On decomposing ideals into products of comaximal ideals

J. W. Brewer and W. J. Heinzer

Department of Mathematical Sciences, Florida Atlantic University, Boca Raton, Florida 33431 *E-mail address: brewer@fau.edu*

Department of Mathematics, Purdue University, West Lafayette, Indiana 47907-1395 *E-mail address: heinzer@math.purdue.edu*

I. Introduction

A classical result in commutative ring theory as recorded by W. Krull in [7, page 24] asserts that if D is a one-dimensional Noetherian domain, then each nonzero ideal A of D is a product of pairwise comaximal primary ideals. Another result along these lines is a variant of the Chinese Remainder Theorem: an ideal A of a commutative ring R is a product of pairwise comaximal primary ideals if and only if R/A is a finite direct sum of rings R_i where (0) is a primary ideal in each R_i. In particular, if $\dim(R/A) = 0$ and A is contained in only

2000 Mathmatics Subject Classification Numbers: 13A15, 13C05, 13F05, 13G05.
Keywords and phrases: comaximal ideals, prime radical, primary ideal, Prüfer domain, Dedekind domain.

finitely many prime ideals, the Chinese Remainder Theorem implies that A is a finite product of primary ideals. This leads us to wonder:

Question. Which integral domains D have the property that each nonzero ideal A of D can be written as a product

$$A = Q_1 \cdot Q_2 \cdot \cdots \cdot Q_n$$

where the Q_i's are pairwise comaximal and have some additional property?

We also consider this question for nonzero principal ideals aD of D. In the case where aD is a principal ideal, a useful observation is that the factors Q_i are invertible ideals.

Obviously, any Dedekind domain has the property; in this case, each Q_i is a power of a maximal ideal. In this paper we settle the question in the following cases: (i) each Q_i has prime radical (Theorem 1); (ii) each Q_i is primary (Theorem 8); and (iii) each Q_i is the power of a prime (Theorem 9).

We became interested in this property from reading the paper [11] by Sáez-Schwedt and Sánchez-Giralda. It is well known in the classical theory of linear dynamical systems over fields that canonical forms exist for controllable systems of any dimension. In the paper [2] of J. Brewer and L. Klingler, it is shown by means of representation-theoretic methods, that canonical forms are not likely to exist over arbitrary principal ideal domains. The problem treated in [11] is to try to determine a canonical form for two-dimensional controllable systems over principal ideal domains (and Dedekind domains). In [11], the property above for all nonzero principal ideals was the property the authors needed in order to carry out their successful program.

Questions similar to the ones considered here, but without the co-

maximality, have received attention by others. We thank the referee for suggesting the following papers and the references listed there [1], [8], [9]. We also thank Laszlo Fuchs for several helpful comments and for sending us the article [3] concerning ideal theory in Prüfer domains of finite character. As Professor Fuchs noted when we exchanged articles, our paper contains a result identical with one of the theorems in [3]. We arrived at the theorem from completely different directions.

II. Main Results

We begin with the weakest requirement on the factors and the case of nonzero principal ideals.

Theorem 1 *Let D be an integral domain. The following are equivalent:*

1. *Each nonzero principal ideal aD of D can be written in the form $aD = Q_1 \cdot Q_2 \cdot \cdots \cdot Q_n$, where each Q_i has prime radical and the Q_i's are pairwise comaximal.*
2. *D has the following properties:*
   - *(a) For any maximal ideal M of D, the set of prime ideals of D contained in M is linearly ordered under inclusion.*
   - *(b) Each nonzero principal ideal of D has only finitely many minimal primes.*

Proof. $(1) \Rightarrow (2)$: Suppose that M is a maximal ideal of D and that $P_1$and P_2 are incomparable prime ideals of D contained in M. Let a be in P_1 not P_2 and b be in P_2 not P_1. The principal ideal abD can be written in the form $Q_1 \cdot Q_2 \cdot \cdots \cdot Q_n$, where each Q_i has prime radical and the Q_i's are pairwise comaximal. Since $abD \subseteq M$, some Q is contained in M, say Q_1. Moreover, since the Q_i's are pairwise comaximal, only Q_1 is contained in M. Thus, since P_1 and P_2 are contained in M and contain abD, $Q_1 \subseteq P_1 \cap P_2$. But, Q_1 cannot have prime radical, for suppose that $\sqrt{Q_1} = P$, prime. Then P is contained in $P_1 \cap P_2$. Also, P must contain one of a or b. But this is a contradiction to the choice of a and b and it follows that condition (a) holds.

