THE COHEN-MACAULAY AND GORENSTEIN PROPERTIES OF RINGS ASSOCIATED TO FILTRATIONS

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Abstract. Let $(R, \mathbf{m})$ be a Cohen-Macaulay local ring and let $\mathcal{F} = \{F_i\}_{i\in\mathbb{Z}}$ be an F_1-good filtration of ideals in R. If F_1 is $\mathbf{m}$-primary we obtain sufficient conditions in order that the associated graded ring $G(\mathcal{F})$ be Cohen-Macaulay. In the case where R is Gorenstein, we use the Cohen-Macaulay result to establish necessary and sufficient conditions for $G(\mathcal{F})$ to be Gorenstein. We apply this result to the integral closure filtration $\mathcal{F}$ associated to a monomial parameter ideal of a polynomial ring to give necessary and sufficient conditions for $G(\mathcal{F})$ to be Gorenstein. Let $(R, \mathbf{m})$ be a Gorenstein local ring and let F_1 be an ideal with $\mathrm{ht}(F_1) = g > 0$. If there exists a reduction J of $\mathcal{F}$ with $\mu(J) = g$ and reduction number $u := r_J(\mathcal{F})$, we prove that the extended Rees algebra $R^{'}(\mathcal{F})$ is quasi-Gorenstein with $\mathbf{a}$-invariant b if and only if $J^n : F_u = F_{n+b-u+g-1}$ for every $n \in \mathbb{Z}$. Furthermore, if $G(\mathcal{F})$ is Cohen-Macaulay, then the maximal degree of a homogeneous minimal generator of the canonical module $\omega_{G(\mathcal{F})}$ is at most g and that of the canonical module $\omega_{R^{'}(\mathcal{F})}$ is at most $g-1$; moreover, $R^{'}(\mathcal{F})$ is Gorenstein if and only if $J^u : F_u = F_u$. We illustrate with various examples cases where $G(\mathcal{F})$ is or is not Gorenstein.

1. Introduction

All rings we consider are assumed to be commutative with an identity element. A *filtration* $\mathcal{F} = \{F_i\}_{i\in\mathbb{N}}$ on a ring R is a descending chain $R = F_0 \supset F_1 \supset F_2 \supset \cdots$ of ideals such that $F_iF_j \subseteq F_{i+j}$ for all $i, j \in \mathbb{N}$. It is sometimes convenient to extend the filtration by defining $F_i = R$ for all integers $i \leq 0$.

Let t be an indeterminate over R. Then for each filtration $\mathcal{F}$ of ideals in R, several graded rings naturally associated to $\mathcal{F}$ are :

(1) The Rees algebra $R(\mathcal{F}) = \bigoplus_{i\geq 0} F_it^i \subseteq R[t]$,
(2) The extended Rees algebra $R^{'}(\mathcal{F}) = \bigoplus_{i\in\mathbb{Z}} F_it^i \subseteq R[t, t^{-1}]$,
(3) The associated graded ring $G(\mathcal{F}) = \frac{R^{'}(\mathcal{F})}{(t^{-1})R^{'}(\mathcal{F})} = \bigoplus_{i\geq 0} \frac{F_i}{F_{i+1}}$.

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If $\mathcal{F}$ is an I-adic filtration, that is, $\mathcal{F} = \{I^i\}_{i\in\mathbb{Z}}$ for some ideal I in R, we denote $R(\mathcal{F}), R^{'}(\mathcal{F})$, and $G(\mathcal{F})$ by $R(I), R^{'}(I)$, and $G(I)$, respectively.

In this paper we examine the Cohen-Macaulay and Gorenstein properties of graded rings associated to filtrations $\mathcal{F}$ of ideals. We establish

(1) sufficient conditions for $G(\mathcal{F})$ to be Cohen-Macaulay,
(2) necessary and sufficient conditions for $G(\mathcal{F})$ to be Gorenstein, and
(3) necessary and sufficient conditions for $R^{'}(\mathcal{F})$ to be quasi-Gorenstein.

These results extend those given in [HKU] in the case where $\mathcal{F}$ is an ideal-adic filtration.

Let $(R, \mathbf{m})$ be a d-dimensional Cohen-Macaulay local ring and let $\mathcal{F} = \{F_i\}_{i\in\mathbb{Z}}$ be an F_1-good filtration, where F_1 is $\mathbf{m}$-primary. Assume that J is a reduction of $\mathcal{F}$ with $\mu(J) = d$ and let $u := r_J(\mathcal{F})$ denote the reduction number of $\mathcal{F}$ with respect to J. In Theorem 3.12, we prove that $G(\mathcal{F})$ is Cohen-Macaulay, if $J : F_{u-i} = J + F_{i+1}$ for all i with $0 \le i \le u-1$. If R is Gorenstein, we prove in Theorem 4.3 that $G(\mathcal{F})$ is Gorenstein $\iff$ $J : F_{u-i} = J + F_{i+1}$ for $0 \le i \le u-1$ $\iff$ $J : F_{u-i} = J + F_{i+1}$ for $0 \le i \le \lfloor \frac{u-1}{2} \rfloor$. If R is regular with $d \ge 2$ and $G(\mathcal{F})$ is Cohen-Macaulay, we prove in Theorem 4.7 that $G(\mathcal{F}/J)$ has a nonzero socle element of degree $\le d-2$. We deduce in Corollary 4.9 that if $G(\mathcal{F})$ is Gorenstein and $F_{i+1} \subseteq \mathbf{m} F_i$ for all $i \ge d-1$, then $r_J(\mathcal{F}) \le d-2$.

Let J be a monomial parameter ideal of a polynomial ring $R = k[x_1, \ldots, x_d]$ over a field k. In Section 5 we consider the integral closure filtration $\mathcal{F} := \{\overline{J^n}\}_{n\ge 0}$ associated to J. If $J = (x_1^{a_1}, \ldots, x_d^{a_d})R$ and L is the least common multiple of $a_1, \ldots, a_d$, Theorem 5.6 states that $G(\mathcal{F})$ is Gorenstein if and only if $\sum_{i=1}^d \frac{L}{a_i} \equiv 1 \mod L$. Corollary 5.7 asserts that the following three conditions are equivalent: (i) $\sum_{i=1}^d \frac{L}{a_i} = L+1$, (ii) $G(\mathcal{F})$ is Gorenstein and $r_J(\mathcal{F}) = d-2$, (iii) the Rees algebra $R(\mathcal{F})$ is Gorenstein. Example 5.13 demonstrates the existence of monomial parameter ideals for which the associated integral closure filtration $\mathcal{E}$ is such that $G(\mathcal{E})$ and $R(\mathcal{E})$ are Gorenstein and $\mathcal{E}$ is not an ideal-adic filtration.

In Section 6 we consider a d-dimensional Gorenstein local ring $(R, \mathbf{m})$ and an F_1-good filtration $\mathcal{F} = \{F_i\}_{i\in\mathbb{Z}}$ of ideals in R, where $\operatorname{ht}(F_1) = g > 0$. Assume there exists a reduction J of $\mathcal{F}$ with $\mu(J) = g$ and reduction number $u := r_J(\mathcal{F})$. In Theorem 6.1, we prove that the extended Rees algebra $R^{'}(\mathcal{F})$ is quasi-Gorenstein with $\mathbf{a}$-invariant b if and only if $(J^n : F_u) = F_{n+b-u+g-1}$ for every $n \in \mathbb{Z}$. If $G(\mathcal{F})$ is Cohen-Macaulay, we prove in Theorem 6.2 that the maximal degree of a homogeneous minimal generator of the canonical module $\omega_{G(\mathcal{F})}$ is at most g and that of the canonical module $\omega_{R'(\mathcal{F})}$ is at most $g-1$. With the same hypothesis, we prove in Theorem 6.3 that $R'(\mathcal{F})$ is Gorenstein if and only if $J^u : F_u = F_u$.

In Section 7 we present and compare properties of various filtrations.

2. Preliminaries

Definition 2.1. Let $\mathcal{F} = \{F_i\}_{i\in\mathbb{Z}}$ be a filtration of ideals in R and let I be an ideal of R.

(1) The filtration $\mathcal{F}$ is called *Noetherian* if the Rees ring $R(\mathcal{F})$ is Noetherian.

(2) The filtration $\mathcal{F}$ is called an *I-good filtration* if $IF_i \subseteq F_{i+1}$ for all $i \in \mathbb{Z}$ and $F_{n+1} = IF_n$ for all $n >> 0$. The filtration $\mathcal{F}$ is called a *good filtration* if it is an I-good filtration for some ideal I in R.

(3) A *reduction* of a filtration $\mathcal{F}$ is an ideal $J \subseteq F_1$ such that $JF_n = F_{n+1}$ for all large n. A *minimal reduction* of $\mathcal{F}$ is a reduction of $\mathcal{F}$ minimal with respect to inclusion.

(4) If $J \subseteq F_1$ is a reduction of $\mathcal{F}$, then

$$r_J(\mathcal{F}) = \min\{r \mid F_{n+1} = JF_n \quad \text{for all} \quad n \geq r\}$$

is the *reduction number* of $\mathcal{F}$ with respect to J.

(5) If L is an ideal of R, then $\mathcal{F}/L$ denotes the filtration $\{(F_i + L)/L\}_{i\in\mathbb{Z}}$ on R/L. The filtration $\mathcal{F}/L$ is Noetherian, resp. good, if $\mathcal{F}$ is Noetherian, resp. good.

Remark 2.2. If the filtration $\mathcal{F}$ is Noetherian, then R is Noetherian and $R'(\mathcal{F})$ is finitely generated over R [BH, Propositon 4.5.3]. Moreover, $\dim R'(\mathcal{F}) = \dim R + 1$ and $\dim G(\mathcal{F}) \leq \dim R$, with $\dim G(\mathcal{F}) = \dim R$ if F_1 is contained in all the maximal ideals of R [BH, Theorem 4.5.6]. Furthermore, one has $\dim R(\mathcal{F}) = \dim R + 1$, if F_1 is not contained in any minimal prime ideal $\mathbf{p}$ in R with $\dim(R/\mathbf{p}) = \dim(R)$ (cf. [Va]). Assume the ring R is Noetherian, then the filtration $\mathcal{F} = \{F_i\}_{i\in\mathbb{Z}}$ is a good filtration $\iff$ it is an F_1-good filtration, and $\mathcal{F}$ is an F_1-good filtration $\iff$ there exists an integer k such that $F_n \subseteq (F_1)^{n-k}$ for all n $\iff$ the Rees algebra $R(\mathcal{F})$ is a finite $R(F_1)$-module [B, Theorem III.3.1.1 and Corollary III.3.1.4].

If $\mathcal{F} = \{F_i\}_{i\in\mathbb{Z}}$ is a filtration on R, then we have

$$R(F_1) = \bigoplus_{n\geq 0} F_1^n t^n \subseteq R(\mathcal{F}) = \bigoplus_{n\geq 0} F_n t^n \subseteq R[t].$$

If R is Noetherian and $\mathcal{F} = \{F_i\}_{i\in\mathbb{Z}}$ is an F_1-good filtration, then $R(\mathcal{F})$ is a finite $R(F_1)$-module, and hence $R(\mathcal{F})$ is integral over $R(F_1)$. Thus, in this case, we have $F_1^n \subseteq F_n \subseteq \overline{F_1^n}$, for all $n \geq 0$, where $\overline{F_1^n}$ denotes the integral closure of F_1^n. Notice also that if $\mathcal{F}$ is an F_1-good filtration, then J is a reduction of $\mathcal{F} \iff J$ is a reduction of F_1.

The proof of Remark 2.3 is straightforward using the definition of an F_1-good filtration.

Remark 2.3. Let $(R, \mathbf{m})$ be a Noetherian local ring and let $\mathcal{F} = \{F_i\}_{i\in\mathbb{Z}}$ be a F_1-good filtration of R. Set

$$\begin{aligned} R(\mathcal{F})_+ &= \bigoplus_{i\geq 1} F_i t^i, \\ R(\mathcal{F})_+(1) &= \bigoplus_{i\geq 0} F_{i+1} t^i, \\ G(\mathcal{F})_+ &= \bigoplus_{i\geq 1} G_i, \quad \text{where} \quad G_i = F_i/F_{i+1} \quad i \geq 0. \end{aligned}$$

Then we have the following:

(1) $\sqrt{F_1 \cdot R(\mathcal{F})} = \sqrt{R(\mathcal{F})_+(1)}$.
(2) $\sqrt{F_i t^i \cdot R(\mathcal{F})} = \sqrt{R(\mathcal{F})_+}$ for each $i \geq 1$.
(3) $\sqrt{G_i \cdot G(\mathcal{F})} = \sqrt{G(\mathcal{F})_+}$ for each $i \geq 1$.
(4) $(G(\mathcal{F})_+)^n \subseteq \bigoplus_{i\geq n} G_i = G_n \cdot G(\mathcal{F})$ for all $n >> 0$.

We use Lemma 2.4 in Section 6.

Lemma 2.4. *Let $(R, \mathbf{m})$ be a Noetherian local ring and let $\mathcal{F} = \{F_i\}_{i\in\mathbb{Z}}$ be an F_1-good filtration of ideals in R. Let $G := G(\mathcal{F}) = \bigoplus_{i\geq 0} F_i/F_{i+1} = \bigoplus_{i\geq 0} G_i$ and $G_+ := \bigoplus_{i\geq 1} F_i/F_{i+1}$. If $\operatorname{grade} G_+ \geq 1$, then for each integer $n \geq 1$ we have:*

(1) *$F_{n+i} : F_i = F_n$ for all $i \geq 1$.*
(2) *$F_n = \cap_{j\geq 1}(F_{n+j} : F_j) = \cup_{j\geq 1}(F_{n+j} : F_j)$.*

Proof. (1) For a fixed $i \geq 1$ we have $G_+^m \subseteq G_i G$ for some $m >> 0$ by Remark 2.3. Therefore $\operatorname{grade} G_i G \geq 1$. It is clear that $F_n \subseteq F_{n+i} : F_i$. Assume there exists $b \in (F_{n+i} : F_i) \setminus F_n$. Then $b \in F_j \setminus F_{j+1}$ for some j with $0 \leq j \leq n-1$, and $0 \neq b^* = b + F_{j+1} \in F_j/F_{j+1} = G_j$. Since $b \in (F_{n+i} : F_i)$, we have $b^* G_i = 0$, and so $b^* G_i G = 0$. This is a contradiction.

(2) Item (2) is immediate from item (1). $\square$

The I-adic filtration $\mathcal{F} = \{I^i\}_{i\in\mathbb{Z}}$ is an I-good filtration. We describe in Examples 2.5 and 2.6 other examples of good filtrations.

Example 2.5. Let I be a proper ideal of a Noetherian ring R. If I contains a non-zero-divisor, then Ratliff and Rush consider in [RR] the following ideal associated to I :

$$\widetilde{I} = \bigcup_{i\geq 1}(I^{i+1} : I^i).$$

The ideal $\widetilde{I}$ is now called the *Ratiliff-Rush* ideal associated to I, or the *Ratliff-Rush closure* of I. It is characterized as the largest ideal having the property that $(\widetilde{I})^n = I^n$ for all sufficiently large positive integers n. Moreover, for each positive integer s

$$\widetilde{I^s} = \bigcup_{i\geq 1}(I^{i+s} : I^i),$$

and there exists a positive integer n such that $\widetilde{I^k} = I^k$ for all integers $k \geq n$ [RR, (2.3.2)]. Consequently, $\mathcal{F} = \{\widetilde{I^i}\}_{i\in\mathbb{N}}$ is a Noetherian I-good filtration.

Example 2.6. Let $(R, \mathbf{m})$ be a Noetherian local ring with $\dim R = d$ and let I be an $\mathbf{m}$-primary ideal. The function $H_I(n) = \lambda(R/I^n)$ is called the Hilbert-Samuel function of I. For sufficiently large values of n, $\lambda(R/I^n)$ is a polynomial $P_I(n)$ in n of degree d, the Hilbert-Samuel polynomial of I. We write this polynomial in terms of binomial coefficients:

$$P_I(n) = e_0(I)\binom{n+d-1}{d} - e_1(I)\binom{n+d-2}{d-1} + \cdots + (-1)^d e_d(I).$$

The coefficients $e_i(I)$ are integers and are called the Hilbert coefficients of I. In particular, the leading coefficient $e_0(I)$ is a positive integer called the multiplicity of I.

As was first shown by Shah in [Sh], if $(R, \mathbf{m})$ is formally equidimensional of dimension $d > 0$ with $|R/\mathbf{m}| = \infty$, then for each integer k in $\{0, 1, \ldots, d\}$ there exists a unique largest ideal $I_{\{k\}}$ containing I and contained in the integral closure $\overline{I}$ such that

$$e_i(I_{\{k\}}) = e_i(I) \quad \text{for} \quad i = 0, 1, \ldots, k.$$

We then have the chain of ideals

$$I = I_{\{d+1\}} \subseteq I_{\{d\}} \subseteq \cdots \subseteq I_{\{1\}} \subseteq I_{\{0\}} = \overline{I}. \tag{1}$$

The ideal $I_{\{k\}}$ is called the k^{th} coefficient ideal of I, or the e_k-ideal associated to I. The ideal $I_{\{0\}}$ is the integral closure $\overline{I}$ of I, and if I contains a regular element, then $I_{\{d\}}$ is the Ratliff-Rush closure of I.

