WELL-CENTERED OVERRINGS OF AN INTEGRAL DOMAIN

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Abstract. Let A be an integral domain with field of fractions K. We investigate the structure of the overrings $B \subseteq K$ of A that are well-centered on A in the sense that each principal ideal of B is generated by an element of A. We consider the relation of well-centeredness to the properties of flatness, localization and sublocalization for B over A. If $B = A[b]$ is a simple extension of A, we prove that B is a localization of A if and only if B is flat and well-centered over A. If the integral closure of A is a Krull domain, in particular, if A is Noetherian, we prove that every finitely generated flat well-centered overring of A is a localization of A. We present examples of (non-finitely generated) flat well-centered overrings of a Dedekind domain that are not localizations.

1. Introduction.

All rings we consider here are assumed to be commutative with unity. If R is a ring, we denote by $\mathcal{U}(R)$ the multiplicative group of units of R. If A is an integral domain with field of fractions K, we refer to a subring B of K with $A \subseteq B$ as an *overring* of A.

Fix an integral domain A with field of fractions K and an overring B of A.

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We say that B is *well-centered on* A if for each $b \in B$ there exists a unit $u \in B$ such that $ub = a \in A$. Thus, B is well-centered on A iff each element of B is an associate in B of an element of A iff each principal ideal of B is generated by an element of A.

The overring B of A is a *localization* of A if $B = S^{-1}A = A_S$, where S is a multiplicatively closed subset of nonzero elements of A. Thus B is a localization of A iff $B = A_{\mathcal{U}(B)\cap A}$. A localization of A is both flat over A and well-centered on A. Conversely, we prove in Theorem 4.3 that a simple flat well-centered overring of an integral domain A is a localization of A. If the integral closure of A is the intersection of a family of valuation domains of finite character, we prove in Theorem 4.15 that every finitely generated flat well-centered overring of A is a localization of A. Thus every finitely generated flat well-centered overring of an integral domain A which is either Krull or Noetherian is a localization of A (Corollary 4.16). On the other hand, we establish in Theorem 3.16 the existence of non-finitely generated flat well-centered overrings of a Dedekind domain that are not localizations.

The overring B of A is a *sublocalization* of A if B is an intersection of localizations of A. Thus B is a sublocalization of A if and only if there exists a family $\{S_\lambda\}_{\lambda\in\Lambda}$ of multiplicatively closed subsets of nonzero elements of A such that $B = \bigcap_{\lambda\in\Lambda} A_{S_\lambda}$. It is well-known [32], [12] that a sublocalization B of A is an intersection of localizations of A at prime ideals. Indeed $\bigcap_{\lambda\in\Lambda} A_{S_\lambda} = \bigcap\{A_P : P \in \operatorname{Spec} A \text{ and } P \cap S_\lambda = \emptyset \text{ for some } \lambda \in \Lambda\}$ (see Discussion (2.1)).

A sublocalization B of A need be neither well-centered on A nor flat over A. We discuss in §2 the sublocalization condition in relation to the properties of flatness and well-centeredness for an overring B of A. We give in Corollary 2.8 necessary and sufficient conditions for each sublocalization overring of a Noetherian domain A to be a localization of A.

We prove in Theorem 3.6 that every finitely generated well-centered overring of an integrally closed domain is flat and therefore, in particular, a sublocalization. In Example 3.24 we establish the existence of a non-archimedean well-centered overring of a factorial domain.

Our interest in the well-centered property of an overring of an integral domain A arose from conversations that the first author had with Jack

Ohm a number of years ago. The property arises naturally in relation to results established by Ohm in Theorem 5.1 and Example 5.3 of [26]. M. Griffin in [16, page 76] defines well-centeredness of a valuation v with ring B containing the domain A in a manner equivalent to the definition of B being well-centered on A given above. We thank Muhammad Zafrullah for pointing out to us this reference to Griffin. We also thank the referee for several helpful suggestions that have improved the paper.

2. When a sublocalization is flat or a localization.

Interesting work on the structure of flat overrings of an integral domain has been done by Richman in [32] and Akiba in [1]. Richman observes that an overring B of A is a flat A-module if and only if $B_M = A_{M\cap A}$ for every maximal (or equivalently prime) ideal M of B [32, Theorem 2]. In particular, if B is a flat overring of A then B is a sublocalization of A. The converse of this result, however, is not true in general. We indicate below methods for obtaining sublocalizations B of A that fail to be flat over A.

Discussion 2.1. (1) If B is a flat overring of A, then every ideal J of B is extended from A. Indeed, for each maximal ideal M of B we have $B_M = A_{M\cap A}$, hence $JB_M = JA_{M\cap A} = (J \cap A)A_{M\cap A} = ((J \cap A)B)B_M$. Thus $J = (J \cap A)B$. It is not true, however, that a flat overring B of an integral domain A need be well-centered on A (cf. Proposition 3.13 and Example 4.6). The distinction is that principal ideals of a flat overring B need not be the extension of principal ideals of A.

(2) If S is a multiplicatively closed subset of an integral domain A with $0 \notin S$, then
$$A_S = \bigcap\{A_P : P \in \operatorname{Spec} A \text{ and } P \cap S = \emptyset\}.$$
Therefore if $\{S_\lambda\}_{\lambda\in\Lambda}$ is a family of multiplicatively closed sets of nonzero elements of A and $B = \bigcap_{\lambda\in\Lambda} A_{S_\lambda}$, then
$$B = \bigcap\{A_P : P \in \operatorname{Spec} A \text{ and } P \cap S_\lambda = \emptyset \text{ for some } \lambda \in \Lambda\}.$$
Thus B is a sublocalization over A if and only if
$$B = \bigcap\{A_P : P \in \operatorname{Spec} A \text{ and } B \subseteq A_P\}.$$
In contrast with this characterization of a sublocalization, the condition for each $P \in \operatorname{Spec} A$ that either $PB = B$ or $B \subseteq A_P$ is, in general, stronger

than the sublocalization property. Indeed, by [32, Theorem 1], this latter property is equivalent to flatness of B over A. Thus every flat overring is a sublocalization. Hence every flat overring of an integrally closed domain is again integrally closed [32, Corollary, page 797]. Also from Richman's characterization that B is a flat overring of A iff for each $Q \in \operatorname{Spec} B$, we have $B_Q = A_{Q \cap A}$ [32, Theorem 2], it follows that if B is a quasilocal flat overring of A, then B is a localization of A.

(3) A useful observation is that if an overring $B \subseteq K$ of A has one of the properties of being flat, well-centered, a localization, or a sublocalization over A, then for each subring C of B with $A \subseteq C$, it follows that B as an extension of C is, respectively, flat, well-centered, a localization, or a sublocalization. This is easily seen in each case.

(4) If B is a flat overring of A and C is a subring of B with $A \subseteq C$ such that B is integral over C, then $C = B$. For in this case B is a flat integral overring of C, so by [32, Prop. 2], $C = B$.

(5) The localization, well-centered and flatness properties are transitive in the sense that if B is an overring of A and C is an overring of B, then one of these properties holding for B over A and for C over B implies the property also holds for C over A.

(6) The localization and flatness properties also behave well with respect to compositum in the following sense: for an arbitrary overring $C \subseteq K$ of A, if B is a localization or a flat overring of A, then $C[B]$ is, respectively, a localization or a flat overring of C. For if $B = S^{-1}A$, then $C[B] = S^{-1}C$, while for flatness if $Q \in \operatorname{Spec} C[B]$ and $P = Q \cap B$, then $B_P = A_{P \cap A}$ implies $C[B]_Q = C_{Q \cap C}$.

It would be interesting to know precise conditions for a Noetherian integral domain to admit a non-Noetherian sublocalization overring. In Corollary 2.8, we describe the class of Noetherian domains A for which each sublocalization over A is a localization of A. In particular, a Noetherian domain in this class does not admit a non-Noetherian sublocalization overring.

We begin with more general considerations. We use $\operatorname{Rad} I$ to denote the radical of an ideal I.

