $\mathbf{m}$-adic completion $\widehat{R}$ have the same dimension. A theorem of Ratliff (Theorem 2.1) which is crucial for our work states that R is universally catenary if and only if $R/\mathbf{p}$ is formally equidimensional for each minimal prime $\mathbf{p}$ of R [Ra, Theorem 2.6].

Section 2 of this paper contains several results concerning conditions for a Noetherian local ring $(R, \mathbf{m})$ to be universally catenary. First, Ratliff's theorem leads to the observation (Proposition 2.2) that a Henselian Noetherian local ring having geometrically normal formal fibers is universally catenary.

Suppose now that $(R, \mathbf{m})$ is a Noetherian local integral domain having geometrically normal formal fibers. It follows (Corollary 2.3) that R^h also has geometrically normal formal fibers and thus by (2.2) is universally catenary; moreover, if the derived normal ring $\overline{R}$ is local, then R is universally catenary. In Theorem 2.6, we show R is universally catenary if and only if the set Γ is empty, where

$$\Gamma := \{W \in \operatorname{Spec}(R^h) \mid \dim(R^h/W) < \dim(R/(W \cap R))\}.$$

We also observe that Γ has a "going down" property.

In Theorem 2.7 we prove for R as above that R is catenary but is not universally catenary if and only if Γ is nonempty and each prime W in Γ has dimension one. Thus, as we observe in Corollary 2.8, if R is catenary but not universally catenary, this is signaled by the existence of dimension one minimal primes of the $\mathbf{m}$-adic completion $\widehat{R}$ of R. If R is catenary, each minimal prime of $\widehat{R}$ having dimension different from $\dim(R)$ must have dimension one.

In Section 3 we provide examples to illustrate the results of Section 2. We apply a construction involving power series, homomorphic images and intersections. The use of power series to construct interesting examples of Noetherian integral domains has a rich history [Az], [BR1], [BR2], [H], [O1], [O2], [R1], [R2], [R3]. We give a brief review of relevant notation and results from our earlier papers describing this technique [HRW1], [HRW2], [HRW3], [HRW4]. The construction begins with a Noetherian domain which may be taken to be a "standard" Noetherian domain such as a polynomial ring in several indeterminates over a field. In Theorem 3.6 we extend this construction by proving that in certain circumstances it is possible to transfer the flatness, Noetherian and computability properties of integral domains associated with ideals $I_1, \dots, I_n$ to the integral domain associated with their intersection $I = I_1 \cap \dots \cap I_n$.

We apply these concepts in Examples 4.1 - 4.3 to produce Noetherian local domains which are not universally catenary. In Remark 4.4, we specifiy precisely which of these rings are catenary. These domains illustrate the results of Section 2, because in Section 5 we prove that they have geometrically regular formal fibers.

The books of Matsumura [M1], [M2] and the book of Nagata [N2] are good references for our terminology.

We would like to thank M. Brodmann and R. Sharp for raising a question on catenary/universally catenary rings which motivated our work in this paper.

2 Geometric normality of formal fibers

Throughout this section $(R, \mathbf{m})$ is a Noetherian local ring, usually a domain.

We use the following interesting result proved by Ratliff in [Ra, Theorem 2.6] relating the universally catenary property to properties of the completion:

2.1 Theorem. (Ratliff) A Noetherian local ring $(R, \mathbf{m})$ is universally catenary if and only if $R/\mathbf{p}$ is formally equidimensional for every minimal prime ideal $\mathbf{p} \in \operatorname{Spec}(R)$.

2.2 Proposition. If $(R, \mathbf{m})$ is a Henselian Noetherian local ring having geometrically normal formal fibers, then R is universally catenary, and for each $P \in \operatorname{Spec} R$, the extension $P\widehat{R}$ of P to the $\mathbf{m}$-adic completion of R is also prime.

Proof. By Theorem 2.1, to show R is universally catenary, it suffices to show every minimal prime $\mathbf{p}$ of R is formally equidimensional. By passing from R to $R/\mathbf{p}$, we may assume that R is an integral domain. We prove that $\widehat{R}$ is also an integral domain, so, in particular, the zero ideal of R is formally equidimensional. Since R has normal formal fibers, the completion $\widehat{R}$ of R is reduced. Hence the derived normal ring $\overline{R}$ of R is a finitely generated R-module [N2, (32.2)]. Moreover, since R is Henselian, $\overline{R}$ is local [N2, (43.12)].

The completion $\widehat{\overline{R}}$ of $\overline{R}$ is $\widehat{R} \otimes_R \overline{R}$ [N2, (17.8)]. Since the formal fibers of R are geometrically normal, the formal fibers of $\overline{R}$ are also geometrically normal. It follows that $\widehat{\overline{R}}$ is normal [M2, Corollary, page 184], and hence an integral domain because $\overline{R}$ is local. Since $\widehat{R}$ is a flat R-module, $\widehat{R}$ is a subring of $\widehat{\overline{R}}$. Therefore $\widehat{R}$ is an integral domain and so R is formally equidimensional.
□

2.3 Corollary. Suppose R is a Noetherian local domain having geometrically normal formal fibers. Then

(1) R^h is universally catenary.
(2) If the derived normal ring $\overline{R}$ of R is again local, then R is universally catenary.