Associated to I and the chain of coefficient ideals given in (1), we have a chain of filtrations

$$\mathcal{F}_{d+1} \subseteq \mathcal{F}_d \subseteq \cdots \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_0, \tag{2}$$

where the filtration $\mathcal{F}_k := \{(I^n)_{\{k\}}\}_{n\in\mathbb{Z}}$, for each k such that $0 \leq k \leq d+1$. In particular, $\mathcal{F}_{d+1} = \{I^n\}_{n\in\mathbb{Z}}$ is the I-adic filtration, and $\mathcal{F}_0 = \{\overline{I^n}\}_{n\in\mathbb{Z}}$ is the filtration given by the integral closures of the powers of I. If I contains a non-zero-divisor, then $\mathcal{F}_d = \{\widetilde{I^n}\}_{n\in\mathbb{Z}}$ is the filtration given by the Ratliff-Rush ideals associated to the powers of I. The filtration $\mathcal{F}_1 = \{(I^n)_{\{1\}}\}_{n\in\mathbb{Z}}$ is called the e_1-closure filtration. In this connection, see also [C1], [C2] and [CPV]. If R is also assumed to be analytically unramified, then each of the filtrations $\mathcal{F}_k := \{(I^n)_{\{k\}}\}_{n\in\mathbb{Z}}$ is an I-good filtration. This follows because the integral closure of the Rees ring $R(I) = R[It]$ in the polynomial ring $R[t]$ is the graded ring $\bigoplus_{n\geq 0} \overline{I^n}t^n$, and a well-known result of Rees [R], [SH, Theorem 9.1.2] implies that $\bigoplus_{n\geq 0} \overline{I^n}t^n$ is a finite $R(I)$-module. Thus $\{\overline{I^n}\}_{n\in\mathbb{Z}}$ is a Noetherian I-good filtration. Moreover, if R is analytically unramified and contains a field and if $(I^n)^*$ denotes the tight closure of I^n, then $\mathcal{F} = \{(I^n)^*\}_{n\in\mathbb{Z}}$ is an I-good filtration.

3. The Cohen-Macaulay property for $G(\mathcal{F})$

Let $(R, \mathbf{m})$ be a Noetherian local ring and let $\mathcal{F} = \{F_i\}_{i\in\mathbb{Z}}$ be a Noetherian filtration on R. For an element $x \in F_1$, let x^* denote the image of x in $G(\mathcal{F})_1 = F_1/F_2$. The element x is called *superficial for* $\mathcal{F}$ if there exists a positive integer c such that $(F_{n+1} : x) \cap F_c = F_n$ for all $n \geq c$. In terms of the associated graded ring $G(\mathcal{F})$, the element x is superficial for $\mathcal{F}$ if and only if the n-th homogeneous component $[0 :_{G(\mathcal{F})} x^*]_n$ of the annihilator of x^* in $G(\mathcal{F})$ is zero for all $n >> 0$. If $\operatorname{grade} F_1 \geq 1$ and x is superficial for $\mathcal{F}$, then x is a regular element of R. For if $u \in R$ and $ux = 0$, then $(F_1)^c u \subseteq \bigcap_n (F_{n+1} : x) \cap F_c = \bigcap_n F_n = 0$. Since $\mathcal{F}$ is a Noetherian filtration, it follows that $u = 0$. A sequence $x_1, \ldots, x_k$ of elements of F_1 is called a *superficial sequence for* $\mathcal{F}$ if x_1 is superficial for $\mathcal{F}$, and x_i is superficial for $\mathcal{F}/(x_1, \ldots, x_{i-1})$ for $2 \leq i \leq k$.

The following well-known fact is useful in working with filtrations.

Fact 3.1. If x^* is a regular element of $G(\mathcal{F})$, then x is a regular element of R and $G(\frac{\mathcal{F}}{(x)}) \cong G(\mathcal{F})/(x^*)$.

We record in Proposition 3.2 a result of Huckaba and Marley that involves what is now called Sally's machine, cf. [RV, Lemma 1.8].

Proposition 3.2. ([HM, Lemma 2.1 and Lemma 2.2]) *Let $(R, \mathbf{m})$ be a Noetherian local ring, let $\mathcal{F} = \{F_i\}_{i \in \mathbb{Z}}$ be a Noetherian filtration on R, and let $x_1, \ldots, x_k$ be a superficial sequence for $\mathcal{F}$. Then the following assertions are true:*

(1) *If* $\operatorname{grade}\big(G(\mathcal{F})_+\big) \geq k$*, then $x_1^*, \ldots, x_k^*$ is a $G(\mathcal{F})$-regular sequence.*
(2) *If* $\operatorname{grade}\big(G\big(\frac{\mathcal{F}}{x_1, \ldots, x_k}\big)_+\big) \geq 1$*, then* $\operatorname{grade}\big(G(\mathcal{F})_+\big) \geq k+1$.

The following result of Huckaba and Marley generalizes to filtrations a result of Valabrega and Valla [VV, Corollary 2.7].

Proposition 3.3. ([HM, Proposition 3.5]) *Let $(R, \mathbf{m})$ be a Noetherian local ring, let $\mathcal{F} = \{F_i\}_{i \in \mathbb{Z}}$ be a Noetherian filtration on R, and let $x_1, \cdots, x_k$ be elements of F_1. The following two conditions are equivalent*:

(1) *$x_1^*, \ldots, x_k^*$ is a $G(\mathcal{F})$-regular sequence.*
(2) (*i*) *$x_1, \ldots, x_k$ is an R-regular sequence, and*
   (*ii*) *$(x_1, \ldots, x_k)R \cap F_i = (x_1, \ldots, x_k)F_{i-1}$ for all $i \geq 1$.*

Remark 3.4. Let $(R, \mathbf{m})$ be a Noetherian local ring and let $\mathcal{F} = \{F_i\}_{i \in \mathbb{Z}}$ be a filtration on R. If there exists a reduction J of $\mathcal{F}$ such that $JF_n = F_{n+1}$ for all $n \geq 1$, then $F_n = F_1^n$ for all n, that is, $\mathcal{F}$ is the F_1-adic filtration.

Proof. For every $n \geq 2$ we have $F_n = JF_{n-1} = J^2F_{n-2} = \cdots = J^{n-1}F_1 \subseteq F_1^n$. □

Corollary 3.5. *Let $(R, \mathbf{m})$ be a Cohen-Macaulay local ring and let $\mathcal{F} = \{F_i\}_{i \in \mathbb{Z}}$ be an F_1-good filtration on R, where F_1 is $\mathbf{m}$-primary. If there exists a reduction J of $\mathcal{F}$ with $\mu(J) = \dim R$ and $JF_n = F_{n+1}$ for all $n \geq 1$, then the associated graded ring $G(\mathcal{F})$ is Cohen-Macaulay.*

Proof. Remark 3.4 implies that $\mathcal{F}$ is the F_1-adic filtration. Hence $G(\mathcal{F})$ is Cohen-Macaulay by [S1, Theorem 2.2] or [VV, Proposition 3.1]. □

Proposition 3.6 is a result proved by D.Q. Viet([Vi, Corollary 2.1]). It generalizes to filtrations a result of Trung and Ikeda ([TI, Theorem 1.1]), and is in the nature of the well-known result of Goto-Shimoda ([GS]).

Let $\mathfrak{a}(G(\mathcal{F})) = \max\{n \mid [\mathrm{H}^d_{\mathfrak{M}}(G(\mathcal{F}))]_n \neq 0\}$ denote the $\mathfrak{a}$-invariant of $G(\mathcal{F})$ ([GW, (3.1.4)]), where $\mathfrak{M}$ is the maximal homogeneous ideal of $R(\mathcal{F})$ and $\mathrm{H}^i_{\mathfrak{M}}(G(\mathcal{F}))$ is the i-th graded local cohomology module of $G(\mathcal{F})$ with respect to $\mathfrak{M}$.

Proposition 3.6. ([Vi, Corollary 2.1]) *Let $(R, \mathbf{m})$ be a d-dimensional Cohen-Macaulay local ring and let $\mathcal{F} = \{F_i\}_{i\in\mathbb{Z}}$ be an F_1-good filtration on R, where F_1 is $\mathbf{m}$-primary. Then the following conditions are equivalent:*

1. *$R(\mathcal{F})$ is Cohen-Macaulay.*
2. *$G(\mathcal{F})$ is Cohen-Macaulay with $\mathfrak{a}(G(\mathcal{F})) < 0$.*

Remark 3.7. Let $(R, \mathbf{m})$ be a d-dimensional Cohen-Macaulay local ring and let $\mathcal{F} = \{F_i\}_{i\in\mathbb{Z}}$ be an F_1-good filtration on R, where F_1 is $\mathbf{m}$-primary. Assume that there exists a reduction J of $\mathcal{F}$ with $\mu(J) = d$. If $R(\mathcal{F})$ is Cohen-Macaulay, then Proposition 3.6 implies that $\mathfrak{a}(G(\mathcal{F})) < 0$. Since $r_J(\mathcal{F}) = r_{(0)}(\mathcal{F}/J) = \mathfrak{a}(G(\mathcal{F}/J)) = \mathfrak{a}(G(\mathcal{F})) + d$, it follows that $r_J(\mathcal{F}) < d$.

Proposition 3.8. *Let $(R, \mathbf{m})$ be a d-dimensional regular local ring and let $\mathcal{F} = \{F_i\}_{i\in\mathbb{Z}}$ be an F_1-good filtration on R, where F_1 is $\mathbf{m}$-primary. Assume there exists a reduction J of $\mathcal{F}$ with $\mu(J) = d$. If $G(\mathcal{F})$ is Cohen-Macaulay, then $r_J(\mathcal{F}) < d$.*

Proof. We have $R(F_1) = \oplus_{n\geq 0} F_1^n t^n \subseteq R(\mathcal{F}) = \oplus_{n\geq 0} F_n t^n \subseteq R[t]$. Since $\mathcal{F} = \{F_i\}_{i\in\mathbb{Z}}$ is an F_1-good filtration, $R(\mathcal{F})$ is a finite $R(F_1)$-module, and thus $R(\mathcal{F})$ is integral over $R(F_1)$. Hence we have $F_1^n \subseteq F_n \subseteq \overline{F_1^n}$, for all $n \geq 0$. Since J is a minimal reduction of F_1, it follows that $\overline{F_1^n} \subseteq J$, for every $n \geq d$ by the Briançon-Skoda theorem ([LS, Theorem 1]). Therefore we have $F_n = F_n \cap J$ for $n \geq d$. Since $G(\mathcal{F})$ is Cohen-Macaulay, Proposition 3.3 shows that $F_n \cap J = JF_{n-1}$. Thus $r_J(\mathcal{F}) < d$. $\square$

Remark 3.9. Let $(R, \mathbf{m})$ be a 2-dimensional Cohen-Macaulay local ring and let $\mathcal{F} = \{F_i\}_{i\in\mathbb{Z}}$ be an F_1-good filtration on R, where F_1 is $\mathbf{m}$-primary.

1. If $R(\mathcal{F})$ is Cohen-Macaulay, then Remark 3.7 and Remark 3.4 imply that $\mathcal{F} = \{F_i\}_{i\in\mathbb{Z}}$ is the F_1-adic filtration.
2. If R is also regular and $G(\mathcal{F})$ is Cohen-Macaulay, then Proposition 3.8 and Remark 3.4 imply that $\mathcal{F} = \{F_i\}_{i\in\mathbb{Z}}$ is the F_1-adic filtration.

Let $(R, \mathbf{m})$ be a d-dimensional Cohen-Macaulay local ring and let $\mathcal{F} = \{F_i\}_{i\in\mathbb{Z}}$ be an F_1-good filtration on R, where F_1 is $\mathbf{m}$-primary. Assume that J is a reduction of $\mathcal{F}$ with $\mu(J) = d$ and let $r_J(\mathcal{F}) = u$ denote the reduction number of $\mathcal{F}$ with respect to J. We determine sufficient conditions for $G(\mathcal{F})$ to be Cohen-Macaulay involving the reduction number u and residuation with respect to J. The dimension one case plays a crucial role, so we consider this case first.

Theorem 3.10. *Let $(R, \mathbf{m})$ be a one-dimensional Cohen-Macaulay local ring and let $\mathcal{F} = \{F_i\}_{i\in\mathbb{Z}}$ be an F_1-good filtration, where F_1 is $\mathbf{m}$-primary. Assume there exists a reduction $J = xR$ of $\mathcal{F}$ with reduction number $r_J(\mathcal{F}) = u$ such that*

$$J : F_{u-i} = J + F_{i+1} \text{ for all } i \text{ with } 0 \le i \le u-1.$$

Then the following two assertions are true:

1. $F_u : F_{u-i} = F_i$ *for* $1 \le i \le u$, *and*
2. $G(\mathcal{F})$ *is a Cohen-Macaulay ring.*

Proof. Notice that $J^j F_u = F_{j+u} = F_j F_u$ for all $j \ge 0$. (*)
To establish item (1), we first prove the following claim.

Claim 3.11. $F_i \subseteq F_u : F_{u-i} \subseteq J + F_i$ *for* $1 \le i \le u$.

Proof of Claim. For $1 \le i \le u$, we have

$$\begin{aligned}
F_i \subseteq F_u : F_{u-i} &\subseteq F_u F_u : F_{u-i} F_u \\
&= J^u F_u : J^{u-i} F_u && \text{by } (*) \\
&= J^i F_u : F_u && \text{since } J = (x) \text{ with } x \text{ a regular element} \\
&\subseteq J^i : F_u \\
&= (J^{i+1} : J) : F_u && \text{since } J = (x) \text{ with } x \text{ regular} \\
&= J^{i+1} : J F_u \\
&= J^{i+1} : F_{u+1} \\
&\subseteq J^{i+1} : J^i F_{u-(i-1)} && \text{since } J^i F_{u-(i-1)} \subseteq F_{u+1} \\
&= J : F_{u-(i-1)} && \text{since } J = (x) \text{ with } x \text{ regular} \\
&= J + F_i && \text{by assumption.}
\end{aligned}$$

This establishes Claim 3.11.

For the proof of (1), we use induction on i. If $i = 1$, the assertion is clear in view of Claim 3.11. Assume that $i \geq 2$. Then we have

$$\begin{aligned}
F_u : F_{u-i} &= (J + F_i) \cap (F_u : F_{u-i}) && \text{by Claim 3.11} \\
&= [J \cap (F_u : F_{u-i})] + [F_i \cap (F_u : F_{u-i})] && \text{since } F_i \subseteq F_u : F_{u-i} \\
&= J((F_u : F_{u-i}) : J) + F_i && \text{since } J = (x) \text{ and } F_i \subseteq F_u : F_{u-i} \\
&= J(F_u : JF_{u-i}) + F_i \\
&\subseteq J(F_uF_u : JF_{u-i}F_u) + F_i \\
&= J(J^uF_u : F_{u+u+1-i}) + F_i && \text{by } (*) \\
&\subseteq J(J^uF_u : J^uF_{u-(i-1)}) + F_i && \text{since } J^uF_{u-(i-1)} \subseteq F_{u+u+1-i} \\
&= J(F_u : F_{u-(i-1)}) + F_i && \text{since } J = (x) \\
&= JF_{i-1} + F_i && \text{by the induction hypothesis} \\
&= F_i.
\end{aligned}$$

This establishes item (1).

For item (2), we show that $J \cap F_i = JF_{i-1}$ for $1 \leq i \leq u$. It is clear that $J \cap F_i \supseteq JF_{i-1}$. We prove that $J \cap F_i \subseteq JF_{i-1}$. For $1 \leq i \leq u$, we have

$$\begin{aligned}
J \cap F_i &= J(F_i : J) && \text{since } J = (x) \text{ with } x \text{ regular} \\
&\subseteq J(F_iF_u : JF_u) \\
&= J(J^iF_u : JF_u) && \text{by } (*) \\
&\subseteq J(J^iF_u : J^iF_{u-(i-1)}) && \text{since } J^iF_{u-(i-1)} \subseteq JF_u \\
&= J(F_u : F_{u-(i-1)}) && \text{since } J = (x) \text{ with } x \text{ regular} \\
&= JF_{i-1} && \text{by item (1).}
\end{aligned}$$

By Proposition 3.3, $G(\mathcal{F})$ is Cohen-Macaulay. □

Theorem 3.12 is the main result of this section.

Theorem 3.12. *Let $(R, \mathbf{m})$ be a d-dimensional Cohen-Macaulay local ring and let $\mathcal{F} = \{F_i\}_{i\in\mathbb{Z}}$ be an F_1-good filtration, where F_1 is $\mathbf{m}$-primary. Assume that J is a reduction of $\mathcal{F}$ with $\mu(J) = d$, and let $u := r_J(\mathcal{F})$ denote the reduction number of $\mathcal{F}$ with respect to J. If*

$$J : F_{u-i} = J + F_{i+1} \text{ for all } i \text{ with } 0 \leq i \leq u - 1,$$

then the associated graded ring $G(\mathcal{F})$ is Cohen-Macaulay.

