PROPERTIES OF THE FIBER CONE OF IDEALS IN LOCAL RINGS

William J. Heinzer
Department of Mathematics, Purdue University
West Lafayette, Indiana 47907 USA
E-mail: heinzer@math.purdue.edu

Mee-Kyoung Kim
Department of Mathematics, Sungkyunkwan University
Jangangu Suwon 440-746, Korea
E-mail: mkkim@math.skku.ac.kr

ABSTRACT. For an ideal I of a Noetherian local ring $(R, \mathbf{m})$ we consider properties of I and its powers as reflected in the fiber cone $F(I)$ of I. In particular, we examine behavior of the fiber cone under homomorphic image $R \to R/J = R'$ as related to analytic spread and generators for the kernel of the induced map on fiber cones $\psi_J : F_R(I) \to F_{R'}(IR')$. We consider the structure of fiber cones $F(I)$ for which $\ker \psi_J \neq 0$ for each nonzero ideal J of R. If $\dim F(I) = d > 0$, $\mu(I) = d+1$ and there exists a minimal reduction J of I generated by a regular sequence, we prove that if $\operatorname{grade}(G_+(I)) \geq d-1$, then $F(I)$ is Cohen-Macaulay and thus a hypersurface.

1. Introduction

For an ideal I in a Noetherian local ring $(R, \mathbf{m})$, the *fiber cone* of I is the graded ring

$$F(I) = \bigoplus_{n \geq 0} F_n = \bigoplus_{n \geq 0} I^n / \mathbf{m}\, I^n \cong R[It]/ \mathbf{m}\, R[It],$$

where $R[It]$ is the Rees ring of I and $F_n = I^n / \mathbf{m}\, I^n$. We sometimes write $F_R(I)$ to indicate we are considering the fiber cone of the ideal I of the ring R. In terms of the height, $\operatorname{ht}(I)$, of I and the dimension, $\dim R$, of R, one always has the inequalities $\operatorname{ht}(I) \leq \dim F(I) \leq \dim R$.

For an arbitrary ideal $I \subseteq \mathbf{m}$ of $(R, \mathbf{m})$, the fiber cone $F(I)$ has the attractive property of being a finitely generated graded ring over the residue field $k := R/\mathbf{m}$

Date: January 9, 2003.
1991 *Mathematics Subject Classification.* Primary: 13A30, 13C05; Secondary: 13E05, 13H15.
Key words and phrases. fiber cone, analytic spread , multiplicity, Cohen-Macaulay ring.
We thank Bernd Ulrich for helpful conversations about the topics considered here.
The second author is supported by Korea Research Foundation Grant (KRF-2000-015-DP0005).

that is generated in degree one, i.e., $F_n = F_1^n$ for each positive integer n, so $F(I) = k[F_1]$.

It is well known in this setting that the Hilbert function $H_F(n)$ giving the dimension of $I^n/\mathbf{m}I^n$ as a vector space over k is defined for n sufficiently large by a polynomial $h_F(X) \in \mathbb{Q}[X]$, the *Hilbert polynomial* of $F(I)$ [Mat, Corollary, page 95], [AM, Corollary 11.2]. A simple application of Nakayama's lemma, [Mat, Theorem 2.2], shows that the cardinality of a minimal set of generators of I^n, $\mu(I^n)$, is equal to $\lambda(I^n/\mathbf{m}I^n)$, the value of the Hilbert function $H_F(n)$ of $F(I)$.

An interesting invariant of the ideal I is its analytic spread, denoted $\ell(I)$, where the *analytic spread* of I is by definition the dimension of the fiber cone, $\ell(I) = \dim F(I)$ [NR]. The analytic spread measures the asymptotic growth of the minimal number of generators of I^n as a function of n. In relation to the degree of the Hilbert polynomial, we have the equality $\ell(I) = 1 + \deg h_F(X)$. An ideal $J \subseteq I$ is said to be a *reduction* of I if there exists a positive integer n such that $JI^n = I^{n+1}$. It then follows that $J^iI^n = I^{n+i}$ for every postive integer i. If J is a reduction of I, then J requires at least $\ell(I)$ generators. If the residue field $R/\mathbf{m}$ is infinite, then minimal reductions of I correspond to Noether normalizations of $F(I)$ in the sense that $a_1, \ldots, a_r \in I - I^2$ generate a minimal reduction of I if and only if their images $\overline{a_i} \in I/\mathbf{m}I \subseteq F(I)$ are algebraically independent over $R/\mathbf{m}$ and $F(I)$ is integral over the polynomial ring $(R/\mathbf{m})[\overline{a_1}, \ldots, \overline{a_r}]$. In particular, if $R/\mathbf{m}$ is infinite, then there exist $\ell(I)$-generated reductions of I,

For a positive integer s, the fiber cone $F(I^s)$ of the ideal I^s embeds in the fiber cone $F(I) = \oplus_{n=0}^{\infty} F_n$ of I by means of $F(I^s) \cong \oplus_{n=0}^{\infty} F_{ns}$. This isomorphism makes $F(I)$ a finitely generated integral extension of $F(I^s)$. Thus $\dim F(I) = \dim F(I^s)$ and $\ell(I) = \ell(I^s)$.

We are particularly interested in conditions that imply the fiber cone $F(I)$ is a hypersurface. Suppose $\dim F(I) = d > 0$ and $\mu(I) = d + 1$. If I has a reduction generated by a regular sequence and if $\operatorname{grade}(G_+(I)) \geq d - 1$, we prove in Theorem 5.6 that $F(I)$ is a hypersurface. We have learned from Bernd Ulrich that this result also follows from results in the paper [CGPU] of Corso-Ghezzi-Polini-Ulrich.

A useful property of the analytic spread $\ell(I)$ is that it gives an upper bound on the number of elements needed to generate I up to radical. This property of generation up to radical behaves well with respect to analytic spread of a homomorphic image in the following sense:

Lemma 1.1. *Suppose $I \subseteq \mathbf{m}$ is an ideal of a Noetherian local ring $(R, \mathbf{m})$, where $R/\mathbf{m}$ is infinite. Let $a \in I$ and let $R' := R/aR$ and $I' := IR'$. If $a_1', \ldots, a_s' \in R'$*

are such that $\operatorname{rad}(a'_1, \dots, a'_s)R' = \operatorname{rad} I'$ *and if* $a_i \in R$ *is a preimage of* a'_i, *then* $\operatorname{rad} I = \operatorname{rad}(a_1, \dots, a_s, a)R$. *In particular, if* $\ell(I') = s$, *then* I *can be generated up to radical by* $s + 1$ *elements.*

Proof. Assume that $\operatorname{rad}(a'_1, \dots, a'_s)R' = \operatorname{rad} I'$. If $x \in \operatorname{rad} I$, then for some positive integer n, we have $x^n = y \in I$. Hence the image y' of y in R' is in $\operatorname{rad}(a'_1, \dots, a'_s)R'$. Therefore y and hence also x is in $\operatorname{rad}(a_1, \dots, a_s, a)R$. □

Examples given by Huckaba in [Hu, Examples 3.1 and 3.2] establish the surprising fact of the existence of 3-generated height-2 prime ideals I of a 3-dimensional regular local ring R for which $\dim F(I) = 3 = \dim R$ and for which there exists a principal ideal $J = xR \subseteq I$ such that if $R' := R/xR$ and $I' := IR'$, then $\dim F_{R'}(I') = 1 < \dim R' = \dim R - 1$. This result of Huckaba shows that a statement analogous to Lemma 1.1 for reductions, rather than generators up to radical, is false, that is, it is possible that $I' = I/aR$ has an s-generated reduction while every reduction of I requires at least $s + 2$ generators.