Discussion 2.2. If R is a ring, we define $P \in \operatorname{Spec} R$ to be an *associated prime* of an ideal I of R if there exists $a \in R$ such that P is a minimal prime over $(I :_R a) = \{r \in R : ra \in I\}$ [4, page 289], [21, page 92], [5]. An integral domain A has the representation

$$A = \bigcap\{A_P : P \text{ is an associated prime of a principal ideal of } A\}$$

[5, Prop. 4]. Moreover, if each principal ideal of A has only finitely many associates primes, then by [5, Prop. 4] for S a multiplicatively closed subset of A, we have

$$A_S = \bigcap\{A_P : P \text{ is an associated prime of a principal ideal and } P \cap S = \emptyset\}.$$

Lemma 2.3. *Let P be a prime ideal of an integral domain A. Then the following three properties are equivalent:*

(1) *For each family $\mathcal{Q}$ of prime ideals of A, if $P \subseteq \bigcup_{Q \in \mathcal{Q}} Q$, then $P \subseteq Q$ for some $Q \in \mathcal{Q}$.*
(2) *For each family $\mathcal{Q}$ of minimal primes over principal ideals of A, if $P \subseteq \bigcup_{Q \in \mathcal{Q}} Q$, then $P \subseteq Q$ for some $Q \in \mathcal{Q}$.*
(3) *P is the radical of a principal ideal.*

Proof. (1 $\Longrightarrow$ (2)

Obvious.

(2) $\Longrightarrow$ (1)

Let $P \subseteq \bigcup_{Q \in \mathcal{Q}} Q$, where $\mathcal{Q}$ is a set of prime ideals. Thus P is contained in the union of the set $\mathcal{M}$ of all minimal primes over principal ideals contained in one of the primes $Q \in \mathcal{Q}$. Hence P is contained in some prime in $\mathcal{M}$ which is contained in a prime $Q \in \mathcal{Q}$.

(1) $\Longrightarrow$ (3)

Let $\mathcal{Q}$ be the set of prime ideals of A that do not contain P. Thus $P \nsubseteq \bigcup_{Q \in \mathcal{Q}} Q$. Let c be an element in $P \setminus \bigcup_{Q \in \mathcal{Q}} Q$. Since P and Ac are contained in the same prime ideals, it follows that $P = \mathrm{Rad}(Ac)$.

(3) $\Longrightarrow$ (1)

Assume that $P = \mathrm{Rad}(Ac)$ for some element $c \in A$. Let $\mathcal{Q}$ be a family of prime ideals of A so that $P \subseteq \bigcup_{Q \in \mathcal{Q}} Q$. Thus $c \in Q$ for some prime ideal $Q \in \mathcal{Q}$, which implies that $P \subseteq Q$.

□

We generalize below the theorem for Dedekind domains stated on page 257 of [11] (see [15]).

Theorem 2.4. *Let A be an integral domain with field of fractions K, and let $\mathcal{P}$ be a set of prime ideals in A. Consider the sublocalization $B = \bigcap_{P\in\mathcal{P}} A_P$. The following are equivalent:*

(1) *B is a localization of A.*
(2) *If $x \in K \setminus A$, and $(A :_A x) \subseteq \bigcup_{P\in\mathcal{P}} P$, then $(A :_A x) \subseteq P$ for some $P \in \mathcal{P}$.*
Moreover, if each principal ideal of A has only finitely many associated primes, then the following condition is equivalent to the two conditions above:
(3) *If Q is an associated prime of a principal ideal such that $Q \subseteq \bigcup_{P\in\mathcal{P}} P$, then $Q \subseteq P$ for some $P \in \mathcal{P}$.*

Proof. $(1) \Longrightarrow (2):$ Assume that $B = A_S$ for some multiplicative subset S of A. Let $x \in K$ such that $(A :_A x) \subseteq \bigcup_{P\in\mathcal{P}} P$, thus $(A :_A x) \cap S = \emptyset$, hence $x \notin A_S = B$. Thus there exists a prime $P \in \mathcal{P}$ such that $x \notin A_P$. It follows that $(A :_A x) \subseteq P$.

$(2) \Longrightarrow (1):$ Let $S = A \setminus (\bigcup_{P\in\mathcal{P}} P)$. We prove that $B = A_S$. If $s \in S$, then s is a unit in A_P for all $P \in \mathcal{P}$, hence s is a unit in B. It follows that $A_S \subseteq B$. On the other hand let $b \in B \setminus A$, thus $(A :_A b) \nsubseteq P$ for all $P \in \mathcal{P}$. By assumption $(A :_A b) \nsubseteq \bigcup_{P\in\mathcal{P}} P)$, that is, $(A :_A b) \cap S \neq \emptyset$. It follows that $b \in A_S$.

Assume now that each principal ideal of A has only finitely many associated primes.

$(2) \Longrightarrow (3):$ Since principal ideals in A have only finitely many associated primes, an associated prime of a principal ideal is of the form $\operatorname{Rad}(A :_A x)$ for some $x \in K$ [17, Prop. 3.5].

$(3) \Longrightarrow (2):$ Let $x \in K$ such that $(A :_A x) \subseteq \bigcup_{P\in\mathcal{P}} P$. By assumption, there are only finitely many prime ideals $Q_1, \dots, Q_n$ minimal over $(A :_A x)$. If none of the primes Q_i is contained in $\bigcup_{P\in\mathcal{P}} P$, then choose an element $t_i \in Q_i \setminus \bigcup_{P\in\mathcal{P}} P$ for each i. Thus for some positive integer m, we have $(\prod_{i=1}^n t_i)^m \notin \bigcup_{P\in\mathcal{P}} P$, a contradiction. Hence at least one of the ideals Q_i is contained in $\bigcup_{P\in\mathcal{P}} P$, which implies that $(A :_A x)$ is contained in $\bigcup_{P\in\mathcal{P}} P$. $\square$

Theorem 2.5. *Let A be an integral domain with field of fractions K. Each sublocalization over A is a localization of A if and only if for each $x \in K \setminus A$, the ideal* $\mathrm{Rad}(A :_A x)$ *is the radical of a principal ideal.*

Moreover, if each principal ideal of A has only finitely many associated primes, then each sublocalization of A is a localization iff each associated prime of a principal ideal is the radical of a principal ideal.

Proof. If each ideal of the form $\mathrm{Rad}(A :_A x)$ is the radical of a principal ideal, then each sublocalization of A is a localization by Theorem 2.4.

Conversely, assume that each sublocalization of A is a localization of A. Let $x \in K \setminus A$. By Theorem 2.4, $(A :_A x)$ is not contained in the union of the prime ideals not containing $(A :_A x)$. Let c be an element in $(A :_A x)$ that does not belong to this union. Thus $(A :_A x)$ and Ac are contained in the same prime ideals, which implies that $\mathrm{Rad}(A :_A x) = \mathrm{Rad}(Ac)$.

Assume now that each principal ideal of A has only finitely many associated primes, and that each sublocalization of A is a localization. Let P be a prime associated with a principal ideal of A. By Theorem 2.4, P is not contained in a union of primes not containing P. Hence, by Lemma 2.3, P is the radical of a principal ideal.

Conversely, if each principal ideal of A has only finitely many associated primes and if each associated prime of a principal ideal is the radical of a principal ideal, then each sublocalization of A is a localization by Theorem 2.4. □

We apply the above results to various classes of integral domains. In Corollary 2.7 we describe the class of Mori domains and the class of semi-Krull domains for which each sublocalization is a localization. In Corollary 2.8 we characterize the Noetherian domains having this property.

We recall that A is a *Mori domain* if A satisfies the ascending chain condition on integral divisorial ideals [2]. In particular, a Mori domain satisfies the ascending chain condition on principal ideals (a.c.c.p.). Examples of Mori domains include factorial and Krull domains as well of course as Noetherian domains. An integral domain A is *semi-Krull* [23], if $A = \bigcap_P A_P$, where P ranges over the set of height-one primes of A, this intersection has finite character, and for each height-one prime P, every nonzero ideal of A_P contains a power of PA_P.