Let a be a nonzero element of D. Write $aD = Q_1 \cdot Q_2 \cdot \cdots \cdot Q_n$, where each Q_i has prime radical P_i and the Q_i's are pairwise comaximal. If P is a minimal prime of aD, then P contains some Q_i and hence, $P = P_i$. It follows that $\{P_1, P_2, \ldots, P_n\}$ is the set of minimal prime ideals of aD. This proves condition (b).

(2) $\Rightarrow$ (1) : Suppose that aD is a nonzero principal ideal of D and that $P_1, P_2, \ldots, P_n$ are the minimal primes of aD. By condition (a), the P_i's, are pairwise comaximal. Hence, there exist x_i in P_i and y_i in P_j for $j \neq i$ such that $x_i + y_i = 1$. Let $D_i = D[1/y_i]$, the localization of D at the powers of the element y_i. If $Q_i = aD_i \cap D$, then Q_i has radical P_i. Let M be a maximal ideal of D. Then either $aD \subseteq M$ or $aD \not\subseteq M$. If $aD \not\subseteq M$, then $aD_M \cap D = D$. If $aD \subseteq M$, then $M \supset P_i$ for some i, and hence $y_i \notin M$; it follows that D_M is a term in the intersection $D[1/y_i] = \bigcap D_P$, where P is prime and $y_i \notin P$. Thus, $D \subseteq \bigcap_{i=1}^{n} D_i \cap D[1/a] \subseteq \bigcap_{M \text{ max}} D_M = D$. Now, for every ideal A of D, $A = \bigcap_{M \text{ max}} AD_M \cap D$. It follows that $aD = Q_1 \cap Q_2 \cap \cdots \cap Q_n$. Moreover the Q_i's are pairwise comaximal, so their intersection is their product. This completes the proof. ■

The proof of Theorem 1 can be easily modified to prove

Theorem 2 *Let D be an integral domain. The following are equivalent:*

1. *Each nonzero ideal A of D can be written in the form $A =$*

$Q_1 \cdot Q_2 \cdot \dots \cdot Q_n$, where each Q_i has prime radical and the Q_i's are pairwise comaximal.

2. *D has the following properties:*
   (a) *For any maximal ideal M of D, the set of prime ideals of D contained in M is linearly ordered under inclusion.*
   (b) *Each nonzero ideal of D has only finitely many minimal primes.*

Remark 3 Note that condition 2(a) above is equivalent to the following: If P and Q are nonzero prime ideals of D, then either P and Q are comaximal or P and Q are comparable. Viewed in this way, conditions 2(a) and 2(b) are topological conditions on the prime spectrum of D with the Zariski topology. More specifically, 2(a) says that any two proper closed irreducible subsets of $Spec(D)$ are either comparable or disjoint; 2(b) says that each closed subset of $Spec(D)$ has only finitely many irreducible components. The paper [5] of M. Hochster establishes the existence of many examples of integral domains having these two properties.

There are situations in which Theorem 1 and Theorem 2 can be combined into one result as the following proposition illustrates.

Proposition 4 *Suppose that the integral domain D has the property that each nonzero prime ideal of D is contained in only finitely many maximal ideals. The following are equivalent:*

1. *Each nonzero ideal A of D can be written in the form $A = Q_1 \cdot Q_2 \cdot \dots \cdot Q_n$, where each Q_i has prime radical and the Q_i's are pairwise comaximal.*
2. *Each nonzero principal ideal aD of D can be written in the form $aD = Q_1 \cdot Q_2 \cdot \dots \cdot Q_n$, where each Q_i has prime radical and the Q_i's are pairwise comaximal.*

3. D has the following properties:

(a) For any maximal ideal M of D, the set of prime ideals of D contained in M is linearly ordered under inclusion.

(b) Each nonzero principal ideal of D has only finitely many minimal primes.

