BUILDING NOETHERIAN DOMAINS INSIDE AN IDEAL-ADIC COMPLETION

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Abstract. Suppose a is a nonzero nonunit of a Noetherian integral domain R. An interesting construction introduced by Ray Heitmann addresses the question of how ring-theoretically to adjoin a transcendental power series in a to the ring R. We apply this construction, and its natural generalization to finitely many elements, to exhibit Noetherian extension domains of R inside the (a)-adic completion R^* of R. Suppose $\tau_1, \dots, \tau_s \in aR^*$ are algebraically independent over K, the field of fractions of R. Starting with $U_0 := R[\tau_1, \dots, \tau_s]$, there is a natural sequence of nested polynomial rings U_n between R and $A := K(\tau_1, \dots, \tau_s) \cap R^*$. It is not hard to show that if $U := \cup_{n=0}^{\infty} U_n$ is Noetherian, then A is a localization of U and $R^*[1/a]$ is flat over U_0. We prove, conversely, that if $R^*[1/a]$ is flat over U_0, then U is Noetherian and $A := K(\tau_1, \dots, \tau_s) \cap R^*$ is a localization of U. Thus the flatness of $R^*[1/a]$ over U_0 implies the intersection domain A is Noetherian.

1. Introduction. Suppose a is a nonzero nonunit of a Noetherian integral domain R. The (a)-adic completion R^* of R is isomorphic to the ring $R[[x]]/(x-a)$ [N, (17.5), page 55]. Thus elements of the (a)-adic completion may be regarded as formal power series in a. Of course if R is already complete in its (a)-adic topology, then $R = R^*$, but often it is the case that there are elements of R^* that are transcendental over R. An interesting construction first introduced by Ray Heitmann in [H, page 126] addresses the question of how ring-theoretically to adjoin a transcendental (over R) power series in a to the ring R. We have made use of this construction of Heitmann in [HRW3] in a local or semilocal context. Our purpose here is to consider this construction in the more general context of an arbitrary Noetherian integral domain.

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There are numerous articles in the literature that have relevance for the building of Noetherian domains inside an ideal-adic completion, for example [BR1], [BR2], [HRS], [H1], [H2], [H3], [L], [N2], [O1], [O2], [R1], [R2], [R3], and [W].

Let R be a Noetherian integral domain with field of fractions K and let a be a nonzero nonunit of R. We are interested in the structure of certain intermediate integral domains between R and $R^* := \widehat{(R,(a))} = R[[x]]/(x-a)$, the (a)-adic completion of R. We are particularly interested in domains of the form $A := L \cap R^*$, where L is an intermediate field between K and the total ring of fractions of R^*. It is often difficult to compute this intersection ring A. Thus we seek conditions in order that A be realizable as a localization of a directed union of polynomial ring extensions of R.

This intersection construction inside the completion of R with respect to a principal ideal yields interesting Noetherian rings which are directed unions of localized polynomial rings, as we see below. By contrast, taking the analogous construction inside the completion with respect to a maximal ideal, even of an excellent local domain seems less likely to give Noetherian intersection domains. In [HRW1], it is shown for a countable excellent local domain $(R, \mathbf{m})$ of dimension at least two that there exist infinitely many algebraically independent elements $\tau_1, \tau_2, \dots$ in the $\mathbf{m}$-adic completion $\widehat{R}$ of R such that the corresponding intersection domain is a localized polynomial ring in infinitely many variables over R; that is, $\widehat{R} \cap K(\tau_1, \tau_2, \dots) = R[\tau_1, \tau_2, \dots]_{(\mathbf{m}, \tau_1, \tau_2, \dots)}$.