A nonzero prime ideal of a Mori domain or a semi-Krull domain is an associated prime of a principal ideal iff it is a prime divisorial ideal (see [2, Theorem 3.2] and [3, Theorem 1.7]). Thus by Discussion 2.2, if A_S is a localization of a Mori domain A or a semi-Krull domain A, then $A_S = \bigcap_{P\in\mathcal{P}} A_P$, where $\mathcal{P}$ is the set of prime divisorial ideals $P \in \operatorname{Spec} A$ such that $P \cap S = \emptyset$. Therefore if B is a sublocalization over A, then B has the form $B = \bigcap_{P\in\mathcal{P}} A_P$, where $\mathcal{P}$ is a set of prime divisorial ideals in A.

Theorem 2.4 implies:

Corollary 2.6. *Let A be a Mori domain or a semi-Krull domain and let $\mathcal{P}$ be a set of prime ideals in A. Consider the sublocalization $B = \bigcap_{P\in\mathcal{P}} A_P$. The following are equivalent:*

(1) *B is a localization of A.*
(2) *If Q is a prime divisorial ideal of A and $Q \subseteq \bigcup_{P\in\mathcal{P}} P$, then $Q \subseteq P$ for some $P \in \mathcal{P}$.*

Theorem 2.5 implies:

Corollary 2.7. *Let A be a Mori domain or a semi-Krull domain. Each sublocalization over A is a localization of A if and only if each prime divisorial ideal of A is the radical of a principal ideal.*

Corollary 2.8. *Let A be a Noetherian integral domain. Each sublocalization over A is a localization of A if and only if each associated prime of a principal ideal of A is the radical of a principal ideal. In particular, if A has these equivalent properties, then nonzero principal ideals of A have no embedded associated primes.*

A Krull domain has torsion divisor class group iff each prime divisorial ideal (that is, prime ideal of height one) is the radical of a principal ideal. Hence Corollary 2.7 implies:

Corollary 2.9. *A Krull domain A has torsion divisor class group if and if every sublocalization over A is a localization of A.*

Corollary 2.10. *Let A be a one-dimensional integral domain. If each maximal ideal of A is the radical of a principal ideal, then every sublocalization over A is a localization of A. The converse holds if A has Noetherian prime spectrum.*

Proof. A commutative ring has Noetherian spectrum iff each prime ideal is the radical of a finitely generated ideal [27]. Thus a one-dimensional integral domain has Noetherian spectrum iff each nonzero element is contained in only finitely many maximal ideals iff principal ideals have only finitely many associated primes. Thus Corollary 2.10 follows from Theorem 2.5. □

Question 2.11. *What (Noetherian) integral domains A have the property that every sublocalization extension is flat?*

For a one-dimensional integral domain with Noetherian spectrum we give in Theorem 2.12 a complete answer to Question 2.11.

Theorem 2.12. *Suppose A is a one-dimensional integral domain with Noetherian spectrum. Then every sublocalization over A is flat over A.*

Proof. Let B be a sublocalization over A. We may assume that $B \subsetneq K$, where K is the field of fractions of A. By Discussion 2.1 (2), there exists a family $\{P_\alpha\}$ of prime ideals of A such that $B = \bigcap_\alpha A_{P_\alpha}$. Since $\dim A = 1$, we may assume that each P_α is a maximal ideal of A. Let $Q_\alpha = P_\alpha A_{P_\alpha} \cap B$. We have $B_{Q_\alpha} = A_{P_\alpha}$ and $B = \bigcap_\alpha B_{Q_\alpha}$. Since A has Noetherian spectrum, the family $\{B_{Q_\alpha}\}$ has finite character in the sense that a nonzero element of B is a unit in all but finitely many of the B_{Q_α}. To prove that B is flat over A, we show for each maximal ideal Q of B that $B_Q = A_{Q \cap A}$. Let $P = Q \cap A$ and let $S = A \setminus P$. By [18, Lemma 1.1] we have $S^{-1}B = \bigcap_\alpha (S^{-1}B_{Q_\alpha})$. Since B_{Q_α} is a one-dimensional quasilocal domain, $S^{-1}B_{Q_\alpha}$ is either B_{Q_α} if $S \cap Q_\alpha \neq \emptyset$ or K otherwise. Since $A_{P_\alpha} = B_{Q_\alpha}$, we see that Q_α is the unique prime of B lying over P_α. Thus if $Q \neq Q_\alpha$, then $S \cap Q_\alpha$ is nonempty and $S^{-1}B_{Q_\alpha} = K$. If this were true for each α, then $S^{-1}B = \bigcap_\alpha S^{-1}B_{Q_\alpha} = K$, but clearly $S^{-1}B \subseteq B_Q$, a contradiction. Hence $Q = Q_\alpha$ for some α and therefore $A_P = B_Q$. □

3. Properties of flat and well-centered overrings.

Richman observes [32, Theorem 3] that a flat overring of a Noetherian domain is Noetherian. There exist Noetherian integral domains with non-Noetherian sublocalizations that are ideal transforms ([7] and [8, Theorem 3.2]). If B is a non-Noetherian ideal transform of a Noetherian domain A, then B is not flat over A by the result of Richman mentioned above.

Proposition 3.1 shows that B with these properties also fails to be well-centered on A.

Proposition 3.1. *A well-centered extension of a Noetherian domain is Noetherian.*

Proof. If B is well-centered on A, then every ideal of B is the extension of an ideal of A. Thus if A is Noetherian, then every ideal of B is finitely generated and B is also Noetherian. □

We observe in Theorem 3.6 that a finitely generated well-centered overring of an integrally closed domain is a flat extension. In the proof of this result we use Proposition 3.2 which holds for arbitrary well-centered extension rings.

Proposition 3.2. *Let S be a well-centered extension ring of a ring R. If M is a maximal ideal of R such that $MS \neq S$, then MS is a maximal ideal of S.*

Proof. We have a natural embedding $R/M \hookrightarrow S/MS$. Moreover the fact that S is well-centered over R implies that S/MS is well-centered over R/M. Since a well-centered extension of a field is a field, S/MS is a field and MS is a maximal ideal of S. □

For an extension ring S of a ring R, we consider the following condition that is in general weaker than the well-centered property.

Definition 3.3. *An extension ring S of a ring R is said to be almost well-centered on R if for each $s \in S$ there exists a positive integer n depending on s and an element $u \in \mathcal{U}(S)$ such that $us^n \in R$.*

The following remark concerning almost well-centered extensions is clear.

Remark 3.4. *If S is an almost well-centered extension ring of a ring R, then for each ideal J of S we have* $\operatorname{Rad} J = \operatorname{Rad}(J \cap R)S$.

In view of Remark 3.4, we have the following analogue of Proposition 3.2.

Proposition 3.5. *Let S be an almost well-centered extension ring of a ring R. If M is a maximal ideal of R such that $MS \neq S$, then* $\operatorname{Rad} MS$ *is a maximal ideal of S.*

Theorem 3.6. *If B is a finitely generated almost well-centered overring of A and if A is integrally closed in B, then B is flat over A. In particular, every finitely generated almost well-centered overring of an integrally closed domain A is flat over A.*

Proof. Let Q be a maximal ideal of B and let $P = Q \cap A$. By Proposition 3.5, $\mathrm{Rad}(PB) = Q$. The Peskine-Evans version of Zariski's Main Theorem [30], [9] implies there exists $s \in A \setminus P$ such that $A_s = B_s$. In particular, $A_P = B_Q$. Thus B is flat over A. □

Proposition 3.7. *If $B = A[u]$ is a simple overring of A, where u is a unit of B, and if A is integrally closed in B, then B is a localization of A.*

Proof. Since $u^{-1} \in B$ it follows that u^{-1} is integral over A [19, Theorem 15]. Thus $u^{-1} \in A$ and B is a localization of A. □

Corollary 3.8. *A simple almost well-centered overring of an integrally closed domain is a localization.*

Proof. Let $B = A[b]$ be a simple almost well-centered overring of an integrally closed domain A. By Theorem 3.6, B is flat over A. Since B is almost well-centered over A, there exist a positive integer n and a unit $u \in \mathcal{U}(B)$ such that $ub^n = a \in A$. Thus B is a flat integral overring of $A[b^n] = A[u]$. By Discussion 2.1 (4), $B = A[u]$ and B is a localization of A. □

Theorem 3.6 and Corollary 3.8 may fail if A is not integrally closed. We use Proposition 3.9 to show in Example 3.10 the existence of Noetherian integral domains that admit simple proper well-centered integral overrings. Corollary 2.8 shows that in an integral domains having this property there are principal ideals with embedded associated prime ideals.