Proof. That (1) implies (2) is obvious and that (2) is equivalent to (3) is the content of Theorem 1. Thus, we have only to prove that (3) implies (1). Since condition (2a) is the same in both theorems, we have to verify that each nonzero ideal A of D has only finitely many minimal primes. Pick a nonzero element $a \in A$ and let $P_1, P_2, \ldots, P_n$ be the minimal primes of aD. By hypothesis, each P_i is contained in only finitely many maximal ideals, say P_i is contained in $M_{i1}, \ldots, M_{ik_i}$. Let P be a minimal prime of A. Then $aD \subseteq P$ and $P \supseteq P_j$ for some j. It follows that $P \subseteq M_{jl}$ for some l between $j1$ and jk_j. (P might be contained in more that one such M_{jl}, but that causes no difficulty.) If Q is another minimal prime of A that contains P_j, then $Q \subseteq M_{ji}$ for $i \neq l$ since the primes contained in a given maximal ideal are comparable. Therefore, there are at most k_j distinct minimal primes of A that contain P_j. Since each minimal prime of A must contain some P_j, it follows that there are at most $k_1 + k_2 + \cdots k_n$ minimal primes of A. ■

We record the following corollary to Theorem 2.

Corollary 5 *Let D be a Prüfer domain. If each nonzero element of D belongs to only finitely many maximal ideals of D, then each nonzero ideal of D can be written in the form $Q_1 \cdot Q_2 \cdot \cdots \cdot Q_n$, where each Q_i has prime radical and the Q_i's are pairwise comaximal.*

Proof. If D is a Prüfer domain, then each localization of D at a prime ideal is a valuation domain. In particular, the prime ideals of

D contained in a given maximal ideal of D are linearly ordered under inclusion. Let A be a nonzero ideal of D with P a minimal prime of A. Since each nonzero element of D belongs to only finitely many maximal ideals, the same is true for A. Let those maximal ideals be $M_1, M_2, \dots, M_k$. Then $P \subseteq M_j$ for some j. Since the prime ideals of D contained in a given maximal ideal of D are linearly ordered, any M_i can contain at most one minimal prime of A. Thus, A has at most k minimal primes and the result follows from Theorem 2. ■

Remark 6 Laszlo Fuchs has pointed out to us that the proof of Corollary 5 also applies for a non-Prüfer domain D: if each nonzero element of D belongs to only finitely many maximal ideals and if the prime ideals contained in a given maximal ideal of D are linearly ordered under inclusion, then each nonzero ideal of D can be written in the form $Q_1 \cdot Q_2 \cdot \cdots \cdot Q_n$, where each Q_i has prime radical and the Q_i's are pairwise comaximal.

There exist Prüfer domains which satisfy the hypothesis of Theorem 1 and which have nonzero elements contained in infinitely many maximal ideals. For example, if $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{Z}$ the ring of integers,and X an indeterminate, let M be the maximal ideal $X\mathbb{Q}[X]_{(X)}$ of $\mathbb{Q}[X]_{(X)}$. Then the domain $D = \mathbb{Z} + M$ is a two-dimensional Prüfer domain having a unique prime ideal M of height one and having infinitely many maximal ideals corresponding to the maximal ideals of $\mathbb{Z}$. The elements of M are contained in infinitely many maximal ideals of D while the elements of D not in M are in only finitely many prime ideals. Thus, D satisfies the hypothesis of Theorem 1

We next present an example to illustrate the fact that Theorems 1 and 2 cannot always be combined. Thus, for the case when the ideals Q_i are only assumed to have prime radical, the condition for all principal ideals is not equivalent to the condition for all ideals. More specifically, we construct a Prüfer domain D having the following two properties: Each nonzero principal ideal of D has only finitely many minimal primes; D has a nonzero ideal having infinitely many minimal primes. Since a Prüfer domain has the property that any two prime ideals contained in a given maximal ideal are comparable, this example effectively separates Theorem 1 from Theorem 2. Thus, for this Prüfer domain D, each nonzero principal ideal aD can be written in the form $aD = Q_1 \cdot Q_2 \cdot \cdots \cdot Q_n$, where each Q_i has prime radical and the Q_i's are pairwise comaximal, but there exists a nonzero ideal A of D which cannot be so written.

