IDEALS HAVING A ONE-DIMENSIONAL FIBER CONE

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Dedicated to Jim Huckaba on the occasion of his retirement

Abstract. For a regular ideal I having a principal reduction in a Noetherian local ring $(R, \mathfrak{m})$ we consider properties of the powers of I as reflected in the fiber cone $F(I)$ and the associated graded ring $G(I)$ of I. In particular, we examine the postulation number of $F(I)$ and compare it with the reduction number of I, and the postulation number of $G(I)$ when the latter is meaningful. We discuss a sufficient condition for $F(I)$ to be Cohen–Macaulay and consider for a fixed R what is possible for the reduction number $r(I)$ of I and the multiplicity of $F(I)$.

1. Introduction.

Given an ideal I in a Noetherian local ring $(R, \mathfrak{m})$, information on properties of I^n as n grows is encoded in various graded rings related to I. These graded rings are built by using and manipulating the I-adic filtration, $\{I^n\}_{n\geq 0}$ on R. Among these rings are:

(i) the *Rees algebra* of I, $R[It] = \oplus_{n\geq 0} I^n$,

(ii) the *associated graded ring* of I, $G(I) = \oplus_{n\geq 0} I^n/I^{n+1} = \oplus_{n\geq 0} G_n \cong R[It]/IR[It]$, and

(iii) the *fiber cone* of I, $F(I) = \oplus_{n\geq 0} I^n/\mathfrak{m}I^n = \oplus_{n\geq 0} F_n \cong R[It]/\mathfrak{m}R[It]$.

These are a few of the *blowup algebras* of I, (see [V1]), where by this term one refers to those algebraic objects that are related to the concept of blowing up a variety along a subvariety.

The graded rings $G(I)$ and $F(I)$ are both *homogeneous,* or *standard,* in the sense that they are generated by their forms of degree one over their subring of elements of degree zero. If I is $\mathfrak{m}$-primary, the Hilbert function giving the length,

$\lambda(I^n/I^{n+1})$, of I^n/I^{n+1} as an R-module is $H_G(n)$, where $H_G(X)$ is the Hilbert function of the associated graded ring $G(I)$. For an arbitrary ideal $I \subseteq \mathfrak{m}$ of $(R, \mathfrak{m})$, the fiber cone $F(I)$ has the attractive property of being a finitely generated graded ring over the residue field $K := R/\mathfrak{m}$. It is well known in this setting that the Hilbert function $H_F(n)$ giving the dimension of $I^n/\mathfrak{m}I^n$ as a vector space over K is defined for n sufficiently large by a polynomial $h_F(X) \in \mathbb{Q}[X]$, the *Hilbert polynomial* of $F(I)$ [Mat, Corollary, page 95], [AM, Corollary 11.2]. A simple application of Nakayama's lemma, [Mat, Theorem 2.2], shows that the cardinality of a minimal set of generators of I^n, $\mu(I^n)$, is equal to $\lambda(I^n/\mathfrak{m}I^n)$, the Hilbert function $H_F(n)$ of $F(I)$. For these reasons $G(I)$ and $F(I)$ are good objects to analyze and compare when studying the asymptotic properties of I.

If I is $\mathfrak{m}$-primary it is natural to ask about the relationship of the Hilbert function $H_G(X)$ and Hilbert polynomial $h_G(X)$ of $G(I)$ with the Hilbert function $H_F(X)$ and Hilbert polynomial $h_F(X)$ of $F(I)$. We begin such an investigation here in the one-dimensional case. We also consider, with no restriction on the dimension of R, properties of the fiber cone $F(I)$ of ideals I that have principal reductions in R.

Thus, in certain aspects, this paper is a continuation of our work in [DGH]. If I is a regular ideal having a principal reduction, we analyze in [DGH] the mutual relations among the following, where $\mathbb{N}$ denotes the nonnegative integers:

(1) $r = r(I) = \min\{n \in \mathbb{N} \mid I^{n+1} = xI^n$ for some x $\in I\}$, the *reduction number* of I,

(2) $k = k(I) = \min\{n \in \mathbb{N} \mid \widetilde{I} = (I^{n+1} : I^n)\}$, the *Ratliff–Rush number* of I,

(3) $h = h(I) = \min\{n \in \mathbb{N} \mid \widetilde{I^m} = I^m$ for all $m \geq n\}$, the *asymptotic Ratliff–Rush number* of I.

In defining k and h, we are using the *Ratliff–Rush closure* of I and of its powers, namely $\widetilde{I^k} := \cup_{n \in \mathbb{N}}(I^{n+k} :_R I^n)$. This concept was introduced by L. J. Ratliff and D. E. Rush in [RR] where it was also observed that $\widetilde{I^m} = I^m$ for all sufficiently large integers m [RR, Remark (2.3)]. This motivates our definition of h. We have $h(I) = 0$ if and only if $G(I)$ contains a regular homogeneous element of positive degree. If $h(I) > 0$, then $h(I) \geq 2$.

It is shown in [RV] in the case of the maximal ideal of a one-dimensional local

Noetherian ring and in a more general setting in [DGH, (2.1) and (2.2)] that if I has a principal reduction, then $h \leq r$ and $k \leq \max\{0, r-1\}$. If R is a reduced Noetherian ring with total ring of fractions $Q(R)$ and if the integral closure $\overline{R}$ of R in $Q(R)$ is a finitely generated R-module, it is shown in [DGH, (3.2) and (3.10)] that if I is contained in the conductor of $\overline{R}$ into R, then $k = \max\{0, r-1\}$, and, if $I \neq \widetilde{I}$, then $h = r$.

Suppose I is an $\mathfrak{m}$-primary ideal of a Noetherian local ring $(R, \mathfrak{m})$. The *postulation number* $n(I)$ of I is the largest integer n such that $H_G(n) \neq h_G(n)$, where $H_G(X)$ and $h_G(X)$ are, respectively, the Hilbert function and Hilbert polynomial of $G(I)$. Thus $H_G(n)$ is the length of I^n/I^{n+1} as an R-module. T. Marley shows in [Mar, Theorem 2] that if $(R, \mathfrak{m})$ is a d-dimensional Cohen–Macaulay local ring with infinite residue field and I is an $\mathfrak{m}$-primary ideal such that $G(I)$ has depth at least $d-1$, then $r(I) = n(I) + d$. In the case where $I = \mathfrak{m}$ this had been shown by J. Sally [Sa, Proposition 3]. Note that the condition on the depth of $G(I)$ is vacuous if $d = 1$, a case also considered by A. Ooishi in [O, Proposition 4.10]. Thus if I is an $\mathfrak{m}$-primary ideal in a one-dimensional Cohen–Macaulay local ring $(R, \mathfrak{m})$ with $R/\mathfrak{m}$ infinite, then $n(I) = r(I) - 1$.

We define the *fiber postulation number* $fp(I)$ of I to be the largest integer n such that $H_F(n) \neq h_F(n)$, where $H_F(X)$ and $h_F(X)$ are, respectively, the Hilbert function and Hilbert polynomial of the fiber cone $F(I)$. Suppose I is a nonprincipal regular ideal having a principal reduction in a Noetherian local ring $(R, \mathfrak{m})$. We observe in Proposition 2.2 that $fp(I) \leq r(I) - 1$, where $r(I)$ is the reduction number of I. Thus if I is a nonprincipal $\mathfrak{m}$-primary ideal of a one-dimensional Cohen–Macaulay local ring, then the fiber postulation number $fp(I)$ is less than or equal to the postulation number $n(I)$ of I.