Proposition 3.9. *Let B be an integral domain of the form $B = K + M$, where K is a field and M is a nonzero maximal ideal of B. If A is a subring of B such that $M \subset A$, then B is well-centered on A.*

Proof. Let $b \in B$. Then $b = k + m$, where $k \in K$ and $m \in M$. If $k = 0$, then $b \in A$. If $k \neq 0$, then k is a unit of B and $a := b/k = 1 + (m/k) \in A$. Hence B is well-centered over A. □

Example 3.10. *A simple well-centered integral (thus not flat) proper overring B of a Noetherian integral domain A such that B is a sublocalization of A. Moreover, each height-one prime of A is the radical of a principal ideal.*

Let $E = F(c)$ be a simple proper finite algebraic field extension, let B be the localized polynomial ring $E[X,Y]_{(X,Y)}$, let $M = (X,Y)B$, and let $A = F + M$. Then A is Noetherian and $B = A[c]$ is a simple, proper integral extension of A. Hence B is not flat as an A-module [32, Prop. 2]. Proposition 3.9 implies that B is well-centered on A.

Since B is factorial, B is the intersection of the rings B_Q as Q ranges over the nonzero principal prime ideals of B. For such Q we have $Q \subsetneq M \subset A$, thus $B \subseteq A_Q$, so $B_Q = A_Q$. It follows that B is a sublocalization over A. Since B is a unique factorization domain, each height-one prime of B is principal. Since $M \subset A$, each height-one prime of A is the radical of a principal ideal. $\square$

The following example where B is not well-centered on A illustrates restrictions on generalizing Proposition 3.9. The original Example 3.11, as noticed by Jung-Chen Liu and her student Jing-Ping Tsai, is wrong. We provide here a correct example for the same statement (actually, a little improvement since in the new version M is a maximal ideal of both A and B).

Example 3.11. *Integral domains of the form $A = A_0 + M \subseteq B = B_0 + M$, where A_0, B_0 are subrings of A and B, respectively, and M is a maximal ideal of B such that B is not almost well-centered on A.*

Let X be an indeterminate over the field $\mathbb{Q}$ of rational numbers and let $B = B_0 = \mathbb{Q}[X]$. Let $M = (X^2-2)B, A_0 = \mathbb{Q}$ and $A = A_0 + M$. Clearly, M is a maximal ideal of both A and $B = B_0 + M$. No power of $X+1$ is associate in B with an element of A; otherwise, since $\mathcal{U}(B) = \mathbb{Q} \setminus \{0\}$, we obtain that some power of $X+1$ is in A: $(X+1)^n \in \mathbb{Q} + (X^2-2)\mathbb{Q}[X]$. Put $X \to \sqrt{2}$ in order to obtain the contradiction $(\sqrt{2}+1)^n \in \mathbb{Q}$ (for the binomial expansion implies $(\sqrt{2}+1)^n \notin \mathbb{Q}$). Thus B is not almost well-centered over A. $\square$

Discussion 3.12. Let B be an overring of an integral domain A and let $S = \mathcal{U}(B) \cap A$. Then $B = B_S$ is a well-centered overring of A_S if and only if B is a well-centered overring of A. Moreover, $\mathcal{U}(A_S) = \mathcal{U}(B_S) \cap A_S$, and B

is a localization of A if and only if $A_S = B$. Thus in considering the question of whether an overring B of an integral domain A is a localization of A, by passing from the ring A to its localization $A_{\mathcal{U}(B)\cap A}$, we may assume that $\mathcal{U}(B) \cap A = \mathcal{U}(A)$. The localization question is then reduced to the question of whether $A = B$. In general, if B is a well-centered overring of A which properly contains A, then $\mathcal{U}(A) \subsetneq \mathcal{U}(B)$. For if $b \in B \setminus A$ and $u \in \mathcal{U}(B)$ is such that $ub \in A$, then $u^{-1} \notin A$ so $u \in \mathcal{U}(B) \setminus \mathcal{U}(A)$.

If A is a Dedekind domain, then every overring B of A is a flat A-module, thus a sublocalization over A. Moreover, we have:

Proposition 3.13. *Let A be a Dedekind domain. The following conditions are equivalent:*

(1) *A has torsion divisor class group.*
(2) *Every overring of A is a localization of A.*
(3) *Every overring of A is well-centered on A.*
(4) *A has no proper simple overring with the same set of units.*

Proof. (1) $\iff$ (2) : By Corollary 2.9, (2) holds if and only if each maximal ideal of A is the radical of a principal ideal, and this is equivalent to (1).

It is clear that (2) $\implies$ (3) and (3) $\implies$ (4). Thus it remains to show:

(4) $\implies$ (2) : Assume that (2) does not hold. Then A has a maximal ideal P that is not the radical of a principal ideal. We claim that $B = A[P^{-1}]$ is a simple flat overring of A with $\mathcal{U}(B) = \mathcal{U}(A)$. Indeed, if $b \in P^{-1} \setminus A$, we have $B = A[b]$ since both of these rings are equal to $\bigcap\{A_Q : Q \in \operatorname{Spec} A \text{ and } Q \neq P\}$. Suppose there exists an element $u \in \mathcal{U}(B) \setminus \mathcal{U}(A)$. Then u is not a unit in A_P, but either u or u^{-1} is in A_P. We may assume that $u \in A_P$, thus $u \in PA_P$. Then $u \in A$ and $\operatorname{Rad} uA = P$, a contradiction. $\square$

We show in Theorem 4.5 that if B is a finitely generated overring of a Dedekind domain A, then B is a localization of A iff B is well-centered on A iff B is almost well-centered on A. However, for overrings of a Dedekind domain having nontorsion class group, we present in Theorem 3.16 examples of well-centered overrings that are not localizations and examples of almost well-centered overrings that are not well-centered.

If A is a Dedekind domain, we denote its class group by $\mathcal{C}(A)$; if I is a nonzero fractional ideal of A, we denote the ideal class of I by $\mathcal{C}_A(I)$, and if $\mathcal{P}$ is a subset of $\operatorname{Max} A$, we denote the set $\{\mathcal{C}_A(P) \mid P \in \mathcal{P}\}$ by $\mathcal{C}_A(\mathcal{P})$. The complement of a subset $\mathcal{P}$ of $\operatorname{Max} A$ is denoted by $\mathcal{P}^c$. We denote the submonoid generated by a subset S of a monoid by $\mathcal{M}(S)$, and the subgroup generated by a subset S of a group by $\mathcal{G}(S)$. Thus, if S is a set of nonzero fractional ideals of a Dedekind domain A viewed as a subset of the ideal monoid of A, we have $\mathcal{M}(C_A(S)) = \mathcal{C}_A(\mathcal{M}(S))$.

We recall that if A is a Dedekind domain, and B is an overring of A, then there exists a unique set of maximal ideals $\mathcal{P}$ in A such that $B = \bigcap\{A_P : P \in \mathcal{P}\}$. The overring B of A can also be described as the compositum of the overrings $A[Q^{-1}]$ such that $Q \in \operatorname{Max} A \setminus \mathcal{P}$. Thus for each $Q \in \operatorname{Max} A$ we have $QB = B$ if and only if $Q \in \mathcal{P}^c$.