Proof. We may assume that $R/\mathbf{m}$ is infinite. There is nothing to prove if $d = 0$. If $d = 1$, then $G(\mathcal{F})$ is Cohen-Macaulay by Theorem 3.10. Assume that $d \geq 2$. There exists elements $x_1, \ldots, x_d$ that form a minimal generating set for J and a superficial sequence for $\mathcal{F}$. Set $\overline{R} := R/(x_1, \ldots, x_{d-1})$, $\overline{\mathbf{m}} := \mathbf{m}/(x_1, \ldots, x_{d-1})$, and $\overline{\mathcal{F}} := \mathcal{F}/(x_1, \ldots, x_{d-1}) = \{\overline{F_i}\}_{i\in\mathbb{Z}}$ where $\overline{F_i} = F_i\overline{R}$ for all $i \in \mathbb{Z}$. Then $(\overline{R}, \overline{\mathbf{m}})$ is a 1-dimensional Cohen-Macaulay local ring and $\overline{\mathcal{F}} = \{\overline{F_i}\}_{i\in\mathbb{Z}}$ is an $\overline{F_1}$-good filtration, where $\overline{F_1}$ is $\overline{\mathbf{m}}$-primary. Since J is a minimal reduction of $\mathcal{F}$ with $u := r_J(\mathcal{F})$, $\overline{J} \cdot \overline{F_n} = \overline{F_{n+1}}$ for all $n \geq u$, and hence $\overline{J} = (\overline{x_d})$ is a minimal reduction of $\overline{\mathcal{F}}$ and $\overline{u} := r_{\overline{J}}(\overline{\mathcal{F}}) \leq u$. Finally, we need to check that $\overline{J} : \overline{F_{\overline{u}-i}} = \overline{J} + \overline{F_{i+1}}$ for $0 \leq i \leq \overline{u} - 1$. Since $\overline{u} \leq u$, we have

$$\overline{J} : \overline{F_{\overline{u}-i}} \subseteq \overline{J} : \overline{F_{u-i}} \subseteq \overline{J : F_{u-i}} = \overline{J + F_{i+1}} = \overline{J} + \overline{F_{i+1}}.$$

The other inclusion is shown as follows:

$$(\overline{J} + \overline{F_{i+1}}) \cdot \overline{F_{\overline{u}-i}} = \overline{J} \cdot \overline{F_{\overline{u}-i}} + \overline{F_{i+1}} \cdot \overline{F_{\overline{u}-i}} \subseteq \overline{J} \cdot \overline{F_{\overline{u}-i}} + \overline{F_{\overline{u}+1}} \subseteq \overline{J},$$

and hence $\overline{J} + \overline{F_{i+1}} \subseteq \overline{J} : \overline{F_{\overline{u}-i}}$. By Theorem 3.10, $G(\overline{\mathcal{F}})$ is Cohen-Macaulay. Since $\dim(G(\overline{\mathcal{F}})) = 1$, we have $\operatorname{grade}\big(G(\frac{\mathcal{F}}{(x_1, \cdots, x_{d-1})})_+\big) = 1$, and thus by Proposition 3.2 (2), $\operatorname{grade}(G(\mathcal{F})_+) = d$. Therefore $G(\mathcal{F})$ is Cohen-Macaulay. □

Remark 3.13. The sufficient conditions given in Theorem 3.12 in order that $G(\mathcal{F})$ be Cohen-Macaulay are not necessary conditions. For example, with $R = k[[t^5, t^6, t^9]]$ and $\mathbf{m} = (t^5, t^6, t^9)R$ as in [HKU, Example 3.6], then $G(\mathbf{m})$ is Cohen-Macaulay and the ideal $J = t^5R$ is a minimal reduction of $\mathbf{m}$ with reduction number $r_J(\mathbf{m}) = 3$. However, $t^9 \in (J : \mathbf{m}^2) \setminus J + \mathbf{m}^2$.

4. The Gorenstein property for $G(\mathcal{F})$

In this section, we give a necessary and sufficient condition for $G(\mathcal{F})$ to be Gorenstein. We first state this in dimension zero. Among the equivalences in Theorem 4.2, the equivalence of (1) and (3) are due to Goto and Iai [GI, Proposition, 2.4]. We include elementary direct arguments in the proof. We use the floor function $\lfloor x \rfloor$ to denote the largest integer that is less than or equal to x.

Lemma 4.1. *Let $(R, \mathbf{m})$ be a zero-dimensional Gorenstein local ring and let $\mathcal{F} = \{F_i\}_{i\in\mathbb{Z}}$ be an F_1-good filtration. Assume that $F_u \neq 0$ and $F_{u+1} = 0$, that is, $u = r_{(0)}(\mathcal{F})$. Let $G := G(\mathcal{F}) = \bigoplus_{i=0}^{u} F_i/F_{i+1} = \bigoplus_{i=0}^{u} G_i$ and let $S := \operatorname{Soc}(G) = \bigoplus_{i=0}^{u} S_i$ denote the socle of G. Then the following hold:*

(1) $S_i = \frac{F_i \cap (F_{i+1}:\mathbf{m}) \cap (F_{i+2}:F_1) \cap \cdots \cap (F_{i+u+1}:F_u)}{F_{i+1}}$ *for* $0 \leq i \leq u$.

(2) $S_u = (0 : \mathbf{m}) \cap F_u$.

(3) $S_u \cong R/\,\mathbf{m}$.

Proof. (1): We may assume that $u > 0$. Let $k := R/\,\mathbf{m}$ and write $\mathfrak{M} := \mathbf{m}\,/F_1 \bigoplus G_+$ for the unique maximal homogeneous ideal of G. For $0 \leq i \leq u$ we have

$$\begin{aligned} S_i &= 0 :_{G_i} \mathfrak{M} \\ &= (0 :_{F_i/F_{i+1}} \mathbf{m}\,/F_1) \cap (0 :_{F_i/F_{i+1}} F_1/F_2) \cap \cdots \cap (0 :_{F_i/F_{i+1}} F_u/F_{u+1}) \\ &= \frac{F_i}{F_{i+1}} \cap \frac{(F_{i+1} : \mathbf{m})}{F_{i+1}} \cap \frac{(F_{i+2} : F_1)}{F_{i+1}} \cap \cdots \cap \frac{(F_{i+u+1} : F_u)}{F_{i+1}}. \end{aligned}$$

(2): $S_u = F_u \cap (0 : \mathbf{m})$, because $F_{u+i} = 0$ for $i \geq 1$ and $0 : \mathbf{m} \subseteq 0 : F_1 \subseteq \cdots \subseteq 0 : F_u$.
(3): Since $S_u = 0 :_{F_u} \mathbf{m} \subseteq 0 :_{F_u} F_1 = F_u \neq 0$ and $(R, \mathbf{m})$ is a zero-dimensional Gorenstein local ring, we have $S_u \cong k$. □

Theorem 4.2. *Let $(R, \mathbf{m})$ be a zero-dimensional Gorenstein local ring and let $\mathcal{F} = \{F_i\}_{i \in \mathbb{Z}}$ be an F_1-good filtration. Assume that $F_u \neq 0$ and $F_{u+1} = 0$, that is, $u = r_{(0)}(\mathcal{F})$. Let $G := G(\mathcal{F}) = \bigoplus_{i=0}^{u} F_i/F_{i+1} = \bigoplus_{i=0}^{u} G_i$ and let $S := \mathrm{Soc}(G) = \bigoplus_{i=0}^{u} S_i$ denote the socle of G. The following are equivalent*:

(1) *$G(\mathcal{F})$ is Gorenstein.*

(2) $S_i = 0$ *for* $0 \leq i \leq u - 1$.

(3) $0 : F_{u-i} = F_{i+1}$ *for* $0 \leq i \leq u - 1$.

(4) $0 : F_{u-i} = F_{i+1}$ *for* $0 \leq i \leq \lfloor \frac{u-1}{2} \rfloor$.

(5) $\lambda(G_i) = \lambda(G_{u-i})$ *for* $0 \leq i \leq \lfloor \frac{u-1}{2} \rfloor$.

Proof. (1) $\Longleftrightarrow$ (2): $G(\mathcal{F})$ is Gorenstein if and only if $\dim_k S = 1$ if and only if $S_i = 0$ for $0 \leq i \leq u - i$, by Lemma 4.1.(3).
(2) $\Longrightarrow$ (3): Suppose that $S_i = 0$ for $0 \leq i \leq u - 1$. Then $S = S_u \cong k$, by Lemma 4.1.(3). Hence there exists $0 \neq s^* \in S_u$ such that $S = s^* k$. Let $0 \leq i \leq u - 1$. The containment " $\supseteq$ " is clear, because $F_{u+1} = 0$. To see the other containment, we assume that $0 : F_{u-j} \nsubseteq F_{j+1}$ for some j with $0 \leq j \leq u-1$. In this case there exists an element $\beta \in 0 : F_{u-j}$, but $\beta \notin F_{j+1}$, and hence we can choose an integer υ with $0 \leq \upsilon \leq j$ such that $\beta \in F_\upsilon \setminus F_{\upsilon+1}$. Hence $0 \neq \beta^* = \beta + F_{\upsilon+1} \in F_\upsilon/F_{\upsilon+1}$. Since the graded ring G is an essential extension of $\mathrm{Soc}(G)$, we have $\beta^* G \cap \mathrm{Soc}(G) \neq 0$. Then there exists a non-zero element ξ such that $\xi \in \beta^* G \cap \mathrm{Soc}(G)$. Since $S = S_u = s^* k$, we can express $s^* = \beta^* \omega^* = \beta\omega + F_{u+1}$, for some $\omega \in F_{u-\upsilon}$. Then $\beta\omega \neq 0$, because $s^* \neq 0$. This is impossible, because $\beta \in 0 : F_{u-j}$ and $\omega \in F_{u-\upsilon} \subseteq F_{u-j}$, as $\upsilon \leq j$.

(3) $\Longrightarrow$ (4): This is clear.

(4) $\Longrightarrow$ (5): For $0 \le i \le \lfloor \frac{u-1}{2} \rfloor$, we have

$$\begin{aligned}\lambda(G_{u-i}) &= \lambda(F_{u-i}/F_{u-i+1}) \\ &= \lambda(R/F_{u-i+1}) - \lambda(R/F_{u-i}) \\ &= \lambda(0:F_{u-i+1}) - \lambda(0:F_{u-i}) && \text{by [BH, Proposition 3.2.12]} \\ &= \lambda(F_i) - \lambda(F_{i+1}) && \text{by condition (4)} \\ &= \lambda(F_i/F_{i+1}) = \lambda(G_i).\end{aligned}$$

(5) $\Longrightarrow$ (3): For $0 \le i \le u-1$, we have

$$\begin{aligned}\lambda(F_{i+1}) &= \lambda(F_{i+1}/F_{u+1}) \quad \text{since } F_{u+1} = 0 \\ &= \lambda(G_{i+1}) + \lambda(G_{i+2}) + \cdots + \lambda(G_u) \\ &= \lambda(G_{u-(i+1)}) + \lambda(G_{u-(i+2)}) + \cdots + \lambda(G_{u-u}) && \text{by condition (5)} \\ &= \lambda(R/F_{u-i}) = \lambda(0:F_{u-i}) && \text{by [BH, Proposition 3.2.12].}\end{aligned}$$

Since $F_{u+1} = 0$, we have $F_{i+1} \subseteq 0 : F_{u-i}$ for $0 \le i \le u-1$. We conclude that $F_{i+1} = 0 : F_{u-i}$, because these two ideals have the same length.

(3) $\Longrightarrow$ (2): Let $0 \le i \le u-1$. By Lemma 4.1.(1), we have

$$\begin{aligned}S_i &= \frac{F_i \cap (F_{i+1}:\mathbf{m}) \cap (F_{i+2}:F_1) \cap \cdots \cap (F_u : F_{u-(i+1)}) \cap (F_{u+1}:F_{u-i}) \cap \cdots \cap (F_{i+u+1}:F_u)}{F_{i+1}} \\ &\subseteq \frac{F_{u+1}:F_{u-i}}{F_{i+1}} \\ &= \frac{0:F_{u-i}}{F_{i+1}} && \text{since } F_{u+1} = 0 \\ &= \frac{F_{i+1}}{F_{i+1}} && \text{by condition (3).}\end{aligned}$$

Hence $S_i = 0$ for $0 \le i \le u-1$. □

Theorem 4.3. *Let $(R, \mathbf{m})$ be a d-dimensional Gorenstein local ring and let $\mathcal{F} = \{F_i\}_{i\in\mathbb{Z}}$ be an F_1-good filtration, where F_1 is $\mathbf{m}$-primary. Assume there exists a minimal reduction J of $\mathcal{F}$ such that $\mu(J) = d$, and let $u := r_J(\mathcal{F})$ denote the reduction number of $\mathcal{F}$ with respect to J. The following are equivalent:*

(1) *$G(\mathcal{F})$ is Gorenstein.*
(2) *$J : F_{u-i} = J + F_{i+1}$ for $0 \le i \le u-1$.*
(3) *$J : F_{u-i} = J + F_{i+1}$ for $0 \le i \le \lfloor \frac{u-1}{2} \rfloor$.*

Proof. The equivalence of items (2) and (3) follows from the double annihilator property in the zero-dimensional Gorenstein local ring R/J, see, for example [BH, (3.2.15), p.107]. To prove the equivalence of (1) and (2), by Theorem 3.12, we may assume that $G(\mathcal{F})$ is Cohen-Macaulay. Choose $x_1, \dots, x_d$ in F_1 such that $J = (x_1, \dots, x_d)R$ and $x_1, \dots, x_d$ is a superficial sequence for $\mathcal{F}$. Since $G(\mathcal{F})$ is Cohen-Macaulay, the leading forms $x_1^*, \dots, x_d^*$ in F_1/F_2 are a $G(\mathcal{F})$-regular sequence by Proposition 3.2, and hence we have the isomorphism

$$G(\mathcal{F})/(x_1^*, \dots, x_d^*) \cong G(\mathcal{F}/J)$$

as graded R-algebras. Set $\overline{R} := R/J$, $\overline{\mathbf{m}} := \mathbf{m}/J$, and $\overline{\mathcal{F}} := \mathcal{F}/J = \{\overline{F_i}\}_{i \in \mathbb{Z}}$, where $\overline{F_i} = F_i\overline{R}$ for all $i \in \mathbb{Z}$. Then $(\overline{R}, \overline{\mathbf{m}})$ is a zero-dimensional Gorenstein local ring and $\overline{\mathcal{F}}$ is a $\overline{F_1}$-good filtration with $\overline{F_{u+1}} = 0$ and $\overline{F_u} \neq 0$. To show the last equality suppose that $\overline{F_u} = 0$. In this case $F_u \subseteq J$, and hence $F_u = F_u \cap J = JF_{u-1}$, as $G(\mathcal{F})$ is Cohen-Macaulay. This is impossible since $u := r_J(\mathcal{F})$. Now we have

$$\begin{aligned} G(\mathcal{F}) \quad \text{is Gorenstein} &\iff G(\overline{\mathcal{F}}) \quad \text{is Gorenstein} \\ &\iff 0 : \overline{F_{u-i}} = \overline{F_{i+1}} \quad \text{for} \quad 0 \le i \le u-1 \quad \text{by Theorem 4.2} \\ &\iff J : F_{u-i} = J + F_{i+1} \quad \text{for} \quad 0 \le i \le u-1. \end{aligned}$$

This completes the proof of Theorem 4.3. □

The following is an immediate consequence of Theorem 4.3 for the case of reduction number two.

Corollary 4.4. *Let $(R, \mathbf{m})$ be a d-dimensional Gorenstein local ring and let $\mathcal{F} = \{F_i\}_{i \in \mathbb{Z}}$ be an F_1-good filtration, where F_1 is $\mathbf{m}$-primary. Assume there exists a minimal reduction J of $\mathcal{F}$ such that $\mu(J) = d$ and that $r_J(\mathcal{F}) = 2$. Then:*

$$G(\mathcal{F}) \quad \textit{is Gorenstein} \iff J : F_2 = F_1.$$

Corollary 4.5 deals with the problem of lifting the Gorenstein property of associated graded rings. Notice we are not assuming that $G(\mathcal{F})$ is Cohen-Macaulay.

Corollary 4.5. *Let $(R, \mathbf{m})$ be a d-dimensional Cohen-Macaulay local ring and let $\mathcal{F} = \{F_i\}_{i \in \mathbb{Z}}$ be an F_1-good filtration, where F_1 is $\mathbf{m}$-primary. Assume there exists a minimal reduction J of $\mathcal{F}$ such that $\mu(J) = d$ and that $F_u \nsubseteq J$ for $u := r_J(\mathcal{F})$. Set $\overline{R} := R/J$ and $\overline{\mathcal{F}} := \mathcal{F}/J = \{F_i\overline{R}\}_{i \in \mathbb{Z}}$. If $G(\overline{\mathcal{F}})$ is Gorenstein, then $G(\mathcal{F})$ is Gorenstein.*

Proof. If $G(\overline{\mathcal{F}})$ is Gorenstein, then $\overline{R}$ is Gorenstein, and hence R is also Gorenstein, because $(R, \mathbf{m})$ is Cohen-Macaulay. The condition $F_u \nsubseteq J$ implies that $\overline{F_u} \neq 0$ and

$\overline{F_{u+1}} = 0$. Hence $r_J(\mathcal{F}) = r_{(0)}(\overline{\mathcal{F}})$. The assertion now follows from Theorem 4.2 and Theorem 4.3. □

The following theorem is a special case of a result of Goto and Nishida that characterizes the Gorenstein property of the Rees algebra $R(\mathcal{F})$.