For each integer $n \geq 3$, we exhibit in Example 2.3 the existence of a one-dimensional Cohen–Macaulay local domain R_n and an ideal I_n primary for the maximal ideal of R_n such that $fp(I_n) = 0$ while $r(I_n) = n - 1$. This shows that the difference $r(I) - fp(I)$ can be arbitrarily large for ideals I and rings R as in Propositon 2.2. For the ideals I_n of Example 2.3 the fiber postulation number $fp(I_n)$ is strictly smaller than the postulation number $n(I)$. It follows that the fiber cone $F(I_n)$ is not Cohen–Macaulay. Indeed, as we observe in Proposition 3.2: if $F(I)$ is Cohen–Macaulay, then $fp(I) = r(I) - 1$. This and a higher dimensional analogue stating that if $F(I)$ is Cohen–Macaulay, then $fp(I) = r(I) - \ell$, where ℓ is the analytic spread of I, are consequences of results of T. Cortadellas and S. Zarzuela [CZ] and C. D'Cruz, K. N. Raghavan and J. Verma [DRV] on the structure of the Hilbert function of a Cohen–Macaulay fiber cone $F(I)$. One can use [BH, Proposition 4.1.12] to obtain the asserted relation between the reduction number and fiber postulation number.

In Example 2.4, we exhibit a 3-dimensional Cohen–Macaulay local domain $(R, \mathfrak{m})$ of multiplicity 3 having, for each positive integer k, an ideal I_k such that: (i) I_k has a principal reduction, (ii) I_k is minimally generated by $k+3$ elements, and (iii) the fiber cone $F(I_k)$ has multiplicity 3 and is not Cohen–Macaulay.

We observe in Remark 4.2 that the ideal I_3 of Example 2.3 provides an example of a one-dimensional Cohen–Macaulay local domain $(R, \mathfrak{m})$ and an $\mathfrak{m}$-primary ideal I such that the associated graded ring $G(I)$ is Cohen–Macaulay, while the fiber cone $F(I)$ is not Cohen–Macaulay.

In Example 4.6 we exhibit a 3-dimensional Cohen–Macaulay local domain $(R, \mathfrak{m})$ of multiplicity 2 with the following property. For each positive integer k, the ring R has a stable ideal I_k having a principal reduction such that the fiber cone $F(I_k)$ of I_k is Cohen–Macaulay and has multiplicity $k+2$. Therefore, in contrast with the case where $\dim R = 1$, for R of higher dimension there may exist no upper bound on the multiplicity of $F(I)$ as I varies over the ideals of R that have a principal reduction.

In §5 we consider a question raised by S. Huckaba [Hu1,Question 2.6] as to whether the multiplicity of a quasi-unmixed analytically unramified Noetherian local ring containing an infinite field of characteristic different from 2 is strictly larger than the reduction number of each regular ideal of analytic spread one of the ring. Using an interesting observation shown to us by Craig Huneke (Proposition 5.1) we obtain a positive answer to Huckaba's question in several special cases. For example, in Corollary 5.7 we confirm a positive answer to the question for the case of a 2-dimensional Noetherian analytically irreducible local domain. However, the general case of Huckaba's question remains open.

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2. The postulation number of a one-dimensional fiber cone.

Suppose I is a regular ideal of a Noetherian local ring $(R, \mathfrak{m})$. If there exists $x \in I$ and an integer $n \geq 0$ such that $xI^n = I^{n+1}$, then xR is said to be a *principal reduction of* I. In terms of the fiber cone $F(I)$ this is reflected in the fact that if $\overline{x} \in I/\mathfrak{m}I$ denotes the image of x, then $F(I)$ is a finitely generated integral extension of its polynomial subring $K[\overline{x}]$, where $K := R/\mathfrak{m}$. Indeed, if K is infinite, the converse also holds: if the fiber cone $F(I)$ is one-dimensional, then using Noether normalization, there exists $\overline{x} \in I/\mathfrak{m}I$ such that $F(I)$ is integral over $K[\overline{x}]$. If $x \in I$ is a preimage of $\overline{x}$, then xR is a principal reduction of I.

D. G. Northcott and D. Rees in their famous paper [NR] introduce reductions of ideals and associate to an ideal I a homogeneous ideal in a polynomial ring over K, the *null form ideal of* I. The null form ideal of I is the kernel in a presentation of the fiber cone $F(I)$ as a homomorphic image of a polynomial ring over K. K. Shah in [S] coined the term 'fiber cone' for this ring.

If I is a regular ideal having a principal reduction, the reduction number $r(I)$ is known to be independent of the principal reduction x [Hu1, page 504]. In this situation, $r(I)$ is also equal to the reduction number of the algebra $F(I)$. If $m_1, \ldots, m_s$ is a minimal set of homogeneous module generators for $F(I)$ over $(R/\mathfrak{m})[\overline{x}]$, then $r(I) = \max\{\deg m_i\}$, see [V2].

The Hilbert function $H_F(X)$ defined so that $H_F(n) = \lambda(I^n/mI^n)$ is given for all sufficiently large n by a polynomial $h_F(X)$. Since, in this case, $F(I) = \oplus_{n \geq 0} I^n/mI^n$ is a one-dimensional homogeneous graded ring over a field, $h_F(X)$ is a constant f_0. Here f_0 is a positive integer. It is the multiplicity of the graded ring $F(I)$ and the number of elements in a minimal generating set of I^n for all sufficiently large n.

It is of interest to also consider the first iterated Hilbert function of $F(I)$. This is the function $H_F^1(n) = \sum_{j=0}^{n} H_F(j) = \sum_{j=0}^{n} \lambda(I^j/\mathfrak{m}I^j)$. It is a polynomial function for $n > fp(I)$. The associated Hilbert polynomial is $h_F^1(X) = f_0(X+1) - f_1 = f_0X - (f_1 - f_0)$. In analogy with the Hilbert coefficients of $G(I)$ in the case where I is $\mathfrak{m}$-primary, it is of interest to have sufficient conditions in order that $H_F^1(0) - h_F^1(0) = 1 - f_0 + f_1 \geq 0$. In other words, sufficient conditions in order that $f_1 \geq f_0 - 1$. Notice that for $n \geq r-1$, where $r = r(I)$ one has

$$H_F^1(n) = \sum_{j=0}^{n} \lambda(I^j/\mathfrak{m}I^j) = H_F^1(r-1) + (n-(r-1))f_0 = nf_0 - ((r-1)f_0 - H_F^1(r-1)).$$

It follows that $f_1 - f_0 = (r-1)f_0 - H_F^1(r-1)$ and $f_1 = rf_0 - H_F^1(r-1)$. We observe in Example 2.4 below that this last equality implies that f_1 can be negative and that for a fixed ring R there may be no lower bound on the value of f_1 as we vary over ideals of R having a principal reduction. In the case, however, where the fiber cone $F(I)$ is Cohen–Macaulay, as we observe in Remark 2.1, it follows from classical results that $f_1 \geq f_0 - 1$.