In [HRW2], [HRW3] and the present paper, we study the following element-wise form of the problem. Let $\tau_1, \dots, \tau_s \in aR^*$ be elements which are algebraically independent over K. Starting with $U_0 := R[\tau_1, \dots, \tau_s]$, we define a sequence of nested polynomial rings U_n in s variables over R inside $A := K(\tau_1, \dots, \tau_s) \cap R^*$. In [HRW3] we consider in the case where R is a semilocal Noetherian integral domain and a is an element of the Jacobson radical of R the condition that the embedding $U_0 \to R^*[1/a]$ is flat. Our goal here is to examine flatness of the embedding $U_0 \to R^*[1/a]$ in a more general context, and to prove the following theorem.[1]

Theorem 1.1. Suppose R is a Noetherian domain, $a \in R$ is a nonzero nonunit, and $\tau_1, \dots, \tau_s$ are elements of the (a)-adic completion R^* of R that are algebraically

[1]This result generalizes [HRW3, Theorem 2.12].

independent over R.[2] Then the following conditions are equivalent:

(1) The ring $R^*[1/a]$ is flat over $U_0 = R[\tau_1, \dots, \tau_s]$.

(2) The directed union $U := \cup_{n=0}^{\infty} U_n$ is Noetherian.[3]

Moreover, if these equivalent conditions hold, then the integral domain $A := K(\tau_1, \dots, \tau_s) \cap R^*$ is a localization of U, and hence A is Noetherian.

Remark 1.2. An example given in [HRW3, (4.4)] shows that it is possible for A to be a localization of U and yet for A, and therefore also U, to fail to be Noetherian. Thus the equivalent conditions of (1.1) are not implied by the property that A is a localization of U.

We present in (2.5) an example that is a modification of [HRW2, Example 2.1] to show that A being Noetherian does not imply that U is Noetherian.

The following diagram displays the situation concerning possible implications between A being a localization of U and A or U being Noetherian:

$$\begin{tikzcd}
\boxed{R^*[1/a] \text{ is flat over } U_0 = R[\tau_1, \dots, \tau_s]} \arrow[r, Leftrightarrow] \arrow[d, Rightarrow, shift left] & \boxed{U \text{ Noetherian}} \arrow[d, Rightarrow, shift left] \\
\boxed{A \text{ is a localization of } U} \arrow[u, Rightarrow, shift left, "/" marking] \arrow[r, Rightarrow, shift left, "/" marking] & \boxed{A \text{ Noetherian}} \arrow[u, Rightarrow, shift left, "/" marking] \arrow[l, Rightarrow, shift left, "/" marking]
\end{tikzcd}$$

2. The general setting.

(2.1) Let R be a Noetherian integral domain of dimension $d > 0$ with fraction field K. Let a be a nonzero element nonunit of R, let $R^* := (\widehat{R, (a)})$ be the (a)-adic completion of R and let $R^*_a := R^*[1/a]$. Suppose $\tau_1, \dots, \tau_s \in aR^*$ are regular elements[4] of R^* that are algebraically independent over K. We consider the polynomial ring

$$U_0 := R[\tau_1, \dots, \tau_s].$$

For every $\gamma \in R^*$ and every $n > 0$, we define the n^{th}-endpiece γ_n with respect

[2] We say that elements are algebraically independent over an integral domain if they are algebraically independent over its fraction field.

[3] Heitmann in [H, page 126] considers the case where there is one transcendental element τ and defines the corresponding extension U to be a *simple PS-extension of R for a*. Heitmann proves in this case that a certain monomorphism condition on a sequence of maps is equivalent to U being Noetherian [H, Theorem 1.4].

[4] We say an element of a ring is a *regular element* if it is not a zero divisor.

to a of γ to be

$$\gamma_n := \sum_{j=n+1}^{\infty} c_j a^{j-n}, \text{ where } \gamma := \sum_{j=1}^{\infty} c_j a^j \text{ with each } c_j \in R. \tag{2.1.1}$$

In particular, we represent each of the τ_i by a power series expansion in a; we use these representations to obtain for each positive integer n the n^{th}-endpieces τ_{in} and the corresponding n^{th}-polynomial ring U_n: For $1 \le i \le s$, and $\tau_i := \sum_{j=1}^{\infty} r_{ij} a^j$, where the $r_{ij} \in R$, $\tau_{in} := \sum_{j=n+1}^{\infty} r_{ij} a^{j-n}, U_n := R[\tau_{1n}, \dots, \tau_{sn}]$, for each $n \in \mathbb{N}$. We have a birational inclusion of polynomial rings $U_n \subset U_{n+1}$. We define

$$U := \cup_{n=0}^{\infty} U_n = \varinjlim U_n \qquad \text{and} \qquad A := K(\tau_1, \dots, \tau_s) \cap R^*. \tag{2.1.2}$$

It is readily seen that A is a birational extension of U. We say that the τ_i have *good limit-intersecting behavior* if A is a localization of U.

