THE GORENSTEIN AND COMPLETE INTERSECTION PROPERTIES OF ASSOCIATED GRADED RINGS

Dedicated to Wolmer Vasconcelos on the occasion of his 65th birthday.

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Abstract. Let I be an $\mathbf{m}$-primary ideal of a Noetherian local ring $(R,\mathbf{m})$. We consider the Gorenstein and complete intersection properties of the associated graded ring $G(I)$ and the fiber cone $F(I)$ of I as reflected in their defining ideals as homomorphic images of polynomial rings over R/I and $R/\mathbf{m}$ respectively. In case all the higher conormal modules of I are free over R/I, we observe that: (i) $G(I)$ is Cohen-Macaulay iff $F(I)$ is Cohen-Macaulay, (ii) $G(I)$ is Gorenstein iff both $F(I)$ and R/I are Gorenstein, and (iii) $G(I)$ is a relative complete intersection iff $F(I)$ is a complete intersection. In case $(R,\mathbf{m})$ is Gorenstein, we give a necessary and sufficient condition for $G(I)$ to be Gorenstein in terms of residuation of powers of I with respect to a reduction J of I with $\mu(J) = \dim R$ and the reduction number r of I with respect to J. We prove that $G(I)$ is Gorenstein if and only if $J : I^{r-i} = J + I^{i+1}$ for $0 \le i \le r-1$. If $(R,\mathbf{m})$ is a Gorenstein local ring and $I \subseteq \mathbf{m}$ is an ideal having a reduction J with reduction number r such that $\mu(J) = \mathrm{ht}(I) = g > 0$, we prove that the extended Rees algebra $R[It, t^{-1}]$ is quasi-Gorenstein with $\mathbf{a}$-invariant a if and only if $J^i : I^r = I^{i+a-r+g-1}$ for every $i \in \mathbb{Z}$. If, in addition, $\dim R = 1$, we show that $G(I)$ is Gorenstein if and only if $J^i : I^r = I^i$ for $1 \le i \le r$.

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1. Introduction

For an ideal $I \subseteq \mathbf{m}$ of a Noetherian local ring $(R, \mathbf{m})$, several graded rings naturally associated to I are:

(1) the symmetric algebra $\mathrm{Sym}_R(I)$ and the Rees algebra $\mathcal{R} = R[It] = \bigoplus_{i\geq 0} I^i t^i$ (considered as a subalgebra of the poynomial ring $R[t]$),

(2) the extended symmetric algebra $\mathrm{Sym}_R(I, t^{-1})$ and the extended Rees algebra $R[It, t^{-1}] = \bigoplus_{i\in\mathbb{Z}} I^i t^i$ (using the convention that $I^i = R$ for $i \leq 0$),

(3) the symmetric algebra $\mathrm{Sym}_{R/I}(I/I^2)$ and the associated graded ring $G(I) = R[It, t^{-1}]/(t^{-1}) = R[It]/IR[It] = \bigoplus_{i\geq 0} I^i/I^{i+1}$, and

(4) the symmetric algebra $\mathrm{Sym}_{R/\mathbf{m}}(I/\mathbf{m}\, I)$ and the fiber cone $F(I) = R[It]/\mathbf{m}\, R[It] = \bigoplus_{i\geq 0} I^i/\mathbf{m}\, I^i$.

These graded rings encode information about I and its powers. The *analytic spread* of I, denoted $\ell(I)$, is the dimension of the fiber cone $F(I)$ of I. An ideal $J \subseteq I$ is said to be a *reduction* of I if there exists a nonnegative integer k such that $JI^k = I^{k+1}$. It then follows that $J^j I^k = I^{k+j}$ for every nonnegative integer j. These concepts were introduced by Northcott and Rees in [NR]. If J is a reduction of I, then J requires at least $\ell(I)$ generators. Reductions of I with $\ell(I)$ generators are necessarily minimal reductions in the sense that no properly smaller ideal is a reduction of I. They correspond to Noether normalizations of $F(I)$ in the sense that $a_1, \ldots, a_\ell$ generate a reduction of I with $\ell = \ell(I)$ if and only if their images $\overline{a_i} \in I/\mathbf{m}\, I \subseteq F(I)$ are algebraically independent over $R/\mathbf{m}$ and $F(I)$ is integral over the polynomial ring $(R/\mathbf{m})[\overline{a_1}, \ldots, \overline{a_\ell}]$. In particular, if $R/\mathbf{m}$ is infinite, then there exist reductions of I generated by $\ell(I)$ elements.