Remark 2.1: Suppose $(R, \mathfrak{m})$ is a Noetherian local ring and I is a regular ideal of R having a principal reduction. If the fiber cone $F(I)$ is Cohen–Macaulay, then a result of Northcott implies that $f_1 \geq f_0 - 1$. For the one-dimensional homogeneous graded ring $F(I)$ has the same Hilbert function as the local ring $(S, \mathfrak{n})$ obtained by localizing $F(I)$ at its homogeneous maximal ideal. If $F(I)$ is Cohen–Macaulay, then $(S, \mathfrak{n})$ is a one-dimensional Cohen–Macaulay local ring. Northcott defines normalized Hilbert coefficients e_0 and e_1 of $\mathfrak{n}$ such that $\lambda(S/\mathfrak{n}^{n+1}) = e_0(n+1) - e_1$ for $n >> 0$, and proves [No, page 211] that $e_1 \geq e_0 - 1$. Since $\lambda(S/\mathfrak{n}^{n+1}) = H_F^1(n) = f_0(n+1) - f_1$ for $n >> 0$, we have $e_0 = f_0, e_1 = f_1$ and thus $f_1 \geq f_0 - 1$.

We are interested in comparing the fiber postulation number $fp(I)$ of I with other invariants associated to I. We start by observing in Proposition 2.2 an upper bound for $fp(I)$ in terms of the reduction number of I.

Proposition 2.2: *Suppose $(R, \mathfrak{m})$ is a Noetherian local ring and I is a nonprincipal regular ideal of R having a principal reduction xR. Then $fp(I) \leq r(I) - 1$, where $r(I) = r$ is the reduction number of I.*

Proof: Notice that $I^s/\mathfrak{m}I^s = x^{s-r}I^r/\mathfrak{m}x^{s-r}I^r \cong I^r/\mathfrak{m}I^r$ for all $s \geq r = r(I)$. Thus $H_F(n) = \lambda(I^r/\mathfrak{m}I^r)$ for every $n \geq r$. It follows that $h_F(X) = f_0 = \lambda(I^r/\mathfrak{m}I^r)$ and that $H_F(n) = h_F(n)$ for every $n \geq r$. Therefore $fp(I) \leq r(I) - 1$. $\square$

With notation as in the proof of Proposition 2.2, observe that $fp(I) < r(I) - 1$ if and only if $h_F(r-1) = H_F(r-1) = f_0$. Thus $fp(I) < r(I) - 1$ if and only if I^{r-1} is minimally generated by f_0 elements. We use this observation in order to construct in Example 2.3 for each integer $n \geq 3$ an example of a one-dimensional Cohen–Macaulay local domain R_n and an ideal I_n primary for the maximal ideal of R_n such that $fp(I_n) = 0$ while $r(I_n) = n - 1$.

As is the case for our examples in §4 of [DGH], the rings R_n in Example 2.3 are complete one-dimensional local Cohen–Macaulay domains of the form $R_n = K[\![t^s : s \in S_n]\!]$, i.e. formal power series in the indeterminate t with coefficients in a field K and exponents from an additive submonoid S_n of the nonnegative integers that contains all sufficiently large integers. The formal power series ring $K[\![t]\!]$ is a finitely generated R_n-module having the same fraction field as R_n and is thus the integral closure of R_n. To establish the asserted properties of the rings R_n and ideals I_n in Example 2.3 we work directly with the additive monoid S_n and the corresponding semigroup ideal I_n of S_n.

Example 2.3: Fix an integer $n \geq 3$, and consider the numerical semigroup S_n generated by the elements $a := 2n$, $b := 4n-1$, $d := n(2n-1)$, and $c_h := (n+h)(2n-1) + 1$, where $h = 3 \dots n$. Thus $S_n =< a, b, d, c_3, \dots , c_n >$. We prove that these $n + 1$ elements are the minimal generating set for S_n. In considering the general case, it is useful to keep in mind the first few examples, $S_3 =< 6, 11, 15, 31 >$, $S_4 =< 8, 15, 28, 50, 57 >$ and $S_5 =< 10, 19, 45, 73, 82, 91 >$.

Claim 2.3.1: The monoid S_n is minimally generated by a, b, d, c_3, $c_4, \dots , c_n$.

Proof: We have, modulo $2n$, that $b \equiv -1 \equiv 2n - 1$, $d \equiv -n \equiv n$ and $c_h \equiv -n - h + 1 \equiv n - (h - 1)$. The smallest integer x in $< a, b >$ such that $x \equiv n$ is $x = nb = 4n^2 - n$, which is larger than d; hence $d \notin < a, b >$ and a, b, d are minimal generators of the submonoid $< a, b, d >$.

Now we consider c_h. If $h > 3$ and if $3 \leq i < h$, then $c_h - c_i \notin < a, b, d >$. In fact, we have $c_h - c_i = (h-i)(2n-1) < d$; hence $c_h - c_i \notin < a, b, d > \Longleftrightarrow c_h - c_i \notin < a, b >$. But $c_h - c_i \equiv -h + i \pmod{2n}$ and the smallest integer x in $< a, b >$ such that $x \equiv -h+i$ is $x = (h-i)b = (h-i)(4n-1)$ which is larger than $c_h - c_i = (h-i)(2n-1)$. It follows that $c_h - c_i \notin < a, b, d >$ for every $i = 3, \ldots, h-1$. Thus, for every $h > 3$, $c_h \notin < a, b, d, c_3, \ldots, c_{h-1} >$ if and only if $c_h \notin < a, b, d >$.

Therefore, in order to prove that S_n is minimally generated by $a, b, d, c_3, c_4, \ldots, c_n$, it suffices to show that $c_h \notin < a, b, d >$, for every $h = 3, \ldots, n$. Since $c_h \equiv n - (h-1) \pmod{2n}$, we compute the smallest element in $< a, b, d >$ that is congruent to $n - (h-1)$ modulo $2n$ and observe that it is larger than c_h. We have that $a\alpha + b\beta + d\delta \equiv -\beta + n\delta \pmod{2n}$. Hence we want to compute the minimum of the set $H = \{b\beta + d\delta \mid \beta, \delta \geq 0,\ -\beta + n\delta \equiv n - (h-1)\}$.

We claim that $\min H = (h-1)b + d$ (i.e. we have the minimum for $\beta = (h-1)$ and $\delta = 1$). Assume, by way of contradiction, that there exists an integer $b\beta + d\delta \equiv n - (h-1)$ such that $b\beta + d\delta < (h-1)b + d$. This implies $\beta < (h-1)$ or $\delta = 0$.

If $\delta = 0$, then $b\beta \equiv -\beta \equiv n - (h-1)$, that is $\beta \equiv -n + (h-1) \equiv n + (h-1)$; this implies $\beta = n + h - 1 + m(2n)$, with $m \in \mathbb{N}$. But $nb > d$, thus $(n + h - 1 + m(2n))b > (h-1)b + d$, a contradiction.