We observe the following properties of (a)-adic completions and an implication of this concerning good limit-intersecting behavior.

Proposition 2.2 (cf. [HRW2],[HRW3, (2.2)]**).** *Assume the notation and setting of (2.1), and let U^* and A^* denote the (a)-adic completions of U and A. Then*

(1) *$a^kU = a^kA \cap U = a^kR^* \cap U$ for each positive integer k.*
(2) *$U^* = A^* = R^*$, so $R/aR = U/aU = A/aA = R^*/aR^*$.*
(3) *If U is Noetherian, then R^* is flat over U and A is the localization of U at the multiplicative system $1 + aU$ of U.*

Proof. We have $R \subseteq U \subseteq A \subseteq R^*$. Since R is Noetherian, R^* is flat over R [M1, Theorem 8.8, page 60]. Moreover, a^kR is closed in the (a)-adic topology on R, so we have $a^kR^* \cap R = a^kR$ for each positive integer k [ZS, Theorem 8, page 261]. Furthermore, $A = R^* \cap K(\tau_1, \dots, \tau_s)$ implies $a^kA = a^kR^* \cap A$. It is clear that $a^kU \subseteq a^kR^* \cap U$, thus for (1) and (2) it suffices to show $a^kR^* \cap U \subseteq a^kU$. Moreover, if $aR^* \cap U = aU$, it follows that $a^kR^* \cap U = a^kR^* \cap aU = a(a^{k-1}R^* \cap U)$, and by induction we see that $a^kR^* \cap U = a^kU$. Thus we show $aR^* \cap U \subseteq aU$.

Let $g \in aR^* \cap U$. Then there is a positive integer n with $g \in U_n = R[\tau_{1n}, \dots, \tau_{sn}]$. Write $g = r_0 + g_0$ where $g_0 \in (\tau_{1n}, \dots, \tau_{sn})U_n$ and $r_0 \in R$. From the definition of τ_{in}, we have $\tau_{in} = a\tau_{in+1} + a_{in}a$, where $a_{in} \in R$, for each i with $1 \le i \le s$. Thus $r_0 \in aR^* \cap R = aR$, $\tau_{in}U_n \subseteq aU_{n+1}$ and $g \in aU$. This completes the proof of (1) and (2). If U is Noetherian, then $U^* = R^*$ is flat over U. Let S be the multiplicative system $1 + aU$ and let $B = S^{-1}U$. Then B is Noetherian, the (a)-adic completion of B is R^* and R^* is faithfully flat over B [M1, Theorem 8.14, page 62]. Therefore $B = K(\tau_1, \dots, \tau_s) \cap R^* = A$. $\square$

With the notation and setting of (2.1), the representation of the τ_i as power series in a with coefficients in R is, in general, not unique. However, as we observe in (2.3), the rings U and U_n are uniquely determined by the τ_i.