Theorem 4.6. (Goto and Nishida [GN]) *Let $(R, \mathbf{m})$ be a Gorenstein local ring of dimension $d \geq 2$ and let $\mathcal{F} = \{F_i\}_{i\in\mathbb{Z}}$ be an F_1-good filtration, where F_1 is $\mathbf{m}$-primary. Let J be a reduction of $\mathcal{F}$ with $\mu(J) = d$. The following are equivalent:*

(1) *The Rees algebra $R(\mathcal{F})$ is Gorenstein.*
(2) *The associated graded ring $G(\mathcal{F})$ is Gorenstein and $\mathfrak{a}(G(\mathcal{F})) = -2$.*
(3) *The associated graded ring $G(\mathcal{F})$ is Gorenstein and $r_J(\mathcal{F}) = d - 2$.*

In Theorem 4.7 and Corollary 4.9, we generalize to the case of filtrations results of Herrmann-Huneke-Ribbe [HHR, Theorem 2.5]

Theorem 4.7. *Let $(R, \mathbf{m})$ be a regular local ring of dimension $d \geq 2$ and let $\mathcal{F} = \{F_i\}_{i\in\mathbb{Z}}$ be an F_1-good filtration, where F_1 is $\mathbf{m}$-primary. Let J be a reduction of $\mathcal{F}$ with $\mu(J) = d$ and $r_J(\mathcal{F}) = u$. If $G(\mathcal{F})$ is Cohen-Macaulay, then $G(\mathcal{F}/J)$ has a nonzero homogeneous socle element of degree $\leq d - 2$.*

Proof. We have

$$F_j \subseteq F_j : \mathbf{m} \subseteq F_j : F_1 = F_{j-1} \quad \text{for all integers} \quad j, \tag{3}$$

where the last equality holds by Lemma 2.4(1) because $G(\mathcal{F})$ is Cohen-Macaulay. Since J is a reduction of $\mathcal{F}$ with $r_J(\mathcal{F}) = u$, we have $F_j \subseteq J^{j-u}$ for all $j \geq u$, hence

$$F_j : \mathbf{m} \subseteq J^{j-u} : \mathbf{m} \subseteq J^{j-u} : J = J^{j-u-1} \subseteq J,$$

whenever $j \geq u + 1$. Thus there exists an integer $k \geq 1$ such that

$$F_k : \mathbf{m} \nsubseteq F_k + J \quad \text{and} \quad F_j : \mathbf{m} \subseteq F_j + J, \quad \text{for all} \quad j \geq k + 1. \tag{4}$$

Let $v \in (F_k : \mathbf{m}) + J \setminus F_k + J$, then $v \in F_{k-1} + J \setminus F_k + J$ by (3). Thus the image $\overline{v}$ of v in R/J has the property that its leading form $\overline{v}^* \in G(\mathcal{F}/J)$ is a nonzero element in $\left[G(\mathcal{F}/J)\right]_{k-1}$.

Claim 4.8. : $\overline{v}^* \in \operatorname{Soc}\left(G(\mathcal{F}/J)\right)$.

Proof of Claim. Let α be any homogeneous element in $\mathfrak{N}$, where $\mathfrak{N}$ is the unique maximal (homogeneous) ideal of the zero-dimensional graded ring $G(\mathcal{F}/J)$. We show that $\alpha \cdot \overline{v}^* = 0$. We have two cases :

(Case i) : Assume that $\deg \alpha = n \geq 1$. Write $\alpha = y + (F_{n+1} + J)$, where $y \in F_n$. Then we have

$$\begin{aligned} \alpha \cdot \overline{v}^* &= yv + (F_{n+k} + J) \\ &= 0, \end{aligned}$$

since $yv \in F_n((F_k : \mathbf{m}) + J) \subseteq (F_n F_k : \mathbf{m}) + J \subseteq (F_{n+k} : \mathbf{m}) + J \subseteq F_{n+k} + J$, where the last inequality holds by (4).

(Case ii) : Assume that $\deg \alpha = 0$. Then $\alpha = z + (F_1 + J)$, where $z \in \mathbf{m}$, and we have

$$\begin{aligned} \alpha \cdot \overline{v}^* &= zv + (F_k + J) \\ &= 0, \end{aligned}$$

where the last equality holds because $v \in (F_k : \mathbf{m}) + J$ and $z \in \mathbf{m}$. This completes the proof of Claim 4.8.

Since $\mathcal{F}$ is an F_1- good filtration, we have $F_1^n \subseteq F_n \subseteq \overline{F_1^n}$ for all $n \geq 0$, where $\overline{F_1^n}$ denotes the integral closure of F_1^n. Hence $\overline{F_n} \subseteq \overline{F_1^n}$ for all $n \geq 0$. We have

$$F_d : \mathbf{m} \subseteq F_d : \mathbf{m}^{d-1} \subseteq \overline{F_d} : \mathbf{m}^{d-1} \subseteq \overline{F_1^d} : \mathbf{m}^{d-1} \subseteq J,$$

where the last inclusion follows from a result of Lipman [L, Corollary 1.4.4]. Hence we have

$$F_j : \mathbf{m} \subseteq F_d : \mathbf{m} \subseteq J \quad \text{for all} \quad j \geq d.$$

Thus by (4), we have $k \leq d-1$. Therefore $\deg \overline{v}^* = k-1 \leq d-2$. Since $\overline{v}^* \in \operatorname{Soc}(G(\overline{\mathcal{F}}))$ by Claim 4.8, the proof of Theorem 4.7 is complete. □

Corollary 4.9. *Let $(R, \mathbf{m})$ be a regular local ring of dimension $d \geq 2$ and let $\mathcal{F} = \{F_i\}_{i \in \mathbb{Z}}$ be an F_1-good filtration, where F_1 is $\mathbf{m}$-primary. Let J be a reduction of $\mathcal{F}$ with $\mu(J) = d$. If $F_{i+1} \subseteq \mathbf{m} F_i$ for each $i \geq d-1$ and $G(\mathcal{F})$ is Gorenstein, then $r_J(\mathcal{F}) \leq d-2$.*

Proof. Since $G(\mathcal{F})$ is Gorenstein, Proposition 3.3 shows that $G(\mathcal{F}/J)$ is Gorenstein, as well. Hence Theorem 4.7 implies that $[G(\mathcal{F}/J)]_i = 0$ for all $i \geq d-1$. Thus for $i \geq d-1$ we have

$$0 = [G(\mathcal{F}/J)]_i = \frac{F_i + J}{F_{i+1} + J} \cong \frac{F_i}{F_{i+1} + (J \cap F_i)} = \frac{F_i}{F_{i+1} + JF_{i-1}},$$

where the last equality holds again by Proposition 3.3. Thus for all $i \geq d-1$, we have

$$F_i = F_{i+1} + JF_{i-1}, \tag{5}$$

and hence by Nakayama's Lemma, $F_i = JF_{i-1}$ since $F_{i+1} \subseteq \mathbf{m} F_i$. Therefore $r_J(\mathcal{F}) \le d-2$. □

5. Integral closure filtrations of monomial parameter ideals

In this section we examine the integral closure filtration $\mathcal{F}$ associated to a monomial parameter ideal in a polynomial ring. We use Theorem 4.3 to give necessary and sufficient conditions in order that $G(\mathcal{F})$ be Gorenstein. We demonstrate that $G(\mathcal{F})$ and even $R(\mathcal{F})$ may be Gorenstein and yet $\mathcal{F}$ is not an ideal-adic filtration.

Setting 5.1. Let $R := k[x_1, \dots, x_d]$ be a polynomial ring in $d \ge 1$ variables over the field k. Let $a_1, \dots, a_d$ be positive integers and let $J := (x_1^{a_1}, \dots, x_d^{a_d})R$ be a monomial parameter ideal. Let $L := \mathrm{LCM}\{a_1, \dots, a_d\}$ denote the least common multiple of the integers $a_1, \dots, a_d$, and let $\mathcal{F} := \{\overline{J^n}\}_{n \in \mathbb{Z}}$ be the integral closure filtration associated to J. The ideal J has a unique Rees valuation v that is defined as follows: $v(x_i) := L/a_i$ for each i with $1 \le i \le d$. Then for every polynomial $f \in R$ one defines $v(f)$ to be the minimum of the v-value of a nonzero monomial occuring in f (cf. [SH, (10.18), p. 209]). The Rees valuation v determines the integral closure $\overline{J^n}$ of every power J^n of J. We have $\overline{J^n} = \{f \in R \mid v(f) \ge nL\}$. Each of the ideals $\overline{J^n}$ is again a monomial ideal. Let $\mathbf{m} := (x_1, \dots, x_d)R$ denote the graded maximal ideal of R. Notice that $s := x_1^{a_1 - 1} \cdots x_d^{a_d - 1} \in (J : \mathbf{m}) \setminus J$ is a socle element modulo J. Since R is Gorenstein and J is a parameter ideal, we have $(J, s)R = J : \mathbf{m}$, and $s \in K$ for each ideal K of R that properly contains J.

Remark 5.2. The filtrations $\mathcal{F} = \{\overline{J^n}\}_{n \ge 0}$ of Setting 5.1 may also be described as the integral closure filtrations associated to zero-dimensional monomial ideals having precisely one Rees valuation [SH, Theorem 10.3.5].

Lemma 5.3. *Let the notation be as in Setting 5.1. For each integer k, let $I_k := \{f \in R \mid v(f) \ge k\}$. We have :*

(1) *Let $\alpha \in R$ be a monomial, then $\alpha \notin J \iff s \in \alpha R$.*
(2) *Let K be a monomial ideal, then $K \subseteq J \iff s \notin K$.*
(3) *Each I_k is a monomial ideal, and $I_k \subseteq J \iff k \ge v(s) + 1$.*
(4) *The reduction number $r_J(\mathcal{F})$ satisfies $r_J(\mathcal{F}) = u \iff s \in \overline{J^u} \setminus \overline{J^{u+1}}$.*

Proof. For item (1), let $K = (J, \alpha)R$. If $\alpha \notin J$ then $s \in K$. Since K is a monomial ideal, s is a multiple of some monomial generator of K. Since $s \notin J$, we must have

s is a multiple of α. Conversely, if $s \in \alpha R$ then $\alpha \notin J$ because $s \notin J$. Items (2) and (3) follow from item (1). For item (4), a theorem of Hochster implies that $R(\mathcal{F})$ is Cohen-Macaulay [H, Theorem 1], [BH, Theorem 6.3.5(a)]. Therefore $G(\mathcal{F})$ is Cohen-Macaulay, which gives $r_J(\mathcal{F}) = s_J(\mathcal{F}) := \min\{n \mid \overline{J^{n+1}} \subseteq J\}$. Hence by item (2), we have item (4). ☐

Proposition 5.4. *Let the notation be as in Setting 5.1. Write*

$$v(x_1) + v(x_2) + \cdots + v(x_d) = jL + p, \quad \textit{where} \quad j \geq 0 \quad \textit{and} \quad 1 \leq p \leq L.$$

Then the reduction number satisfies $r_J(\mathcal{F}) = d - (j+1)$.

Proof. Observe that

$$\begin{aligned} v(s) &= dL - (v(x_1) + v(x_2) + \cdots + v(x_d)) \\ &= dL - (jL + p) \qquad \text{by hypothesis} \\ &= (d-j)L - p. \end{aligned}$$

Therefore $(d-(j+1))L \leq v(s) < (d-j)L$ and hence $s \in \overline{J^{d-(j+1)}} \setminus \overline{J^{d-j}}$. Thus $r_J(\mathcal{F}) = d - (j+1)$ by Lemma 5.3(4). ☐

Lemma 5.5. *Let the notation be as in Setting 5.1 and let* $\sum_{k=1}^{d} v(x_k) = jL + p$, *where* $j \geq 0$ *and* $1 \leq p \leq L$. *The following are equivalent :*

(1) *The associated graded ring* $G(\mathcal{F})$ *is Gorenstein.*
(2) *For every integer* $i \geq 0$ *and every monomial* $\alpha \in R$ *with* $s \in \alpha R$ *one has*

$$v(\alpha) \leq (i+1)L - 1 \iff v(\alpha) \leq (i+1)L - p.$$

Proof. Let $u := r_J(\mathcal{F})$. Proposition 5.4 shows that $v(s) = (u+1)L - p$. For any monomial $\alpha \in R$ one has

$$\begin{aligned} \alpha \notin J + \overline{J^{i+1}} &\iff \alpha \notin J \quad \text{and} \quad \alpha \notin \overline{J^{i+1}} \\ &\iff s \in \alpha R \quad \text{and} \quad v(\alpha) \leq (i+1)L - 1. \end{aligned}$$

Here we have used Lemma 5.3(1) and the fact that $\overline{J^{i+1}}$ is a monomial ideal.

Likewise,

$$\begin{aligned} \alpha \notin J : \overline{J^{u-i}} &\iff \alpha \overline{J^{u-i}} \nsubseteq J \\ &\iff s \in \alpha \overline{J^{u-i}} \\ &\iff s \in \alpha R \quad \text{and} \quad \frac{s}{\alpha} \in \overline{J^{u-i}} \\ &\iff s \in \alpha R \quad \text{and} \quad v(s) - v(\alpha) \geq (u-i)L \\ &\iff s \in \alpha R \quad \text{and} \quad v(\alpha) \leq (i+1)L - p. \end{aligned}$$

Thus, item (2) above holds if and only if $J + \overline{J^{i+1}} = J : \overline{J^{u-i}}$ for every $i \geq 0$ or, equivalently, for $0 \leq i \leq u-1$. But this means that $G(\mathcal{F})$ is Gorenstein according to Theorem 4.3. □

We thank Paolo Mantero for showing us that $G(\mathcal{F})$ is Gorenstein implies $\sum_{k=1}^{d} v(x_k) \equiv 1 \mod L$ as stated in Theorem 5.6.

Theorem 5.6. *Let the notation be as in Setting 5.1. Then we have*

$$G(\mathcal{F}) \quad \textit{is Gorenstein} \qquad \Longleftrightarrow \qquad \sum_{k=1}^{d} v(x_k) \equiv 1 \mod L.$$

Proof. If $p = 1$, then $G(\mathcal{F})$ is Gorenstein according to Lemma 5.5. To show the converse notice that for $i >> 0$, $(i+1)L-1$ is in the numerical semigroup generated by the relatively prime integers $v(x_1), \ldots, v(x_d)$. As $L = a_k v(x_k)$, we may subtract a multiple of L to obtain $(i+1)L - 1 = c_1 v(x_1) + \cdots + c_d v(x_d)$ for some integer i and c_k integers with $0 \leq c_k \leq a_k - 1$. Clearly $i \geq 0$. Write $\alpha := x_1^{c_1} \cdots x_d^{c_d}$. Now $\alpha \in R$ is a monomial with $s \in \alpha R$ and $v(\alpha) = (i+1)L - 1$. If $G(\mathcal{F})$ is Gorenstein then by Lemma 5.5, $v(\alpha) \leq (i+1)L - p$. Therefore $p \leq 1$, which gives $p = 1$. □

Corollary 5.7. *Let the notation be as in Setting 5.1 and assume that $d \geq 2$. The following are equivalent* :

(1) $\sum_{k=1}^{d} v(x_k) = L + 1$.
(2) *$G(\mathcal{F})$ is Gorenstein and $r_J(\mathcal{F}) = d - 2$.*
(3) *The Rees algebra $R(\mathcal{F})$ is Gorenstein.*

Proof. The equivalence of items (1) and (2) follows from Proposition 5.4 and Theorem 5.6, whereas the equivalence of items (2) and (3) is a consequence of Theorem 4.6. □

Remark 5.8. Assume notation as in Setting 5.1. Since $G(\mathcal{F})$ is Cohen-Macaulay, Proposition 3.8 implies that the maximal value of the reduction number $r_J(\mathcal{F})$ is $d-1$. For every dimension d, the minimal value of $r_J(\mathcal{F})$ is zero as can be seen by taking $a_1 = \cdots = a_{d-1} = 1$. If $d \geq 2$ and all the exponents a_k are assumed to be greater than or equal to 2, then the inequalities $L/2 \geq L/a_k$ along with Lemma 5.3 imply that the possible values of the reduction number $u := r_J(\mathcal{F})$ are all integers u such that $\lfloor \frac{d}{2} \rfloor \leq u \leq d-1$.

Setting 5.9. Let the notation be as in Setting 5.1. Let e be a positive integer and let $y_1, \ldots, y_e$ be indeterminates over R. Let $S := R[y_1, \ldots, y_e]$. Let $b_1, \ldots, b_e$ be positive integers and let $K := (J, y_1^{b_1}, \ldots, y_e^{b_e})S$ be a monomial parameter ideal of S. Let $\mathcal{E} := \{\overline{K^n}\}_{n\geq 0}$ denote the integral closure filtration associated to the ideal K. Let w denote the Rees valuation of K, and let $t := x_1^{a_1-1} \cdots x_d^{a_d-1} y_1^{b_1-1} \cdots y_e^{b_e-1}$ denote the socle element modulo the ideal K.

Remark 5.10 records several basic properties relating to the filtrations $\mathcal{F}$ and $\mathcal{E}$.

Remark 5.10. Assume notation as in Setting 5.1 and 5.9. Then the following hold:

(1) For each positive integer n we have

$$J^n = K^n \cap R \qquad (\overline{J})^n = (\overline{K})^n \cap R \qquad \overline{J^n} = \overline{K^n} \cap R.$$

(2) If $\mathcal{E}$ is an ideal-adic filtration, then $\mathcal{F}$ is an ideal-adic filtration.

(3) The reduction numbers satisfy the inequality $r_J(\mathcal{F}) \leq r_K(\mathcal{E})$.