If $\beta < (h-1)$, then $b\beta + d\delta \equiv -\beta + n\delta \equiv n - (h-1)$. But this implies that $n(\delta - 1) \equiv \beta - (h-1)$, which is a contradiction, since $n(\delta-1) \equiv n$ or $n(\delta - 1) \equiv 0$ and $-n < \beta - (h-1) < 0$.

We conclude that $(h-1)b + d$ is the minimum element $x \in < a, b, d >$ such that $x \equiv n - (h-1)$. Since for every $h = 3, \ldots, n$, we have $(h-1)b + d = (h-1)(4n - 1) + n(2n-1) = 2n^2 + n(4(h-1)-1) - h + 1 > 2n^2 + n(2(h-1)+1) - h + 1 = c_h$, it follows that $c_h \notin < a, b, d >$. Therefore S_n is minimally generated by a, b, d, $c_3, \ldots, c_n$. $\square$

The *Frobenius number* $g(S_n)$ of S_n is by definition the largest integer not belonging to S_n. We next compute $g(S_n)$: since c_n is a generator of S_n, we have $c_n - a \notin S_n$. Moreover all the elements of the set $\{c_n - a + 1, c_n - a + 2, \ldots, c_n\}$ belong to S_n:

(1) $c_n - a + 1 = (2n-1)(2n-1) + 1 = c_{n-1}$;

(2) for all $m = 2, \dots n-3$, $c_n - a + m = 2n(2n-1) + 1 - 2n + m \equiv m + 1 \equiv n - (n-m) + 1 \equiv c_{n-m} \pmod a$ and $c_n - a + m > c_{n-m}$;

(3) $c_n - a + n - 2 = 2n(2n-1) - n - 1 \equiv b + d \pmod a$ and $c_n - a + n - 2 > b + d$;

(4) $c_n - a + n - 1 = 2n(2n-1) - n \equiv d \pmod a$ and $c_n - a + n - 2 > d$;

(5) for every $m = 0, \dots, n-2$, $c_n - a + n + m = 2n(2n-1) + 1 - n + m = (2m+1))a + (n-m-1)b$;

(6) $c_n - a + 2n - 1 = 2n(2n-1) = 2d$;

(7) $c_n - a + 2n = c_n$.

Therefore S_n contains all integers greater than or equal to $c_n - a + 1 = c_{n-1}$ and $g(S_n) = c_n - a$.

Consider the semigroup ideal $I_n :=< a, b, c_3, \dots, c_h >$ of S_n.

Claim 2.3.2: For every integer $m \geq 1$, the ideal mI_n is minimally generated by n elements.

Proof: Since the elements $a, b, c_3, \dots, c_n$ are part of the minimal set of generators of S_n, it follows at once that I_n is minimally generated by a, b, $c_3, \dots c_n$. We prove that $2I_n$ is minimally generated by $2a$, $a+b$, $2b$, $a+c_4, \dots, a+c_n$. We have the following relations:

$$(R_1) \qquad a + c_3 = 2b + d$$

and, for every $h = 4, \dots n$,

$$(R_2) \qquad a + c_h = b + c_{h-1}.$$

Moreover, since $b + c_n - 2a > c_n - a = g(S_n)$, for some $s \in S_n$ we have

$$(R_3) \qquad b + c_n = 2a + s.$$

The same is true for $c_h + c_k$, for all $h, k = 3, \dots, n$, since $c_h + c_k \geq 2c_3 > b + c_n$: for some $t_{h,k} \in S_n$,

$$(R_4) \qquad c_h + c_k = 2a + t_{h,k}.$$

It follows that $2I_n =< 2a, a+b, 2b, a+c_4, \dots, a+c_n >$. We check that this is the minimal set of generators (i.e. there are not other relations). Clearly we have

$a+b-2a = 2b-(a+b) = b-a \notin S_n$. Notice that $y(b-a) \notin S_n$ for every $y < n$: in fact, $y(b-a) < d$ and $y(b-a) \notin < a, b >$, since $y(b-a) < yb$ which is the smallest integer x in $< a, b >$ such that $x \equiv -y \pmod a$. It follows that $2b-2a \notin S_n$. Moreover, for every $h = 4, \dots, n$, since c_h is part of the minimal set of generators of I_n, we have $a+c_h-2a = c_h-a \notin S_n$, $a+c_h-(a+b) = c_h-b \notin S_n$, $a+c_h-2b = b+c_{h-1}-2b = c_{h-1}-b \notin S_n$ and, for $k < h$, $a+c_h-(a+c_k) = c_h-c_k \notin S_n$. Therefore $2I_n$ is minimally generated by $2a$, $a+b$, $2b$, $a+c_4, \dots, a+c_n$.

Proceeding by induction on m, we assume that $2 \le m \le n-2$ and that for every k with $2 \le k \le m$, the semigroup ideal kI_n is minimally generated by ka, $(k-1)a+b$, $\dots$, kb, $(k-1)a+c_{k+2}, \dots, (k-1)a+c_n$. We prove that $(m+1)I_n$ is minimally generated by $(m+1)a$, $ma+b$, $\dots$, $(m+1)b$, $ma+c_{m+3}, \dots, ma+c_n$ (in the case $m = n-2$, the inequality $m+3 > n$ means that $(m+1)I_n = (n-1)I_n$ is minimally generated by $(n-1)a$, $(n-2)a+b$, $\dots$ $(n-1)b$).

Using relations (R_1) and (R_2), notice that $a+(m-1)a+c_{m+2} = a+(m-1)b+c_3 = (m+1)b+d$. Moreover, for every $h = m+2, \dots, n-1$, we have $b+(m-1)a+c_h = ma+c_{h+1}$ (using (R_2)) and $b+(m-1)a+c_n = (m+1)a+s$ (using (R_3)).

Now we consider the elements $c_k+ra+sb$ and $c_k+(m-1)a+c_h$ (where $k = 3, \dots n$, $r+s = m$ and $s > 0$). We easily get, by (R_4), that $c_k+(m-1)a+c_h = (m+1)a+t_{k,h}$; moreover, if $k+s \le n$, using (R_2), we get $c_k+ra+sb = c_{k+s}+ma$, while, if $k+s > n$, using (R_2) and (R_3), $c_k+ra+sb = c_n+(r+n-k)a+(s-(n-k))b = (r+n-k+2)a+(s-(n-k)-1)b+s$. Hence $(m+1)I_n = < ma, (m-1)a+b, \dots, mb, (m-1)a+c_{m+2}, \dots, (m-1)a+c_n >$.

In order to prove that $(m+1)I_n$ is minimally generated by these elements, we show that there are no other relations. If $r+s = u+v = m+1$ and $s > v$, the difference $ra+sb-(ua+vb) = (s-v)(b-a) \notin S_n$, since $s-v < n$. Moreover, for every $h \ge m+3$ and $r+s = m+1$, $ma+c_h-(ra+sb) = (m-r)a+c_h-(m-r+1)b \notin S_n$, since $(m-r)a+c_h$ is part of the minimal set of generators of $(m-r+1)I_n$. Analogously, if $k < h$, $ma+c_h-(ma+c_k) = c_h-c_k \notin S_n$, since c_h is part of a minimal set of generators of I_n.