Example 7 We are going to construct a Prüfer domain D having for each positive integer n precisely two primes of height n, M_n which is maximal, and P_n which is nonmaximal. This gives rise to the diagram below describing the prime ideal lattice of D (the union of the P_n is a unique maximal ideal of infinite height).

$$\begin{array}{ccc} \vdots & & \vdots \\ M_4 & & P_4 \\ & \nwarrow & \uparrow \\ M_3 & & P_3 \\ & \nwarrow & \uparrow \\ M_2 & & P_2 \\ & \nwarrow & \uparrow \\ M_1 & & P_1 \\ & \nwarrow\!\nearrow & \\ & (0) & \end{array}$$

We construct D to be the union of a chain of Prüfer subdomains D_n, where D_n is n-dimensional and D_n has two maximal ideals of height n and has for each positive integer $m < n$ two primes of height

m, exactly one of which is maximal. The construction is such that there exists an inclusion map of D_n into D_{n+1} so that the two maximal ideals of D_{n+1} of height $n+1$ both lie over the same maximal ideal of D_n. Moreover there is also a nonmaximal prime of D_{n+1} of height n lying over this same maximal ideal of D_n. For each $m < n$, the maximal ideal of D_{n+1} of height m lies over the maximal ideal of D_n of height m and the nonmaximal prime of D_{n+1} of height m lies over the nonmaximal prime of D_n of height m.

Let $A := \cap_{n=1}^{\infty} M_n$. We show that each of the M_n's is a minimal prime of A, so that A has infinitely many minimal primes. This follows from the fact that $A_n := A \cap D_n$ is the Jacobson radical of D_n and D_n has precisely $n+1$ maximal ideals, one of height i for each i between 1 and $n-1$ and two of height n. We have $A_n D_{n+1}$ properly contained in A_{n+1} and $A = \cup_{n=1}^{\infty} A_n$. As we go up from D_n to D_{n+1}, our ideal A is picking up more minimal primes and in the union has infinitely many. Now each principal ideal of D has the form aD, where a is in D_n for some n. Then aD_n has at most $n+1$ minimal primes and by the construction, aD has at most $n+1$ minimal primes. The point is that in passing from D_n to D_{n+1}, there are three prime ideals of D_{n+1} lying over the same maximal ideal of height n of D_n. If a is contained in one of them, then it is in all three and has precisely one of them as a minimal prime.

We now spell out a way to construct such a Prüfer domain D. (We will be intersecting finite families of valuation domains, and using the theorem on independence of valuations as given by Nagata in [10, (11.11), page 38].) For k a field and $x_1, x_2, \ldots$ indeterminates over k, we construct a chain D_n of the type discussed above, where D_n has fraction field $k(x_1, \ldots, x_n)$: Let D_1 be the intersection of the

DVRs $(k[x_1])_{(x_1)} = V_1$ and $(k[x_1])_{(x_1-1)} = W_1$. Let

$$M_{11} = (x_1 - 1)D_1 \text{ and } P_{11} = x_1 D_1.$$

denote the maximal ideals of D_1. We construct D_2 on the field $k(x_1, x_2)$ as an intersection of three valuation domains. We denote by $W_1(x_2)$ the valuation domain $(k(x_2)[x_1])_{(x_1-1)}$, i.e., $W_1[x_2]$ localized at the extension of the maximal ideal of W_1 to this polynomial ring. This valuation domain $W_1(x_2)$ is sometimes called the Gaussian or trivial extension of W_1 to the simple transcendental field extension generated by x_2. We define valuation domains W_2 and V_2 of rank 2 on $k(x_2, y_2)$ both of which extend V_1. Let N_2 denote the maximal ideal of $V_1(x_2)$ and let

$$V_2 = (k[x_2])_{(x_2)} + N_2 \text{ and } W_2 = (k[x_2])_{(x_2-1)} + N_2.$$

Define $D_2 := W_1(x_2) \cap W_2 \cap V_2$. Using [10, (11.11)], we see that D_2 has two maximal ideals of height 2,

$$M_{22} := (x_2 - 1)D_2 \text{ and } P_{22} := x_2 D_2,$$

and two prime ideals of height one, $M_{21} := (x_1 - 1)D_2$ which is maximal, and $P_{21} := N_2 \cap D_2$ which contains x_1 and is not maximal. Observe that M_{22}, P_{22} and P_{21} all have the property that their intersection with D_1 is P_{11}.