These interesting examples are the original motivation for our interest in the behavior of analytic spread in a homomorphic image.

2. Behavior of the fiber cone under homomorphic image.

Setting 2.1. *Let* $J \subseteq \mathbf{m}$ *be an ideal of a Noetherian local ring* $(R, \mathbf{m})$, *let* $R' := R/J$, *and let* $\mathbf{m}' = \mathbf{m}/J$. *For an ideal* $I \subseteq \mathbf{m}$ *of* R *let* $I' = (I + J)/J = IR'$ *denote the image of* I *in* R'. *There is a canonical surjective ring homomorphism of the fiber cone* $F_R(I)$ *of* I *onto the fiber cone* $F_{R'}(I')$.

We have $R[It] = \bigoplus_{n \geq 0} I^n t^n$ and $R'[I't] = \bigoplus_{n \geq 0} (I')^n t^n$. Since

$$(I')^n = (I^n + J)/J \cong I^n/(I^n \cap J),$$

there is a canonical surjective homomorphism of graded rings $\phi_J : R[It] \to R'[I't]$, with $\ker \phi_J = \bigoplus_{n \geq 0} (I^n \cap J)t^n$.

Since $F_R(I) = R[It]/\mathbf{m} R[It]$ and $F_{R'}(I') = R'[I't]/\mathbf{m}' R'[I't]$, the homomorphism $\phi_J : R[It] \to R'[I't]$ induces a surjective homomorphism $\psi_J : F_R(I) \to F_{R'}(I')$ which preserves grading. This is displayed in the following commutative diagram for which the rows are exact and the column maps are surjective:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \oplus_{n\geq 0}(I^n\cap J)t^n & \longrightarrow & R[It] & \xrightarrow{\phi_J} & R'[I't] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \oplus_{n\geq 0}\frac{(I^n\cap J)+\mathbf{m}\, I^n}{\mathbf{m}\, I^n} & \longrightarrow & \frac{R[It]}{\mathbf{m}\, R[It]} & \xrightarrow{\psi_J} & \frac{R'[I't]}{\mathbf{m}'\, R'[I']} & \longrightarrow & 0 \end{array}$$

Since we are interested in the behavior of the fiber cone under homomorphic image, we are especially interested in

$$\ker\psi_J = \oplus_{n\geq 0}\frac{(I^n\cap J)+\mathbf{m}\, I^n}{\mathbf{m}\, I^n}$$

Remark 2.2. *Let $(R,\mathbf{m})$ be a Noetherian local ring and let $I\subseteq\mathbf{m}$ be an ideal of R. Suppose $J_1\subseteq J_2\subseteq\mathbf{m}$ are ideals of R. Let $R_i := R/J_i$, $i=1,2$, and let $\psi_i : F_R(I)\to F_{R_i}(IR_i)$ denote the canonical surjective homomorphisms on fiber cones as in (2.1). Then $R_2\cong R_1/J'$, where $J'=J_2/J_1$, and there exists a canonical surjective homomorphism $\psi' : F_{R_1}(IR_1)\to F_{R_2}(IR_2)$ such that $\psi_2=\psi'\circ\psi_1$.*

With notation as in (2.1), if J is a nilpotent ideal of R, then $\ker\psi_J$ is a nilpotent ideal of $F_R(I)$. For suppose $x\in J$ is such that $x^s=0$. If $\overline{x}\in\frac{(I^n\cap J)+\mathbf{m}\, I^n}{\mathbf{m}\, I^n}=F_n$ is the image of x in $F(I)$, then by definition $\overline{x}^s$ is the image of x^s in F_{sn}, so $\overline{x}^s=0$. Thus for J a nilpotent ideal of R, we have $\dim F_R(I)=\dim F_{R'}(I')$ and $\ell(I)=\ell(I')$.

Applying this to the situation considered in (2.2), if s is a positive integer, $J_1=J^s$ and $J_2=J$, then with $\psi' : F_{R_1}(IR_1)\to F_{R_2}(IR_2)$ as in (2.2), it follows that $\ker\psi'$ is a nilpotent ideal and in this situation $\dim F_{R_1}(IR_1)=\dim F_{R_2}(IR_2)$. In particular for the examples of Huckaba [Hu, Examples 3.1 and 3.2] mentioned in the end of Section 1, going modulo a power x^nR of the ideal xR also reduces the dimension of the fiber cone $F(I)$ from 3 to 1.

Proposition 2.3. *With notation as in Setting 2.1, we have the following implications of Remark 2.2.*

(1) *If $J'\subseteq J$ are ideals of R and if $\ker\psi_J=0$, then $\ker\psi_{J'}=0$.*
(2) *$\ker\psi_J=0$ if and only if $\ker\psi_{xR}=0$ for each $x\in J$.*
(3) *For $x\in\mathbf{m}$, we have $\ker\psi_{xR}=0$ if and only if $(I^n:x)=(\mathbf{m}\, I^n:x)$ for each $n\geq 0$.*

Proof. Statements (1) and (2) are clear in view of (2.2) and the description of $\ker\psi_J$ given in (2.1). For statement (3), we use that $I^n\cap xR=x(I^n:x)$. Thus $0=\ker\psi_{xR}=\oplus_{n\geq 0}\frac{(I^n\cap xR)+\mathbf{m}\, I^n}{\mathbf{m}\, I^n}\iff(I^n\cap xR)\subseteq\mathbf{m}\, I^n$ for each $n\iff x(I^n:x)\subseteq\mathbf{m}\, I^n$ for each $n\iff(I^n:x)\subseteq(\mathbf{m}\, I^n:x)$ for each n. This last statement is equivalent to $(I^n:x)=(\mathbf{m}\, I^n:x)$ for each n. $\square$

Proposition 2.4. *Let $(R, \mathbf{m})$ be a Noetherian local ring and let $I \subseteq \mathbf{m}$ be an ideal of R. Suppose J_1 and J_2 are ideals of R such that $\operatorname{rad} J_1 = \operatorname{rad} J_2$. Let $R_i := R/J_i$, $i = 1, 2$, and let $\psi_i : F_R(I) \to F_{R_i}(IR_i)$ denote the canonical surjective homomorphisms on fiber cones as in (2.1). Then $\dim F_{R_1}(IR_1) = \dim F_{R_2}(IR_2)$ and $\ell(IR_1) = \ell(IR_2)$.*

Proof. Since $\operatorname{rad}(J_1 + J_2) = \operatorname{rad} J_1 = \operatorname{rad} J_2$, it suffices to consider the case where $J_1 \subseteq J_2$. With notation as in (2.2), $\ker \psi'$ is a nilpotent ideal. Thus $\dim F_{R_1}(IR_1) = \dim F_{R_2}(IR_2)$ and $\ell(IR_1) = \ell(IR_2)$. □

As we remarked in Section 1, the dimension of the fiber cone $F(I)$ of an ideal I is the same as the dimension of the fiber cone $F(I^n)$ of a power I^n of I. Hence, with notation as in (2.1), we have $\dim F_{R'}(IR') = \dim F_{R'}(I^n R')$ and $\ell(IR') = \ell(I^n R')$ for each positive integer n.