It follows that, for every $m = 2, \dots, n-1$, mI_n is minimally generated by

ma, $(m-1)a+b$, $\dots$, mb, $(m-1)a+c_{m+2}, \dots, (m-1)a+c_n$. In particular $(n-1)I_n = < (n-1)a, (n-2)a+b, \dots, (n-1)b >$. Now, since

$$(R_5) \qquad nb = na + d,$$

we have, with similar arguments as above, that nI_n is minimally generated by na, $(n-1)a+b$, $\dots$, nb, that is $nI_n = (n-1)I_n + a$. Hence, for every $m \geq n$, also mI_n is minimally n-generated; moreover the reduction number of I_n is $r(I_n) = n-1$. This completes the proof of Claim 2.3.2 and the presentation of Example 2.3. □

In Example 2.4, we exhibit a 3-dimensional Cohen–Macaulay local domain $(R, \mathfrak{m})$ of multiplicity 3 having, for each positive integer k, an ideal I_k such that I_k has a principal reduction, I_k is minimally generated by $k+3$ elements, and the fiber cone $F(I_k)$ has multiplicity 3 and is not Cohen–Macaulay.

Example 2.4: Let $(S, \mathfrak{n})$ be a two-dimensional regular local ring and let $R = S[t^3, t^4, t^5]_{(\mathfrak{n}, t^3, t^4, t^5)}$, where t is an indeterminate over S. For k a fixed positive integer, let $I_k = (t^3, t^4, \mathfrak{n}^k t^5)R$. Since $\mathfrak{n}^k$ is minimally generated by $k+1$ elements as an ideal in S, we see that I_k is minimally generated by $k+3$ elements. Notice that $t^3 I_k \subsetneq I_k^2 = (t^6, t^7, t^8)$ and that $I_k^3 = t^3 I_k^2$. Thus t^3 is a principal reduction of I_k and I_k has reduction number $r(I_k) = 2$. Moreover, I_k^2 is minimally generated by t^6, t^7, t^8, so the multiplicity of the fiber cone $F(I_k)$ is $f_0 = 3$ and $f_1 = 6 - (1+k+3)$. Since k can be arbitrarily large, there is no lower bound on f_1 as we vary over ideals of R having a principal reduction.

Since $t^3 \mathfrak{n}^k t^5 \subseteq \mathfrak{m} I_k^2$, where $\mathfrak{m}$ is the maximal ideal of R, we see that the image of t^3 in $I_k/\mathfrak{m} I_k = F_1$ in $F(I_k)$ is a zero-divisor. Therefore Remark 3.1 as given below implies that $F(I_k)$ is not Cohen–Macaulay and thus has depth zero.

3. The Cohen–Macaulay property of the fiber cone.

Interesting work on the Cohen–Macaulay property of the fiber cone $F(I)$ has been done by K. Shah in [S], by T. Cortadellas and S. Zarzuela in [CZ] and by C. D'Cruz, K. N. Raghavan and J. K. Verma in [DRV]. In particular, we use the following:

Remark 3.1: As Shah notes in [S], the freeness lemma of Hironaka [N, (25.16)] implies that $F(I)$ is Cohen–Macaulay if and only if $F(I)$ is a free module over one (or equivalently every) Noether normalization subring. If x is a principal reduction of I and $\overline{x}$ denotes the image of x in $I/\mathfrak{m}I = F_1$, then $K[\overline{x}]$ is a Noether normalization of $F(I)$. Since $K[\overline{x}]$ is a principal ideal domain, $F(I)$ is a free $K[\overline{x}]$-module if and only if it is a torsionfree $K[\overline{x}]$-module. Since $F(I)$ is graded and $\overline{x}$ is homogeneous, we see that $F(I)$ is torsionfree as a $K[\overline{x}]$-module if and only if $\overline{x}$ is a regular element of $F(I)$. Therefore $F(I)$ is Cohen–Macaulay if and only if $\overline{x}$ is a regular element of $F(I)$.

It follows from Remark 3.1 that for the ideal $I_n = (t^a, t^b, t^{c_3}, \dots, t^{c_n})$ of the ring $R_n = K[\![t^s : s \in S_n]\!]$ given in Example 2.3, the fiber cone $F(I_n)$ is not Cohen-Macaulay: the relation (R_1) of Example 2.3 implies that $t^a t^{c_3} \in I_n^2\mathfrak{m}$. Therefore the image in $F(I_n)$ of t^a (which is a principal reduction of I_n) is a zerodivisor in $F(I_n)$. The fact that $F(I_n)$ is not Cohen-Macaulay also follows from Proposition 3.2.

Proposition 3.2 as given below is a consequence of results of Cortadellas and Zarzuela [CZ] and D'Cruz, Raghavan and Verma [DRV] on the structure of the Hilbert function of a Cohen–Macaulay fiber cone $F(I)$. One can use [BH, Proposition 4.1.12] to see the asserted relation between the fiber postulation number and reduction number from their results. We give a direct elementary proof of the result.

Proposition 3.2: *Suppose $(R, \mathfrak{m})$ is a Noetherian local ring and I is a nonprincipal regular ideal of R having a principal reduction xR. If $F(I)$ is Cohen–Macaulay, then $fp(I) = r(I) - 1$.*

Proof: Let $\overline{x}$ denote the image of x in $I/\mathfrak{m}I = F_1$. As noted in Remark 3.1, $F(I)$ is Cohen–Macaulay if and only if $\overline{x}$ is a regular element of $F(I)$. By Proposition 2.2, $fp(I) \le r(I) - 1$, and by definition of $r(I) = r$, we have $xI^{r-1} \subsetneq I^r$.

Assume, by way of contradiction, that $fp(I) < r-1$. This means that $\dim F_{r-1} = f_0 = \dim F_r$. Since $\overline{x}$ is a regular element of $F(I)$, $f_0 = \dim \overline{x}F_{r-1}$. Since $\overline{x}F_{r-1} \subseteq$

F_r, this implies $\overline{x}F_{r-1} = F_r$. However, this implies $xI^{r-1} = I^r$, a contradiction. $\square$

Remark 3.3: The converse of Proposition 3.2 is not true in general. For example, let $R = K[t^4, t^5, t^{11}]_{(t^4, t^5, t^{11})}$, where t is an indeterminate over the field K. Let $I = \mathfrak{m}$, the maximal ideal of R. Since the image of $t^4 \in I/I^2$ is a zero-divisor, $F(I) = G(I)$ is not Cohen–Macaulay. On the other hand, $fp(I) = n(I) = 2$ and $r(I) = 3$, so $fp(I) = r(I) - 1$.

Other examples illustrating the failure of the converse of Proposition 3.2 are the ideals I_k of Example 2.4. In these examples one has $fp(I_k) = 1$ and $r(I_k) = 2$.