We construct D_3 on $k(x_1, x_2, x_3)$ as an intersection of four valuation domains. Let N_3 denote the maximal ideal of $V_2(x_3)$. Define

$$V_3 := (k[x_3])_{(x_3)} + N_3 \text{ and } W_3 = (k[x_3])_{(x_3-1)} + N_3.$$

Define $D_3 := W_1(x_2, x_3) \cap W_2(x_3) \cap W_3 \cap V_3$. Using [10, (11.11)], we see that D_3 has two maximal ideals of height 3,

$$M_{33} := (x_3 - 1)D_3 \text{ and } P_{33} := x_3 D_3,$$

two prime ideals of height two, $M_{32} := (x_2 - 1)D_3$ which is maximal, and $P_{32} := N_3 \cap D_3$ which contains x_2 and is not maximal, and two prime ideals of height one, $M_{31} := (x_1 - 1)D_3$ which is maximal and P_{31} which is the contraction to D_3 of the height-one prime of V_3.

Let $t \geq 3$ be a fixed positive integer and assume for each positive integer $n \leq t$ we have constructed $n+1$ valuation domains

$$W_1(x_2, \dots, x_n), W_2(x_3, \dots, x_n), \dots, W_i(x_{i+1}, \dots, x_n), \dots, W_n \text{ and } V_n$$

on the field $k(x_1, \dots, x_n)$ such that W_m has rank m for each m with $1 \leq m \leq n$, V_n has rank n, and

$$D_n := W_1(x_2, \dots, x_n) \cap W_2(x_3, \dots, x_n) \cap \dots \cap W_n \cap V_n$$

has the following properties:

1. D_n has two maximal ideals of height n, $M_{nn} = (x_n - 1)D_n$ and $P_{nn} = (x_n)D_n$.
2. For each positive integer $m < n$, D_n has two prime ideal of height m, $M_{nm} = (x_m - 1)D_n$, which is maximal, and P_{nm} which is nonmaximal.
3. For positive integers $m \leq n \leq s \leq t$, the prime ideal P_{sm} of D_s intersects D_n in P_{nm}.

It is clear that we can continue the construction by defining $t+2$ valuation domains on $k(x_1, \dots, x_{t+1})$ as follows. Let N_{t+1} denote the maximal ideal of $V_t(x_{t+1})$ and define $V_{t+1} = k[x_{t+1}]_{(x_{t+1})} + N_{t+1}$ and $W_{t+1} = k[x_{t+1}]_{(x_{t+1}-1)} + N_{t+1}$. For $1 \leq i \leq t$, define the Gaussian extension $W_i(x_{i+1}, \dots, x_{t+1})$. Then D_{t+1} defined as the intersection of these $t+2$ valuation domains on $k(x_1, \dots, x_{t+1})$ gives the properties listed above for positive integers $m \leq n \leq s \leq t+1$. Therefore, by induction, we have the stated properties for all positive integers $m \leq n \leq s$.

We define $D = \cup_{n=1}^{\infty} D_n$. For each positive integer m, D has two prime ideals of height m, $M_m := \cup_{n=m}^{\infty} M_{nm}$, which is maximal, and

$P_m := \cup_{n=m}^{\infty} P_{nm}$, which is nonmaximal. D also has a unique maximal ideal $M_\infty := \cup_{n=1}^{\infty} P_{nn}$ having infinite height. This completes the construction of the example.

By [10, (11.11)], each of the D_n is a Bezout domain. Therefore D is also a Bezout domain.

If we strengthen the assumption on the factors, we get stronger conditions on the domain. In addition, we also get that the condition on all nonzero ideals is equivalent to the condition on all principal ideals.