3. The associated graded ring and the fiber cone.

The associated graded ring of the ideal I plays a role in the behavior of the fiber cone of the image of I modulo a principal ideal as we illustrate in Proposition 3.1 and Example 3.2.

Proposition 3.1. *Let $I \subseteq \mathbf{m}$ be an ideal of a Noetherian local ring $(R, \mathbf{m})$. For $x \in \mathbf{m}$, let x^* denote the image of x in the associated graded ring $G(I) = R[It]/IR[It]$ and let $\overline{x}$ denote the image of x in the fiber cone $F(I)$. If x^* is a regular element of $G(I)$, then $F(I)/\overline{x}F(I) \cong F_{R'}(I')$, where $R' = R/xR$ and $I' = IR'$.*

Proof. There exists a positive integer s such that $x \in I^s - I^{s+1}$. Since x^* is a regular element of $G(I)$ with $\deg x^* = s$, we have $(I^n \cap xR) = xI^{n-s}$ for every $n \geq 0$, where $I^{n-s} := R$ if $n - s \leq 0$. Hence we have

$$[\ker \psi_{xR}]_n = \frac{(I^n \cap xR) + \mathbf{m}\, I^n}{\mathbf{m}\, I^n} = \frac{xI^{n-s} + \mathbf{m}\, I^n}{\mathbf{m}\, I^n} = [\overline{x}F(I)]_n,$$

for every $n \geq 0$. Therefore $F(I)/\overline{x}F(I) \cong F_{R'}(I')$. □

With notation as in Proposition 3.1, the following example shows that for $x \in I - \mathbf{m}\, I$ such that $\overline{x}$ is a regular element of $F(I)$, it may happen that $\overline{x}F(I) \subsetneq \ker \psi_{xR}$ and $F_{R'}(I') \ncong F_R(I)/\overline{x}F(I)$, where $R' = R/xR$ and $I' = IR'$. Proposition 3.1 implies that for such an example $x^* \in G(I)$ is necessarily a zero divisor.

Example 3.2. Let k be a field and consider the subring $R := k[[t^3, t^4, t^5]]$ of the formal power series ring $k[[t]]$. Thus $R = k + t^3 k[[t]]$ is a complete Cohen-Macaulay one-dimensional local domain. Let $I = (t^3, t^4)R$. An easy computation implies $I^3 = t^3 I^2$. Hence $t^3 R$ is a principal reduction of I. Since I is 2-generated, it follows from [DGH, Proposition 3.5] that $F(I)$ is Cohen-Macaulay and in fact a complete intersection. Let X, Y be indeterminates over k and define a k-algebra homomorphism $\phi : k[X, Y] \to F(I)$ by setting $\phi(X) = \overline{t^3}$ and $\phi(Y) = \overline{t^4}$. Then $\ker \phi = Y^3 k[X, Y]$ and $F(I) \cong k[X, Y]/Y^3 k[X, Y]$. Thus $\overline{t^3}$ is a regular element of $F(I)$ and $F(I)/\overline{t^3} F(I) \cong k[Y]/Y^3 k[Y]$. Let $J = t^3 R$, $R' = R/J$ and $I' = IR'$. Since $t^8 \in (I^2 \cap J)$, we have $\phi(Y^2) = \overline{t^8} \in \ker \psi_J$ and $F_{R'}(I') \cong k[Y]/Y^2 k[Y]$. Thus $F(I)/\overline{t^3} F(I) \ncong F_{R'}(I')$. In fact, we have $\ker \psi_J = (\overline{t^3}, \overline{t^8}) F(I)$ and $\overline{t^8} \notin \overline{t^3} F(I)$.

We list several observations and questions concerning the dimension of fiber cones and their behavior under homomorphic image.

Discussion 3.3. Let $I \subseteq \mathbf{m}$ be an ideal of a Noetherian local ring $(R, \mathbf{m})$. If $J \subseteq \mathbf{m}$ is an ideal of R and $R' = R/J$, then there exists a surjective ring homomorphism $\chi_J : G_R(I) = R[It]/IR[It] \to G_{R'}(IR')$ of the associated graded ring $G_R(I)$ of I onto the associted graded ring $G_{R'}(IR')$ of IR' [K, page 150].

We have the following commutative diagram involving the associated graded rings and fiber cones for which the vertical maps α and β are surjective:

$$\begin{array}{ccc}
G_R(I) = R[It]/IR[It] = \oplus_{n \geq 0} I^n / I^{n=1} & \xrightarrow{\chi_J} & R'[I't]/I'R'[I't] = \oplus (I')^n/(I')^{n+1} \\
\alpha \Big\downarrow & & \beta \Big\downarrow \\
F_R(I) = R[It]/\mathbf{m}\, R[It] \oplus_{n \geq 0} I^n / \mathbf{m}\, I^n & \xrightarrow{\psi_J} & R'[I't]/\mathbf{m}'\, R'[I'] = \oplus_{n \geq 0} (I')^n / \mathbf{m} (I')^n
\end{array}$$

If J is nonzero, then $\ker \chi_J \neq 0$. It can happen, however, that J is nonzero and yet $\ker \psi_J = 0$. This is possible even in the case where I is $\mathbf{m}$-primary. In an example exhibiting this behavior, commutativity of the diagram above implies one must have $\ker \chi_J \subseteq \ker \alpha$.

Example 3.4. Let k be a field and let $R = k[x, y]_{(x,y)}$, where $x^2 = xy = 0$. Let $I = yR$ and let $J = xR$. Then $\ker \psi_J = \bigoplus_{n \geq 0} \frac{(xR \cap y^n R) + \mathbf{m}\, y^n R}{\mathbf{m}\, y^n R} = 0$, but $J = xR \neq 0$.

A reason for the existence of examples such as Example 3.4 is given in Proposition 3.5.

Proposition 3.5. *Suppose $(R, \mathbf{m})$ is a Noetherian local ring and I is an $\mathbf{m}$-primary ideal. If the fiber cone $F(I)$ is an integral domain, then $\ker \psi_J = 0$ for every ideal*

J of R such that $\dim(R/J) = \dim R$. In particular, if I is $\mathbf{m}$-primary and $F(I)$ is an integral domain, then there exists a prime ideal J of R such that $\ker \psi_J = 0$.