Discussion 3.4: Shah in [S] proves several interesting results on the Cohen–Macaulay property of $F(I)$. In [S, Theorem 1] he proves that if I is an ideal of a Noetherian local ring $(R, \mathfrak{m})$ and if I is integral over an ideal generated by a regular sequence $\underline{x}$ such that $I^2 = (\underline{x})I$, then $F(I)$ is Cohen–Macaulay. As Shah notes, it was proved earlier by C. Huneke and J. Sally in [HS, Proposition 3.3] that if $(R, \mathfrak{m})$ is a Cohen–Macaulay local ring and I is an $\mathfrak{m}$-primary ideal such that $I^2 = (\underline{x})I$, where $\underline{x}$ is a regular sequence, then $F(I)$ is Cohen–Macaulay. It is immediate from these results that, if I is a regular ideal having a principal reduction xR and if $r(I) \leq 1$, then the fiber cone $F(I)$ is Cohen–Macaulay. In [S, Theorem 2], Shah proves that if I is an ideal of a Noetherian local ring $(R, \mathfrak{m})$ and if I is integral over an ideal generated by a regular sequence $\underline{x}$ such that $I^3 = (\underline{x})I^2$, $I^2 \cap (\underline{x}) = I(\underline{x})$, and $I^2\mathfrak{m} = I(\underline{x})\mathfrak{m}$, then $F(I)$ is Cohen–Macaulay. In [CZ; Theorem 3.2], Cortadellas and Zarzuela generalize Shah's result and show that if I is an ideal and J a minimal reduction of I generated by a regular sequence with $J \cap I^n = JI^{n-1}$ for all $1 \leq n \leq r_J(I)$, then $F(I)$ is Cohen–Macaulay if and only if $J \cap \mathfrak{m}I^n = J\mathfrak{m}I^{n-1}$ for all $1 \leq n \leq r_J(I)$.

Proposition 3.5: *Suppose I is a regular ideal having a principal reduction (x) in a Noetherian local ring $(R, \mathfrak{m})$. If I is 2-generated, then $F(I)$ is Cohen–Macaulay.*

Proof: Consider a presentation ϕ of $F(I)$ as a graded K-algebra homomorphic image of the polynomial ring $K[X, Y]$, where $\phi(X)$ and $\phi(Y)$ are the images in

$I/\mathfrak{m}I = F_1$ of two generators of I. To show $F(I)$ is Cohen–Macaulay it suffices to show that $\ker\phi$ is principal. By a result of P. Eakin and A. Sathaye [ES, page 440], if I^n is generated by less than $n+1$ elements, then $xI^n = I^{n+1}$. Hence if $r = r(I)$ is the reduction number of I, then for each positive integer $n \leq r$ the ideal I^n is minimally generated by $n+1$ elements. Thus the multiplicity of $F(I)$ is $r+1$ and $\dim F_n = r+1$ for each $n \geq r$. Therefore the minimal degree of a form $f \in \ker\phi$ is $r+1$ and there exists a form $f \in \ker\phi$ with $\deg f = r+1$. It follows that $\ker\phi$ is generated by f. Therefore $F(I)$ is a complete intersection and hence Cohen–Macaulay. $\square$

Corollary 3.6: *Suppose $(R, \mathfrak{m})$ is a one-dimensonal Cohen–Macaulay local ring of multiplicity 2. For each $\mathfrak{m}$-primary ideal I of R, the fiber cone $F(I)$ is Cohen–Macaulay.*

Proof: Since R has multiplicity 2, the integral closure of R has at most 2 maximal ideals and each $\mathfrak{m}$-primary ideal I of R is 2-generated. Therefore I has a principal reduction and by Proposition 3.5, $F(I)$ is Cohen–Macaulay. $\square$

Question 3.7: For which one-dimensional Cohen–Macaulay local rings $(R, \mathfrak{m})$ is it true that $F(I)$ is Cohen–Macaulay for every $\mathfrak{m}$-primary ideal I? Is this true if R has multiplicity 3 ?

Example 3.8: Let $R = K[\![t^3, t^4, t^5]\!]$. Then $K[\![t]\!]$ is the integral closure $\overline{R}$ of R and the maximal ideal $\mathfrak{m} = (t^3, t^4, t^5)R$ is the conductor of $\overline{R}$ into R. If I is an $\mathfrak{m}$ primary ideal of R, then either (i) I is principal, or (ii) I is minimally 2-generated, or (iii) I is minimally 3-generated and $I = I\overline{R}$. In this third case, $I = t^s\overline{R}$ for some integer $s \geq 3$. It follows that $t^s I = I^2$ and I has reduction number one as an ideal of R. Therefore the fiber cone $F(I)$ is Cohen–Macaulay for each $\mathfrak{m}$-primary ideal I of R.

Remark 3.9: For R as in Example 3.8, there exist $\mathfrak{m}$-primary ideals I of R such that the associated graded ring $G(I)$ is not Cohen–Macaulay. For example, $I = (t^3, t^4)$ is not a Ratliff-Rush ideal since $I \subsetneq \mathfrak{m}$ and $I^2 = \mathfrak{m}^2$. Therefore I is an example of an ideal for which $F(I)$ is Cohen–Macaulay and $G(I)$ is not Cohen–Macaulay.

Remark 3.10: Suppose $(R, \mathfrak{m})$ is a one-dimensional Cohen–Macaulay local ring and I is an $\mathfrak{m}$-primary ideal having a principal reduction $xR = J$ with reduction number r. As a special case of [DRV, Theorem 2.1] it follows that the fiber cone $F(I)$ is Cohen–Macaulay if and only if the Hilbert series $P(t) = \sum_{n=0}^{\infty} \lambda(I^n/\mathfrak{m}I^n)t^n$ has the form $h(t)/(1-t)$, where $h(t) = \sum_{i=0}^{r} \lambda(I^i/(JI^{i-1} + \mathfrak{m}I^i))$. It is not true, however, that the Cohen–Macaulay property of the fiber cone $F(I)$ is determined by its Hilbert series $P(t)$. For example, if $R = K[\![t^6, t^{11}, t^{15}, t^{31}]\!]$ and $I = (t^6, t^{11}, t^{31})$ are as in Example 2.3, where $R = K[\![t^s : s \in S_3]\!]$ and I corresponds to the semigroup ideal I_3 of S_3, $F(I)$ is not Cohen–Macaulay and the Hilbert series for I is $P(t) = (1 + 2t)/(1 - t)$. This is also the Hilbert series of a Cohen–Macaulay fiber cone; for example it is the Hilbert series of the maximal ideal of $R = K[\![x, y, z]\!]$, where $y^2 = yz = z^2 = 0$.

4. The multiplicity of the fiber cone.

Suppose I is a regular ideal of R having a principal reduction xR. In this situation, the blowing up scheme Proj $R[It]$ of Spec R with respect to I is affine and has the form Spec $R[I/x]$. Therefore we refer to the ring $R^I := R[I/x]$ as the *blowing–up ring* of I. It is simple to prove that $R^I := R[I/x] = \bigcup_{n \geq 0} I^n/x^n = I^r/x^r$ where $r = r(I)$. Moreover, the blowing up ring R^I of I is also the blowing up ring of each power I^n of I.