(4) The Rees valuation w restricted to R defines a valuation that is equivalent to the Rees valuation v, that is, these two valuations determine the same valuation ring.

Corollary 5.11. *Assume notation as in Setting 5.1 and 5.9. For each monomial parameter ideal J of R there exists an extension $S = R[y_1, \ldots, y_e]$ and a monomial parameter ideal $K = (J, y_1^{b_1}, \ldots, y_e^{b_e})S$ such that $G(\mathcal{E})$ is Gorenstein where $\mathcal{E} = \{\overline{K^n}\}_{n\geq 0}$ is the integral closure filtration associated to K.*

Proof. Let $J = (x_1^{a_1}, \ldots, x_d^{a_d})R$, let L be the least common multiple of $a_1, \ldots, a_d$ and let v denote the Rees valuation of J. Write $\sum_{k=1}^{d} v(x_k) = jL + p$, where $j \geq 0$ and $1 \leq p \leq L$. If $p = 1$, then $G(\mathcal{F})$ is Gorenstein by Theorem 5.6 and we can take $S = R$. If $p > 1$, let $e = L - p + 1$ and let $S = R[y_1, \ldots, y_e]$ and $K = (J, y_1^L, \ldots, y_e^L)S$. Then $w(y_k) = 1$ for each k with $1 \leq k \leq e$. Also w restricted to R is equal to v and we have

$$\sum_{k=1}^{d} w(x_k) + \sum_{k=1}^{e} w(y_k) = jL + p + L - p + 1 = (j+1)L + 1.$$

Therefore $G(\mathcal{E})$ is Gorenstein by Theorem 5.6. □

Remark 5.12. With the notation of Corollary 5.11, we have :

(1) If $\sum_{k=1}^{d} v(x_k) = jL + p$, where $1 \leq p \leq L$, then from the construction used in the proof of Corollary 5.11 one may obtain for each positive m a

polynomial extension S and a monomial parameter ideal K of S such that $r_K(\mathcal{E}) = \dim S - (j+m)$, where $\mathcal{E} = \{\overline{K^n}\}_{n\geq 0}$.

(2) If $\sum_{k=1}^{d} v(x_k) \leq L$, then by Corollary 5.7 there exists a monomial parameter ideal $K = (J, y_1^{b_1}, \ldots, y_e^{b_e})S$ such that the Rees algebra $R(\mathcal{E})$ is Gorenstein.

Example 5.13 demonstrates the existence of monomial parameter ideals K such that the integral closure filtration $\mathcal{E} = \{\overline{K^n}\}_{n\geq 0}$ has the following properties:

(1) The reduction number satisfies $r_K(\mathcal{E}) = d - 2$.
(2) The associated graded ring $G(\mathcal{E})$ and the Rees algebra $R(\mathcal{E})$ are Gorenstein.
(3) The filtration $\mathcal{E}$ is not an ideal-adic filtration.

Example 5.13. Let $R = k[x_1, x_2, x_3]$ and let $J = (x_2^2, x_2^3, x_3^7)R$. Then $L = 42$ and $v(x_1) = 21, v(x_2) = 14$ and $v(x_3) = 6$. Thus $\sum_{i=1}^{3} v(x_i) = 41 = L - 1$. Hence $G(\mathcal{F})$ is not Gorenstein. Notice that $r_J(\mathcal{F}) = 2$ and

$$\overline{J} \;=\; (J,\; x_1x_3^4,\; x_1x_2x_3^2,\; x_1x_2^2,\; x_2x_3^5,\; x_2^2x_3^3)R.$$

The element $x_1x_2^2x_3^6 \in \overline{J^2} \;\setminus\; (\overline{J})^2$. Hence the filtration $\mathcal{F} = \{\overline{J^n}\}_{n\geq 0}$ is not an ideal-adic filtration. Let $S = R[y_1, y_2]$ and let $K = (J, y_1^{42}, y_2^{42})S$. Then we have $w(y_1) = w(y_2) = 1$ and $w(x_i) = v(x_i)$ for each i. Hence the sum of the w-values of the variables is equal to $L + 1$. Therefore $G(\mathcal{E})$ is Gorenstein. Notice that also the Rees algebra $R(\mathcal{E})$ is Gorenstein by Corollary 5.7.

Alternatively, one could let $S = R[y_1]$ and let $K = (J, y_1^{21})S$. Again the sum of the w-values of the variables is $L + 1$, so $R(\mathcal{E})$ and $G(\mathcal{E})$ are Gorenstein. In both cases $r_K(\mathcal{E})$ is the dimension of S minus two. In the previous case $r_K(\mathcal{E}) = 3$ and in this case $r_K(\mathcal{E}) = 2$.

6. The Quasi-Gorenstein Property for $R^{'}(\mathcal{F})$

Let $(R, \mathbf{m})$ be a d-dimensional Gorenstein local ring and let $\mathcal{F} = \{F_i\}_{i\in\mathbb{Z}}$ be an F_1-good filtration in R, where $\operatorname{ht}(F_1) = g > 0$. Assume there exists a reduction J of $\mathcal{F}$ with $\mu(J) = g$ and reduction number $u := r_J(\mathcal{F})$. In Theorem 6.1, we prove that the extended Rees algebra $R^{'}(\mathcal{F})$ is quasi-Gorenstein with $\mathbf{a}$-invariant b if and only if $J^n : F_u = F_{n+b-u+g-1}$ for every $n \in \mathbb{Z}$. If $G(\mathcal{F})$ is Cohen-Macaulay, we prove in Theorem 6.2 that the maximal degree of a homogeneous minimal generator of the canonical module $\omega_{G(\mathcal{F})}$ is at most g and that of the canonical module $\omega_{R^{'}(\mathcal{F})}$ is at most $g-1$. With the same hypothesis, we prove in Theorem 6.3 that $R^{'}(\mathcal{F})$ is Gorenstein if and only if $J^u : F_u = F_u$.

Theorem 6.1. *Let $(R, \mathbf{m})$ be a d-dimensional Gorenstein local ring and let $\mathcal{F} = \{F_i\}_{i\in\mathbb{Z}}$ be an F_1-good filtration of ideals in R. Let F_1 be an equimultiple ideal of R with $\operatorname{ht} F_1 = g > 0$ and $J = (x_1, x_2, \cdots, x_g)R \subseteq F_1$ be a minimal reduction of $\mathcal{F}$. Let $R^{'}(\mathcal{F}) = \bigoplus_{i\in\mathbb{Z}} F_i t^i$. Then the following assertions are true.*

1. *$R^{'}(\mathcal{F})$ has the canonical module $\omega_{R^{'}(\mathcal{F})} = \bigoplus_{i\in\mathbb{Z}}(J^{i+u} : F_u)t^{i+(g-1)}$.*
2. *$R^{'}(\mathcal{F})$ is quasi-Gorenstein with $\mathfrak{a}$-invariant $b \iff J^i : F_u = F_{i+b-u+g-1}$ for all $i \in \mathbb{Z}$.*

Proof. (1) Let $K := \operatorname{Quot}(R)$ denote the total ring of quotients of R. Let $A := R[Jt, t^{-1}] \subseteq \mathcal{C} := R^{'}(\mathcal{F}) = \bigoplus_{i\in\mathbb{Z}} F_i t^i$. Notice that $G(J) \cong A/t^{-1}A$, where t^{-1} is a homogeneous A-regular element of degree -1. Since $J = (x_1, x_2, \cdots, x_g)R$ is generated by a regular sequence, $G(J) \cong (R/J)[X_1, X_2, \cdots, X_g]$ is a standard graded polynomial ring in g-variables over a Gorenstein local ring R/J, whence A is Gorenstein and $\omega_A \cong A(-g+1) \cong At^{g-1}$. Since $\mathcal{C}$ is a finite extension of A and $\operatorname{Quot}(A) = \operatorname{Quot}(\mathcal{C}) = K(t)$ ($\because g > 0$), we have that

$$
\begin{aligned}
\omega_{\mathcal{C}} \cong \operatorname{Ext}^0_A(\mathcal{C}, \omega_A) &= \operatorname{Hom}_A(\mathcal{C}, A(-g+1)) \\
&\cong \operatorname{Hom}_A(\mathcal{C}, At^{g-1}) \\
&\cong \operatorname{Hom}_A(\mathcal{C}, A)t^{g-1} \\
&\cong (A :_{K(t)} \mathcal{C})t^{g-1} \\
&= (A :_{R[t,t^{-1}]} \mathcal{C})t^{g-1},
\end{aligned}
$$

where the last equality holds because

$$A :_{K(t)} \mathcal{C} \subseteq A :_{K(t)} A \subseteq A \subseteq R[t, t^{-1}].$$

We have $\bigoplus_{i\in\mathbb{Z}}[\omega_{\mathcal{C}}]_i t^i = \bigoplus_{i\in\mathbb{Z}}[A :_{R[t,t^{-1}]} \mathcal{C}]_i t^{i+g-1}$. Since J is complete intersection and $J^{i+j+1} : J = J^{i+j}$ for all i and j, we have

$$[\omega_{\mathcal{C}}]_i = [A :_{R[t,t^{-1}]} \mathcal{C}]_i = \cap_j (J^{i+j} : F_j) = J^{i+u} : F_u,$$

for all $i \in \mathbb{Z}$. Therefore $\omega_{\mathcal{C}} = \bigoplus_{i\in\mathbb{Z}}[\omega_{\mathcal{C}}]_i t^i = \bigoplus_{i\in\mathbb{Z}}(J^{i+u} : F_u)t^{i+g-1}$.
(2) $\mathcal{C}$ is quasi-Gorenstein with $b := \mathfrak{a}(\mathcal{C})$ if and only if

$$\begin{aligned}\omega_{\mathcal{C}} \cong \mathcal{C}(b) &\Longleftrightarrow \bigoplus_{i\in\mathbb{Z}}[\omega_{\mathcal{C}}]_i t^i = \bigoplus_{i\in\mathbb{Z}}[\mathcal{C}]_{i+b}t^i \\ &\Longleftrightarrow \bigoplus_{i\in\mathbb{Z}}(J^{i+u} : F_u)t^{i+g-1} = \bigoplus_{i\in\mathbb{Z}} F_{i+b}t^i \\ &\Longleftrightarrow \bigoplus_{i\in\mathbb{Z}}(J^{i} : F_u)t^{i+(g-1)-u} = \bigoplus_{i\in\mathbb{Z}} F_{i}t^{i-b} \\ &\Longleftrightarrow J^i : F_u = F_{i+b+(g-1)-u} \quad \text{for all} \quad i \in \mathbb{Z}\,.\end{aligned}$$

This completes the proof of Theorem 6.1. □

Theorem 6.2. *Let $(R, \mathbf{m})$ be a d-dimensional Gorenstein local ring and let $\mathcal{F} = \{F_i\}_{i\in\mathbb{Z}}$ be an F_1-good filtration of ideals in R, where F_1 is an equimultiple ideal with $\operatorname{ht} F_1 = g > 0$ and $J = (x_1, x_2, \cdots, x_g)R \subseteq F_1$ is a minimal reduction of $\mathcal{F}$. Assume that the associated graded ring $G(\mathcal{F})$ is Cohen-Macaulay. Then :*

1. *The maximal degree of a homogeneous minimal generator of $\omega_{G(\mathcal{F})}$ is $\leq g$.*
2. *The maximal degree of a homogeneous minimal generator of $\omega_{R'(\mathcal{F})}$ is $\leq g-1$.*

Proof. (1) Since $J = (x_1, x_2, \cdots, x_g)R$ is an R-regular sequence, $(R/J, \mathbf{m}/J)$ is a Gorenstein local ring of dimension $d - g$. We may assume that $(R/J, \mathbf{m}/J)$ is complete. By Cohen's Structure Theorem [BH, Theorem A.21, page 373], there exists a regular local ring T that maps surjectively onto R/J, say $T \xrightarrow{\phi} R/J$, and hence $R/J \cong T/K$, where $K = \ker\phi$. Let

$$c := \operatorname{codim} K = \dim T - \dim T/K = \dim T - \dim R/J.$$

Then $\dim T = (d-g)+c$. Notice that $G(J) = \bigoplus_{i\geq 0} J_i/J_{i+1} \cong (R/J)[X_1, X_2, \cdots, X_g]$ is a polynimial ring in g-variables over R/J. Let $S = T[X_1, X_2, \cdots, X_g]$. Then we have

$$S \longrightarrow G(J) \longrightarrow G(\mathcal{F}).$$

Since $G(\mathcal{F})$ is a finite $G(J)$-module, $G(\mathcal{F})$ is a finite S-module and by assumption $G(\mathcal{F})$ is Cohen-Macaulay. The graded version of the Auslander-Buchbaum formula implies that $\operatorname{pd}_S G(\mathcal{F}) = c$. Let $\mathbb{H}_\bullet$ be a homogeneous minimal free resolution of $G(\mathcal{F})$ over S

$$\mathbb{H}_\bullet : 0 \longrightarrow H_c \longrightarrow H_{c-1} \longrightarrow \cdots \longrightarrow H_1 \longrightarrow H_0 \longrightarrow G(\mathcal{F}) \longrightarrow 0.$$

Notice that $H_c \neq 0$. Let $\mathbb{E}_\bullet := \operatorname{Hom}_S(\mathbb{H}_\bullet, \omega_S) = \operatorname{Hom}_S(\mathbb{H}_\bullet, S(-g))$. It follows [BH, Corollary 3.3.9] that

$$\mathbb{E}_\bullet : 0 \longrightarrow E_c \longrightarrow E_{c-1} \longrightarrow \cdots \longrightarrow E_1 \longrightarrow E_0 \longrightarrow \omega_{G(\mathcal{F})} \longrightarrow 0.$$

is a homogeneous minimal free resolution of $\omega_{G(\mathcal{F})}$ over S, where

$$E_i = \operatorname{Hom}_S(H_{c-i}, \omega_S) = \operatorname{Hom}_S(H_{c-i}, S(-g))$$

for $0 \leq i \leq c$. Since $H_c = \bigoplus_j^{\text{finite}} S(-j)^{\beta_{cj}} (\neq 0)$, we have

$$E_0 = \operatorname{Hom}_S(H_c, S(-g)) \quad = \bigoplus_j^{\text{finite}} \operatorname{Hom}_S(S,S)(j-g)^{\beta_{cj}} = \bigoplus_j^{\text{finite}} S(j-g)^{\beta_{cj}}.$$

Thus the maximal degree of a homogeneous minimal generator of $\omega_{G(\mathcal{F})}$ is $\leq g-j$ and this is $\leq g$ since $j \geq 0$.

(2) Let $\mathcal{C} = R^{'}(\mathcal{F})$. Since $G(\mathcal{F}) \cong \mathcal{C}\,/t^{-1}\,\mathcal{C}$ and t^{-1} is a non-zero-divisor of $\mathcal{C}$, we have

$$G(\mathcal{F}) \text{ is Cohen-Macaulay} \Longleftrightarrow \mathcal{C} \text{ is Cohen-Macaulay.}$$

By [BH, Corollary 3.6.14], we have

$$\omega_{G(\mathcal{F})} = \omega_{\mathcal{C}\,/t^{-1}\,\mathcal{C}} \cong \Big(\omega_{\mathcal{C}}/t^{-1}\omega_{\mathcal{C}}\Big)(\deg t^{-1}) = \Big(\omega_{\mathcal{C}}/t^{-1}\omega_{\mathcal{C}}\Big)(-1).$$

That is, we have

$$\bigoplus_{i\in\mathbb{Z}} [\omega_{G(\mathcal{F})}]_i = \Big(\omega_{\mathcal{C}}/t^{-1}\omega_{\mathcal{C}}\Big)(-1) \quad = \bigoplus_{i\in\mathbb{Z}} \Big[(\omega_{\mathcal{C}}/t^{-1}\omega_{\mathcal{C}})(-1)\Big]_i = \bigoplus_{i\in\mathbb{Z}} \Big[\omega_{\mathcal{C}}/t^{-1}\omega_{\mathcal{C}}\Big]_{i-1}.$$

Letting $\varrho(-)$ denote maximal degree of a minimal homogeneous generator, by (1), we have

$$\varrho(\omega_{G(\mathcal{F})}) \leq g \Longleftrightarrow \varrho\Big(\omega_{\mathcal{C}}/t^{-1}\omega_{\mathcal{C}}\Big) \leq g-1.$$

Since t^{-1} is a non-zero-divisor on $\omega_{\mathcal{C}}$, the graded version of Nakayama's lemma ([BH, Exercise 1.5.24])implies that $\varrho\ (\omega_{\mathcal{C}}) \leq g-1$. □

Theorem 6.3. *Let $(R, \mathbf{m})$ be a d-dimensional Gorenstein local ring and let $\mathcal{F} = \{F_i\}_{i\in\mathbb{Z}}$ be an F_1-good filtration of ideals in R. Let F_1 be an equimultiple ideal of R with $\operatorname{ht} F_1 = g > 0$, let $J = (x_1, \cdots, x_g)R \subseteq F_1$ be a minimal reduction of $\mathcal{F}$, and let $u := r_J(\mathcal{F})$ be the reduction number of the filtration $\mathcal{F}$ with respect to J. Let $\mathcal{C} := R^{'}(\mathcal{F}) = \bigoplus_{i\in\mathbb{Z}} F_i t^i$. If $G(\mathcal{F})$ is Cohen-Macaulay, then the following conditions are equivalent.*

1. *$R^{'}(\mathcal{F})$ is quasi-Gorenstein.*
2. *$R^{'}(\mathcal{F})$ is Gorenstein.*
3. *$J^u : F_u = F_u$.*

Proof. Since $G(\mathcal{F})$ is Cohen-Macaulay, items (1) and (2) are equivalent.