Proposition 2.3 (cf. [HRW3, (2.3)]**).** *Assume the notation and setting of (2.1). Then U and the U_n are independent of the representation of the τ_i as power series in a with coefficients in R.*

Proof. For $1 \le i \le s$, assume that τ_i and $\omega_i = \tau_i$ have representations

$$\tau_i = \sum_{j=1}^{\infty} a_{ij}a^j \qquad \text{and} \qquad \omega_i = \sum_{j=1}^{\infty} b_{ij}a^j,$$

where each $a_{ij}, b_{ij} \in R$. We define the n^{th}-endpieces τ_{in} and ω_{in} as in (2.1.1):

$$\tau_{in} = \sum_{j=n+1}^{\infty} a_{ij}a^{j-n} \qquad \text{and} \qquad \omega_{in} = \sum_{j=n+1}^{\infty} b_{ij}a^{j-n}.$$

Then we have

$$\tau_i = \Sigma_{j=1}^{\infty} a_{ij}a^j = \Sigma_{j=1}^{n} a_{ij}a^j + a^n\tau_{in} = \Sigma_{j=1}^{\infty} b_{ij}a^j = \Sigma_{j=1}^{n} b_{ij}a^j + a^n\omega_{in} = \omega_i.$$

Therefore, for $1 \le i \le s$ and each positive integer n,

$$a^n\tau_{in} - a^n\omega_{in} = \Sigma_{j=1}^{n} b_{ij}a^j - \Sigma_{j=1}^{n} a_{ij}a^j, \qquad \text{and so} \qquad \tau_{in} - \omega_{in} = \frac{\Sigma_{j=1}^{n}(b_{ij} - a_{ij})a^j}{a^n}.$$

Since $\Sigma_{j=1}^{n}(b_{ij} - a_{ij})a^j \in R$ is divisible by a^n in R^* and since $a^nR = R \cap a^nR^*$ because a^nR is closed in the (a)-adic topology, it follows that a^n divides the sum $\Sigma_{j=1}^{n}(b_{ij} - a_{ij})a^j$ in R. Therefore $\tau_{in} - \omega_{in} \in R$. It follows that U_n and $U = \cup_{n=1}^{\infty} U_n$ are independent of the representation of the τ_i. $\square$

Remark 2.4. With notation as in (2.1), if the embedding $U_0 = R[\tau_1, \dots, \tau_s] \to R^*[1/a]$ is flat, then every nonzero element of U_0 is a regular element of R^*.

Example 2.5. (cf. [HRW2, Example 2.1]) In $\mathbb{Q}[[x,y]]$, the power series ring in the two variables x and y over the rational numbers, let $\gamma := e^x - 1$ and $\tau := e^y - 1$; take γ_n to be the n^{th}-endpiece of γ with respect to x and take τ_n to be the n^{th}-endpiece of τ with respect to y, as described in (2.1). Set $R := \cup_{n\in\mathbb{N}}\mathbb{Q}[x,y,\gamma_n]_{(x,y,\gamma_n)}$. Then $R = \mathbb{Q}[y]_{(y)}[[x]] \cap \mathbb{Q}(x,y,\gamma)$ is an excellent two-dimensional regular local domain. Now define U in the (y)-adic completion of R using the endpieces τ_n as above. Then $U \supseteq V := \cup_{n\in\mathbb{N}}\mathbb{Q}[x,y,\gamma_n,\tau_n]$. The ring $A := \mathbb{Q}[[x,y]] \cap \mathbb{Q}(x,y,\gamma,\tau)$ is Noetherian but is different from $B := \cup\mathbb{Q}[x,y,\gamma_n,\tau_n]_{(x,y,\gamma_n,\tau_n)}$. The ring B is the localization of U at the multiplicative system $1+yU$, and the rings B and U are not Noetherian. It follows that A is not a localization of U.

Proof. Consider the element $\theta = \frac{\gamma-\tau}{x-y} \in A$. If θ is an element of B, then

$$\gamma - \tau \in (x-y)B \cap V = (x-y)V.$$

Now

$$V = \cup_{n\in\mathbb{N}}\mathbb{Q}[x,y,\gamma_n,\tau_n] \subseteq \mathbb{Q}[x,y,\gamma,\tau][1/x,1/y] \subseteq \mathbb{Q}[x,y,\gamma,\tau]_{(x-y)},$$

and so

$$\gamma - \tau \in (x-y)\mathbb{Q}[x,y,\gamma,\tau]_{(x-y)} \cap \mathbb{Q}[x,y,\gamma,\tau] = (x-y)\mathbb{Q}[x,y,\gamma,\tau],$$

but this contradicts the fact that x, y, γ, τ are algebraically independent over $\mathbb{Q}$.