In particular, if R is a normal Noetherian local domain having geometrically normal formal fibers, then R is universally catenary.

Proof. For item (1), the Henselization R^h of R is again a Noetherian local domain having geometrically normal formal fibers. For (2), by [N2, (43.20)], R is formally equidimensional and hence universally catenary. □

We use the following result relating the catenary property to the height of maximal ideals of the derived normal ring.

2.4 Proposition. Let $(R, \mathbf{m})$ be a Noetherian local domain of dimension d and let $\overline{R}$ be the derived normal ring of R. If $\overline{R}$ contains a maximal ideal $\overline{\mathbf{m}}$ with $\operatorname{ht}(\overline{\mathbf{m}}) = r \notin \{1, d\}$, then R is not catenary.

Proof. Since $\overline{R}$ has only finitely many maximal ideals [N2, (33.10)], there exists $b \in \overline{\mathbf{m}}$ such that b is in no other maximal ideal of $\overline{R}$. Let $R' = R[b]$ and $\mathbf{m}' = \overline{\mathbf{m}} \cap R'$. By the Going Up Theorem[M2,(9.3)], $\operatorname{ht}(\mathbf{m}') = r \notin \{1, d\}$. Since R' is a finitely generated R-module there exists a nonzero $a \in \mathbf{m}$ such that $aR' \subseteq R$. It follows that $R[1/a] = R'[1/a]$. The maximal ideals of $R[1/a]$ have the form $PR[1/a]$, where $P \in \operatorname{Spec}(R)$ is maximal with respect to not containing a. Since there are no prime ideals strictly between P and $\mathbf{m}$ [M2, (13.5)], if $\operatorname{ht}(P) = h$, then there exists in R a saturated chain of prime ideals through P of length $h + 1$. Thus to show R is not catenary, it suffices to establish the existence of a maximal ideal of $R[1/a]$ having height different from $d-1$. Since $R[1/a] = R'[1/a]$, the maximal ideals of $R[1/a]$ correspond to $P' \in \operatorname{Spec}(R')$ maximal with respect to not containing a. Since $\operatorname{ht}(\mathbf{m}') > 1$, there exists $c \in \mathbf{m}'$ such that c is not in any minimal prime of aR' nor in any maximal ideal of R' other than $\mathbf{m}'$. Hence there exist prime ideals of R' containing c and not containing a. Let $P' \in \operatorname{Spec}(R')$ be maximal with respect to $c \in P'$ and $a \notin P'$. Then $P' \subseteq \mathbf{m}'$, so $\operatorname{ht}(P') \leq r - 1 < d - 1$. Therefore R is not catenary. □

2.5 Remark. For $(R, \mathbf{m})$ a Noetherian local domain, it is well known that the maximal ideals of the derived normal ring $\overline{R}$ of R are in one-to-one correspondence with the minimal primes of the Henselization R^h of R [N2, (43.20)]. Moreover, if a maximal ideal $\overline{\mathbf{m}}$ of $\overline{R}$ corresponds to a minimal prime Q of R^h, then the derived normal ring of R^h/Q is the Henselization of $\overline{R}_{\overline{\mathbf{m}}}$ [N2, Ex. 2, page 188], [N1]. Therefore $\operatorname{ht}(\overline{\mathbf{m}}) = \dim(R^h/Q)$.

2.6 Theorem. Suppose $(R, \mathbf{m})$ is a Noetherian local integral domain having geometrically normal formal fibers. Consider the set

$$\Gamma := \{W \in \operatorname{Spec}(R^h) \mid \dim(R^h/W) < \dim(R/(W \cap R))\}.$$

(1) For $\mathbf{p} \in \operatorname{Spec}(R)$, $R/\mathbf{p}$ is not universally catenary if and only if there exists $W \in \Gamma$ such that $\mathbf{p} = W \cap R$. The set Γ is empty if and only if R is universally catenary.

(2) If $\mathbf{p} \subseteq \mathbf{q}$ in $\operatorname{Spec}(R)$ and if there exists $W \in \Gamma$ with $W \cap R = \mathbf{q}$, then there also exists $W' \in \Gamma$ with $W' \cap R = \mathbf{p}$.

(3) If $W \in \Gamma$ and Q is a minimal prime of R^h such that $Q \subseteq W$, then Q is also in Γ, that is, $\dim(R^h/Q) < \dim(R^h) = \dim(R)$.

Proof. For item (1), we use that the map of $R/\mathbf{p}$ to its $\mathbf{m}$-adic completion $\widehat{R}/\mathbf{p}\widehat{R}$ factors through $R^h/\mathbf{p}R^h$. Therefore, by Theorem 2.1, $R/\mathbf{p}$ is universally catenary if and only if $R^h/\mathbf{p}R^h$ is equidimensional if and only if there does not exist $W \in \Gamma$ with $W \cap R = \mathbf{p}$. To prove item (2), observe that if $R/\mathbf{p}$ is universally catenary, then $R/\mathbf{q}$ is also universally catenary [M2, Theorem 31.6].