Lemma 4.1: *The multiplicity f_0 of $F(I)$ is the minimal number of generators of R^I as an R-module. Thus $f_0 = \lambda(R^I/\mathfrak{m}R^I)$.*

Proof: We know that $f_0 = \lambda(I^r/\mathfrak{m}I^r)$. In other words the multiplicity of $F(I)$ is given by the cardinality of a minimal set of generators for I^r as an R-module. Moreover, the fact that x is a regular element of R implies that if $\{a_1, \ldots, a_{f_0}\}$ is a minimal set of generators for I^r, then $\{a_1/x^r, \ldots, a_{f_0}/x^r\}$ is a minimal set of generators of I^r/x^r as an R-module. $\square$

Remark 4.2: Suppose $(R, \mathfrak{m})$ is a one-dimensional Cohen–Macaulay local ring and I is an $\mathfrak{m}$-primary ideal having a principal reduction xR. The associated graded ring $G(I)$ is Cohen–Macaulay if and only if $I^n = \widetilde{I^n}$ for each $n \in \mathbb{N}$. Moreover, $\widetilde{I^n} = I^n R^I \cap R$, where $R^I = R[I/x]$ is the blowing up ring of I. We use this to observe that $G(I)$ is Cohen–Macaulay if $R = K[\![t^6, t^{11}, t^{15}, t^{31}]\!]$ and $I = (t^6, t^{11}, t^{31})$ are as in Example 2.3, where $R = K[\![t^s : s \in S_3]\!]$ and I corresponds to the semigroup ideal I_3 of S_3. Since $r(I) = 2$, it follows from [DGH, Proposition 2.2] that the asymptotic Ratliff-Rush number $h(I) \leq 2$. Thus $\widetilde{I^n} = I^n$ for each $n \geq 2$. Hence it suffices to show that $I = IR^I \cap R$. Observe that $R^I = R[t^5] = K[\![t^5, t^6]\!]$, $IR^I = t^6 R^I$, and $\lambda(R/I) = 2$. Thus it suffices to show that $t^{15} \notin t^6 K[\![t^5, t^6]\!]$, and this is clear since $t^9 \notin K[\![t^5, t^6]\!]$.

We remark that for this ring $R = K[\![t^6, t^{11}, t^{15}, t^{31}]\!]$, the associated graded ring $G(\mathfrak{m}) = F(\mathfrak{m})$ of the maximal ideal $\mathfrak{m}$ is not Cohen–Macaulay. This is readily seen from the fact that the image of t^6 in $\mathfrak{m}/\mathfrak{m}^2$ in $G(\mathfrak{m})$ is a zero divisor of $G(\mathfrak{m})$ since $t^{31} \notin \mathfrak{m}^2$ and $t^6 t^{31} = t^{15}(t^{11})^2 \in \mathfrak{m}^3$.

Discussion 4.3: Suppose $(R, \mathfrak{m})$ is a one-dimensional Cohen–Macaulay local ring of multiplicity e. It is well known that every $\mathfrak{m}$-primary ideal I of R can be generated by e elements [SV, Theorem 1.1]. Therefore $F(I)$ has multiplicity at most e for each $\mathfrak{m}$-primary ideal I of R. Moreover, high powers of $\mathfrak{m}$ require e generators and $F(\mathfrak{m}) = G(\mathfrak{m})$ has multiplicity e. Also there exist $\mathfrak{m}$-primary principal ideals of R and it is clear that if I is a principal $\mathfrak{m}$-primary ideal, then $F(I)$ has multiplicity one. Thus the integers 1 and e are the multiplicity of $F(I)$ for $\mathfrak{m}$-primary ideals I of R. It is natural to ask what integers f with $1 \leq f \leq e$ are realized as the multiplicity of $F(I)$ for some $\mathfrak{m}$-primary ideal I of R. We exhibit in Example 4.4 the existence, for each positive integer $e \geq 2$, of a one-dimensional Cohen–Macaulay local domain $(R, \mathfrak{m})$ having multiplicity e such that for each $\mathfrak{m}$-primary ideal I of R, either I is principal and $F(I)$ has multiplicity one, or else $F(I)$ has multiplicity e. On the other hand, we observe in Example 4.5 the existence of a one-dimensional Cohen–Macaulay local domain $(R, \mathfrak{m})$ of multiplicity e such that for every integer f with $1 \leq f \leq e$ there exists an $\mathfrak{m}$-primary ideal I with $F(I)$ having multiplicity

f.

Example 4.4: Fix a positive integer $e \geq 2$ and let K/E be an algebraic extension of fields such that $[K : E] = e$ and such that there are no fields properly between E and K. Let t be an indeterminate over K, let $S = K[t]_{(t)}$, and $R = E + tS$. Then S is the integral closure of R and S is the only subring of S that properly contains R. Hence if I is a nonprincipal $\mathfrak{m}$-primary ideal of R, then $R^I = S$, so $F(I)$ has multiplicity e.

Example 4.5: Fix a positive integer $e \geq 2$ and let K be a field. Let t be an indeterminate over K and let $R = K[t^e, t^{e+1}, \dots, t^{2e-1}]_{(t^e, t^{e+1}, \dots, t^{2e-1})}$. Fix an integer k with $1 \leq k < e$. Let $I = (t^e, t^{2e-k}, t^{2e-k+1}, \dots, t^{2e-1})$. Then R^I as an R-module is minimally generated by the $k+1$ elements $1, t^{e-k}, t^{e-k+1}, \dots, t^{e-1}$. Therefore by Lemma 4.1 $F(I)$ has multiplicity $f := k+1$

In Example 4.6 we exhibit a 3-dimensional Cohen–Macaulay local domain $(R, \mathfrak{m})$ of multiplicity 2 with the following property. For each positive integer k, the ring R has a stable ideal I_k having a principal reduction such that the fiber cone $F(I_k)$ of I_k is Cohen–Macaulay and has multiplicity $k+2$. Therefore, in contrast with the case where $\dim R = 1$, for R of higher dimension there may exist no upper bound on the multiplicity of $F(I)$ as I varies over the ideals of R that have a principal reduction.

Example 4.6: Let $(S, \mathfrak{n})$ be a two-dimensional regular local ring and let $R = S[t^2, t^3]_{(\mathfrak{n}, t^2, t^3)}$, where t is an indeterminate over S. Fix a positive integer k, and let $I = (t^2, \mathfrak{n}^k t^3)R$. Since $\mathfrak{n}^k$ is minimally generated by $k+1$ elements as an ideal in S, we see that I is minimally generated by $k+2$ elements. Moreover, $I^2 = (t^4, \mathfrak{n}^k t^5) = t^2 I$. Thus I has a principal reduction and is a stable ideal, i.e., $r(I) = 1$. It follows that $F(I)$ is Cohen–Macaulay and has multiplicity $f_0 = k+2$.

5. A bound for the reduction number.

Suppose $(R, \mathfrak{m})$ is a one-dimensional Cohen–Macaulay local ring of multiplicity e.

It is well known from results of Sally-Vasconcelos [SV] and Eakin-Sathaye [ES] that $e-1$ is then an upper bound for the reduction number $r(I)$ of $\mathfrak{m}$-primary ideals of R having a principal reduction. In a higher dimensional situation, S. Huckaba proves in [Hu1, Theorem 2.5] that if $(R, \mathfrak{m})$ is a quasi-unmixed analytically unramified local ring containing an infinite field of characteristic $\neq 2$ and if the multiplicity of R is 2, then each regular ideal I of R of analytic spread one has reduction number $r(I) \leq 1$. Now the two-dimensional regular local domain S of Example 4.6 can be chosen so that R as in Example 4.6 satisfies all these properties. It then follows that for ideals I of R having a principal reduction there is no upper bound on the multiplicity of $F(I)$, but the reduction number $r(I) \leq 1$ for each I.