(1) $\Longrightarrow$ (3) : Since $G(\mathcal{F})$ is Cohen-Macaulay and $G(\mathcal{F}) = \mathcal{C}/t^{-1}\mathcal{C}$, we have $\mathfrak{a}(G(\mathcal{F})) = \mathfrak{a}(\mathcal{C}) + \deg(t^{-1}) = b-1$. By [HZ, Theorem 3.8], $u = r_J(\mathcal{F}) = \mathfrak{a}(G(\mathcal{F})) + \ell(\mathcal{F}) = b-1+g$, where $\ell(\mathcal{F})$ is analytic spread of $\mathcal{F}$. By Theorem 6.1 (2), we have that $J^i : F_u = F_i$ for all $i \in \mathbb{Z}$. In particular, $J^u : F_u = F_u$.

(3) $\Longrightarrow$ (1) : Suppose that $J^u : F_u = F_u$. Let $b = \mathfrak{a}(\mathcal{C})$. Then we have

$$\mathcal{C}(b) = \bigoplus_{i\in\mathbb{Z}}[\mathcal{C}]_{i+b}t^i = \bigoplus_{i\in\mathbb{Z}}[\mathcal{C}]_{i+b+(g-1)}t^{i+(g-1)} = \bigoplus_{i\in\mathbb{Z}}[\mathcal{C}]_{i+u}t^{i+(g-1)} = \bigoplus_{i\in\mathbb{Z}} F_{i+u}t^{i+(g-1)}.$$

By Theorem 6.1 (1), we have

$$\omega_{\mathcal{C}} = \bigoplus_{i\in\mathbb{Z}}(J^{i+u} : F_u)t^{i+(g-1)}.$$

To see $\omega_{\mathcal{C}} \cong \mathcal{C}(b)$, we use :

Claim 6.4. : *$J^{i+u} : F_u = F_{i+u}$ for all $i \in \mathbb{Z}$.*

Proof of Claim. $\supseteq$: For all $i \in \mathbb{Z}$, we have $F_{i+u} \cdot F_u \subseteq F_{i+u+u} = J^{i+u}F_u \subseteq J^{i+u}$, and hence $F_{i+u} \subseteq J^{i+u} : F_u$.

$\subseteq$: We have three cases : (Case i) $i \leq -u$, (Case ii) $-u+1 \leq i \leq -1$, and (Case iii) $i \geq 0$.

Case i : Suppose that $i \leq -u$. Then we have $J^{i+u} : F_u = R : F_u = R = F_{i+u}$.

Case ii : Suppose that $-u+1 \leq i \leq -1$. It is enough to show that $J^{u-j} : F_u \subseteq F_{u-j}$ for $1 \leq j \leq u-1$. In fact, let $\alpha \in J^{u-j} : F_u$ for some j with $1 \leq j \leq u-1$. Then we have $\alpha F_u \subseteq J^{u-j}$, and hence $\alpha J^j F_u \subseteq J^j J^{u-j} = J^u$. Thus we have $\alpha J^j \subseteq J^u : F_u = F_u$, by assumption (3). Therefore we have

$$\begin{aligned}
\alpha &\in F_u : J^j \\
&\subseteq F_u \cdot F_n : J^j F_n \quad \text{for} \quad n >> u \quad (\because J^j F_u = F_{u+j} \quad \text{for all} \quad j \geq 0) \\
&\subseteq F_{u+n} : F_{j+n} \\
&\subseteq F_{u-j} \quad \text{by Lemma 2.4.}
\end{aligned}$$

Case iii : Suppose that $i \geq 0$. It is clear for the case where $i = 0$, by assumption. To complete the case (iii), we use :

Claim 6.5. : *$J^{i+u} : F_u \subseteq J^i(J^u : F_u)$ for all $i \geq 1$.*

Proof of Claim. Since $\omega_{\mathcal{C}}$ is a finite $\mathcal{C}$-module and $\mathcal{C}$ is a finite $A := R[Jt, t^{-1}]$-module, we have that $\omega_{\mathcal{C}}$ is a finite A-module. Let $\{\alpha_1, \alpha_2, \cdots, \alpha_h\}$ be a minimal set of homogeneous generator of $\omega_{\mathcal{C}}$ over A and let $\deg \alpha_j = n_j$ for $1 \leq j \leq h$. By

Theorem 6.2 (2), $\deg \alpha_j \leq g-1$ for $1 \leq j \leq h$. That is, $(g-1)-n_j \geq 0$ for $1 \leq j \leq h$. Hence we have

$$[\omega_{\mathcal{C}}]_{g-1} = \sum_{j=1}^{h} [A]_{(g-1)-n_j}\alpha_j = \sum_{j=1}^{h} J^{(g-1)-n_j}\alpha_j,$$

$$[\omega_{\mathcal{C}}]_{g} = \sum_{j=1}^{h} [A]_{g-n_j}\alpha_j = \sum_{j=1}^{h} J^{(g-1)-n_j}J\alpha_j = J\sum_{j=1}^{h} J^{(g-1)-n_j}\alpha_j = J[\omega_{\mathcal{C}}]_{g-1},$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$[\omega_{\mathcal{C}}]_{g+i} = \sum_{j=1}^{h} [A]_{(g+i)-n_j}\alpha_j = \sum_{j=1}^{h} J^{(g-1)-n_j}J^{i+1}\alpha_j = J^{i+1}\sum_{j=1}^{h} J^{(g-1)-n_j}\alpha_j = J^{i+1}[\omega_{\mathcal{C}}]_{g-1}.$$

Thus $[\omega_{\mathcal{C}}]_{(g-1)+i} = J^i[\omega_{\mathcal{C}}]_{g-1}$ for all $i \geq 0$, and hence $J^{i+u} : F_u = J^i(J^u : F_u)$, which completes the proof of Claim 6.5. The Claim 6.4 implies that

$$\bigoplus_{i\in\mathbb{Z}} (J^{i+u} : F_u)t^{i+(g-1)} = \bigoplus_{i\in\mathbb{Z}} F_{i+b}t^i.$$

Thus $\omega_{\mathcal{C}} \cong \mathcal{C}(b)$, where $b = \mathfrak{a}(\mathcal{C})$. This completes the proof of Theorem 6.3.

□

Corollary 6.6. *Let $(R, \mathbf{m})$ be a d-dimensional Gorenstein local ring and let $\mathcal{F} = \{F_i\}_{i\in\mathbb{Z}}$ be an F_1-good filtration of ideals in R such that F_1 is an equimultiple ideal with $\operatorname{ht} F_1 = g > 0$ and $J = (x_1, \cdots, x_g)R \subseteq F_1$ is a minimal reduction of $\mathcal{F}$ with $u := r_J(\mathcal{F})$. Let $\mathcal{C} := R^{'}(\mathcal{F}) = \bigoplus_{i\in\mathbb{Z}} F_it^i$. Then the following conditions are equivalent.*

1. *$G(\mathcal{F})$ is Gorenstein.*
2. *$R^{'}(\mathcal{F})$ is Gorenstein.*
3. *$G(\mathcal{F})$ is Cohen-Macaulay and $J^u : F_u = F_u$.*

Proof. Since $G(\mathcal{F}) \cong \mathcal{C}/t^{-1}\mathcal{C}$ and t^{-1} is a non-zero-divizor of $\mathcal{C}$, we have (1) $\Longleftrightarrow$ (2), and Theorem 6.3 implies (2) $\Longleftrightarrow$ (3). □

Taking the I-adic fitration $\mathcal{F} = \{I^i\}_{i\in\mathbb{Z}}$, we get the usual definition of reduction number with respect to a minimal reduction of the ideal(i.e., $r_J(I) = r_J(\mathcal{F})$). As another consequence of Theorem 6.3, we obtain a result of Goto and Iai.

Corollary 6.7. ([GI, Theorem 1.4]) *Assume that $(R, \mathbf{m})$ is a Gorenstein local ring and let I be an equmultiple ideal with $\operatorname{ht} I \geq 1$. Let $r = r_J(I)$ be a reduction number with respect to a minimal reduction J of I. Then the following two conditions are equvalent.*

(1) *$G(I)$ is a Gorenstein ring.*
(2) *$G(I)$ is a Cohen-Macaulay ring and $J^r : I^r = I^r$.*

Remark 6.8. Let $(R, \mathbf{m})$ be a Cohen-Macaulay local ring with $\dim R = 1$ and let I be an $\mathbf{m}$-primary ideal. As described in Example 2.5, the Ratliff-Rush filtration $\mathcal{F} = \{\widetilde{I^i}\}_{i\in\mathbb{Z}}$ is an I (and $\widetilde{I}$)-good filtration. Since the ideals $\widetilde{I^i}$ are Ratiliff-Rush ideals, $G(\mathcal{F})_+ = \bigoplus_{i\geq 1}\widetilde{I^i}/\widetilde{I^{i+1}}$ contains a non-zero-divisor, and hence, since $\dim G(\mathcal{F}) = 1$, $G(\mathcal{F})$ is Cohen-Macaulay. Let $J = xR$ be a principal reduction of I. The reduction number $r_J(\mathcal{F})$ is independent of the principal reduction J by [HZ, Proposition 3.6]. Let $s_J(I) = \min\{i \mid I^{i+1} \subseteq J\}$ denote the *index of nilpotency* of I with respect to J. An easy computation shows that $r_J(I) \geq r_J(\mathcal{F}) \geq s_J(I)$.

For R of dimension one, we have the following corollary to Theorem 6.3.

Corollary 6.9. *Let $(R, \mathbf{m})$ be a Gorenstein local ring with $\dim R = 1$, let I be an $\mathbf{m}$-primary ideal, and let $\mathcal{F} = \{\widetilde{I^i}\}_{i\in\mathbb{Z}}$ denote the Ratliff-Rush filtration associated to I. Let $J = xR$ be a principal reduction of I and set $r = r_J(I)$ and $u = r_J(\mathcal{F})$. Then the following conditions are equivalent.*
(1) *$G(\mathcal{F}) = \bigoplus_{i\geq 0}\widetilde{I^i}/\widetilde{I^{i+1}}$ is Gorenstein.*
(2) *$\mathcal{C} := R'(\mathcal{F}) = \bigoplus_{i\in\mathbb{Z}}\widetilde{I^i}t^i$ is Gorenstein.*
(3) *$J^r : \widetilde{I^u} = \widetilde{I^u}$.*
(4) *$J^r : I^r = \widetilde{I^u}$.*

Proof. (1) $\Longleftrightarrow$ (2) : Notice that $G(\mathcal{F}) \cong \mathcal{C}/t^{-1}\mathcal{C}$ and t^{-1} is a non-zero-divisor of $\mathcal{C}$.
(2) $\Longleftrightarrow$ (3) : Apply Corollary 6.6.
(2) $\Longrightarrow$ (4) : Suppose that $\mathcal{C} = \bigoplus_{i\in\mathbb{Z}}\widetilde{I^i}t^i$ is Gorenstein. Then $\mathcal{C}$ is quasi-Gorenstein with $\mathfrak{a}(\mathcal{C}) = r_J(\mathcal{F}) = u$. We have that

$$\omega_{\mathcal{C}} \cong \bigoplus_{i\in\mathbb{Z}}(J^{i+r} : \widetilde{I^r})t^i = \bigoplus_{i\in\mathbb{Z}}(J^{i+r} : I^r)t^i,$$

since $I^i = \widetilde{I^i}$ for all $i \geq r$. Hence $J^r : I^r = \widetilde{I^{r+b-r}} = \widetilde{I^u}$, where $u = \mathfrak{a}(\mathcal{C}) = b$.
(4) $\Longrightarrow$ (2) : Suppose that $J^r : I^r = \widetilde{I^u}$. We have that

$$\omega_{\mathcal{C}} \cong \bigoplus_{i\in\mathbb{Z}}(J^{i+r} : \widetilde{I^r})t^i = \bigoplus_{i\in\mathbb{Z}}(J^{i+r} : I^r)t^i.$$

To see that $\mathcal{C}$ is Gorenstein, it suffices to show that $\omega_{\mathcal{C}} \cong \mathcal{C}(u)$. That is, we need to prove the following claim :

Claim 6.10. : *$J^{i+r} : I^r = \widetilde{I^{i+u}}$ for all $i \in \mathbb{Z}$.*

Proof of Claim : Notice that $r := r_J(I) \geq u := r_J(\mathcal{F})$. There is nothing to show in the case where $r = u$, and hence we consider only the case where $r > u$.

$\supseteq$: Since $\widetilde{I^{i+u}} I^r = \widetilde{I^{i+u}} \widetilde{I^r} \subseteq \widetilde{I^{i+u+r}} = I^{i+u+r} = J^{i+r} I^u \subseteq J^{i+r}$, we have $\widetilde{I^{i+u}} \subseteq J^{i+r} : I^r$ for all $i \in \mathbb{Z}$.

$\subseteq$: Let $p := r - u \geq 1$. We have four cases : (Case i) $i \leq -r$, (Case ii) $-r+1 \leq i \leq -r+p (= -u)$, (Case iii) $-u+1 \leq i \leq -1$, and (Case iv) $i \geq 0$.

Case i : Suppose that $i \leq -r$. Then $J^{i+r} : I^r = R : I^r = R = I^{i+u}$, since $r > u$.

Case ii : Suppose that $-r+1 \leq i \leq -r+p$. It is enough to show that $J^j : I^r \subseteq \widetilde{I^{j+u-r}}$ for all $1 \leq j \leq p$. In fact, let $\alpha \in J^j : I^r$ for all $1 \leq j \leq p$. Then $\alpha I^r \subseteq J^j$, and hence $\alpha J^{r-j} I^r \subseteq J^{r-j} J^j = J^r$. Thus we have $\alpha J^{r-j} \subseteq J^r : I^r = \widetilde{I^u}$, by assumption (4). Therefore

$$\begin{aligned}
\alpha \in \widetilde{I^u} : J^{r-j} &\subseteq \widetilde{I^u} I^r : J^{r-j} I^r \\
&\subseteq \widetilde{I^{u+r}} : J^{r-j} I^r \\
&= I^{u+r} : I^{2r-j} \\
&\subseteq \widetilde{I^{j+u-r}} \quad \text{by the fact : } \quad \widetilde{I^k} = \cup_{n \geq 1} (I^{n+k} : I^n).
\end{aligned}$$

Case iii : Suppose that $-u+1 \leq i \leq -1$. It is enough to show that $J^{r-j} : I^r \subseteq \widetilde{I^{u-j}}$ for all $1 \leq j \leq u-1$. In fact, let $\alpha \in J^{r-j} : I^r$ for all $1 \leq j \leq u-1$. Then $\alpha I^r \subseteq J^{r-j}$, and hence $\alpha J^j I^r \subseteq J^j J^{r-j} = J^r$. Thus we have $\alpha J^j \subseteq J^r : I^r = \widetilde{I^u}$, by assumption (4). Therefore

$$\begin{aligned}
\alpha \in \widetilde{I^u} : J^j &\subseteq \widetilde{I^u} I^r : J^j I^r \\
&\subseteq \widetilde{I^{u+r}} : J^j I^r \\
&= I^{u+r} : I^{r+j} \\
&\subseteq \widetilde{I^{u-j}} \quad \text{by the fact : } \quad \widetilde{I^k} = \cup_{n \geq 1} (I^{n+k} : I^n).
\end{aligned}$$

Case iv : Suppose that $i \geq 0$. The claim is clear in the case where $i = 0$. For $i > 0$, we have

$$\begin{aligned}
J^{i+r} : I^r &= J^i (J^r : I^r) \\
&= J^i \widetilde{I^u} \quad \text{by assumption (4)} \\
&= \widetilde{I^{i+u}}.
\end{aligned}$$

This completes the proof of Claim 6.10.

By Claim 6.10, we have

$$\omega_{\mathcal{C}} = \bigoplus_{I \in \mathbb{Z}} (J^{i+r} : I^r) t^i = \bigoplus_{I \in \mathbb{Z}} \widetilde{I^{i+u}} t^i \cong \bigoplus_{i \in \mathbb{Z}} [\mathcal{C}]_{i+u} t^i = \mathcal{C}(u).$$

Thus $\mathcal{C} = \bigoplus_{I \in \mathbb{Z}} \widetilde{I^i} t^i$ is quasi-Gorenstein with $\mathfrak{a}(\mathcal{C}) = u$. This completes the proof of Corollary 6.9. $\square$

7. Examples of filtrations

We first present three examples of one-dimensional Gorenstein local domains constructed as follows. Let k be a field and let $0 < n_1 < n_2 < n_3$ be integers with $\mathrm{GCD}(n_1, n_2, n_3) = 1$. Consider the subring $R = k[[s^{n_1}, s^{n_2}, s^{n_3}]]$ of the formal power series ring $k[[s]]$. Notice that R is a numerical semigroup ring associated to the numerical semigroup $H = \langle n_1, n_2, n_3 \rangle$. The *Frobenius number* of a numerical semigroup H is the largest integer not in H.