It remains to prove item (3). Suppose there exists $P \subseteq W$ in $\operatorname{Spec}(R^h)$ with $\dim(R^h/P) = \dim(R^h)$. Let $\mathbf{w} = W \cap R$. Since R^h is flat over R with zero-dimensional fibers, $\operatorname{ht}(W) = \operatorname{ht}(\mathbf{w})$ [M2, Theorem 15.1]. By Proposition 2.3.1, R^h is universally catenary. Therefore $\operatorname{ht}(W/P) + \dim(R^h/W) = \dim(R^h/P) = \dim(R^h) \geq \operatorname{ht}(W) + \dim(R^h/W)$, and so $\operatorname{ht}(W/P) = \operatorname{ht}(W)$. Since $\operatorname{ht}(\mathbf{w}) + \dim(R/\mathbf{w}) \leq \dim(R) = \dim(R^h)$, it follows that $\dim(R^h/W) \geq \dim(R/\mathbf{w})$, so $W \notin \Gamma$. □

2.7 Theorem. Let $(R, \mathbf{m})$ and Γ be as in Theorem 2.6. Then R is catenary but not universally catenary if and only if Γ is nonempty and each prime $W \in \Gamma$ has dimension one. In this case, each $W \in \Gamma$ is a minimal prime of R^h.

Proof. Assume that R is catenary but not universally catenary. By Theorem 2.6, the set Γ is nonempty and there exist minimal primes Q of R^h such that $\dim(R^h/Q) < \dim(R^h)$. By Remark 2.5, if a maximal ideal $\overline{\mathbf{m}}$ of $\overline{R}$ corresponds to a minimal prime Q of R^h, then $\operatorname{ht}(\overline{\mathbf{m}}) = \dim(R^h/Q)$. Since R is catenary, Proposition 2.4 implies the height of each maximal ideal of the derived normal ring $\overline{R}$ of R is either one or $\dim(R)$. Therefore $\dim(R^h/Q) = 1$ for each minimal prime Q of R^h for which $\dim(R^h/Q) \neq \dim(R^h)$. Part (3) of Theorem 2.6 implies each $W \in \Gamma$ is a minimal prime of R^h and of dimension one.

For the converse, assume that Γ is nonempty and each prime $W \in \Gamma$ has dimension one. Then R is not universally catenary by part (1) of Theorem 2.6 and by part (3) of Theorem 2.6, each prime of Γ is a minimal prime of R^h and therefore lies over (0) in R. To show R is catenary, it suffices to show for each nonzero nonmaximal prime ideal $\mathbf{p}$ of R that $\operatorname{ht}(\mathbf{p}) + \dim(R/\mathbf{p}) = \dim(R)$ [M2, Theorem 31.4]. Let $P \in \operatorname{Spec}(R^h)$ be a minimal prime of $\mathbf{p}R^h$. Since R^h is flat over R with zero-dimensional fibers, $\operatorname{ht}(\mathbf{p}) = \operatorname{ht}(P)$. Let Q be a minimal prime of R^h with $Q \subseteq P$. Then $Q \notin \Gamma$. For by assumption every prime of Γ has dimension one, so if Q were in Γ, then $Q = P$. But $P \cap R = \mathbf{p}$, which is nonzero, and $Q \cap R = (0)$. Therefore $Q \notin \Gamma$ and hence $\dim(R^h/Q) = \dim(R^h)$. Since R^h is catenary, it follows that $\operatorname{ht}(P) + \dim(R^h/P) = \dim(R^h)$. Since $P \notin \Gamma$, we have $\dim(R/\mathbf{p}) = \dim(R^h/P)$. Therefore $\operatorname{ht}(\mathbf{p}) + \dim(R/\mathbf{p}) = \dim(R)$ and R is catenary. □

2.8 Corollary. If R has geometrically normal formal fibers and is catenary but not universally catenary, then there exist in the $\mathbf{m}$-adic completion $\widehat{R}$ of R minimal prime ideals $\widehat{q}$ such that $\dim(\widehat{R}/\widehat{q}) = 1$.

Proof. By Theorem 2.7, each prime ideal $Q \in \Gamma$ has dimension one and is a minimal prime of R^h. Moreover, $Q\widehat{R} := \widehat{q}$ is a minimal prime of $\widehat{R}$. Since $\dim(R^h/Q) = 1$, we have $\dim(\widehat{R}/\widehat{q}) = 1$. □

3 A method for constructing examples

In this section we give a brief review of relevant notation and results from our earlier papers describing a method for constructing examples. The construction begins with a Noetherian domain which may be taken to be a "standard" Noetherian domain such as a polynomial ring in several indeterminates over a field. In Theorem 3.6 we extend our previous results; we use Theorem 3.6 in Section 4 to obtain examples with larger dimensions and more minimal primes.