Proof. Let $R' := R/J$. Since I is $\mathbf{m}$-primary, $\dim F(IR') = \dim R'$. Thus $\dim R' = \dim R$ implies $\dim F(IR') = \dim F(I)$. Since $F(I)$ is an integral domain, it follows that $\ker \psi_J = 0$. The last statement follows becaues there exists a prime ideal J of R such that $\dim R = \dim(R/J)$. $\square$

Propositon 3.5 and Example 3.4 show that with notation as in (3.1), it can happen that $x^* \in G(I)$ is not a regular element and yet $\ker \psi_{xR} = \overline{x}F(I)$.

In Section 4 we consider fiber cones $F(I)$ such that $\ker \psi_J \neq 0$ for each nonzero ideal J.

4. Maximal fiber cones with respect to homomorphic image.

Suppose $(R, \mathbf{m})$ is a Noetherian local ring and $I \subseteq \mathbf{m}$ is an ideal of R. If J is a nonzero ideal of R such that $\ker \psi_J = \bigoplus_{n \geq 0} \frac{(J \cap I^n) + \mathbf{m} I^n}{\mathbf{m} I^n}$ is the zero ideal of $F(I)$, then we have $F_R(I) = F_{R'}(IR')$, where $R' := R/J$; so the fiber cone $F(I)$ is realized as a fiber cone of a proper homomorphic image R' of R. If there fails to exist such an ideal J, i.e., if $\ker \psi_J \neq 0$ for each nonzero ideal J, then we say that $F(I)$ is a *maximal fiber cone* of R.

We record in Remark 4.1 some immediate consequences of the inequality $\dim F_{R'}(IR') \leq \dim R'$.

Remark 4.1. With notation as in (2.1), we have:

1. If J is such that $\dim R' < \dim R$ and if $\dim F_R(I) = \dim R$, then $\ker \psi_J \neq 0$.
2. If I is $\mathbf{m}$-primary and J is not contained in a minimal prime of R, then $\ker \psi_J \neq 0$.
3. If R is an integral domain and $\dim F(I) = \dim R$, then $F(I)$ is a maximal fiber cone.
4. If R is an integral domain, then $F(I)$ is a maximal fiber cone for every $\mathbf{m}$-primary ideal I of R.

We are interested in describing all the maximal fiber cones of R. Thus we are interested in conditions on I and R in order that there exist a nonzero ideal J of R such that $\ker \psi_J = 0$. In considering this question, by Proposition 2.3, one may assume that $J = xR$ is a nonzero principal ideal. Thus the question can also be phrased:

Question 4.2. Under what conditions on I and R does it follow for each nonzero element $x \in \mathbf{m}$ that $\ker \psi_{xR} \neq 0$?

Discussion 4.3. Information about Question 4.2 is provided by the work of Rees in [R]. In particular, [R, Theorem 2.1] implies that if $x \in \mathbf{m}$ is such that $(I^n : x) = I^n$ for each positive integer n, then $F_R(I) = F_{R'}(I')$, where $R' := R/xR$ and $I' := IR'$. Thus for $x \in \mathbf{m}$ a sufficient condition for $\ker \psi_{xR} = 0$ is that $(I^n : x) = I^n$ for each positive integer n. It is readily seen that this colon condition on x is equivalent to $x \notin I$ and the image of x in the associated graded ring $G(I) = R[It]/IR[It]$ is a regular element. More generally, if $x \in I^s - I^{s+1}$ and if the image x^* of x in $G(I)$ is a regular element, then by Proposition 3.1 $\ker \psi_{xR} = \overline{x}F(I)$. Thus if we also have $x \in \mathbf{m}\, I^s$, then $\ker \psi_{xR} = 0$. Example 3.4 shows that this sufficient condition for $\ker \psi_{xR} = 0$ is not a necessary condition.

Proposition 2.3 gives a necessary and sufficient condition on a principal ideal $J = xR$ in order that $\ker \psi_{xR} = 0$, namely that $(I^n : x) = (\mathbf{m}\, I^n : x)$ for each integer $n \geq 0$. By Proposition 2.3, if $\ker \psi_{xR} = 0$, then also $\ker \psi_{yxR} = 0$ for every $y \in R$.

If $I = yR$ is a non-nilpotent principal ideal of R, we give in Corollary 4.5 necessary and sufficient conditions for $F(I)$ to be a maximal fiber cone.

Proposition 4.4. *Suppose $(R, \mathbf{m})$ is a Noetherian local ring and $I = yR \subseteq \mathbf{m}$ is a non-nilpotent principal ideal of R. For $x \in \mathbf{m}$, we have $\ker \psi_{xR} = 0 \iff y^n \notin xR$ for each positive integer n.*

Proof. We have $\ker \psi_{xR} = 0 \iff (y^n R \cap xR) \subseteq \mathbf{m}\, y^n R$ for each positive integer n, and $y^n \notin xR \iff (y^n R \cap xR) \subsetneq y^n R \iff (y^n R \cap xR) \subseteq \mathbf{m}\, y^n R$. □

Corollary 4.5. *Let $(R, \mathbf{m})$ be a Noetherian local ring and $I = yR \subseteq \mathbf{m}$ be a non-nilpotent principal ideal of R. Then $F(I)$ is a maximal fiber cone if and only if R is a one-dimensional integral domain.*

Proof. By Proposition 4.4, for $x \in \mathbf{m}$ we have $y \in \operatorname{rad} xR \iff \ker \psi_{xR} \neq 0$. Suppose $F(I)$ is a maximal fiber cone. Then by definition, $\ker \psi_{xR} \neq 0$ for each nonzero $x \in \mathbf{m}$. Since y is not nilpotent, there exists a minimal prime P of R such that $y \notin P$. It follows that $P = 0$, for if not, then there exists a nonzero $x \in P$ and $y \in \operatorname{rad} xR \subseteq P$ implies $y \in P$. Thus R is an integral domain. Moreover, this same argument implies y is in every nonzero prime of R. Since R is Noetherian, it follows that $\dim R = 1$. For yR has only finitely many minimal primes and every minimal prime of yR has height one by the Altitude Theorem of Krull [N, page 26] or [Mat, page 100]. If there exists $P \in \operatorname{Spec} R$ with $\operatorname{ht} P > 1$, then the Altitude Theorem of Krull implies P is the union of the height-one primes contained in P. This implies there exist infinitely many height-one primes contained in P. Since y is contained in only finitely many height-one primes, this is impossible. Thus $\dim R = 1$. Since R is local, $\mathbf{m}$ is the only nonzero prime of R.

Conversely, if R is a one-dimensional Noetherian local integral domain, then (4.1) implies that $F(I)$ is a maximal fiber cone for every non-nilpotent principal ideal $I = yR \subseteq \mathbf{m}$. $\square$

Question 4.6. If $F(I)$ is a maximal fiber cone of R, does it follow that $\dim F(I) = \dim R$?