Proposition 3.14. *Let A be a Dedekind domain with field of fractions K and let $B \subsetneq K$ be an overring of A, thus*

$$B = \bigcap\{A_P : P \in \mathcal{P}\}.$$

for a unique subset $\mathcal{P}$ of $\operatorname{Max} A$. *Let J be a nonzero ideal of B. Then $J = IB$ where I is an ideal of A belonging to $\mathcal{M}(\mathcal{P})$). Moreover, we have*

(1) *J is a principal ideal of $B \iff \mathcal{C}_A(I) \in \mathcal{G}(\mathcal{C}_A(\mathcal{P}^c))$.*
(2) *J is an extension of a principal ideal of $A \iff \mathcal{C}_A(I) \in -\mathcal{M}(C_A(\mathcal{P}^c))$.*

Proof. Part (1) follows from [6, Corollary 3]. For part (2), assume first that there exists a principal ideal I_0 of A such that $IB = I_0B$. Since $I \in \mathcal{M}(\mathcal{P})$, it follows that $I_0 = II_1$, where $I_1 \in \mathcal{M}(\mathcal{P}^c)$. Thus $\mathcal{C}_A(I) = -\mathcal{C}_A(I_1) \in -\mathcal{M}(\mathcal{C}_A(\mathcal{P}^c))$.

Conversely, let $\mathcal{C}_A(I) \in -\mathcal{M}(\mathcal{C}_A(\mathcal{P}^c))$. There exists an ideal $I_1 \in \mathcal{M}(P^c)$ such that II_1 is a principal ideal of A. Also $J = (II_1)B$. □

Proposition 3.14 implies:

Corollary 3.15. *Let A be a Dedekind domain with field of fractions K and let $B \subsetneq K$ be an overring of A, thus*

$$B = \bigcap\{A_P : P \in \mathcal{P}\}.$$

for a unique subset $\mathcal{P}$ of $\operatorname{Max} A$. *Then*

(1) *B is a well-centered extension of $A \iff$*

$$(\mathcal{C}_A\,\mathcal{M}(\mathcal{P})) \cap \mathcal{G}(\mathcal{C}_A(\mathcal{P}^c)) \subseteq -\mathcal{C}_A(\mathcal{M}(\mathcal{P}^c)).$$

(2) *B is an almost well-centered extension of A* $\iff$ *each element of*

$M(C_A(\mathcal{P})) \cap \mathcal{G}(\mathcal{C}_A(\mathcal{P}^c))$ *has a positive integer multiple in* $-\mathcal{M}(\mathcal{C}_A(\mathcal{P}^c))$.

Theorem 3.16.

(1) *There exists a Dedekind domain A having a well-centered overring that is not a localization.*

(2) *There exists a Dedekind domain A having an almost well-centered overring that is not well-centered.*

Moreover, in each case the domain A can be chosen so that it has exactly two almost well-centered overrings that are not localizations of A, these two overrings being also the unique almost well-centered overrings D of A such that $\mathcal{U}(D) \cap A = \mathcal{U}(A)$.

Proof. We will use the well known result of Claborn [6] that every Abelian group is the ideal class group of a Dedekind domain, along with the fact that for a countably generated Abelian group G and a nonempty subset S of G, there exists a Dedekind domain A with class group G such that $S = \{\mathcal{C}(P) : P \in \operatorname{Max} A\}$ if and only if S generates G as a monoid [14, Theorem 5].

Let A be a Dedekind domain having ideal class group the infinite cyclic group $\mathbb{Z}$. Define

$$B = \bigcap\{A_Q : Q \in \operatorname{Max} A \text{ and } \mathcal{C}(Q) \leq 0\}.$$

Since the set $\{\mathcal{C}(P) : P \in \operatorname{Max} A\}$ generates $\mathbb{Z}$ as a monoid, there exists $P \in \operatorname{Max} A$ with $\mathcal{C}(P) > 0$. Thus B is a proper overring of A. For a nonzero nonunit $a \in A$, if $aA = P_1^{e_1} \cdots P_n^{e_n}$ is the factorization of the principal ideal aA as a product of maximal ideals, then $0 = e_1\,\mathcal{C}(P_1) + \cdots + e_n\,\mathcal{C}(P_n)$. Therefore $\mathcal{C}(P_i) \leq 0$ for at least one of the P_i. It follows that $A \setminus \mathcal{U}(A) = \bigcup\{Q : Q \in \operatorname{Max} A \text{ and } \mathcal{C}(Q) \leq 0\}$. Since the maximal ideals of B lie over the ideals Q of A with $\mathcal{C}(Q) \leq 0$, we see that $B \setminus \mathcal{U}(B) = \bigcup\{QB : Q \in \operatorname{Max} A \text{ and } \mathcal{C}(Q) \leq 0\}$, hence $\mathcal{U}(B) \cap A = \mathcal{U}(A)$.

By Corollary 3.15, B is almost well-centered on A: indeed, since there exists $P \in \mathcal{P}^c$ with $C_A(P) > 0$, each element of $\mathcal{M}(\mathcal{C}_A(\mathcal{P}))$ has a power in $-\mathcal{M}(\mathcal{C}_A(\mathcal{P}^c))$. Moreover, if there exists $P \in \operatorname{Max} A$ with $\mathcal{C}(P) = 1$, by Corollary 3.15, B is well-centered on A.

To obtain an example where B is almost well-centered but not well-centered on A we argue as follows. By [14, Theorem 8], there exists a Dedekind domain A with class group $\mathbb{Z}$ such that $\{\mathcal{C}(P) : P \in \operatorname{Max} A\} = \{-1, 2, 3\}$. The overring

$$B = \bigcap \{A_Q : Q \in \operatorname{Max} A \text{ and } \mathcal{C}(Q) \leq 0\}$$

is a principal ideal domain, since the primes $P \in \operatorname{Max} A$ such that $PB = B$ generate $\mathbb{Z}$ as a group. Hence for $Q \in \operatorname{Max} A$ with $\mathcal{C}(Q) = -1$, we have $QB = bB$ is a principal ideal that is not generated by an element of A.

Next we show that for each Dedekind domain A with ideal class group $\mathbb{Z}$ as constructed above, there are precisely two proper almost well-centered overrings D of A such that $\mathcal{U}(D) \cap A = \mathcal{U}(A)$. These are the overring B as defined above and $C = \bigcap\{A_P : \mathcal{C}(P) \geq 0\}$. A proof that $A \subsetneq C$, C is almost well-centered over A, and that $\mathcal{U}(C) \cap A = \mathcal{U}(A)$ is similar to that given above to show B has these properties. Moreover, if D is an overring of A such that $\mathcal{U}(D) \cap A = \mathcal{U}(A)$, then either $D \subseteq B$ or $D \subseteq C$. For otherwise, either there exists a $Q \in \operatorname{Max} A$ with $\mathcal{C}(Q) = 0$ such that $QD = D$ or there exist $P, Q \in \operatorname{Max} A$ with $\mathcal{C}(P) = r > 0, \mathcal{C}(Q) = -s < 0$ and $PD = QD = D$. In the first case, $Q = aA$ is principal and $a \in \mathcal{U}(D) \cap A \setminus \mathcal{U}(A)$. In the second case $P^s Q^r = aA$ is principal and again $a \in \mathcal{U}(D) \cap A \setminus \mathcal{U}(A)$.

It remains to show that if $A \subsetneq D \subsetneq B$ or $A \subsetneq D \subsetneq C$, then D is not almost well-centered over A. If $A \subsetneq D \subsetneq B$, then the ideal class group of D is a proper homomorphic image of $\mathbb{Z}$ and hence a finite cyclic group, thus each nonzero ideal of D has a power that is a principal ideal. Since $D \subsetneq B$, there exists $P \in \operatorname{Max} A$ with $\mathcal{C}(P) < 0$ such that $PD \in \operatorname{Max} D$. By Proposition 3.14 (2), no power of PD is an extension of a principal ideal of A. Therefore D is not well-centered on A. The proof that an overring D of A with $A \subsetneq D \subsetneq C$ is not well-centered on A is the same. Thus B and C are the unique proper almost well-centered overrings of A such that every nonunit of A remains a nonunit in the overring.