We use the following result from [HRW2]:

3.1 Theorem. Let R be a Noetherian integral domain with fraction field K. Let x be a nonzero nonunit of R and let R^* denote the (x)-adic completion of R. Suppose I is an ideal of R^* with the property that $\mathbf{p} \cap R = (0)$ for each $\mathbf{p} \in \operatorname{Ass}(R^*/I)$, and set $A := K \cap (R^*/I)$. Then $R \to (R^*/I)_x$ is flat if and only if A is Noetherian and is realizable as a localization of a subring of $R_x = R[1/x]$.

For our constructions we apply Theorem 3.1 and some other results of [HRW2] and [HRW1] to a more specific setting, outlined in (3.2).

3.2 Setting and notation for examples. Let k be a field, let $n \geq s \in \mathbf{N}$, and let $x, y_1, \dots, y_n$ be indeterminates over k. Let $R := k[x, y_1, \dots, y_n]$ and let R^* be the (x)-adic completion of R. Suppose $\tau_1, \dots, \tau_s \in xk[[x]] \subseteq R^*$ are algebraically independent over $k(x, y_1, \dots, y_n)$. Set $I := (y_1 - \tau_1, \dots, y_s - \tau_s)R^*$ and $A := k(x, y_1, \dots, y_n) \cap (R^*/I)$.

The domain A can also be expressed as an intermediate domain between a Noetherian domain and its x-adic completion:

$$A := k(x, y_{s+1}, \dots, y_n, \tau_1, \dots, \tau_s) \cap k[y_{s+1}, \dots, y_n]\,[[x]]. \tag{3.3}$$

It is convenient to also consider a *local* version of (3.2):

3.2′ Local setting and notation for examples. Here R is the *localized* polynomial ring $k[x, y_1, \dots, y_n]_{(x, y_1, \dots, y_n)}$; otherwise this is the same setup as (3.2). Again let

$$A := k(x, y_1, \dots, y_n) \cap (R^*/I), \quad \text{where} \quad I := (y_1 - \tau_1, \dots, y_s - \tau_s)R^*.$$

Then A can be expressed as an intermediate domain between the Noetherian local domain R and its x-adic completion:

$$(3.3') \qquad A := k(x, y_{s+1}, \dots, y_n, \tau_1, \dots, \tau_s) \cap k[y_{s+1}, \dots, y_n]_{(y_{s+1},\dots,y_n)}[[x]].$$

The expressions in (3.3) and (3.3′) represent a special case of the construction, a simpler "intermediate form"— so that we need not pass to a proper homomorphic image of the completion. This was our approach to the construction in [HRW1].

The following proposition describes the situation for the two settings.

3.4 Proposition. ([HRW2, (4.1)]) Assume that R, R^*, I, and A are as in the setting of (3.2) or (3.2′). Then

(1) The canonical map $\alpha : R \to (R^*/I)_x$ is flat.
(2) With the notation of (3.2), A is Noetherian of dimension $n - s + 1$ and is a localization of a polynomial ring in $n - s$ variables over a DVR.
(3) A is a nested union of localizations of polynomial rings in $n+1$ variables over k.
(4) If k has characteristic zero, then A is excellent.

3.5 Examples. Assume the notation of (3.2) or (3.2′).
(1) Let $R := k[y_1, \dots, y_s, x]$ (that is, $n = s$ in (3.2)) and let R^* denote the (x)-adic completion of R. Then $(R^*/I) \cap K = A$ is the DVR obtained by localizing U at the prime ideal xU. In this example $R_x = U_x$ has dimension $s + 1$ and so $\dim(U) = s + 1$, while $\dim(R^*/I) = \dim(A) = 1$.
(2) A modification of Example 1 is to take R to be the $(s + 1)$-dimensional regular local domain $k[y_1, \dots, y_s, x]_{(y_1,\dots,y_s,x)}$. In this case $R_x = U_x$ has dimension s, while we still have $R^*/I \cong k[[x]]$.

With R as in either (1) or (2), each domain A constructed is a directed union of $(s + 1)$-dimensional regular local domains dominated by $k[[x]]$ and having k as a coefficient field. In either case, since $(R^*/I)_x$ is a field, $R \hookrightarrow (R^*/I)_x$ is flat, so we have a nested union of $(s+1)$-dimensional regular local domains whose union is Noetherian, in fact a DVR.
(3) With $R = k[x, y_1, \dots, y_n]_{(x,y_1,\dots,y_n)}$, a localized polynomial ring in $n + 1$ variables, and $d := n - s$, let J be the ideal $(y_1 - \tau_1, \dots, y_s - \tau_s)R^*$. Then $R^* \cong k[y_1, \dots, y_n]_{(y_1,\dots,y_n)}[[x]]$ is an $n + 1$ dimensional regular local domain and $R^*/J \cong k[y_{s+1}, \dots, y_n]_{(y_{s+1},\dots,y_n))}[[x]]$ is a $(d+1)$-dimensional regular local domain. By (3.4.1), $(R^*/J)_x$ is flat over R. If $V = k[[x]] \cap k(x, \tau_1, \dots, \tau_s)$, then V is a DVR and $(R^*/J) \cap K \cong V[y_{s+1}, \dots, y_n]_{(x,y_{s+1},\dots,y_n)}$ is a (d+1)-dimensional regular local domain which is a nested union of $(n+1)$-dimensional regular local domains.