If U were Noetherian, then B would be Noetherian. But the maximal ideal of B is $(x,y)B$, so if B were Noetherian, then it would be a regular local domain with completion $\mathbb{Q}[[x,y]]$. Since the completion of a local Noetherian ring is a faithfully flat extension of it, and since the fraction field of B is $\mathbb{Q}(x,y,\gamma,\tau)$, then B would equal A.

That A is Noetherian follows from [V, Proposition 3]. If A were a localization of U, then A would be a localization of B. But each of A and B has a unique maximal ideal and the maximal ideal of A contains the maximal ideal of B. Therefore $B \subsetneq A$ implies that A is not a localization of B. $\square$

3. The proof of the main theorem.

Proof of Theorem 1.1. Assume that U is Noetherian. By (2.2), the (a)-adic completion U^* of U is equal to R^*. Since U is Noetherian, $U^* = R^*$ is flat over

U [M1, Theorem 8.8]. Therefore the localization $R^*[1/a]$ is flat over U. Since $U[1/a] = U_0[1/a]$, the localization $R^*[1/a]$ is also flat over U_0.

To prove the converse we use results of Heitmann in [H1, Theorem 1.4].

First we show in (3.1) that the flatness condition for $R^*[1/a]$ over U_0 behaves well under certain residue class formations.

Proposition 3.1. *Let R be a Noetherian domain, let a be a nonzero nonunit of R, let R^* be the y-adic completion of R and let $\tau_1, \ldots, \tau_s \in aR^*$ be algebraically independent over R. Suppose that $R_a^* := R^*[1/a]$ is flat over U_0, using the notation of (2.1) and that Q is a prime ideal of R with $a \notin Q$. Assume that Q is the contraction of a prime ideal of R^*. Let $^-$ denote image in R_a^*/QR_a^* and let $(R/Q)^*$ denote the $(\bar{a})$-adic completion of R/Q. Then $(R/Q)_{\bar{a}}^* := (R/Q)^*[\bar{1}/\bar{a}]$ is flat over $(R/Q)[\bar{\tau}_1, \ldots, \bar{\tau}_s]$.*

Proof. The $(\bar{a})$-adic completion $(R/Q)^*$ of R/Q is canonically isomorphic to R^*/QR^*. Therefore $\bar{\tau}_1, \ldots, \bar{\tau}_s$ are regular elements of $(R/Q)^*$. We show $\bar{\tau}_1, \ldots, \bar{\tau}_s$ are algebraically independent over R/Q. Since $R[\tau_1, \ldots, \tau_s] \longrightarrow R_a^*$ is flat, $a \notin Q$, and Q is the contraction of a prime ideal of R^*, we have $QR[\tau_1, \ldots, \tau_s] = QR_a^* \cap R[\tau_1, \ldots, \tau_s]$. Thus

$$R[\tau_1, \ldots, \tau_s]/(QR_a^* \cap R[\tau_1, \ldots, \tau_s]) \cong (R/Q)[\bar{\tau}_1, \ldots, \bar{\tau}_s]$$

is a polynomial ring in s variables $\bar{\tau}_1, \ldots, \bar{\tau}_s$ over R/Q. Therefore $\bar{\tau}_1, \ldots, \bar{\tau}_s$ are algebraically independent over R/Q.

We show flatness of the map:

$$\bar{\phi} : (R/Q)[\bar{\tau}_1, \ldots, \bar{\tau}_s] \longrightarrow R_a^*/QR_a^* = (R/Q)_{\bar{a}}^*.$$