If A has no principal maximal ideals, then B and C as defined in the previous paragraph are the unique almost well-centered overrings of A that are not localizations of A. For if D is a proper well-centered overring of A distinct from B and C, then there exists a localization E of A such that $A \subsetneq E \subseteq D$. Since A has no principal maximal ideals, the ideal class group of E is a proper homomorphic image of $\mathbb{Z}$. Therefore E has finite class group and every overring of E is a localization of E. Thus D is a localization of E and E is a localization of A, so D is a localization of A. $\square$

Proposition 3.17. *Let A be a Dedekind domain such that each ideal class in the class group $C(A)$ of A contains a maximal ideal. If $C(A)$ is torsionfree, then each overring of A is an intersection of two principal ideal domains that are well-centered overrings of A.*

Proof. Let $B = \bigcap_{P \in \mathcal{P}} P$ be an overring of A, where $\mathcal{P}$ is a set of maximal ideals of A. Since $C(A)$ is torsionfree it can be linearly ordered. With respect to a fixed linear order $\geq$ on $C(A)$, define $B^+ = \bigcap_{\{P \in \mathcal{P} \text{ and } C_A(P) \geq 0\}} A_P$ and $B^- = \bigcap_{\{P \in \mathcal{P} \text{ and } C_A(P) \leq 0\}} A_P$. Then $B = B^+ \cap B^-$, the empty intersection being defined as the field of fractions of A. Since each ideal class of A contains a prime ideal, Proposition 3.14 implies that B^+ and B^- are well-centered over A and that each prime ideal of B^+ and B^- is the extension of a principal ideal of A. Thus B^+ and B^- are principal ideal domains that are well-centered overrings of A with $B = B^+ \cap B^-$. $\square$

In §4 we use the following well-known general result characterizing flat overrings, see for example [1, Theorem 1]. The implication (1) $\implies$ (2) in Proposition 3.18 holds without assuming that B is an overring of A, cf. [4, Exer. 22, page 47].

Proposition 3.18. *Assume that B is an overring of an integral domain A and that $S \subseteq B$ is such that $B = A[S]$. Then the following conditions are equivalent:*

(1) *B is a flat extension of A.*
(2) *For any element $s \in S$ we have $(A :_A s)B = B$.*

If B is well-centered over A, then $B = A[\mathcal{U}(B)]$. Thus the following corollary of Proposition 3.18 is immediate.

Corollary 3.19. *Assume that B is a well-centered overring of the integral domain A. Then the following conditions are equivalent:*

(1) *B is a flat extension of A.*
(2) *For each unit $u \in B$ we have $(A :_A u)B = B$.*

We recall that an integral domain B is said to be *Archimedean* if for each nonunit $b \in B$ we have $\bigcap_{n=1}^{\infty} b^n B = (0)$.

Remark 3.20. *If B is a localization of an Archimedean domain A such that the conductor of B in A is nonzero, then $B = A$.*

Indeed, suppose $B = A_S$ and let $0 \neq a \in (A :_A B)$. Then for each $s \in S$ we have $a/s^n \in A$ for all $n \geq 1$. Since A is Archimedean, it follows that s is a unit in A. Hence $B = A$.

Proposition 3.21. *Suppose B is an overring of a Mori integral domain A. If the conductor of B in A is nonzero and B is flat over A, then $A = B$.*

Proof. Since A is Mori and $(A :_A B) \neq 0$, there exists a finite subset F of B such that $(A :_A B) = (A :_A F)$. Since B is flat over A, Proposition 3.18 implies that $(A :_A F)B = B$, hence $(A :_A B) = (A :_A B)B = B$. Therefore $A = B$. □

Example 3.22. *If A is not Mori, the conclusion of Proposition 3.21 need not hold.*

Indeed, let k be a field and let $R = k[X, Y]$ be a polynomial ring over k. Then $B := R[1/Y]$ is a localization of $A := R + XB$. The conductor of B in A contains XB and hence is nonzero. Moreover, $A \subsetneq B$ since $Y^{-1} \in B \setminus A$. □

The following structural result is proved by Querré in [31].

Proposition 3.23. [31] *If A is a Mori domain and B is a sublocalization over A, then B is also Mori. In particular, a flat overring of a Mori domain is again a Mori domain.*

We observe in Proposition 3.1 that a well-centered overring of a Noetherian domain is Noetherian. Example 3.24 shows that in general the Mori property is not preserved by well-centered overrings. Indeed, Example 3.24 establishes the existence of a polynomial ring A over a field and a well-centered overring B of A that is not Archimedean. In particular, B fails to satisfy a.c.c.p. and therefore is not Mori.

Example 3.24. *A well-centered overring of a factorial domain (even of a polynomial ring over a field) is not necessarily Archimedean.*

Let k be a field and let a, c be two independent indeterminates over k. Define

$$T_0 = k[a, c, \{\frac{a}{c^n} : n \geq 1\}].$$

Proceeding inductively, define integral domains T_m for $m \geq 1$ as follows: let $V_m = \{v_{m,t} : t$ is a nonzero nonunit in $T_{m-1}\}$ be a set of independent indeterminates over T_{m-1} and define $T_m := T_{m-1}[\{v_{m,t}, \frac{1}{v_{m,t}} : v_{m,t} \in V_m\}]$. Thus T_m is a domain extension of T_{m-1} obtained by adjoining the indeterminates in V_m along with their inverses. Let $V = \bigcup_{m=1}^{\infty} V_m$ and define W to be the union of the set $\{a, c\}$ with the set $\{tv_{m,t} : v_{m,t} \in V\}]$. The elements of W are algebraically independent over k. Thus $A := k[W]$ is a polynomial ring over the field k. Define $B := \bigcup_{m=1}^{\infty} T_m$. Since T_0 is an overring of $k[a, c]$, we see that B is an overring of A. Since every element of T_{m-1} is an associate in T_m to an element of A, it follows that B is well-centered on A. The domain B is not Archimedean since $a/c^n \in B$ for all positive integers n although $a, c \in B$ and c is not a unit in B; indeed, $c \notin \mathcal{U}(T_0)$ and T_0 is a retract of B under the retraction over T_0 that sends each $v \in V$ to 1. □

4. Finitely generated well-centered extensions.

The structure of a simple flat extension $S = R[s] = R[X]/I$ of a commutative ring R is considered in [32], [33], [34], [28], [29]. Richman in [32, Prop. 3] shows that if A is an integrally closed domain and $B = A[a/b]$ is a simple flat overring of A, then $(a, b)A$ is an invertible ideal of A. We observe in Theorems 4.1 that a simple flat overring B generated by a unit of B is a localization of A. It follows (Corollary 4.2 and Corollary 4.3) that well-centered simple flat overrings are localizations.

Theorem 4.1. *Let A be an integral domain and let $B = A[u]$ be a simple flat overring of A, where u is a unit of B. There exists a positive integer m such that $u^{-r} \in A$ for all integers $r \geq m$. Thus B is a localization of A.*

Proof. Since $u^{-1} \in A[u]$, the element u^{-1} is integral over A, hence $A[u^{-1}]$ is a finitely generated A-module. Since B is a flat extension of A, we have $(A :_A A[u^{-1}])B = B$. Hence there exist $c_0, \dots, c_m \in (A :_A A[u^{-1}])$ with $1 = c_0 + c_1 u + \cdots + c_m u^m$. Thus for each integer $r \geq m$ we have

$$u^{-r} = c_0 u^{-r} + c_1 u^{-r+1} + \cdots + c_m u^{-r+m} \in A.$$

In particular, $u^{-m}, u^{-m-1} \in A$. This implies that $B = A[u^{m+1}]$ is a localization of A. □

Corollary 4.2. *Let $B = A[b]$ be a simple flat overring of an integral domain A. The following are equivalent.*

1. *B is a localization of A.*
2. *B is well-centered on A.*
3. *B is almost well-centered on A.*
4. *The element b is associate in B with an element of A.*
5. *Some power of b is associate in B with an element of A.*

Proof. It is enough to prove (5) $\implies$ (1). Assume for some positive integer n that $b^n = au$ with $a \in A$ and $u \in \mathcal{U}(B)$. Then $b^n \in A[u]$ implies B is a flat integral overring of $A[u]$. Therefore $B = A[u]$ [32, Prop. 2]. Hence by Theorem 4.1, B is a localization of A. □

As an immediate consequence of either Theorem 4.1 or Corollary 4.2 we have:

Corollary 4.3. *If B is a simple flat well-centered overring of an integral domain A, then B is a localization of A.*

We present several additional corollaries of Theorem 4.1 concerning finitely generated flat overrings.