Proposition 4.7. *Suppose $(R, \mathbf{m})$ is a Noetherian local ring and $I \subseteq \mathbf{m}$ is an ideal of R. If $\dim F(I) := n = \operatorname{ht}(I) < \dim R$ and if $F(I)$ is an integral domain, then $F(I)$ is not a maximal fiber cone. In particular, if I is of the principal class, i.e., $I = (a_1, \dots, a_n)R$, where $\operatorname{ht}(I) = n$, and if $\operatorname{ht}(I) < \dim R$, then $F(I)$ is not a maximal fiber cone of R.*

Proof. Choose $x \in \mathbf{m}$ such that x is not in any minimal prime of I. Then $L := (I, x)R$ has height $n+1$. Let $\overline{x}$ denote the image of x in the fiber cone $F_R(L)$. Then $F_R(L)$ is a homomorphic image of a polynomial ring in one variable $F_R(I)[z]$ over $F_R(I)$ by means of a homomorphism mapping $z \to \overline{x}$. Since $\dim F(I) = n$ and $F(I)$ is an integral domain, it follows that $F(I)[z] \cong F(L)$ by means of an isomorphism taking $z \to \overline{x}$. Let $J = xR$ and $R' := R/J$. Then $\operatorname{ht}(IR') = \operatorname{ht}(L/xR) = n$, so $\dim F_{R'}(IR') \geq n$. Since $\psi_J : F_R(I) \to F_{R'}(IR')$ is surjective and $F_R(I)$ is an n-dimensional integral domain, it follows that $\psi_J : F_R(I) \to F_{R'}(IR')$ is an isomorphism. In particular, if I is of the principal class, then $F(I)$ is a polynomial ring in n variables over the field $R/\mathbf{m}$, so $F(I)$ is an integral domain with $\dim F(I) = \operatorname{ht}(I)$. $\square$

If I is generated by a regular sequence, then I is of the principal class. Thus if $F(I)$ is a maximal fiber cone and I is generated by a regular sequence, then by Proposition 4.7, $\dim F(I) = \dim R$.

We observe in Proposition 4.8 a situation where the integral domain hypothesis of Proposition 4.7 applies.

Proposition 4.8. *Let $A = k[X_1, X_2, \cdots, X_d] = \bigoplus_{n=0}^{\infty} A_n$ be a polynomial ring in d variables over a field k and let $\mathbf{m} = (X_1, X_2, \cdots, X_d)A$ denote its homogeneous* *maximal ideal. Suppose $I = (f_1, f_2, \cdots, f_n)A$, where $f_1, f_2, \cdots, f_n$ are homogeneous polynomials all of the same degree t. Let $R = A_{\mathbf{m}}$. Then $F(IR)$ is an integral domain. Thus if $F(IR)$ is a maximal fiber cone, then $\dim F(IR) = d$.*

Proof. We have
$$k[f_1, f_2, \cdots, f_n] = k \oplus I_1 \oplus I_2 \oplus \cdots,$$
where $I_i = I^i \cap A_{it}$ for $i > 0$. Since $I^i/\mathbf{m}I^i \cong I^i \cap A_{it}$ for $i \geq 0$, we have the following isomorphisms:
$$k[f_1, f_2, \cdots, f_n] \cong \oplus_{i=0}^{\infty}(I^i/\,\mathbf{m}\,I^i) \cong \oplus_{i=0}^{\infty}(I^iR/\,\mathbf{m}\,I^iR) = F(IR).$$
Therefore $F(IR)$ is an integral domain. The result now follows from Proposition 4.7. $\square$

Corollary 4.9. *With notation as in Proposition 4.8, if $\dim F(I) = \operatorname{ht} I$ and $F(IR)$ is a maximal fiber cone, then I is $\mathbf{m}$-primary.*

Proof. We have $\dim F(IR) = d$ by Proposition 4.8. Since I is homogeneous ideal and $\operatorname{ht} I = d$, $\mathbf{m}$ is the unique homogeneous minimal prime of I, Therefore I is $\mathbf{m}$-primary. $\square$

Question 4.10. Let $(R, \mathbf{m})$ be a Noetherian local ring and let $I \subseteq \mathbf{m}$ be an ideal of R. If $\dim F(I) = \operatorname{ht} I$ and $F(I)$ is a maximal fiber cone, does it follow that I is $\mathbf{m}$-primary?

Remark 4.11. *Without the assumption in Question 4.10 that $\dim F(I) = \operatorname{ht} I$, it is easy to give examples where $F(I)$ is a maximal fiber cone and yet I is not $\mathbf{m}$-primary. For example, with notation as in Proposition 4.8, if $d > 1$ and $I = (X_1^2, X_1X_2, \ldots, X_1X_d)A$, then $\operatorname{ht}(IR) = 1$, but $\dim F(IR) = d$ and $F(IR)$ is a maximal fiber cone.*

5. When is the fiber cone a hypersurface?

Setting 5.1. Let $I \subseteq \mathbf{m}$ be an ideal of a Noetherian local ring $(R, \mathbf{m})$. In this section we consider the structure of the fiber cone $F(I) = \oplus_{n\geq 0}F_n$ in the case where $\dim F(I) = d > 0$ and $\mu(I) = d + 1$. If $a_1, \ldots, a_{d+1}$ is a basis for $F_1 = I/\,\mathbf{m}\,I$ as a vector space over the field $k := R/\,\mathbf{m}$, then there exists a presentation $\phi : k[X_1, \ldots, X_{d+1}] \to F(I)$ of $F(I)$ as a graded k-algebra homomorphic image of a polynomial ring in $d + 1$ variables over k defined by setting $\phi(X_i) = a_i$, for $i = 1, \ldots, d + 1$. Moreover, $F(I)$ is a hypersurface if and only if $\ker\phi$ is a principal ideal [K, Examples 1.2].

Lemma 5.2. *Let $(R, \mathbf{m})$ be a Noetherian local ring having infinite residue field $R/\mathbf{m} := k$, and let $I \subseteq \mathbf{m}$ be an ideal of R such that $\dim F(I) = d > 0$ and $\mu(I) = d+1$. Let $r = r(I)$ denote the reduction number of I and let $\phi : k[X_1, \dots, X_{d+1}] \to F(I)$ be a presentation of the fiber cone $F(I)$ as in Setting 5.1. Then the minimal degree of a nonzero form $f \in \ker \phi$ is $r+1$.*

Proof. The map ϕ from the graded ring $A = k[X_1, \dots, X_{d+1}] = \oplus_{n \geq 0} A_n$ onto the graded ring $F(I) = \oplus_{n \geq 0} F_n = \oplus_{n \geq 0} (I^n / \mathbf{m} I^n)$ is a surjective graded k-algebra homomorphism of degree 0. Let $K := \ker \phi = \oplus_{n \geq 0} K_n$. For each positive integer n we have a short exact sequence

$$0 \to K_n \to A_n \to F_n \to 0$$

of finite-dimensional vector spaces over k. Since I has reduction number r, it follows from [ES, Theorem, page 440] that $\dim_k F_i = \mu(I^i) = \binom{i+d}{d}$ for $i = 0, 1, \dots, r$ and $\dim_k F_{r+1} = \mu(I^{r+1}) < \binom{r+d+1}{d}$. Since $\dim A_i = \binom{i+d}{d}$ for all i, it follows that $K_i = 0$ for $i = 0, \dots, r$ and $K_{r+1} \neq 0$. Hence the minimal degree of a nonzero form $f \in \ker \phi$ is $r+1$. $\square$