The following theorem shows that in certain circumstances the flatness, Noetherian and computability properties of the integral domains associated with ideals $I_1, \dots, I_n$ of R^* as described in Theorem 3.1 transfer to the integral domain associated to their intersection $I = I_1 \cap \dots \cap I_n$. We show in Section 5 that the property of regularity of formal fibers also transfers in certain cases to the domain associated with an intersection ideal.

3.6 Theorem. Suppose that R is a Noetherian domain, $x \in R$ is a nonzero nonunit, R^* is the (x)-adic completion of R, and $I_1, \dots, I_n$ are ideals of R^* such that, for each i with $1 \leq i \leq n$, each associated prime of R^*/I_i intersects R in (0). Suppose that each $(R^*/I_i)_x$ is a flat R-module and that the localizations at x of the I_i are pairwise comaximal; that is, for all $i \neq j$, $(I_i+I_j)R^*_x = R^*_x$. Let $I := I_1 \cap \cdots \cap I_n$, $A := K \cap (R^*/I)$ and, for $i = 1, 2, \dots n$, let $A_i := K \cap (R^*/I_i)$. Then

(1) Each associated prime of R^*/I intersects R in (0), $(R^*/I)_x$ is flat over R, A is Noetherian, and $A^* = R^*/I$ is the (x)-adic completion of A. Similarly, $A_i^* = R^*/I_i$ is the (x)-adic completion of A_i, for $i = 1, 2, \dots n$.
(2) $A^*_x \cong (A_1^*)_x \oplus \dots \oplus (A_n^*)_x$. If $W \in \operatorname{Spec}(A^*)$ and $x \notin W$, then $(A^*)_W$ is a localization of one of the A_i^*.
(3) $A \subseteq A_1 \cap \cdots \cap A_n$ and, if $\mathbf{w} \in \operatorname{Spec} A$ with $x \notin \mathbf{w}$, then $(A_1)_x \cap \dots \cap (A_n)_x \subseteq A_{\mathbf{w}}$. In particular, $A_x = (A_1)_x \cap \dots \cap (A_n)_x$.

Proof. For (1), since $\operatorname{Ass}(R^*/(I_1 \cap \dots \cap I_n)) \subseteq \operatorname{Ass}(R^*/I_1) \cup \dots \cup \operatorname{Ass}(R^*/I_n)$, the condition on associated primes of Theorem 3.1 holds for the ideal $I = I_1 \cap \dots \cap I_n$. The natural R-algebra homomorphism $\pi : R^* \to (R^*/I_1) \oplus \dots \oplus (R^*/I_n)$ has kernel I. Further, the localization of π at x is onto because for each $i \neq j$, $(I_i + I_j)_x = R^*_x$. Thus $(R^*/I)_x \cong (R^*/I_1)_x \oplus \dots \oplus (R^*/I_n)_x$ is flat over R. Therefore A is Noetherian by Theorem 3.1. By [HRW2, (2.4.4)], $A^* = R^*/I$ is the (x)-adic completion of A.

For (2), the first part is simply that $(R^*/I)_x \cong (R^*/I_1)_x \oplus \dots \oplus (R^*/I_n)_x$. If $W \in \operatorname{Spec}(A^*)$ and $x \notin W$, then $\pi(W_x)$ is a prime ideal of $(R^*/I_1)_x \oplus \dots \oplus (R^*/I_n)_x$, so has the form $(W_i)_x$ in some i^{th} coordinate and $(R^*/I_j)_x$ in all the others, where $W_i \in \operatorname{Spec}(R^*/I_i)$. It follows that A^*_W is a localization of some A^*_i.

Since R^*/I_i is a homomorphic image of R^*/I, it follows that $A \subseteq A_i$ for all $i = 1, 2, \dots, n$. Let $\mathbf{w} \in \operatorname{Spec} A$ with $x \notin \mathbf{w}$. Since $A^* = R^*/I$ is faithfully flat over A, there exists $\mathbf{w}^* \in \operatorname{Spec}(A^*)$ with $\mathbf{w}^* \cap A = \mathbf{w}$. Then $x \notin \mathbf{w}^*$ implies $A^*_{\mathbf{w}^*}$ is some $(A^*_i)_{\mathbf{w}^*_i}$, where $\mathbf{w}^*_i \in \operatorname{Spec}(A^*_i)$. By symmetry, we may assume $A^*_{\mathbf{w}^*} = (A^*_1)_{\mathbf{w}^*_1}$. Let $\mathbf{w}_1 = \mathbf{w}^*_1 \cap A_1$. Since $A_{\mathbf{w}} \hookrightarrow A^*_{\mathbf{w}^*}$ and $(A_1)_{\mathbf{w}_1} \hookrightarrow (A^*_1)_{\mathbf{w}^*_1}$ are faithfully flat, we have

$$A_{\mathbf{w}} = A^*_{\mathbf{w}^*} \cap K = (A^*_1)_{\mathbf{w}^*_1} \cap K = (A_1)_{\mathbf{w}_1} \supseteq (A_1)_x.$$