We consider the Gorenstein property of the associated graded ring $G(\mathcal{F}_i)$ for $i = 0, 1, 2$, where

(1) $\mathcal{F}_0 := \{\overline{\mathbf{m}^i}\}_{i \geq 0}$ is the integral closure filtration associated to $\mathbf{m}$,
(2) $\mathcal{F}_1 := \{\widetilde{\mathbf{m}^i}\}_{i \geq 0}$ is the Ratliff-Rush filtration associated to $\mathbf{m}$,
(3) $\mathcal{F}_2 := \{\mathbf{m}^i\}_{i \geq 0}$ is the $\mathbf{m}$-adic filtration.

The examples below will demonstrate that these filtrations are independent of each other, as far as the Gorenstein property of their associated graded rings is concerned. Notice that $\mathbf{m}^i \subseteq \widetilde{\mathbf{m}^i} \subseteq \overline{\mathbf{m}^i}$ for all $i \geq 0$ and $G(\mathcal{F}_2) = G(\mathbf{m}) = \bigoplus_{i \geq 0} \mathbf{m}^i / \mathbf{m}^{i+1}$. In Examples 7.1, 7.3 and 7.4, we let $S = k[[x, y, z]]$ be the formal power series ring in three variables x, y, z over a field k and $\mathbf{n} := (x, y, z)S$.

Example 7.1. ([GHK, Example 5.5]) Let $R = k[[s^{3m}, s^{3m+1}, s^{6m+3}]]$, where $2 \leq m \in \mathbb{Z}$ and define a homomorphism of k-algebras

$$\varphi : S \longrightarrow R \quad \text{by} \quad \varphi(x) = s^{3m}, \quad \varphi(y) = s^{3m+1}, \text{ and } \quad \varphi(z) = s^{6m+3}.$$

Then the ideal $I = \ker \varphi$ is generated by $f = zx - y^3$ and $g = z^m - x^{2m+1}$, whence R is a complete intersection of dimension one. We have $G(\mathbf{n}) = k[X, Y, Z]$ and $I^* = (XZ, Z^m, Y^3 Z^{m-1}, Y^6 Z^{m-2}, \cdots, Y^{3(m-1)} Z, Y^{3m}) G(\mathbf{n})$. Since $\sqrt{I^* : Z} = (X, Y, Z)$, the associated graded ring

$$G(\mathbf{m}) \cong k[X, Y, Z]/(XZ, Z^m, Y^3 Z^{m-1}, Y^6 Z^{m-2}, \cdots, Y^{3(m-1)} Z, Y^{3m})$$

is not Cohen-Macaulay, see also [GHK, Theorem 5.1], and hence is not Gorenstein. Thus $\mathcal{F}_2 \neq \mathcal{F}_1$, by [HLS, (1.2)]. The reduction number of $\mathbf{m} = (s^{3m}, s^{3m+1}, s^{6m+3})R$ with respect to the principal reduction $J = (s^{3m})R$ is $3m - 1$ and the blowup of $\mathbf{m}$ is $R[\frac{\mathbf{m}}{s^{3m}}] = \frac{\mathbf{m}^{3m-1}}{s^{3m(3m-1)}}$ ([HLS, Fact 2.1]). Since $s = s^{3m+1}/s^{3m} \in \frac{\mathbf{m}}{s^{3m}}$, the blowup of $\mathbf{m}$ is $\overline{R} = k[[s]]$, the integral closure of R. Hence $\mathcal{F}_1 = \mathcal{F}_0$, by [HLS, Corollary 2.7]. Notice that $\widetilde{\mathbf{m}^i} = (s^{3m})^i k[[s]] \cap R$ for all $i \geq 0$. We observe that the reduction number $r_J(\mathcal{F}_1)$ of $\mathcal{F}_1$ with respect to the principal reduction $J = (s^{3m})R$ is $2m$. For $\alpha \in k[[s]]$, we denote by $\operatorname{ord}(\alpha)$ the order of α as a power series in s. Since $\widetilde{\mathbf{m}^i} = \{\alpha \in R \mid \operatorname{ord}(\alpha) \geq (3m)i\}$, and the Frobenius number of the numerical semigroup of R is $6m^2 - 1$, we have $\widetilde{\mathbf{m}^{i+1}} \subseteq J$ and $J\widetilde{\mathbf{m}^i} = \widetilde{\mathbf{m}^{i+1}}$ for every $i \geq 2m$. Furthermore, $s^{6m^2+3m-1} \in \widetilde{\mathbf{m}^{2m}}$, but $s^{6m^2+3m-1} = s^{3m}s^{6m^2-1} \notin J$, which shows $\widetilde{\mathbf{m}^{2m}} \nsubseteq J$. Hence $r_J(\mathcal{F}_1) = 2m$.

Claim 7.2. *$G(\mathcal{F}_1)$ is a Gorenstein ring.*

Proof of Claim. By Corollary 6.9, it suffices to show that

$$J^u : \widetilde{\mathbf{m}^u} = \widetilde{\mathbf{m}^u}, \quad \text{where} \quad u := r_J(\mathcal{F}_1).$$

Since $u := r_J(\mathcal{F}_1) = 2m$, the inclusion "$\supseteq$" is clear. To show the reverse inclusion, it suffices to prove : $\beta \in R \backslash \widetilde{\mathbf{m}^{2m}} \Longrightarrow \beta \notin (J^{2m} : \widetilde{\mathbf{m}^{2m}})$. Let $\beta \in R \backslash \widetilde{\mathbf{m}^{2m}}$, that is, $\beta \in R$ with $\operatorname{ord}(\beta) < 6m^2$. Let $n_\beta := \operatorname{ord}(\beta)$, where $0 \leq n_\beta < 6m^2$. Then $\sigma := s^{6m^2+6m^2-n_\beta-1} \in \widetilde{\mathbf{m}^{2m}}$, since $\operatorname{ord}(\sigma) = 6m^2 + (6m^2 - n_\beta) - 1 \geq 6m^2 + 1 - 1 = 6m^2$. Hence $\beta\sigma = s^{n_\beta} \cdot s^{6m^2+6m^2-n_\beta-1} = s^{6m^2+(6m^2-1)} = (s^{3m})^{2m} \cdot s^{6m^2-1} \notin J^{2m}$, since the Frobenius number of the numerical semigroup of R is $6m^2 - 1$.

Example 7.3. Let $R = k[[s^4, s^6, s^7]]$ and define a homomorphism of k-algebras

$$\varphi : S \longrightarrow R \quad \text{by} \quad \varphi(x) = s^4, \quad \varphi(y) = s^6, \text{ and } \quad \varphi(z) = s^7.$$

Then the ideal $I = \ker \varphi$ is generated by $f = x^3 - y^2$ and $g = z^2 - x^2y$, whence R is a complete intersection of dimension one. We have $G(\mathbf{n}) = k[X, Y, Z]$ and $I^* = (Y^2, Z^2)$. Hence $G(\mathbf{m}) \cong k[X, Y, Z]/(Y^2, Z^2)$ is a Gorenstein ring. In particular $\mathcal{F}_2 = \mathcal{F}_1$ by [HLS, (1.2)]. The reduction number of $\mathbf{m} = (s^4, s^6, s^7)R$ with respect to the principal reduction $J = (s^4)R$ is 2 and the blowup of $\mathbf{m}$ is $R[\frac{\mathbf{m}}{s^4}] = \frac{\mathbf{m}^2}{s^8} = k[[s^2, s^3]]$, which is not equal to the integral closure $\overline{R} = k[[s]]$ of R . Hence $\mathcal{F}_1 \neq \mathcal{F}_0$, by [HLS, Corollary 2.7]. Notice that $\overline{\mathbf{m}^i} = (s^4)^i k[[s]] \cap R$ for all $i \geq 0$. The reduction number $r_J(\mathcal{F}_0)$ of $\mathcal{F}_0$ with respect to the principal reduction $J = (s^4)R$ is 3. Indeed, since $\overline{\mathbf{m}^i} = \{\alpha \in R \mid \operatorname{ord}(\alpha) \geq 4i\}$ we conclude that $\overline{\mathbf{m}^{i+1}} \subseteq J$ for every $i \geq 3$ and hence $J\overline{\mathbf{m}^i} = \overline{\mathbf{m}^{i+1}}$. On the other hand $s^{13} \in \overline{\mathbf{m}^3} \backslash J\overline{\mathbf{m}^2}$. Therefore $r_J(\mathcal{F}_0) = 3$. Since $s^6 \in (J : \overline{\mathbf{m}^2}) \backslash (J + \overline{\mathbf{m}^2})$, we have $J : \overline{\mathbf{m}^2} \neq J + \overline{\mathbf{m}^2}$. Thus $G(\mathcal{F}_0)$ is not Gorenstein by Theorem 4.3.

We thank YiHuang Shen for suggesting to us Example 7.4.

Example 7.4. Let $R = k[[s^6, s^{11}, s^{27}]]$ and define a homomorphism of k-algebras

$$\varphi : S \longrightarrow R \quad \text{by} \quad \varphi(x) = s^6, \quad \varphi(y) = s^{11}, \text{ and } \quad \varphi(z) = s^{27}.$$

Then the ideal $I = \ker \varphi$ is generated by $f = z^2 - x^9$ and $g = xz - y^3$, whence R is a complete intersection of dimension one. We have $G(\mathbf{n}) = k[X, Y, Z]$ and $I^* = (Z^2, ZX, ZY^3, Y^6)$. Since $\sqrt{I^* : X} = (X, Y, Z)$, the associated graded ring

$$G(\mathbf{m}) \cong k[X, Y, Z]/(Z^2, ZX, ZY^3, Y^6)$$

is not a Cohen-Macaulay ring, also see [GHK, Theorem 5.1], and hence is not a Gorenstein ring. Furthermore $\mathcal{F}_2 \neq \mathcal{F}_1$ by [HLS, (1.2)]. The reduction number of $\mathbf{m} = (s^6, s^{11}, s^{27})R$ with respect to the principal reduction $J = (s^6)R$ is 5 and the blowup of $\mathbf{m}$ is $R[\frac{\mathbf{m}}{s^6}] = \frac{\mathbf{m}^5}{s^{30}} = k[[s^5, s^6]]$, which is not equal to the integral closure $\overline{R} = k[[s]]$ of R. Hence $\mathcal{F}_1 \neq \mathcal{F}_0$ by [HLS, Corollary 2.7]. We observe that

$$\begin{aligned} \widetilde{\mathbf{m}^2} &= ks^{27} + \mathbf{m}^2 \\ \widetilde{\mathbf{m}^3} &= ks^{38} + ks^{49} + \mathbf{m}^3 \\ \widetilde{\mathbf{m}^4} &= ks^{49} + \mathbf{m}^4 \qquad \text{and} \\ \widetilde{\mathbf{m}^i} &= \mathbf{m}^i \qquad \text{for every } i \geq 5. \end{aligned}$$

The reduction number $r_J(\mathcal{F}_1)$ of $\mathcal{F}_1$ with respect to the principal reduction $J = (s^6)R$ is 4, since $J\widetilde{\mathbf{m}^i} = \widetilde{\mathbf{m}^{i+1}}$ for every $i \geq 4$, but $s^{49} \notin \widetilde{\mathbf{m}^4} \backslash J\widetilde{\mathbf{m}^3}$. We have that $J + \widetilde{\mathbf{m}^2} \subseteq J : \widetilde{\mathbf{m}^3} \subseteq \mathbf{m}$, where the first inclusion holds since $r_J(\mathcal{F}_1) = 4$. Furthermore $\lambda(\mathbf{m} / J + \widetilde{\mathbf{m}^2}) = 1$, because $\mathbf{m} = ks^{11} + J + \widetilde{\mathbf{m}^2}$. Since the Frobenius number of the numerical semigroup of R is 43 we have $s^{11}s^{38} = s^6 s^{43} \notin J$, and therefore $s^{11} \notin J : \widetilde{\mathbf{m}^3}$. Hence $G(\mathcal{F}_1)$ is Gorenstein by Theorem 4.3. The reduction number $r_J(\mathcal{F}_0)$ of $\mathcal{F}_0$ with respect to the principal reduction $J = (s^6)R$ is 6, since $J\overline{\mathbf{m}^i} = \overline{\mathbf{m}^{i+1}}$ for every $i \geq 6$, but $s^{38} \in \overline{\mathbf{m}^6} \backslash J\overline{\mathbf{m}^5}$. As $s^{17} \in (J : \overline{\mathbf{m}^4}) \backslash (J + \overline{\mathbf{m}^3})$, we obtain $J : \overline{\mathbf{m}^4} \supsetneq J + \overline{\mathbf{m}^3}$. Therefore $G(\mathcal{F}_0)$ is not Gorenstein by Theorem 4.3.

YiHuang Shen proves in [S, Theorem 4.12] that if $(R, \mathbf{m})$ is a numerical semigroup ring with $\mu(\mathbf{m}) = 3$ such that $r_J(\mathbf{m}) = s_J(\mathbf{m})$, then the associated graded ring $G(\mathbf{m})$ is Cohen-Macaulay. The following example given by Lance Bryant shows that this does not hold for one-dimension Gorenstein local rings of embedding dimension three.

Example 7.5. Let $(S, \mathbf{n})$ be a 3-dimensional regular local ring with $\mathbf{n} = (x, y, z)S$ and $S/\mathbf{n} = k$. Let $I = (f, g)$, where $f = x^3 + z^5$ and $g = x^2 y + xz^3$. Put $R := S/I$ and $\mathbf{m} := \mathbf{n}/I$. Then $(R, \mathbf{m})$ is an 1-dimensional Gorenstein local ring. We have

$G(\mathbf{n}) = k[X,Y,Z]$, $f^* = X^3$, and $g^* = X^2Y$. Let $h = -yf + xg$, $\xi_4 = z^3 f - xh$, and $\xi_5 = z^3 g - yh$. Then $h^* = X^2Z^3$, $\xi_4^* = XYZ^5$, and $\xi_5^* = Y^2Z^5 + XZ^6$. let

$$K = (X^3, X^2Y, X^2Z^3, XYZ^5, Y^2Z^5 + XZ^6) \subseteq I^*.$$

Then the Hilbert series of the graded ring $G(\mathbf{n})/K$ is

$$\frac{1 + 2t + 3t^2 + 2t^3 + 2t^4 + t^5 + 2t^6}{1 - t} = 1 + 3t + 6t^2 + 8t^3 + 10t^4 + 11t^5 + 13t^6 + 13t^7 + \cdots$$

and these values are the same as those in the Hilbert series of $G(\mathbf{m}) = G(\mathbf{n})/I^*$, so that $K = I^*$. Since $(I^* : X)$ is primary to the unique homogeneous maximal ideal $(X,Y,Z)G(\mathbf{n})$, $G(\mathbf{m})$ is not Cohen-Macaulay and hence not Gorenstein. Thus $\mathcal{F}_2 \neq \mathcal{F}_1$ by [HLS, (1.2)]. Let $J = (y - z)R$. Then J is a minimal reduction of $\mathbf{m}$. A computation shows that $r_J(\mathcal{F}_2) = r_J(\mathcal{F}_1) = s_J(\mathcal{F}_2) = 6$. By Corollary 6.9, to see that $G(\mathcal{F}_1)$ is Gorenstein, it suffices to show that $(J^6 : \mathbf{m}^6) = \mathbf{m}^6$. To check this, it is enough to show that $\lambda(R/\mathbf{m}^6) = 39 = \frac{(6)(13)}{2}$, where $13 = e(R)$ is the multiplicity of R.

Since R is not reduced, the filtration $\mathcal{F}_0$ is not a good filtration ([SH, Theorem 9.1.2]) so, in particular, $\mathcal{F}_0 \neq \mathcal{F}_1$.

We present examples of 2-dimensonal Gorenstein local rings $(R, \mathbf{m})$ and consider the Gorenstein property of the associated graded rings $G(\mathcal{F}_i)$ for $i = 0, 1, 2, 3$, where

1. $\mathcal{F}_0 := \{\overline{\mathbf{m}^i}\}_{i \geq 0}$ is the integral closure filtration associated to $\mathbf{m}$,
2. $\mathcal{F}_1 := \{(\mathbf{m}^i)_{\{1\}}\}_{i \geq 0}$ is the e_1-closure filtration associated to $\mathbf{m}$,
3. $\mathcal{F}_2 := \{\widetilde{\mathbf{m}^i}\}_{i \geq 0}$ is the Ratliff-Rush filtration associated to $\mathbf{m}$,
4. $\mathcal{F}_3 := \{\mathbf{m}^i\}_{i \geq 0}$ is the $\mathbf{m}$-adic filtration.

Notice that $\mathbf{m}^i \subseteq \widetilde{\mathbf{m}^i} \subseteq (\mathbf{m}^i)_{\{1\}} \subseteq \overline{\mathbf{m}^i}$ for all $i \geq 0$ and $G(\mathcal{F}_3) = G(\mathbf{m}) = \bigoplus_{i \geq 0} \mathbf{m}^i / \mathbf{m}^{i+1}$.

Lemma 7.6 is useful in considering the e_1-closure filtration in a 2-dimensional Noetherian local ring $(R, \mathbf{m})$. For an $\mathbf{m}$-primary ideal F of R, let $P_F(s)$ denote the Hilbert-Samuel polynomial having the property that $\lambda(R/F^s) = P_F(s)$ for all $s >> 0$. We write

$$P_F(s) = e_0(F)\binom{s+1}{2} - e_1(F)\binom{s}{1} + e_2(F).$$

Lemma 7.6. *Let $(R, \mathbf{m})$ be a 2-dimensional Noetherian local ring and let $\mathcal{F} = \{F_i\}_{i \in \mathbb{Z}}$ be an F_1-good filtration, where F_1 is an $\mathbf{m}$-primary ideal. If there exists a positive integer c such that $\lambda(F_i/F_1^i) < c$ for all $i \geq 0$, then the Hilbert coefficients* *of the polynomials* $P_{F_1^i}(s)$ *and* $P_{F_i}(s)$ *satisfy*

$$e_0(F_1^i) = e_0(F_i) \quad \textit{and} \quad e_1(F_1^i) = e_1(F_i) \quad \textit{for all} \quad i \geq 0.$$

Therefore $(F_1^i)_{\{1\}} = (F_i)_{\{1\}}$ *for all* $i \geq 0$.