Remark 5.3. Let $A = k[X_1, \dots, X_n]$ be a polynomial ring in n variables $X_1, \dots, X_n$ over a field k. For an ideal K of A, it is well known that $\operatorname{ht}(K) = 1$ if and only if $\dim(A/K) = n-1$ [K, Corollary 3.6, page 53]. Moreover, K is principal if and only if $\operatorname{ht}(P) = 1$ for each associated prime P of K. If $K = (g_1, \dots, g_m)A$ and g is a greatest common divisor of $g_1, \dots, g_m$, then $K = gJ$, where $\operatorname{ht}(J) > 1$. Thus K is principal if and only if $J = A$. If K is homogeneous, then $g_1, \dots, g_m$ may be taken to be homogeneous; it then follows that g is homogeneous and $K = gJ$, where J is homogeneous with $\operatorname{ht}(J) > 1$. If $K = \operatorname{rad} K$, then each associated prime of K is a minimal prime and K is principal if and only if $\operatorname{ht}(P) = 1$ for each minimal prime P of K.

Proposition 5.4. *Let $(R, \mathbf{m})$ be a Noetherian local ring with infinite residue field $k = R/\mathbf{m}$ and let $I \subseteq \mathbf{m}$ be an ideal of R such that $\dim F(I) = d > 0$ and $\mu(I) = d+1$. Let $\phi : A = k[X_1, \dots, X_{d+1}] \to F(I)$ be a presentation of $F(I)$ as a graded homomorphic image of a polynomial ring as in Setting 5.1. Let $f \in K := \ker \phi$ be a nonzero homogeneous form of minimal degree. Then the following are equivalent.*

(1) $\ker \phi = fA$, *i.e.*, $F(I)$ *is a hypersurface.*
(2) $\operatorname{ht} P = 1$ *for each* $P \in \operatorname{Ass} K$.
(3) $F(I)$ *is a Cohen-Macaulay ring.*
(4) $\deg f = e(F(I))$, *the multiplicity of* $F(I)$.

Proof. That (1) is equivalent to (2) is observed in Remark 5.2. It is clear that (1) implies (3) and it follows from [BH, (2.2.15) and (2.1.14)] that (3) implies (2). To see the equivalence of (3) and (4), we use [DRV, Theorem 2.1]. By Lemma 5.2, $\deg f = r+1$, where r is the reduction number of I.

Since $\dim F(I) = d$, there exists a minimal reduction $J = (x_1, \dots, x_d)R$ of I and $y \in I$ such that $I = J + yR$. By [DRV, Theorem 2.1], $F(I)$ is Cohen-Macaulay if and only if

$$e(F(I)) = \sum_{n=0}^{r} \lambda(\frac{I^n}{JI^{n-1} + \mathbf{m}\, I^n}).$$

Since for $0 \leq n \leq r$, $\lambda(\frac{I^n}{JI^{n-1}+\mathbf{m}\, I^n}) = 1$, the sum on the right hand side of the displayed equation is $r+1 = \deg f$. This proves the equivalence of (3) and (4). $\square$

Remark 5.5. With notation as in Proposition 5.4, we have the following inequality $e(F(I)) \leq \deg f$, where $e(F(I))$ is the multiplicity of $F(I)$. Hence by Proposition 5.4, $F(I)$ is not Cohen-Macaulay $\iff$ $e(F(I)) < \deg f$.

Proof. Let $J = (x_1, \dots, x_d)R$ be a minimal reduction of I. Then $JF(I)$ is generated by a homogeneous system of parameters for $F(I)$ and

$$\lambda(\frac{F(I)}{JF(I)}) \quad = \quad \sum_{n=0}^{r} \lambda(\frac{I^n}{JI^{n-1} + \mathbf{m}\, I^n}).$$

Let $\mathcal{M}$ denote the maximal homogeneous ideal of $F(I)$. Then

$$e(F(I)) = e(F(I)_{\mathcal{M}}) \leq \lambda(\frac{F(I)_{\mathcal{M}}}{JF(I)_{\mathcal{M}}}) = \lambda(\frac{F(I)}{JF(I)}).$$

Thus $e(F(I)) \leq \deg f = r+1$. Hence by Proposition 5.4, $F(I)$ is not Cohen-Macaulay if and only if $e(F(I)) < \deg f$. $\square$

Theorem 5.6. *Let $(R, \mathbf{m})$ be a Noetherian local ring with infinite residue field $k = R/\mathbf{m}$ and let $I \subsetneq \mathbf{m}$ be an ideal of R such that $\dim F(I) = d > 0$ and $\mu(I) = d+1$. Suppose there exists a minimal reduction J of I generated by a regular sequence. Assume that* $\operatorname{grade}(G_+(I)) \geq d-1$. *Then $F(I)$ is Cohen-Macaulay and thus a hypersurface.*

Proof. For $x \in R$, let x^* denote the image of x in $G(I)$ and let $\overline{x}$ denote the image of x in $F(I)$. There exists a minimal reduction $J = (x_1, \dots, x_d) \subseteq I$ and $x_{d+1} \in I$ such that

(I) $\{x_1, \dots, x_d\}$ is a regular sequence in R.
(II) $\{x_1, \dots, x_d, x_{d+1}\}$ is a minimal set of generators of I.
(III) $\{x_1^*, \dots, x_{d-1}^*\}$ is a regular sequence in $G(I)$.

Let $R' = R/(x_1, \ldots, x_{d-1})R$, let $\mathbf{m}' = \mathbf{m}/(x_1, \ldots, x_{d-1})R$ and let $I' = IR'$. By Condition II, I' is a 2-generated ideal having a principal reduction generated by the image x'_d of x_d. Condition I implies that x'_d is a regular element of R'. Hence by [DGH, Proposition 3.5], $F_{R'}(I')$ is Cohen-Macaulay.

As observed in (2.1), the kernel of the canonical map $\psi : F_R(I) \to F_{R'}(I')$ is

$$\oplus_{n\geq 0}\frac{(I^n \cap (x_1, \ldots, x_{d-1})) + \mathbf{m}\, I^n}{\mathbf{m}\, I^n}.$$

Condition III and Proposition 3.1 imply

$$\ker\psi = \oplus_{n\geq 0}\frac{(x_1, \ldots, x_{d-1})I^{n-1} + \mathbf{m}\, I^n}{\mathbf{m}\, I^n} = (\overline{x_1}, \ldots, \overline{x_{d-1}})F(I).$$

Hence

$$\frac{F(I)}{(\overline{x_1}, \ldots, \overline{x_{d-1}})} \cong F_{R'}(I')$$

and to show $F(I)$ is Cohen-Macaulay, it suffices to show $\{\overline{x_1}, \ldots, \overline{x_{d-1}}\}$ is a regular sequence in $F(I)$. By the generalized Vallabrega-Valla criterion of Cortadellas and Zarzuela [CZ, Theorem 2.8], to show $\{\overline{x_1}, \ldots, \overline{x_{d-1}}\}$ is a regular sequence in $F(I)$, it suffices to show

$$(x_1, \ldots, x_{d-1}) \cap \mathbf{m}\, I^{n+1} = (x_1, \ldots, x_{d-1})\, \mathbf{m}\, I^n, \text{ for all } n \geq 0.$$

" $\supseteq$" is clear. We prove " $\subseteq$" by induction on n.