It follows that $(A_1)_x \cap \dots \cap (A_n)_x \subseteq A_{\mathbf{w}}$. Thus we have $(A_1)_x \cap \dots \cap (A_n)_x \subseteq \cap\{A_{\mathbf{w}} : \mathbf{w} \in \operatorname{Spec} A \text{ and } x \notin \mathbf{w}\} = A_x$. Since $A_x \subseteq (A_i)_x$, for each i, it follows that $A_x = (A_1)_x \cap \dots \cap (A_n)_x$. □

4 Examples which are not universally catenary

In [HRW2, (4.5)] we give an example of a Noetherian domain A for which the completion is two dimensional and has exactly two minimal primes; the first minimal prime has dimension one and the other has dimension two. Thus A is not universally catenary. This is done in such a way that A has geometrically regular formal fibers. We generalize this example in the following.

4.1 Example. We construct a two-dimensional Noetherian local domain so that the completion has any desired number of minimal primes of dimensions one and two. For this, let R be the localized polynomial ring in three variables $R := k[x, y, z]_{(x,y,z)}$, where k is a field of characteristic zero and the field of fractions of R is $K := k(x, y, z)$. Then the (x)-adic completion of R is $R^* := k[y, z]_{(y,z)}[[x]]$. Let $\tau_1, \dots, \tau_s, \beta_1, \beta_2, \dots, \beta_m, \gamma \in xk[[x]]$ be algebraically independent power series over $k(x)$. Now define $Q_i := (z - \tau_i, y - \gamma)R^*$, for i with $1 \le i \le r$, and $P_j := (z - \beta_j)R^*$, for j with $1 \le j \le m$. We apply Theorem 3.6 with $I_i = Q_i$ for $1 \le i \le r$, and $I_{r+j} = P_j$ for $1 \le j \le m$. Then the I_i satisfy the comaximality condition at the localization at x. Let $I := I_1 \cap \ldots \cap I_{r+m}$ and let $A := K \cap (R^*/I)$. For J an ideal of R* containing I, let $\bar{J}$ denote the image of J in R^*/I. Then, for each i with $1 \le i \le r$, $\dim((R^*/I)/\bar{Q}_i) = \dim(R^*/Q_i) = 1$ and, for each j with $1 \le j \le m$, $\dim((R^*/I)/\bar{P}_j) = 2$. Thus $\widehat{A}$ contains r minimal primes of dimension one and m of dimension two.

The integral domain A birationally dominates R and is birationally dominated by each of the A_i. It follows from Corollary 5.3 that A has geometrically regular formal fibers. Since $\dim(A) = 2$, A is catenary.

We show in Example 4.2 that for every integer $n \ge 2$ there is a Noetherian local domain $(A, \mathbf{m})$ of dimension n with geometrically regular formal fibers which is catenary but not universally catenary.

4.2 Example. Let $R = k[x, y_1, \dots, y_n]_{(x,y_1,\dots,y_n)}$ be a localized polynomial ring of dimension $n + 1$ where k is a field of characteristic zero. Let $\sigma, \tau_1, \dots, \tau_n \in xk[[x]]$ be $n + 1$ algebraically independent elements over $k(x)$ and consider in $R^* = k[y_1, \dots, y_n]_{(y_1,\dots,y_n)}[[x]]$ the ideals

$$I_1 = (y_1 - \sigma)R^* \quad \text{and} \quad I_2 = (y_1 - \tau_1, \dots, y_n - \tau_n)R^*.$$

Then the ring

$$A = k(x, y_1, \dots, y_n) \cap (R^*/(I_1 \cap I_2))$$

is the desired example. The completion $\widehat{A}$ of A has two minimal primes $I_1\widehat{A}$ having dimension n and $I_2\widehat{A}$ having dimension one. By Corollary 5.3, A has geometrically regular formal fibers. Therefore the Henselization A^h has precisely two minimal prime ideals P, Q which may be labeled so that $P\widehat{A} = I_1\widehat{A}$ and $Q\widehat{A} = I_2\widehat{A}$. Thus $\dim(A^h/P) = n$ and $\dim(A^h/Q) = 1$. By Theorem 2.7, A is catenary but not universally catenary.

In Example 4.3 we construct, for each $t \in \mathbf{N}$ and for specified nonnegative integers $n_1, \dots, n_t$ with $n_1 \ge 1$, a t-dimensional Noetherian local domain A that birationally dominates a $t + 1$-dimensional regular local domain such that the completion of A has, for each r with $1 \le r \le t$, exactly n_r minimal primes of dimension $t + 1 - r$. In particular, if $n_i > 0$ for some $i \ne 1$, then

A is not universally catenary and is not a homomorphic image of a regular local domain. It follows from Remark 2.5 that the derived normal ring $\overline{A}$ of A has exactly n_r maximal ideals of height $t+1-r$ for each r with $1 \le r \le t$.