Theorem 8 *Let D be an integral domain. The following are equivalent:*

1. *Each nonzero principal ideal aD of D can be written in the form $aD = Q_1 \cdot Q_2 \cdot \cdots \cdot Q_n$, where each Q_i is primary and the Q_i's are pairwise comaximal.*
2. *D has the following properties:*
   - *(a) D is one-dimensional.*
   - *(b) D has Noetherian spectrum.*
3. *Each nonzero ideal A of D can be written in the form $A = Q_1 \cdot Q_2 \cdot \cdots \cdot Q_n$, where each Q_i is primary and the Q_i's are pairwise comaximal.*

Proof. (1) $\Rightarrow$ (2): Let P be a nonzero prime ideal of D. If P is not maximal, let M be a maximal ideal of D that properly contains P and let a be a nonzero element of P. Let b be an element of M not in P. Set $x = ab$. By hypothesis, $xD = Q_1 \cdot Q_2 \cdot \cdots \cdot Q_n$, where each Q_i is primary and the Q_i's are pairwise comaximal. Now, P contains x and so P contains some Q, say Q_1. Let $P_1 = \sqrt{Q_1}$. Thus Q_1 is P_1-primary and $P_1 \subseteq P$. Also, because the Q_i's are pairwise comaximal, M contains at most one minimal prime ideal of xD and that minimal prime must be P_1. Localize at the maximal ideal M. This forces $xD_M = Q_1 D_M$ to be $P_1 D_M$-primary. But this is impossible, for b is not in $P_1 D_M$, but $ab = x$ is in xD_M. If xD_M were $P_1 D_M$-primary, then a must be in xD_M. But $a = x(\frac{d}{s})$, for some $d \in D,\ s \in D \setminus M$ implies $x = x(\frac{d}{s})b$, so $s = db$. This contradicts the fact that b is in M. Hence, D is one dimensional.

To see that D has Noetherian spectrum, we first show that each nonzero ideal of D is contained in only finitely maximal ideals. Let a be a nonzero element of D. Then we can write $aD = Q_1 \cdot Q_2 \cdot \dots \cdot Q_n$, where $\sqrt{Q_i} = M_i$ is prime; in fact, M_i is a maximal ideal since D is one-dimensional. If M is any maximal ideal of D of which a is a member, then $M \supseteq M_j$ for some j. Thus, $\{M_1, M_2, \dots, M_n\}$ is the set of maximal ideals of D containing a. If A is any nonzero ideal of D, then the set of maximal ideals containing A is a subset of the set of maximal ideals containing any nonzero element inside A. In particular, the set of maximal ideals containing A is finite.

To show that D has Noetherian spectrum,we must prove that D satisfies the ascending chain condition on radical ideals. Thus, let $A_1 \subsetneqq A_2 \subsetneqq \dots \subsetneqq$ be an ascending sequence of radical ideals. Now, $\sqrt{A_1} = M_1 \cap M_2 \cap \dots \cap M_k$ for some maximal ideals $M_1, M_2, \dots, M_k$ of D. Since $A_1 \subsetneqq A_2$, the set of maximal ideals of D that contain A_2 is a proper subset of $\{M_1, M_2, \dots, M_k\}$. It follows that the sequence must be finite.

$(2) \Rightarrow (3)$: Let A be a nonzero ideal of D, with $M_1, M_2, \dots M_n$ the maximal ideals containing A. That there are only finitely many follows from the fact that D is one-dimensional and the fact that in a ring with Noetherian spectrum, each nonzero ideal has only finitely many minimal primes. Set $Q_i = AD_{M_i} \cap D$. Then Q_i has radical M_i and A is the intersection of the Q_i's. Since the M_i's are maximal, the Q_i's are primary and comaximal. So, A is also the product of the Q_i's.

(3) $\Rightarrow$ (1) : This is obvious. ■

Finally, we treat the case where the Q's are prime powers.

Theorem 9 *Let D be an integral domain. The following are equivalent:*

1. *Each nonzero principal ideal aD of D can be written in the form $aD = Q_1 \cdot Q_2 \cdot \cdots \cdot Q_n$, where each Q_i is a prime power and the Q_i's are pairwise comaximal.*
2. *D has the following properties:*
   (a) *D is one-dimensional.*
   (b) *D is Noetherian.*
   (c) *D is integrally closed.*
3. *Each nonzero ideal A of D can be written as a product of prime ideals.*
4. *D is a Dedekind domain.*
5. *Each nonzero ideal A of D can be written in the form $A = Q_1 \cdot Q_2 \cdot \cdots \cdot Q_n$, where each Q_i is a prime power and the Q_i's are pairwise comaximal.*