(Case i) $n = 0$: Let $u \in (x_1, \ldots, x_{d-1}) \cap \mathbf{m}\, I$. Thus $u = \sum_{i=1}^{d-1} r_i x_i = \sum_{j=1}^{d+1} \alpha_j x_j$, where $r_i \in R$ and $\alpha_j \in \mathbf{m}$. Therefore

$$(r_1 - \alpha_1)x_1 + \cdots + (r_{d-1} - \alpha_{d-1})x_{d-1} - \alpha_d x_d - \alpha_{d+1}x_{d+1} = 0.$$

Since $\{x_1, \ldots, x_{d+1}\}$ is a minimal generating set for I, each $r_i - \alpha_i \in \mathbf{m}$. Since $\alpha_i \in \mathbf{m}$, $r_i \in \mathbf{m}$. Hence $u = \sum_{i=1}^{d-1} r_i x_i \in \mathbf{m}(x_1, \ldots, x_{d-1})$.

(Case ii) $1 \leq n < r$, where $r = r_J(I)$ is the reduction number of I with respect to J: We have $(x_1, \ldots, x_{d-1}) \cap \mathbf{m}\, I^{n+1} = (x_1, \ldots, x_{d-1}) \cap (I^{n+1} \cap \mathbf{m}\, I^{n+1})$ $= ((x_1, \ldots, x_{d-1}) \cap I^{n+1}) \cap \mathbf{m}\, I^{n+1} = ((x_1, \ldots, x_{d-1})I^n \cap \mathbf{m}\, I^{n+1}$, the last equality by Condition III.

Hence $u \in (x_1, \ldots, x_{d-1}) \cap \mathbf{m}\, I^{n+1}$ implies $u \in ((x_1, \ldots, x_{d-1})I^n \cap \mathbf{m}\, I^{n+1}$. Thus $u = \sum_{i=1}^{d-1} x_i g_i$, where $g_i \in I^n$ and $u = H(x_1, \ldots, x_{d+1})$, where $H(X_1, \ldots, X_{d+1}) \in R[X_1, \ldots, X_{d+1}]$ is a homogeneous polynomial with coefficients in $\mathbf{m}$ of degree $n+1$. Let $G_i(X_1, \ldots, X_{d+1}) \in R[X_1, \ldots, X_{d+1}]$ be a homogeneous polynomial of degree n such that $G_i(x_1, \ldots, x_{d+1}) = g_i$.

Let $\tau : R[X_1, \ldots, X_{d+1}] \to R[It]$, where $\tau(X_i) = x_i t$ be a presentation of the Rees algebra $R[It]$. Consider the following commutative diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker(\tau) & \longrightarrow & R[X_1,\dots,X_{d+1}] & \xrightarrow{\tau} & R[It] & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \pi_3} & & \\ 0 & \longrightarrow & \ker(\phi) & \longrightarrow & (R/\mathbf{m})[X_1,\dots,X_{d+1}] & \xrightarrow{\phi} & F(I) & \longrightarrow & 0 \end{array}$$

Since $\sum_{i=1}^{d-1} x_i g_i - H(x_1,\dots,x_{d+1}) = 0$, the homogeneous polynomial

$$\sum_{i=1}^{d-1} X_i G_i(X_1,\dots,X_{d+1}) - H(X_1,\dots,X_{d+1}) \in \ker\tau.$$

Since $H(X_1,\dots,X_{d+1})$ has coefficients in $\mathbf{m}$, we have

$$0 = \pi_3\tau(\sum_{i=1}^{d-1} X_iG_i - H) = \phi\pi_2(\sum_{i=1}^{d-1} X_iG_i - H) = \phi\pi_2(\sum_{i=1}^{d-1} X_iG_i).$$

Hence $\pi_2(\sum_{i=1}^{d-1} X_iG_i) \in \ker\phi$. Since $\sum_{i=1}^{d-1} X_iG_i$ is of degree $n+1 \leq r$, Lemma 5.2 implies $\pi_2(\sum_{i=1}^{d-1} X_iG_i) = 0$. Therefore the coefficients of $\sum_{i=1}^{d-1} X_iG_i$ are in $\mathbf{m}$. Evaluating this polynomial by mapping $X_i \mapsto x_i$ gives $u = \sum_{i=1}^{d-1} x_ig_i \in (x_1,\dots,x_{d_1})\,\mathbf{m}\,I^n$.

(Case iii) $n \geq r$: Since $n \geq r$, we have $I^{n+1} = JI^n = (x_1,\dots,x_d)I^n$.
Let $u \in (x_1,\dots,x_{d-1}) \cap \mathbf{m}\,I^{n+1} = (x_1,\dots,x_{d-1}) \cap \mathbf{m}(x_1,\dots,x_d)I^n$. Thus $u = \sum_{i=1}^{d-1} r_ix_i = \sum_{j=1}^{d} \alpha_jx_j$, where each $r_i \in R$ and each $\alpha_j \in \mathbf{m}\,I^n$. Hence $\alpha_dx_d = \sum_{i=1}^{d-1}(r_i-\alpha_i)x_i$ and this implies $\alpha_d \in ((x_1,\dots,x_{d-1}) : x_d) = (x_1,\dots,x_{d-1})$, the last equality because of Condition I. Hence

$$\alpha_d \in (x_1,\dots,x_{d-1}) \cap \mathbf{m}\,I^n = (x_1,\dots,x_{d-1})\,\mathbf{m}\,I^{n-1},$$

the last equality because of our inductive hypothesis. Thus $u = \sum_{j=1}^{d} \alpha_jx_j = \sum_{j=1}^{d-1} \alpha_jx_j + \alpha_dx_d \in (x_1,\dots,x_{d-1})\,\mathbf{m}\,I^n + (x_1,\dots,x_{d-1})\,\mathbf{m}\,I^{n-1}I = (x_1,\dots,x_{d-1})\,\mathbf{m}\,I^n$. This completes the proof that $\{\overline{x_1},\dots,\overline{x_{d-1}}\}$ is a regular sequence in $F(I)$, and thus the proof of Theorem 5.6 □

6. The Cohen-Macaulay property of one-dimensional fiber cones

We record in this short section several consequences of a result of D'Cruz, Raghavan and Verma [DRV, Theorem 2.1] for the Cohen-Macaulay property of the fiber cone of a regular ideal having a principal reduction.