4.3 Example. Let $t \in \mathbf{N}$ and for each r with $1 \le r \le t$, let n_r be a nonnegative integer. Assume that $n_1 \ge 1$. We construct a t-dimensional domain A for which $\widehat{A}$ has exactly n_r minimal primes of dimension $t+1-r$ for each r. Let $x, y_1 \dots, y_t$ be indeterminates over a field k of characteristic zero [2]. Let $R = k[x, y_1, \dots, y_t]_{(x,y_1,\dots,y_t)}$, let $R^* = k[y_1, \dots, y_t]\,[[x]]_{(x,y_1,\dots,y_t)}$ denote the (x)-adic completion of R and let K denote the fraction field of R. For every $r, j, i \in \mathbf{N}$ such that $1 \le r \le t$, $1 \le j \le n_r$ and $1 \le i \le r$, choose elements $\{\tau_{rji}\}$ of $xk[[x]]$ which are algebraically independent over $k(x, y_1, \dots, y_t)$.

For each r, j with $1 \le r \le t$ and $1 \le j \le n_r$, define the prime ideal $P_{rj} := (y_1 - \tau_{rj1}, \dots, y_r - \tau_{rjr})$ of height r in R^*. Then each ideal P_{rj} in R^* is an example of the type considered in (3.2).

Thus the $(R^*/P_{rj})_x$ are flat over R. Here, for each r, j, define

$$\begin{aligned} A_{rj} := K \cap (R^*/(P_{rj}) \quad &= k(x, y_1, \dots, y_t) \cap \tfrac{k[y_1,\dots,y_t]\,[[x]]_{(-)}}{(y_1-\tau_{rj1},\dots,y_r-\tau_{rjr})} \\ &\cong k(x, Y_r, \Gamma_{rj}) \cap k[Y_r]\,[[x]]_{(-)} \cong V_{rj}[Y_r]_{(x,Y_r)}, \end{aligned}$$

where $Y_r := \{y_{r+1}, \dots, y_t\}$, $\Gamma_{rj} := \{\tau_{rj1}, \dots, \tau_{rjr}\}$, and $V_{rj} = k(\Gamma_{rj}) \cap k[[x]]$ is a DVR. Then A_{rj} is a $(t+1-r)$-dimensional regular local domain that is a nested union of $(t+1)$-dimensional RLRs.

We take the ideal I to be the intersection of all the prime ideals P_{rj}. Since the $\tau_{rji} \in xk[[x]]$ are distinct, the sum of any two of these ideals P_{rj} and P_{mi}, where we assume $r \le m$, has radical $(x, y_1, \dots, y_m)R^*$, and thus $(P_{rj} + P_{mi})R^*[1/x] = R^*[1/x]$. It follows that the intersection I of the P_{rj} is irredundant and $\operatorname{Ass}(R^*/I) = \{P_{rj} \,|\, 1 \le r \le t, 1 \le j \le n_r\}$. Since $P_{rj} \cap R = (0)$, R injects into R^*/I. Let $A := K \cap (R^*/I)$.

By Theorem 3.6, $R \hookrightarrow (R^*/I)_x$ is flat, A is Noetherian and A is a localization of a subring of $R[1/x]$. In particular, A birationally the $(t+1)$-dimensional regular local domain R and the stated properties hold.

4.4 Remark. By Theorem 2.7, the ring A constructed in Example 4.3 is catenary if and only if each minimal prime of $\widehat{A}$ has dimension either one or t. By taking $n_r = 0$ for $r \notin \{1, t\}$ in Example 4.3, we obtain additional examples of catenary Noetherian local domains A of dimension t having geometrically regular formal fibers for which the completion $\widehat{A}$ has precisely n_t minimal primes of dimension one and n_1 minimal primes of dimension t.

4.5 Remark. We would like to thank L. Avramov for suggesting we consider the depth of the rings constructed in Example 4.3. The catenary rings which arise from this construction all have depth one, but we can use Example 4.3 to construct, for each integer $t \ge 3$ and integer d with $2 \le d \le t-1$,

[2] The characteristic zero assumption implies that each A_{rj} as constructed below is excellent cf. (3.4.4).

an example of a noncatenary Noetherian local domain A of dimension t and depth d having geometrically regular formal fibers. The (x)-adic completion A^* of A has precisely two minimal primes, one of dimension t and one of dimension d. To see this with notation as in Example 4.3, we set $m = t - d + 1$ and take $n_r = 0$ for $r \notin \{1, m\}$ and $n_1 = n_m = 1$. Let

$$P_1 := P_{11} = (y_1 - \tau_{111})R^* \text{ and } P_m := P_{m1} = (y_1 - \tau_{m11}, \dots, y_m - \tau_{m1m})R^*.$$

Consider $A^* = R^*/(P_1 \cap P_m)$ and the short exact sequence

$$0 \longrightarrow \frac{P_1}{P_1 \cap P_m} \longrightarrow \frac{R^*}{P_1 \cap P_m} \longrightarrow \frac{R^*}{P_1} \longrightarrow 0.$$

Since P_1 is principal and not contained in P_m, we have $P_1 \cap P_m = P_1 P_m$ and $P_1/(P_1 \cap P_m) \cong R^*/P_m$. It follows that $\operatorname{depth} A = \operatorname{depth} A^* = \operatorname{depth}(R^*/P_m) = d$; see for example [K, page 103, ex 14]. Moreover, the derived normal ring $\overline{A}$ of A has precisely two maximal ideals one of height t and one of height d.