Huckaba [Hu1, Question 2.6] raises the interesting question of whether an analogous result to [Hu1, Theorem 2.5] holds in the case where R has multiplicity $e > 2$. To illustrate a special case of this question, suppose $(R, \mathfrak{m})$ is a complete Noetherian local domain of dimension d containing an infinite coefficient field K. Then there exist $x_1, \dots x_d \in \mathfrak{m}$ that form a system of parameters for R and generate a reduction of $\mathfrak{m}$. Since R is complete $A := K[[x_1, \dots, x_d]]$ is a d-dimensional regular local subring of R and R is a finitely generated A-module. Moreover, the multiplicity e of R is precisely the degree of the fraction field extension $[Q(R) : Q(A)]$ [ZS, Corollary 2, page 300]. In this situation, $\lambda(R/(x_1, \dots, x_d)R) \geq e$ is the number of generators for R as an A-module and R is Cohen–Macaulay if and only if R is a free A-module if and only if $\lambda(R/(x_1, \dots, x_d)R) = e$. Since R is complete, the integral closure $\overline{R}$ of R is again local and is a finitely generated R-module and thus also a finitely generated A-module. The freeness lemma of Hironaka [N, (25.16)] implies that $\overline{R}$ is a free A-module if and only if $\overline{R}$ is Cohen–Macaulay if and only if $\lambda(\overline{R}/(x_1, \dots, x_d)\overline{R}) = e$. If $\overline{R}$ has residue field K, then e is also the multiplicity of $\overline{R}$, but if the residue field of $\overline{R}$ is a proper extension of the residue field K of R with respect to the canonical inclusion map of R into $\overline{R}$, then $\overline{R}$ has multiplicity less than e.

The following interesting observation shown to us by Craig Huneke proves the existence of a global bound on the reduction number $r(I)$ of ideals having a principal reduction in certain rings R.

Proposition 5.1: *Suppose I is a regular ideal of a ring R and that xR is a principal reduction of I. If there exists an n-generated faithful R-module $M = < m_1, \dots, m_n >$ such that $IM = xM$, then I has reduction number $r(I) \leq n-1$.*

Proof: Let $u_1, \dots, u_n$ be (not necessarily distinct) elements of I. Since $IM = xM$, we obtain n equations $u_i m_i = x \sum_{j=1}^{n} a_{ij} m_j$, where the $a_{ij} \in R$. Rearranging to make the system of equations homogeneous, we obtain a coefficient matrix A that applied to the column vector $(m_1, \dots, m_n)^T$ gives the zero vector. By multiplying A by its adjoint, one sees that $\det(A) \in R$ is in $(0 :_R M) = (0)$. Thus $\det(A) = 0$. The explicit expression of $\det(A)$ shows that $u_1 \cdots u_n \in xI^{n-1}$. In conclusion $I^n = xI^{n-1}$ and $r(I) \leq n-1$. $\square$

In general, if $\overline{R}$ is the integral closure of an integral domain R, then principal ideals of $\overline{R}$ are integrally closed. Thus if I is an ideal of R having a principal reduction xR, then $x\overline{R} = I\overline{R}$. Thus the following corollary to Proposition 5.1 is immediate.

Corollary 5.2: *Suppose R is an integral domain and that the integral closure $\overline{R}$ of R in its fraction field is n-generated as an R-module. Then each ideal I of R having a principal reduction has reduction number $r(I) \leq n-1$*

We record several other results that follow from Proposition 5.1.

Corollary 5.3: *Suppose $(R, \mathfrak{m})$ is a Noetherian local ring and I is a regular ideal of R having a principal reduction xR. Let $R^I = R[I/x]$ denote the blowing up ring of I. Then the reduction number $r(I) \leq \lambda(R^I/\mathfrak{m}R^I) - 1$, so $r(I) \leq f_0 - 1$, where f_0 is the multiplicity of the fiber ring $F(I)$.*

Proof: Since R^I is a faithful R-module and $IR^I = xR^I$, Proposition 5.1 implies the first assertion. The second assertion follows from Lemma 4.1. $\square$

Corollary 5.4: *Suppose $(R, \mathfrak{m})$ is a complete local domain of multiplicity e containing an infinite field. If the integral closure $\overline{R}$ of R is Cohen–Macaulay, then every ideal I of R of analytic spread one has reduction number $r(I) \leq e-1$.*

Proof: Since R is complete, R contains a coefficient field K. If $(x_1, \dots, x_d)$ is a minimal reduction of $\mathfrak{m}$, then $A := K[[x_1, \dots, x_d]]$ is a d-dimensional regular local subring of R, R is a finitely generated A-module and $[Q(R) : Q(A)] = e$. Since $\overline{R}$ is Cohen–Macaulay, $\overline{R}$ is a free A-module on e generators. Therefore $\overline{R}$ is e-generated as an R-module. By Proposition 5.1, the reduction number $r(I) \leq e - 1$ for each ideal I of R having a principal reduction, or equivalently in our setting, each ideal of analytic spread one. □

Remark 5.5: There are other situations to which Proposition 5.1 applies. Suppose $(R, \mathfrak{m})$ is a Noetherian local ring and let $\widehat{R}$ denote the $\mathfrak{m}$-adic completion of R. If I is a regular ideal of R having a principal reduction xR, then $x\widehat{R}$ is a principal reduction of $I\widehat{R}$. Moreover, $xI^n = I^{n+1}$ if and only if $xI^n\widehat{R} = I^{n+1}\widehat{R}$. Thus $r(I) = r(I\widehat{R})$. Also R and $\widehat{R}$ have the same multiplicity. Thus if $(R, \mathfrak{m})$ is an analytically irreducible Noetherian local domain of multiplicity e and if its completion $\widehat{R}$ contains an infinite field and has a Cohen–Macaulay normalization, then each ideal I of R of analytic spread one has reduction number $r(I) \leq e - 1$.

Corollary 5.6: *Suppose $(R, \mathfrak{m})$ is a 2-dimensional complete local domain of multiplicity e containing an infinite field. Then every ideal I of R of analytic spread one has reduction number $r(I) \leq e - 1$.*

Proof: Since a 2-dimensional integrally closed Noetherian domain is Cohen–Macaulay, this follows from Corollary 5.4. □

Corollary 5.7: *Suppose $(R, \mathfrak{m})$ is a 2-dimensional Noetherian analytically irreducible local domain containing an infinite field. If R has multiplicity e, then every ideal I of R of analytic spread one has reduction number $r(I) \leq e - 1$.*

Proof: This follows from Remark 5.5 and Corollary 5.6.

Remark 5.8: It is known that a complete local ring that contains a field and satisfies Serre's condition S_n is Cohen–Macaulay if it has multiplicity $\leq n$ [H2]. Thus Corollary 5.4 yields in a special case the result of Huckaba [Hu1, Theorem 2.5] mentioned above.

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