Proposition 6.1. *Let $(R,\mathbf{m})$ be a Noetherian local ring and let $I \subseteq \mathbf{m}$ be a regular ideal having a principal reduction aR. Let $r = r_{aR}(I)$ be the reduction number of I with respect to aR. Then the following are equivalent.*

(1) *$F(I)$ is a Cohen-Macaulay ring.*

(2) $\lambda(\frac{aI^n+\mathbf{m}\,I^{n+1}}{\mathbf{m}\,I^{n+1}}) = \lambda(\frac{I^n}{\mathbf{m}\,I^n})$ *for* $1 \leq n \leq r-1$.

Proof. (1) $\Rightarrow$ (2). Suppose $F(I)$ is a Cohen-Macaulay ring. Then $\overline{a}(= a+\mathbf{m}\,I)$ is a regular element of $F(I)$ with $\deg \overline{a} = 1$. Let $F(I) = \oplus_{n\geq 0} F_n$, where $F_n = I^n/\mathbf{m}\,I^n$, and consider the graded k-algebra homomorphism $\phi_{\overline{a}} : F_n \to F_{n+1}$ given by $\phi_{\overline{a}}(\overline{x}) = \overline{x}\cdot\overline{a}$, for every $\overline{x} \in F_n$. Since $\overline{a}$ is a regular element of $F(I)$, $\dim_k F_n = \dim_k(\overline{a}F_n)$. For $1 \leq n \leq r-1$, we have

$$\lambda(\frac{aI^n + \mathbf{m}\,I^{n+1}}{\mathbf{m}\,I^{n+1}}) = \lambda(\overline{a}(\frac{I^n}{\mathbf{m}\,I^n})) = \dim_k(\overline{a}F_n) = \dim_k(F_n) = \lambda(\frac{I^n}{\mathbf{m}\,I^n}).$$

(2) $\Rightarrow$ (1). Suppose that $\lambda(\frac{aI^n+\mathbf{m}\,I^{n+1}}{\mathbf{m}\,I^{n+1}}) = \lambda(\frac{I^n}{\mathbf{m}\,I^n})$, for $1 \leq n \leq r-1$. Since a is a non-zero-divisor R, $I^{n+r}/\mathbf{m}\,I^{n+r} \cong I^r/\mathbf{m}\,I^r$, for every $n \geq 1$. Hence $e(F(I)) = \lambda(I^r/\mathbf{m}\,I^r)$. To see the Cohen-Macaulay property of $F(I)$, we use [DRV, Theorem 2.1]. We have the following:

$$\begin{aligned}
\sum_{n=0}^{r} \lambda(\frac{I^n}{aI^{n-1}+\mathbf{m}\,I^n}) &= \lambda(\frac{R}{\mathbf{m}}) + \sum_{n=1}^{r} \lambda(\frac{I^n}{aI^{n-1}+\mathbf{m}\,I^n}) \\
&= \lambda(\frac{R}{\mathbf{m}}) + \sum_{n=1}^{r} [\lambda(\frac{I^n}{\mathbf{m}\,I^n}) - \lambda(\frac{aI^{n-1}+\mathbf{m}\,I^n}{\mathbf{m}\,I^n})] \\
&= \lambda(\frac{R}{\mathbf{m}}) + \sum_{n=1}^{r} [\lambda(\frac{I^n}{\mathbf{m}\,I^n}) - \lambda(\frac{I^{n-1}}{\mathbf{m}\,I^{n-1}})] \\
&= \lambda(\frac{I^r}{\mathbf{m}\,I^r}) \\
&= e(F(I)).
\end{aligned}$$

Hence by [DRV, Theorem 2.1], $F(I)$ is a Cohen-Macaulay ring. $\square$

As an immediate consequence of Proposition 6.1 we have

Corollary 6.2. *Let $(R, \mathbf{m})$ be a Noetherian local ring and I be a regular ideal having a principal reduction aR with $r_{aR}(I) = 2$. If $\mu(I) = n$, then*

$$F(I) \text{ is Cohen-Macaulay} \iff \lambda(\frac{aI + \mathbf{m}\,I^2}{\mathbf{m}\,I^2}) = n.$$

Example 6.3 shows that Proposition 6.1 and Corollary 6.2 may fail to be true without the assumption on the length of $\frac{aI^n+\mathbf{m}\,I^{n+1}}{\mathbf{m}\,I^{n+1}}$.

Example 6.3. *Let k be a field and consider the subring $R = k[[t^3, t^7, t^{11}]]$ of the formal power series ring $k[[t]]$. Let $I = (t^6, t^7, t^{11})R$. An easy computation implies $t^6 I \neq I^2$ and $t^6 I^2 = I^3$. Hence $r_{t^6R}(I) = 2$. Note that $\overline{t^6}F(I)$ is a homogeneous system of parameter of $F(I)$. But $\overline{t^6 t^{11}} = (t^6 + \mathbf{m}\,I)(t^{11} + \mathbf{m}\,I) = t^{17} + \mathbf{m}\,I^2 =$*

0, *and hence* $F(I)$ *is not a Cohen-Macaulay ring. And* $\lambda\left(\frac{t^6I+\mathbf{m}I^2}{\mathbf{m}I^2}\right) = \lambda\left(\frac{I^2}{\mathbf{m}I^2}\right) - \lambda\left(\frac{I^2}{t^6I+\mathbf{m}I^2}\right) = 2 < 3.$

References

[AM] M. F. Atiyah and I. G. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley, London, 1969.

[BH] W. Bruns and J. Herzog, *Cohen–Macaulay Rings*, Cambridge University Press, Cambridge, 1993.

[CGPU] A. Corso, L. Ghezzi, C. Polini and B. Ulrich, Cohen-Macaulayness of special fiber rings, in preparation.

[CZ] T. Cortadellas and S. Zarzuela, On the depth of the fiber cone of filtrations, J. Algebra **198** (1997), 428–445.

[DGH] M. D'Anna, A. Guerrieri and W. Heinzer, Ideals having a one-dimensional fiber cone, Ideal Theoretic Methods in Commutative Algebra, Marcel Dekker, 2001, pp. 155-170.

[DRV] C. D'Cruz, K.N. Raghavan and J. Verma, Cohen-Macaulay fiber cone, Commutative Algebra, Algebraic Geometry and computational methods, (Hanoi 1996), Springer, Singapore 1999, pp. 233-246.

[ES] P. Eakin and A. Sathaye, Prestable ideals, J. Algebra **41** (1976), 439–454.

[Hu] S. Huckaba, Analytic spread modulo an element and symbolic Rees algebras, J. Algebra **128** (1990), 306-320.

[K] E. Kunz, *Introduction to Commutative Algebra and Algebraic Geometry*, Birkhauser, Bosten, 1985.

[Mat] H. Matsumura, *Commutative ring theory*, Cambridge Univ. Press, Cambridge, 1986.

[N] M. Nagata, *Local Rings*, Interscience, New York, 1962.

[NR] D.G. Northcott and D. Rees, Reductions of ideals in local rings, Proc. Camb. Phil. Soc. **50** (1954), 145–158.

[R] D. Rees, Rings associted with ideals and analytic spread, Math. Proc. Camb. Phil. Soc. **89** (1981), 423-432.