Let $\bar{P}$ be a prime ideal of R^*/QR^* with $\bar{a} \notin \bar{P}$. The ideal $\bar{P}$ lifts to a prime ideal P of R^* with $a \notin P$ and $QR^* \subseteq P$. By assumption the map

$$\phi_P : R[\tau_1, \ldots, \tau_s] \longrightarrow R_P^*$$

is flat. The map on the residue class rings:

$$\bar{\phi}_{\bar{P}} : (R/Q)[\bar{\tau}_1, \ldots, \bar{\tau}_s] \longrightarrow (R^*/QR^*)_{\bar{P}}$$

is obtained from ϕ_P by tensoring with $(R/Q)[\tau_1, \ldots, \tau_s]$ over the ring $R[\tau_1, \ldots, \tau_s]$. Hence $\bar{\phi}$ is flat. $\square$

Theorem 3.2. *Assume the notation and setting of (2.1). Also assume that $s = 1$, $\tau := \tau_1$ and that the localization $R^*[1/a]$ is flat over $U_0 = R[\tau]$. Then U is Noetherian and $A = R^* \cap K(\tau)$ is a localization of U.*

We use the same proof as in [H1, Theorem 1.4] and prove first the following lemma.

Lemma 3.3. *With notation as in Theorem 3.2, if P is a nonzero prime ideal of U such that $P \cap R = (0)$, then there exists $f \in P$, $r \in R$ and a positive integer N such that $P = (fU :_U ra^N)$.*

Proof. The localization $D := (R - \{0\})^{-1}U$ of U at the nonzero elements of R is also a localization of the polynomial ring $U_0 := R[\tau]$. Hence PD is a a principal maximal ideal of D and there exists a polynomial $f \in R[\tau]$ such that $PD = fD$.

We use the fact that U is the directed union of the polynomial rings $U_n := R[\tau_n]$, $U = \cup_{n=0}^{\infty} U_n$. Let $P_n = P \cap U_n$. Since $D_{PD} = (U_0)_{P_0}$ and U_0 is Noetherian, there exists $r \in R$ such that $P_0 = (fU_0 :_{U_0} r)$. Also for $g \in U$ there exists a positive integer $b(g)$, depending on g, such that $a^{b(g)}g \in U_0$. Hence for $g \in P$ we have $ra^{b(g)}g \in fU_0$.

The Artin-Rees Lemma [N1, (3.7)] applied to the ideals aR^* and fR^* of the Noetherian ring R^* implies the existence of a positive integer N such that for $m \geq N$ we have

$$fR^* \cap (aR^*)^m = (aR^*)^{m-N}((fR^* \cap (aR^*)^N) = (a^{m-N})R^*(fR^* \cap a^N R^*).$$

We may assume that $b(g) \geq N$.

Suppose $g \in P$. Then $ra^{b(g)}g \in fU_0 \subseteq fU$, so

$$ra^{b(g)}g \in fR^* \cap a^{b(g)}R^* = a^{b(g)-N}R^*(fR^* \cap a^N R^*).$$

Since a is not a zero-divisor in R^*, it follows that $ra^N g \in fR^* \cap a^N R^*$. Thus $ra^N g = ft$, where $t \in R^*$. Since we also have $ra^{b(g)}g \in fU$, it follows that $a^{b(g)-N}ft \in fU$, and therefore $a^{b(g)-N}t \in U$, as f is not a zero-divisor in R^*. Therefore $a^{b(g)-N}t \in a^{b(g)-N}R^* \cap U = a^{b(g)-N}U$ by (2.2.1) and so $t \in U$. Hence for every $g \in P$ we have $g \in (fU :_U ra^N)$. It follows that $P = (fU :_U ra^N)$. $\square$

As in [H1, Lemma 1.5], we have:

Lemma 3.4. *With notation as in Theorem 3.2, if each prime ideal P of U such that $P \cap R \neq (0)$ is finitely generated, then U is Noetherian.*