Proof. Fix $i \geq 1$, we have $(F_1^i)^s \subseteq (F_i)^s \subseteq F_{is}$ for all $s \geq 1$. Our hypothesis implies

$$c > \lambda(F_{is}/(F_1^i)^s) \geq \lambda((F_i)^s/(F_1^i)^s) \geq 0 \quad \text{for all} \quad s \geq 1.$$

For all sufficiently large s, we have

$$\begin{aligned} c > \lambda((F_i)^s/(F_1^i)^s) &= \lambda(R/(F_1^i)^s) - \lambda(R/(F_i)^s) \\ &= P_{F_1^i}(s) - P_{F_i}(s). \end{aligned}$$

Thus $P_{F_1^i}(s) - P_{F_i}(s)$ is a constant polynomial, which implies $e_0(F_1^i) = e_0(F_i)$ and $e_1(F_1^i) = e_1(F_i)$. □

Example 7.7. Let k be a field of characteristic other than 2 and set $S = k[[x, y, z, w]]$ and $\mathbf{n} = (x, y, z, w)S$, where x, y, z, w are indeterminates over k. Let

$$f = x^2 - w^4,$$
$$g = xy - z^3.$$

Let $I = (f, g)S$, $R = S/I$, and $\mathbf{m} = \mathbf{n}/I$. Since f, g is a regular sequence, R is a 2-dimensional Gorenstein local ring. We have:

(1) $\mathcal{F}_3 = \mathcal{F}_2 \neq \mathcal{F}_1 = \mathcal{F}_0$.
(2) $G(\mathcal{F}_3)$ is not Gorenstein and $r_J(\mathcal{F}_3) = 5$, where $J = (y, w)R$.
(3) $G(\mathcal{F}_0)$ is Gorenstein and $r_J(\mathcal{F}_0) = 4$, where $J = (y, w)R$.

Proof. The associated graded ring $G := \mathrm{gr}_{\mathbf{n}}(S) = k[X, Y, Z, W]$ is a polynomial ring in 4 variables over the field k, and $G(\mathcal{F}_3) = G(\mathbf{m}) = G/I^*$, where I^* is the leading form ideal of I in $G = \mathrm{gr}_{\mathbf{n}}(S)$. One computes that

$$I^* = (X^2, XY, XZ^3, Z^6 + Y^2W^4)G.$$

Thus $G/I^* = G(\mathbf{m})$ is a 2-dimensional standard graded ring of depth one. Notice that W is $G(\mathbf{m})$-regular. The ring $G(\mathbf{m})$ is not Cohen-Macaulay, and hence $G(\mathbf{m})$ is not Gorenstein. We also have $\mathcal{F}_3 = \mathcal{F}_2$ by [HLS, (1.2)], and $r_J(\mathbf{m}) = 5$, where $J = (y, w)R$.

Set

$$\begin{aligned} T &= \frac{k[x,y,z,w]}{(x^2-w^4, xy-z^3)}, \\ L_1 &= ((y,z,w)+(x))T, \\ L_2 &= ((y,z,w)^2+(x))T, \\ L_3 &= ((y,z,w)^3+x(z,w))T, \\ L_n &= ((y,z,w)^n+xw^{n-4}(z,w)^2)T, \quad \text{for all} \quad n \geq 4. \end{aligned}$$

Then T is 2-dimensional, Gorenstein, excellent and reduced, since the characteristic of the field k is other than 2. The ring T becomes a positively graded k-algebra if we set

$$\deg(x) = 2, \quad \deg(y) = \deg(z) = \deg(w) = 1.$$

With this grading it turns out that $L_n = \bigoplus_{i \geq n}[T]_i$, for all $n \geq 1$. In particular $L_1^n \subseteq L_n$, and since the image in T of x is integral over L_2^2 it follows that L_n is integral over L_1^n. As T is reduced, the ideal $L_n = \bigoplus_{i \geq n}[T]_i$ is integrally closed, and since T is excellent, L_nR remains integrally closed in R, the completion of T with respect to the homogeneous maximal ideal. We conclude that $\overline{\mathbf{m}^n} = \overline{L_1^n R} = L_n R$ for every $n \geq 1$

The reduction number $r_J(\mathcal{F}_0)$ of $\mathcal{F}_0$ with respect to $J = (y,w)R$ is 4, since $J\overline{\mathbf{m}^i} = \overline{\mathbf{m}^{i+1}}$ for all $i \geq 4$, whereas $xz^2 \in \overline{\mathbf{m}^4} \backslash J\overline{\mathbf{m}^3}$. We have that $J + \overline{\mathbf{m}^2} \subseteq J : \overline{\mathbf{m}^3} \subseteq J + \overline{\mathbf{m}}$, where the first inclusion holds because $r_J(\mathcal{F}_0) = 4$. Notice that $J + \overline{\mathbf{m}^2} = (x,y,w,z^2)R$ and $J + \overline{\mathbf{m}} = (x,y,w,z)R$. This implies that $\lambda(J + \overline{\mathbf{m}}/J + \overline{\mathbf{m}^2}) = 1$. Since $z \cdot xz \notin J$ and $xz \in \overline{\mathbf{m}^3}$, $z \notin J : \overline{\mathbf{m}^3}$ and hence $J : \overline{\mathbf{m}^3} = J + \overline{\mathbf{m}^2}$. Thus $G(\mathcal{F}_0)$ is a Gorenstein ring, by Theorem 4.3. One computes that $\lambda(\overline{\mathbf{m}^i}/\mathbf{m}^i) \leq 3$ for all $i \geq 0$. By Lemma 7.6, we have $(\mathbf{m}^i)_{\{1\}} = (\overline{\mathbf{m}^i})_{\{1\}}$ for all $i \geq 1$. Since $\overline{\mathbf{m}^i} \subseteq (\overline{\mathbf{m}^i})_{\{1\}} \subseteq \overline{\overline{\mathbf{m}^i}}$, it follows that $(\mathbf{m}^i)_{\{1\}} = \overline{\mathbf{m}^i}$ for all $i \geq 1$. That is, $\mathcal{F}_1 = \mathcal{F}_0$. Since $G(\mathcal{F}_0)$ is Gorenstein, but $G(\mathcal{F}_3)$ is not, we also deduce that $\mathcal{F}_0 \neq \mathcal{F}_3$. □

Example 7.8. Let $S = k[[x,y,z,w]]$ be a formal power series ring over a field k and $\mathbf{n} = (x,y,z,w)S$, where x,y,z,w are indeterminates over k. Let

$$\begin{aligned} f &= x^2 - w^5, \\ g &= xy - z^3. \end{aligned}$$

Let $I = (f, g)S$, $R = S/I$, and $\mathbf{m} = \mathbf{n}/I$. Since f, g is a regular sequence, R is a 2-dimensional Gorenstein local ring. Set $\mathcal{F} = \{F_i\}_{i \geq 0}$, where

$$\begin{aligned} F_0 &= R, \\ F_1 &= \mathbf{m}, \\ F_2 &= ((y,z,w)^2 + (x))R, \\ F_3 &= ((y,z,w)^3 + x(z,w))R, \\ F_i &= ((y,z,w)^i + xw^{i-4}(z,w)^2)R, \quad \text{for all} \quad i \geq 4. \end{aligned}$$

Then :

(1) $\mathcal{F}$ is a F_1-good filtration.
(2) $G(\mathbf{m})$ is not Gorenstein and $r_J(\mathbf{m}) = 5$, where $J = (y, w)R$.
(3) $G(\mathcal{F})$ is Gorenstein and $r_J(\mathcal{F}) = 4$, where $J = (y, w)R$ and $G(\mathcal{F})$ is not reduced.
(4) $\mathcal{F} = \{(\mathbf{m}^i)_{\{1\}}\}_{i \geq 0}$ is the e_1-closure filtration associated to $\mathbf{m}$.

Proof. The associated graded ring $G := \mathrm{gr}_{\mathbf{n}}(S) = k[X, Y, Z, W]$ is a polynomial ring in 4 variables over the field k, and $G(\mathbf{m}) = G/I^*$, where I^* is the leading form ideal of I in $G = \mathrm{gr}_{\mathbf{n}}(S)$. One computes that

$$I^* = (X^2, XY, XZ^3, Z^6)G.$$

Thus $G/I^* = G(\mathbf{m})$ is a 2-dimensional standard graded ring of depth one. Notice that W is $G(\mathbf{m})$-regular. The ring $G(\mathbf{m})$ is not Cohen-Macaulay, and hence $G(\mathbf{m})$ is not Gorenstein. Also we have $\mathbf{m}^i = \widetilde{\mathbf{m}^i}$ for all $i \geq 1$, by [HLS, (1.2)] and $r_J(\mathbf{m}) = 5$, where $J = (y, w)R$. One computes that $F_1F_1 \subsetneq F_2$ and $F_iF_j = F_{i+j}$ for all $i, j \geq 1$ with $i + j \geq 3$, by using the relations $x^2 = w^5$ and $xy = z^3$ in R. Hence $\mathcal{F}$ is a F_1-good filtration. The reduction number $r_J(\mathcal{F})$ of $\mathcal{F}$ with respect to $J = (y, w)R$ is 4 and $G(\mathcal{F})$ is a Gorenstein ring, by the same argument in the proof of Example 7.7. $G(\mathcal{F})$ is not reduced, since $x^* \in F_2/F_3$ is a non-zero nilpotent element in $G(\mathcal{F})$. For $x \in F_2 \backslash F_3$, $(x^*)^2 = x^2 + F_5 = w^5 + F_5 = 0$, since $w^5 \in F_5$. One computes that $\lambda(F_i/F_1^i) \leq 3$ for all $i \geq 0$. By Lemma 7.6, we have $(F_1^i)_{\{1\}} = (F_i)_{\{1\}}$ for all $i \geq 1$. Since $G(\mathcal{F})$ is Cohen-Macaulay, the extended Rees ring $R'(\mathcal{F})$ is Cohen-Macaulay and hence satisfies (S_2). Therefore by [CPV, Theorem 4.2], we have $F_i = (F_i)_{\{1\}} = (F_1^i)_{\{1\}} = (\mathbf{m}^i)_{\{1\}}$ for all $i \geq 1$. $\square$

Example 7.9. ([CHRR, Example 5.1]) Let k be a field of characteristic other than 2 or 3 and set $S = k[[x,y,z,w]]$ and $\mathbf{n} = (x,y,z,w)S$, where x,y,z,w are indeterminates over k. Let

$$f = z^2 - (x^3 + y^3),$$
$$g = w^2 - (x^3 - y^3).$$

Let $I = (f,g)S$, $R = S/I$, and $\mathbf{m} = \mathbf{n}/I$. Since f,g is a regular sequence, R is a 2-dimensional Gorenstein local ring. Notice that R is also a normal domain. We have:

(1) $\mathcal{F}_3 = \mathcal{F}_2 = \mathcal{F}_1 \neq \mathcal{F}_0$.
(2) $G(\mathcal{F}_3)$ is Gorenstein and $r_J(\mathcal{F}_3) = 2$, where $J = (x,y)R$.
(3) $G(\mathcal{F}_0)$ is not Gorenstein and $r_J(\mathcal{F}_0) = 3$, where $J = (x,y)R$.

Proof. The associated graded ring $G(\mathbf{n}) = k[X,Y,Z,W]$ is a polynomial ring in 4 variables over the field k, and the associated graded ring $G(\mathcal{F}_3) = G(\mathbf{m}) = G/I^*$, where I^* is the leading form ideal of I in G. One computes that $I^* = (Z^2, W^2)G$. Thus $G/I^* = G(\mathbf{m})$ is Gorenstein. In particular the extended Rees ring $R'(\mathcal{F})$ is Cohen-Macaulay, and hence by [CPV, Theorem 4.2], $\mathcal{F}_3 = \mathcal{F}_2 = \mathcal{F}_1$. Also we have $r_J(\mathbf{m}) = 2$, where $J = (x,y)R$, since $zw \in \mathbf{m}^2 \setminus J\,\mathbf{m}$ and $J\,\mathbf{m}^2 = \mathbf{m}^3$.

Set

$$T = \frac{k[x,y,z,w]}{(z^2 - (x^3+y^3), w^2 - (x^3 - y^3))},$$
$$L_1 = ((x,y) + (z,w))T,$$
$$L_2 = ((x,y)((x,y) + (z,w)) + (zw))T,$$
$$L_n = ((x,y)^{n-1}((x,y) + (z,w)) + (x,y)^{n-3}(zw))T \quad \text{for all} \quad n \geq 3.$$

The ring T becomes a positively graded k-algebra if we set

$$\deg(x) = \deg(y) = 2 \quad \text{and} \quad \deg(z) = \deg(w) = 3.$$

Since the characteristic of the field k is not equal to 2 or 3, the ring T is a 2-dimensional Gorenstein excellent normal domain. Notice that

$$[T]_0 = k,\ [T]_1 = (0),\ [T]_2 = \langle x,y\rangle,\ [T]_3 = \langle z,w\rangle,\ [T]_4 = \langle x,y\rangle^2,$$
$$[T]_{2n-1} = \langle x,y\rangle^{n-2}\langle z,w\rangle, \quad [T]_{2n} = \langle x,y\rangle^n + \langle x,y\rangle^{\lfloor \frac{n}{2} \rfloor}\langle zw\rangle \quad \text{for all} \quad n \geq 3,$$

where $\lfloor * \rfloor$ denotes the floor function, $\langle * \rangle$ stands for k vector space spanned by $*$, and power denotes symmetric power. From this one sees that $L_n = \bigoplus_{i \geq 2n} [T]_i$. In particular $L_1^n \subseteq L_n$, and since the image in T of zw is integral over L_1^3 it follows that L_n is integral over L_1^n. We deduce, as in the proof of Example 7.7, that $\overline{L_1^n} = L_n$, and then $\overline{\mathbf{m}^n} = L_nR$ for every $n \geq 1$. The reduction number $r_J(\mathcal{F}_0)$ of $\mathcal{F}_0$ with respect to $J = (x,y)R$ is 3, since $J\overline{\mathbf{m}^i} = \overline{\mathbf{m}^{i+1}}$ for all $i \geq 3$, but $zw \in \overline{\mathbf{m}^3}\backslash J\overline{\mathbf{m}^2}$. Since z and w are in $J : \mathbf{m}^2$, we obtain $J : \mathbf{m}^2 = \mathbf{m}$. We have $J + \overline{\mathbf{m}^2} = (x,y,zw)R$, whereas $J : \overline{\mathbf{m}^2} = \mathbf{m}$ because z and w are in $J : \overline{\mathbf{m}^2}$. Therefore $J + \overline{\mathbf{m}^2} \subsetneq J : \overline{\mathbf{m}^2}$, and then Theorem 4.3 shows that $G(\mathcal{F}_0)$ is not Gorenstein. In particular $\mathcal{F}_3 \neq \mathcal{F}_0$ since $G(\mathcal{F}_3)$ is Gorenstein. □

Remark 7.10. Let $(R, \mathbf{m})$ be a 2-dimensional regular local ring.

(1) Let $\mathcal{F} = \{F_i\}_{i\in\mathbb{Z}}$ be an F_1-good filtration, where F_1 is $\mathbf{m}$-primary. If $G(\mathcal{F})$ is Gorenstein, then $\mathcal{F}$ is the F_1-adic filtration and F_1 is a complete intersection.

(2) Let I be an $\mathbf{m}$-primary ideal. If $G(\overline{I})$ is Gorenstein, then the coefficient ideal filtrations $\mathcal{F}_3 \subseteq \mathcal{F}_2 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_0$ associated to I are all the same.

Proof. (1): We may assume that the residue field of R is infinite., in which case $\mathcal{F}$ has a reduction J which is a complete intersection. If $G(\mathcal{F})$ is Cohen-Macaulay then $r_J(\mathcal{F}) \leq 1$ according to Proposition 3.8, hence $\mathcal{F}$ is the F_1-adic filtration by Remark 3.4. If in addition $G(\mathcal{F})$ is Gorenstein, we claim that $r_J(I) \neq 1$ for $I = F_1$. Indeed, suppose $r_J(I) = 1$. In this case Theorem 4.3 implies that $J : I = I$, hence $\frac{J:I}{J} = \frac{I}{J}$. However, $\frac{J:I}{J} \cong \mathrm{Hom}_R(R/I, R/J) \cong \mathrm{Ext}^2_R(R/I, R)$, and using a minimal free R-resolution of R/I one sees that the minimal number of generators of the latter module is $\mu(I) - 1$. On the other hand, $\mu(I/J) = \mu(I) - 2$ since J is a minimal reduction of I. This contradiction proves that $r_J(I) = 0$, hence $I = J$ is a complete intersection.

(2): We apply part (1) to the filtration $\mathcal{F} = \{\overline{I^i}\}_{i\in\mathbb{Z}}$ and use the fact that a complete intersection has no proper reduction. □

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