Proof. The equivalence of properties (2) - (5) is standard, see for example [4, Theorem 37.8]. In particular, (3) $\Rightarrow$ (4) is a theorem of Matsusita [6, Page 68]. It is obvious that (5) $\Rightarrow$ (1) Thus it suffices to show:

(1) $\Rightarrow$ (2) : We first prove that D is one-dimensional. Let M be a maximal ideal of D and suppose there exists a nonzero prime ideal P properly contained in M. Let $x \in M$, $x \notin P$ and write $xD = P_1^{e_1} \cdots P_n^{e_n}$, where the P_i are pairwise comaximal prime ideals. It follows that the P_i are invertible and M contains one, and only one, of the P_i, say P_1. Because $x \notin P$, P_1 is not contained in P. Let y

be a nonzero element of P and write $yD = N_1^{f_1} \cdots N_m^{f_m}$, where the N_j are pairwise comaximal prime ideals. Thus the N_j are invertible and P contains one, and only one, of the N_j, say N_1. By Theorem 1, the prime ideals contained in M are linearly ordered with respect to inclusion. Since $N_1 \subseteq P$, we have $N_1 \subsetneq P_1$. It follows that $N_1 D_{P_1} \subsetneq P_1 D_{P_1}$ are nonzero principal prime ideals of D_{P_1}. This is a contradiction. (Suppose R is a quasilocal domain with principal maximal ideal xR and that yR is a principal prime ideal with yR strictly smaller than xR. Write $y = xz$ for some $z \in R$. Since yR is a prime ideal and $x \notin yR$, it follows that $z \in yR$, say $z = yr$. Then $y = xyr$, so $y(1 - xr) = 0$. Since $1 - xr$ is a unit of R, $y = 0$).

Let M be a maximal ideal of D and let a be a nonzero element of M. Then $aD = Q_1 \cdot Q_2 \cdot \cdots \cdot Q_n$, where each $Q_i = M_i^{e_i}$ for some maximal ideal M_i of D. Thus, $M = M_j$ for some j. Since a product of ideals is invertible if and only if each factor is invertible, it follows that M is invertible. In particular, M is finitely generated. So, all prime ideals of D are finitely generated and D is Noetherian by Cohen's Theorem [6, Theorem 8]. Finally, MD_M is principal since M is invertible. Therefore, each prime ideal of the quasi-local ring D_M is principal and by an analog to Cohen's Theorem, [6, Page 8] D_M is a principal ideal domain. It follows that D is integrally closed. ■

This paper was written, composed, and typeset electronically. The difficulty posed by two authors working in different locations and using different interfaces to Tex would have been insurmontable without the help of Fred Richman and to Fred we offer our sincere thanks.

References

1. D. D. Anderson and L. A Mahaney, *Commutative rings in which every ideal is a product of primary ideals*, J. Algebra **106** (1987), 528-535.
2. J. Brewer and L. Klingler, *On feedback invariants for linear dynamical systems,* Linear Algebra and Its Appl. **325** (2001), 209-220.
3. L. Fuchs and E. Mosteig, *Ideal theory in Prüfer domains - an unconventional approach*, (to appear).
4. R. Gilmer, *Multiplicative Ideal Theory*, Marcel Dekker, New York, 1972.
5. M. Hochster, *Prime ideal structure in commutative rings*, Trans. Amer. Math. Soc. **142** (1969), 43-60.
6. I. Kaplansky, *Commutative Rings*, University of Chicago Press, Chicago, 1974.
7. W. Krull, *Idealtheorie* Zweite, ergänzte Auflage, Springer-Verlag, Berlin, 1968.
8. K. Levitz, *A characterization of general Z.P.I. rings*, Proc. Amer. Math. Soc. **32** (1972), 376-380.
9. K. Levitz, *A characterization of general Z.P.I. rings II*, Pacific J. Math. **42** (1972), 147-151.
10. M. Nagata, *Local Rings*, Interscience, New York, 1962.
11. T. Sanchez-Giralda and A. Saez-Schwedt, *Canonical forms for 2-dimensional linear systems over commutative rings*, preprint.