Corollary 4.4. *Let B be a finitely generated flat overring of an integral domain A and let A' denote the integral closure of A in B. If B is a localization of A', then B is a localization of A.*

Proof. Since B is finitely generated over A, if B is a localization of A', then $B = A'[u]$ where $u^{-1} \in A'$. It follows that $B = A[u][A']$ is an integral flat overring of $A[u]$. Therefore $B = A[u]$. By Theorem 4.1, B is a localization of A. □

Theorem 4.5. *Let A be a Prüfer domain with Noetherian spectrum (for example, a Dedekind domain), and let B be a finitely generated overring of A. The following are equivalent.*

1. *B is a localization of A.*
2. *B is well-centered on A.*

(3) *B is almost well-centered on A.*

Proof. It is enough to prove (3) $\implies$ (1). Assume that B is almost well-centered on A. By [13, Corollary 5.6], $B = A[b]$ is a simple extension. Since every overring of a Prüfer domain is flat, we obtain by Corollary 4.2 that B is a localization of A. ☐

In Proposition 3.13, we present examples of Dedekind domains $A \subset B$ such that B is a proper simple flat overring of A and $\mathcal{U}(A) = \mathcal{U}(B)$. Example 4.6 provides a more explicit construction of this type and also shows that the condition that u is a unit in B is essential in Theorem 4.1.

Example 4.6. *An example of a simple flat overring B of an integrally closed domain A such that $A \subsetneq B$ and $\mathcal{U}(A) = \mathcal{U}(B)$.*

Let X, Y and Z be indeterminates over a field k. Set

$$A = k[X, Y, XZ, YZ, \frac{1}{X+YZ}],$$
$$B = k[X, Y, Z, \frac{1}{X+YZ}].$$

Clearly A and B have the same field of fractions $k(X, Y, Z)$ and $B = A[Z]$. To see that $Z \in B \setminus A$, observe that the $k[Z]$-algebra homomorphism defined by setting $Y = \frac{1}{Z}$ and $X = 0$ maps A to k. Also $\mathcal{U}(A) = \mathcal{U}(B) = \{a(X+YZ)^m \mid a \in k \setminus \{0\}, m \in \mathbb{Z}\}$. Since A is a localization of the integrally closed domain $k[X, Y, XZ, YZ]$, we see that A is integrally closed. Finally, B is a flat extension of A by Proposition 3.18 since $XZ, YZ \in A$ and the ideal $(X, Y)B = B$.

Question 4.7. *Under what conditions on A is every finitely generated well-centered overring of A a localization of A?*

If A is Noetherian, it follows from Corollary 4.16 that every finitely generated flat well-centered overring of A is a localization of A. In a situation where Question 4.7 has a positive answer, it follows that the finitely generated overring is actually a simple extension, for if B is a finitely generated overring of A that is a localization of A, then B is a simple extension of A.

Remark 4.8. *Let $B = A[u, v]$ be a flat overring of a domain A, where $v \in \mathcal{U}(B)$. Then $B = A[u, \dfrac{1}{f(u)}]$ for some polynomial $f(X) \in A[X]$.*

Indeed, B is a localization of $A[u]$.

Proposition 4.9. *Let $B = A[u, \frac{1}{u}]$ be a flat overring of a domain A, where $u \in \mathcal{U}(B)$. Then $B = A[u + \frac{1}{u}]$. Moreover, if B is well-centered over A, then B is a localization of A.*

Proof. Let $C = A[u+\frac{1}{u}]$. Since $B = C[u] = C[\frac{1}{u}]$, we obtain by Theorem 4.1 that $u^{-n}, u^n \in C$ for sufficiently large n. Hence $u, \frac{1}{u} \in C$, which implies $C = B$. By Corollary 4.2, if B is well-centered over A, then B is a localization of A. $\square$

We extend Proposition 4.9 as follows:

Proposition 4.10. *Let A be an integral domain and let $B = A[u, \frac{1}{f(u)}]$ be a flat well-centered overring of A, where $f(X)$ is a monic polynomial in $A[X]$, and $u, f(u) \in \mathcal{U}(B)$. Then B is a localization of A.*

Proof. Since f is monic, B is integral over $C := A[f(u), \frac{1}{f(u)}]$. Thus B is flat and integral over C and therefore $B = C$. Thus $B = C$ is flat and well-centered over A. Proposition 4.9 implies that $B = A[f(u) + \frac{1}{f(u)}]$ and that B is a localization of A. $\square$

Question 4.11. *Under what conditions on an integral domain A is every flat overring of A well-centered on A?*

Discussion 4.12. Akiba in [1] constructs an interesting example where A is a 2-dimensional normal excellent local domain, P is a height-one prime of A that is not the radical of a principal ideal, and $B = \bigcup_{n=1}^{\infty} P^{-n}$ is the ideal transform of A at P. Thus $B = \bigcap_Q A_Q$, where the intersection ranges over all the height-one primes of A other than P. Akiba proves that $PB = B$. It follows that B is flat and finitely generated over A, but not a localization of A.

We observe that B is not almost well-centered over A. Indeed, assume that B is almost well-centered over A , and let $b \in B \setminus A$. Thus $ub^m \in A$ for some unit u of B and $m \geq 1$. Hence $u \in \mathcal{U}(A_Q)$ for each height-one prime $Q \neq P$ of A. Since A is normal, we have $b^m \in B \setminus A$, thus $b^m \notin A_P$. It follows that $u \in PA_P$. Therefore $u \in A$ and $\sqrt{uA} = P$. This contradicts the fact that P is not the radical of a principal ideal. We conclude that B is not almost well-centered on A.

We observe that B is not a simple extension of A. Moreover, for every nonzero nonunit $b \in B$ we have $C := A + bB \subsetneq B$. This follows because $PB = B$ implies $\dim B = 1$ and $\dim(B/bB) = 0$. However, $C/bB \cong A/(bB \cap A)$ and $\dim(A/(bB \cap A)) = 1$.

Theorem 4.13. *Let B be a well-centered overring of an integral domain A. If there exist finitely many valuation overrings $V_1, \dots, V_n$ of A such that $A = B \cap V_1 \cap \cdots \cap V_n$, then B is a localization of A.*

Proof. For S a multiplicatively closed subset of A, we have

$$S^{-1}A = S^{-1}B \cap S^{-1}V_1 \cap \cdots \cap S^{-1}V_n,$$

so by replacing A by its localization $(\mathcal{U}(B) \cap A)^{-1}A$, we may assume that $\mathcal{U}(B) \cap A = \mathcal{U}(A)$. If $B \subseteq V_i$, then V_i may be deleted in the representation $A = B \cap (\bigcap_{i=1}^n V_i)$. Thus we may assume that $B \not\subseteq V_i$ for each i. We prove that after these reductions we have $A = B$, i.e., the set $\{V_i\}$ is empty. Assume not, then for each $1 \leq i \leq n$ choose $b_i \in B$ such that $b_i \notin V_i$. By [13, Lemma 5.4], there exist positive integers $e_1, \dots, e_n$ such that $b := b_1^{e_1} + b_2^{e_2} + \cdots b_n^{e_n} \notin V_i$, thus $b^{-1} \in V_i$ for each $i = 1, \dots, n$. Since B is well-centered over A, there exists $u \in \mathcal{U}(B)$ such that $ub \in A$. Since $b \notin V_i$, we have $u \in V_i$ for all i. Therefore $u \in B \cap (\cap_{i=1}^n V_i = A$. It follows that $u \in A \cap \mathcal{U}(B) = \mathcal{U}(A)$ and $u^{-1} \in A$. Hence $b \in A$, a contradiction. □