Suppose $(R, \mathbf{m})$ is a Gorenstein local ring of dimension d and I is an $\mathbf{m}$-primary ideal. We are interested in conditions for the associated graded ring $G(I)$ or the fiber cone $F(I)$ to be Gorenstein. Assume $J = (a_1, \ldots, a_d)R$ is a reduction of I. If $G(I)$ is Cohen-Macaulay, then the images $a_1^*, \ldots, a_d^*$ of $a_1, \ldots, a_d$ in I/I^2 form a regular sequence on $G(I)$, and $G(I)$ is Gorenstein if and only if $G(I)/(a_1^*, \ldots, a_d^*)G(I)$ is Gorenstein. Write $\overline{R} = R/(a_1, \ldots, a_d)R$ and $\overline{I} = I\overline{R}$. Then $\overline{R}$ is a zero-dimensional Gorenstein local ring and $G_{\overline{R}}(\overline{I}) = G(I)/(a_1^*, \ldots, a_d^*)G(I)$. Thus, under the hypothesis that $G(I)$ is Cohen-Macaulay, the question of whether $G(I)$ is Gorenstein reduces to a zero-dimensional setting.

For $J \subseteq I$ a reduction of I, the *reduction number* $r_J(I)$ of I with respect to J is the smallest nonnegative integer r such that $JI^r = I^{r+1}$. If I is an $\mathbf{m}$-primary ideal of a d-dimensional Gorenstein local ring $(R, \mathbf{m})$ and J is a d-generated reduction of I, then the reduction number $r = r_J(I)$ plays an important role in considering the Gorenstein property of $G(I)$. If $r = 0$, then $J = I$ is generated by a regular sequence and $G(I)$ is a polynomial ring in d variables over the zero-dimensional Gorenstein local ring R/I. Thus $G(I)$ is Gorenstein in this case. If $r = 1$ and $d \geq 2$, then a result of Corso-Polini [CP, Cor.3.2] states that $G(I)$ is Gorenstein if and only if

$J : I = I$, that is if and only if the ideal I is self-linked with respect to the minimal reduction parameter ideal J.

In Theorem 3.9 we extend this result of Corso-Polini on the Gorenstein property of $G(I)$ in case I is an $\mathbf{m}$-primary ideal. We prove that $G(I)$ is Gorenstein if and only if $J : I^{r-i} = J + I^{i+1}$ for $0 \leq i \leq r-1$. This also gives an analogue for Gorenstein rings to a well-known result about Cohen-Macaulay rings (see [HM, Lemma 2.2]) that asserts: Suppose I is an $\mathbf{m}$-primary ideal of a Cohen-Macaulay local ring $(R, \mathbf{m})$ of dimension d and the elements $a_1, \dots, a_s$ are a superficial sequence for I. Let $\overline{R} = R/(a_1, \dots, a_s)R$ and $\overline{I} = I\overline{R}$. If $s \leq d-1$, then $G(\overline{I})$ is Cohen-Macaulay implies $G(I)$ is Cohen-Macaulay. Corollary 3.11 says that even for $s = d$ it holds that $G(\overline{I})$ is Gorenstein implies $G(I)$ is Gorenstein if one assumes that $I^r \not\subseteq J = (a_1, \dots, a_d)R$.

Let $B = \oplus_{i \in \mathbb{Z}} B_i$ be a Noetherian $\mathbb{Z}$-graded ring which is *local in the sense that it has a unique maximal homogeneous ideal $\mathcal{M}$ [BH, (1.5.13), p.35]. Notice that $R := B_0$ is a Noetherian local ring and B is finitely generated as an R-algebra [BH, Thm.1.5.5, p.29]. We assume for simplicity that R is Gorenstein. Consider a homogeneous presentation $B \cong S/H$ with $S = R[X_1, \dots, X_n]$ a $\mathbb{Z}$-graded polynomial ring and H a homogeneous ideal of height g. Let $\sigma \in \mathbb{Z}$ be the sum of the degrees of the variables $X_1, \dots, X_n$. We write $\omega_B = \operatorname{Ext}^g_S(B, S)(-\sigma)$ and call this module the *graded canonical module* of B. One easily sees that ω_B is a finitely generated graded B-module that is uniquely determined up to homogeneous B-isomorphisms. The ring B is said to be *quasi-Gorenstein* in case $\omega_B \cong B(a)$ for some $a \in \mathbb{Z}$. If the maximal homogeneous ideal $\mathcal{M}$ of B is a maximal ideal, then the integer a is uniquely determined and is called the $\mathbf{a}$-*invariant* of B. We will use the following facts, which are readily deduced from the above definition of graded canonical modules:

• The localization $(\omega_B)_{\mathcal{M}}$ is the canonical module of the local ring $B_{\mathcal{M}}$.

• The module ω_B satisfies S_2.

• The ring B is (locally) Gorenstein if and only if it is quasi-Gorenstein and (locally) Cohen-Macaulay.

• Let A be a $\mathbb{Z}$-graded subring of B with unique maximal homogeneous ideal $\mathcal{M} \cap A$ and $A_0 = R = B_0$, so that A is Cohen-Macaulay and B is a finitely generated A-module; then $\omega_B \cong \operatorname{Hom}_A(B, \omega_A)$ as graded B-modules.