5 Regularity of morphisms and geometrical regularity of formal fibers

We show in (5.3) that the ring A of Examples 4.1, 4.2 and 4.3 have geometrically regular formal fibers.

5.1 Proposition. Let R, A and I be as in Theorem 3.1. Suppose that, for each $P \in \operatorname{Spec}(R^*/I)$ with $x \notin P$, the morphism $\psi_P : R_{P \cap R} \longrightarrow (R^*/I)_P$ is regular. Then

(1) A is Noetherian and the morphism $A \longrightarrow A^* = R^*/I$ is regular.

(2) If R is semilocal with geometrically regular formal fibers and x is in the Jacobson radical of R, then A has geometrically regular formal fibers.

Proof. Since flatness is a local property (and regularity of a morphism includes flatness), the morphism $\psi_x : R \longrightarrow (R^*/I)_x$ is flat. By Theorem 3.1 and [HRW2, (2.4.4)], A is Noetherian with (x)-adic completion $A^* = R^*/I$. Hence $A \longrightarrow A^*$ is flat.

Let $Q \in \operatorname{Spec}(A)$, let $k(Q)$ denote the fraction field of A/Q and let $Q_0 = Q \cap R$.

Case 1: $x \in Q$. Then $R/Q_0 = A/Q = A^*/QA^*$ and the ring $A^*_{QA^*}/QA^*_{QA^*} = A^* \otimes_A k(Q) = A_Q/QA_Q$ is trivially geometrically regular over $k(Q)$.

Case 2: $x \notin Q$. Let $k(Q) \subseteq L$ be a finite algebraic field extension. We show the ring $A^* \otimes_A L$ is regular. Let $W \in \operatorname{Spec}(A^* \otimes_A L)$ and let $W' = W \cap (A^* \otimes_A k(Q))$. The prime W' corresponds to a prime ideal $P \in \operatorname{Spec}(A^*)$ with $P \cap A = Q$. By assumption the morphism

$$R_{Q_0} \longrightarrow (R^*/I)_P = A^*_P$$

is regular. Since $x \notin Q$ it follows that $R_{Q_0} = U_{Q\cap U} = A_Q$ and that $k(Q_0) = k(Q)$. Thus the ring $A^*_P \otimes_{A_Q} L$ is regular. Therefore $(A^* \otimes_A L)_W$ which is a localization of this ring is regular.

For part (2), since R has geometrically regular formal fibers, so has R^* by [R3]. Hence the morphism $\theta : A^* = R^*/I \longrightarrow \widehat{A} = \widehat{(R^*/I)}$ is regular. By [M1, Thm. 32.1 (i)] and part (1) above, it follows that A has geometrically regular formal fibers, that is, the morphism $A \longrightarrow \widehat{A}$ is regular. □

5.2 Proposition. Assume that R, K, x, R^* are as in Theorem 3.1 and $n \in \mathbf{N}$. Let $I_1, \ldots, I_n$ of R^* be ideals of R^* such that each associated prime of R^*/I_i intersects R in (0), for $i = 1, \ldots, n$. Let $I := I_1 \cap \ldots \cap I_n$. Also assume

(1) R is semilocal with geometrically regular formal fibers and x is in the Jacobson radical of R.
(2) Each $(R^*/I_i)_x$ is a flat R-module and, for each $i \neq j$, the ideals $I_i(R^*)_x$ and $I_jR^*_x$ are comaximal in $(R^*)_x$.
(3) For $i = 1, \ldots, n$, $A_i := K \cap (R^*/I_i)$ has geometrically regular formal fibers.

Then $A := K \cap (R^*/I) = B$ has geometrically regular formal fibers.

Proof. Since R has geometrically regular formal fibers, by (5.1.2), it suffices to show for $W \in$ Spec(R^*/I) with $x \notin W$ that $R_{W_0} \longrightarrow (R^*/I)_W$ is regular, where $W_0 := W \cap R$. As in (3.1), we have $(R^*/I)_x = (R^*/I_1)_x \oplus \ldots \oplus (R^*/I_n)_x$. It follows that $(R^*/I)_W$ is a localization of some R^*/I_i. Suppose $(R^*/I)_W = (R^*/I_i)_{W_i}$, for some i with $1 \leq i \leq n$, and $W_i \in$ Spec(R^*/I_i). Then $R_{W_0} = (A_i)_{W_i \cap A_i}$ and $(A_i)_{W_i \cap A_i} \longrightarrow (R^*/I_i)_{W_i}$ is regular. Thus $R_{W_0} \longrightarrow (R^*/I)_W$ is regular. □

5.3 Corollary. The rings A of Examples 4.1, 4.2 and 4.3 have geometrically regular formal fibers, that is, the morphism $\phi : A \to \widehat{A}$ is regular.

Proof. By the definition of R and the observations given in (5.1), the hypotheses of (5.2) are satisfied. □

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