Proof. By a Theorem of Cohen [N1, (3.4)], it suffices to show each $P \in \mathrm{Spec}(U)$ such that $P \cap R = (0)$ is finitely generated. Let P be a nonzero prime ideal of U such that $P \cap R = (0)$. Since the localization of U at the nonzero elements of R is also a localization of the polynomial ring $U_0 := R[\tau]$, every prime ideal of U properly containing P has a nonzero intersection with R. Therefore the hypothesis implies that U/P is Noetherian. By (3.3), there exist $r \in R$ and $f \in P$ such that $P = (fU :_U ra^N)$. Since ra^N is a nonzero element of R, every prime ideal of U containing ra^N is finitely generated, so U/ra^NU is Noetherian. Therefore $U/(P \cap ra^NU)$ is Noetherian [N1, (3.16)]. Since $ra^N \notin P$ and P is prime, we have $ra^NU \cap P = ra^NP$. Therefore U/ra^NP is Noetherian. We have $ra^NP \subseteq fU \subseteq P$. Hence U/fU, as a homomorphic image of U/ra^NP, is Noetherian, and P/fU is finitely generated. It follows that P is finitely generated. □

Proof of Theorem 3.2. Suppose U is not Noetherian and let $Q \in \mathrm{Spec}(R)$ be maximal with respect to being the contraction to R of a non-finitely generated prime ideal of U. Since $R/aR = U/aU = R^*/aR^*$ by (2.2), we have $a \notin Q$. Since $U = \cup_{n=0}^{\infty} U_n$ and QU_n is prime, we have QU is prime in U. We claim that Q is the contraction of a prime ideal of R^*, for otherwise we have $(Q, a)R = R$. But this means that the image of a in U/QU is a unit which implies that $U/QU = U_0/QU_0$ is Noetherian, and this implies that P is finitely generated. Therefore Q is the contraction of a prime of R^*, and (3.1) implies that, passing to the image $\bar{\tau}$ of τ in U/QU, the localization $(R/Q)^*_{\bar{a}}$ is flat over $(R/Q)[\bar{\tau}]$. But Lemma 3.4 then implies that U/QU is Noetherian. This contradicts the existence of a non-finitely generated prime ideal of U lying over Q in R. We conclude that U is Noetherian. Therefore $U^* = R^*$ is flat over U and if S is the multiplicative system $1 + aU$, then $S^{-1}U = R^* \cap K(\tau)$. □

Remark 3.5. The proof of Theorem 3.2 is essentially due to Ray Heitmann. In his paper [H1] Heitmann defines *simple PS-extensions.* For a regular element x in a ring R and a formal power series in x transcendental over R, a simple PS-extension of R for x is an infinite direct union of simple transcendental extensions of R. If R is Noetherian and T is a simple PS-extension of R, Heitmann proves in

[H1, Theorem 1.4] that a certain monomorphism condition is equivalent to T being Noetherian. Heitmann's monomorphism condition insures that the element f in the proof of Lemma 3.3 is a regular element in R^*. In our situation our flatness condition on the embedding $U_0 \longrightarrow R^*_a$, and hence on $U \longrightarrow R^*_a$, implies the regularity of f in R^*. Thus Proposition 3.1 yields that if $s = 1$ and the embedding $U_0 \longrightarrow R^*_a$ is flat, then the ring $U = \varinjlim R[\tau_n]$ is a simple PS-extension satisfying the monomorphism condition of Heitmann. In view of Theorem 1.1, Heitmann's monomorphism condition on the PS-extension determined by τ is equivalent to τ yielding a flat extension. The flat extension concept however extends to more than one element τ.

Completion of Proof of Theorem 1.1. If U is Noetherian, we have already shown that $R^*[1/a]$ is flat over U_0. Assume, conversely, that $R^*[1/a]$ is flat over $U_0 = R[\tau_1, \dots, \tau_s]$. It follows that $R^*[1/a]$ is flat over $R[\tau_1]$. By Theorem 3.2, $U(1)$, the directed union ring constructed with respect to τ_1 in (2.1) is Noetherian and $R^* \cap K(\tau_1)$ is a localization of $U(1)$. It also follows that $U(1)^*[1/a] = R^*[1/a]$ is flat over $U(1)[\tau_2, \dots, \tau_s]$ (cf. [HRW2, Proposition 5.10]). Hence a simple induction argument implies that U is Noetherian. Hence $U^* = R^*$ is flat over U and A is a localization of U. □

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