Lemma 4.14. *Let B be a finitely generated flat overring of an integral domain A and let C be an integral overring of A. The following conditions are equivalent.*

(1) *B is a localization of A.*
(2) *B is a localization of $C \cap B$.*

Proof. Clearly (1) $\implies$ (2). Assume (2). Then $B = (B \cap C)[u]$, where $u^{-1} \in B \cap C$. Since $B \cap C$ is integral over A, it follows that B is flat and integral over $A[u]$. Therefore $B = A[u]$. By Theorem 4.1, B is a localization of A. □

Theorem 4.15. *Let A be an integral domain for which the integral closure A' has a representation $A' = \bigcap_{V \in \mathcal{V}} V$, where $\mathcal{V}$ is a family of valuation* *overrings of A of finite character. If B is a finitely generated flat well-centered overring of A, then B is a localization of A.*

Proof. Since B is finitely generated over A, we have $B \subseteq V$ for all but finitely many domains $V \in \mathcal{V}$. Let $V_1, \dots, V_n$ be the domains in $\mathcal{V}$ that do not contain B. Thus $A' \cap B = B \cap (\bigcap_{i=1}^n V_i)$. By Theorem 4.13, B is a localization of $A' \cap B$. By Lemma 4.14, B is a localization of A. $\square$

It is well known that the integral closure of a Noetherian domain is a Krull domain [25, (33.10)]. Therefore Corollary 4.16 is an immediate consequence of Theorem 4.15.

Corollary 4.16. *Let A be an integral domain for which the integral closure A' is a Krull domain. If B is a finitely generated flat well-centered overring of A, then B is a localization of A. In particular, a finitely generated flat well-centered overring of a Noetherian integral domain A is a localization of A.*

Discussion 4.17. Let A be an integral domain with field of fractions K. Suppose $B = A[b_1, \dots, b_n]$ is a finitely generated overring of A. Let $I_j = (A :_A b_j)$ be the denominator ideal of b_j and let $I = \cap_{j=1}^n I_j$. The overring $C := \{x \in K : xI^n \subseteq A \text{ for some integer } n \geq 1\}$ is called the I-transform of A. This construction was first introduced by Nagata [24] in his work on the 14-th problem of Hilbert. It is clear that $B \subseteq C$ and that C is the IB-transform of B. Nagata observes [24, Lemma 3, page 58] that there is a one-to-one correspondence between the prime ideals Q of C not containing I and the prime ideals P of A not containing I effected by defining $Q \cap A = P$. Moreover, it then follows that $A_P = C_Q$. In particular, if $IB = B$, then $B = C$ is flat over A and there is a one-to-one correspondence between the prime ideals Q of B and the prime ideals P of A not containing I, the correspondence defined by $Q \cap A = P$. Thus if $B = A[b_1, \dots, b_n]$ is a flat overring of A and $P \in \operatorname{Spec} A$, then the following are equivalent:

(1) $PB = B$.
(2) P contains the ideal $I_j = (A :_A b_j)$ for some $j \in \{1, \dots, n\}$.
(3) P contains the ideal $I = \cap_{j=1}^n I_j$.

Theorem 4.18. *Let B be a well-centered flat overring of an integral domain A. If there exists a finite set $\mathcal{F}$ of height-one prime ideals of A such that $A = B \cap \bigcap_{P \in \mathcal{F}} A_P$, then B is a localization of A.*

Proof. Since $P \in \mathcal{F}$ has height-one, for S a multiplicatively closed subset of A either $S^{-1}A_P = A_P$ or $S^{-1}A_P = K$, the field of fractions of A. Therefore

$$S^{-1}A = S^{-1}B \cap \bigcap_P (S^{-1}A_P),$$

where the intersection is over all $P \in \mathcal{F}$ such that $P \cap S = \emptyset$. By replacing A by its localization $(\mathcal{U}(B) \cap A)^{-1}A$, we may assume that $\mathcal{U}(B) \cap A = \mathcal{U}(A)$ and that $B \not\subseteq A_P$ for each $P \in \mathcal{F}$. After this reduction, we claim that $A = B$, i.e., that $\mathcal{F} = \emptyset$. Suppose $\mathcal{F} \neq \emptyset$. Since B is flat over A, for each $P \in \mathcal{F}$ we have $PB = B$. Let c be a nonzero element in $\bigcap_{P \in \mathcal{F}} P$ and consider the ring B/cB and its subring $R = A/(cB \cap A)$. Since $PB = B$ and since every minimal prime of the ring R is the contraction of a prime ideal of B/cB, we have $cB \cap A \not\subseteq P$ for each $P \in \mathcal{F}$. Thus there exists an element $s \in A \setminus \bigcup_{P \in \mathcal{F}} P$, so that $s/c \in B$. Since B is well-centered over A, there exists $u \in \mathcal{U}(B)$ such that $us/c = a \in A$. Thus $u = ac/s \in A_P$ for all $P \in \mathcal{F}$. Therefore $u \in A \cap \mathcal{U}(B) = \mathcal{U}(A)$. Hence $s/c \in A$, but $s/c \notin A_P$, a contradiction. $\square$

Theorem 4.19. *If each nonzero principal ideal of the integral domain A has only finitely many associated primes and each of these associated primes is of height 1, then every finitely generated flat well-centered overring of A is a localization of A.*

Proof. Let $B = A[b_1, \ldots, b_n]$ be a finitely generated flat well-centered overring of A. To prove that B is a localization of A, we may assume that $\mathcal{U}(B) \cap A = \mathcal{U}(A)$, and then we have to show that $A = B$.

Since B is a sublocalization of A, by [5, Prop. 4], $B = \bigcap_{P \in \mathcal{S}} A_P$, where $\mathcal{S}$ is the set of prime ideals P of height 1 of A so that $PB \neq B$. Let $\mathcal{F}$ be the set of prime ideals of height 1 in A such that $PB = B$. Then $A = B \cap \bigcap_{P \in \mathcal{F}} A_P$. By Discussion 4.17, the set $\mathcal{F}$ is finite. Hence by Theorem 4.18, B is a localization of A. $\square$

Proposition 4.20. *Let B be a well-centered overring of an integral domain A. If S is a multiplicative closed subset of A such that $A = A_S \cap B$ and such that B_S is a localization of A_S, then B is a localization of A.*

Proof. Let $b \in B$. There exists an element $t \in A_S \cap \mathcal{U}(B_S)$ such that $tb \in A_S$. We may assume that $t \in A$, thus $tb \in A_S \cap B = A$. Since $t^{-1} \in B_S$, there exists $s \in S$ such that $st^{-1} \in B$. Since B is well-centered over A, there exists $u \in \mathcal{U}(B)$ such that $ust^{-1} = a \in A$. Then $u = at/s \in A_S \cap B = A$ and $ub = atb/s \in A_S \cap B = A$. We have shown for each $b \in B$ there exists $u \in A \cap \mathcal{U}(B)$ such that $ub \in A$. Therefore B is a localization of A. □

Corollary 4.21. *Let B be a well-centered overring of an integral domain A, let I be a proper ideal of A, and let $S = 1+I$. If for each $b \in B$ and $c \in I$ there exists an integer $n \geq 1$ such that $c^n b \in A$ and if B_S is a localization of A_S, then B is a localization of A.*

Proof. The corollary follows from Proposition 4.20 since $A = A_S \cap B$. □

Theorem 4.22. *Every finitely generated flat well-centered overring of a one-dimensional integral domain A is a localization of A.*

Proof. Let $B = A[b_1, \dots, b_n]$ be a finitely generated flat well-centered overring of A and let $I = \bigcap_{j=1}^{n}(A :_A b_j)$. Then $IB = B$ by flatness. Let $S = 1 + I$. Then IA_S is contained in the Jacobson radical of A_S. Since $\dim A_S \leq 1$, IA_S is contained in every nonzero prime ideal of A_S. Since $IB = B$, it follows by [32, Theorem 2 or 3], that B_S is the field of fractions of A_S. By Corollary 4.21, B is a localization of A. □

An interesting question that remains open is whether a finitely generated flat well-centered overring of an integral domain A is always a localization of A.

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