Notice that a quasi-Gorenstein ring necessarily satisfies S_2. Thus if B is quasi-Gorenstein and $\dim B \leq 2$, then B is Gorenstein. In higher dimensions there do exist examples of quasi-Gorenstein rings that are not Gorenstein. There exists an example of a prime ideal P of height two in a 5-dimensional regular local ring R such that the extended Rees algebra $R[Pt, t^{-1}]$ is quasi-Gorenstein but not Gorenstein [HH, Ex.4.7]. We are interested in classifying quasi-Gorenstein extended Rees algebras and saying more about when such rings are Gorenstein. In Theorem 4.1, we prove that if $(R, \mathbf{m})$ is a Gorenstein local ring and $I \subseteq \mathbf{m}$ is an ideal having a reduction J with reduction number r such that $\mu(J) = \operatorname{ht}(I) = g > 0$, then the extended

Rees algebra $R[It, t^{-1}]$ is quasi-Gorenstein with $\mathbf{a}$-invariant a if and only if $J^i : I^r = I^{i+a-r+g-1}$ for every $i \in \mathbb{Z}$. A natural question here that we consider but resolve only in special cases is whether $R[It, t^{-1}]$ is Gorenstein if it is quasi-Gorenstein. Equivalently, is the associated graded ring $G(I)$ then Gorenstein. We observe that this is true if $\dim R = 1$ or if $\dim R = 2$ and R is regular. If $\dim R = 1$, Corollary 4.5 implies that $G(I)$ is Gorenstein if and only if $J^i : I^r = I^i$ for $1 \leq i \leq r$.

2. Defining ideals and freeness of the higher conormal modules

Let $(R, \mathbf{m})$ be a Noetherian local ring and let $I = (a_1, \dots, a_n)R \subseteq \mathbf{m}$ be an ideal of R. Consider the presentation of the Rees algebra $R[It]$ as a homomorphic image of a polynomial ring over R obtained by defining an R-algebra homomorphism $\tau : R[X_1, \dots, X_n] \to R[It]$ such that $\tau(X_i) = a_i t$ for $1 \leq i \leq n$. Now define $\psi = \tau \otimes R/I : (R/I)[X_1, \dots, X_n] \to G(I)$, where $\psi(X_i) = a_i + I^2 \in I/I^2 = G_1$, and $\phi = \tau \otimes R/\mathbf{m} : (R/\mathbf{m})[X_1, \dots, X_n] \to F(I)$, where $\phi(X_i) = a_i + \mathbf{m} I \in I/\mathbf{m} I = F_1$.

We have the following commutative diagram for which the rows are exact and the column maps are surjective.

$$\begin{tikzcd}
0 \arrow[r] & E := \ker(\tau) \arrow[r] \arrow[d, "\pi_1"'] & R[X_1, \dots, X_n] \arrow[r, "\tau"] \arrow[d, "\pi_2"'] & R[It] \arrow[r] \arrow[d, "\pi_3"'] & 0 \\
0 \arrow[r] & L := \ker(\psi) \arrow[r] \arrow[d, "\pi_1'"'] & (R/I)[X_1, \dots, X_n] \arrow[r, "\psi"] \arrow[d, "\pi_2'"'] & G(I) \arrow[r] \arrow[d, "\pi_3'"'] & 0 \\
0 \arrow[r] & K := \ker(\phi) \arrow[r] & (R/\mathbf{m})[X_1, \dots, X_n] \arrow[r, "\phi"] & F(I) \arrow[r] & 0.
\end{tikzcd}$$

The ideal $E = 0$ if and only if $n = 1$ and a_1 is a regular element of R, while $L = 0$ if and only if $a_1, \dots, a_n$ form a regular sequence, see for example [K, Cor.5.13, p.154]. A necessary and sufficient condition for $K = 0$ is that $a_1, \dots, a_n$ be what Northcott and Rees [NR] term *analytically independent.*

Let v be the maximal degree of a homogeneous minimal generator of the ideal E. The integer $N_{\mathcal{R}}(I) := \max\{1, v\}$ is called the *relation type* of the Rees algebra $\mathcal{R} = R[It]$ with respect to the given generating set $a_1, \dots, a_n$. The relation type of $R[It]$ may also be defined by considering the kernel N of the canonical homomorphism from the symmetric algebra $\mathrm{Sym}_R(I)$ onto $R[It]$; then $N_{\mathcal{R}}(I) = \max\{1, w\}$, where w denotes the maximal degree of a homogeneous minimal generator of N. Since the symmetric algebra $\mathrm{Sym}_R(I)$ is independent of the choice of generators for I, it follows that the relation type $N_{\mathcal{R}}(I)$ of $R[It]$ is independent of the generating set of I. The ideal I is said to be of *linear type* if $N_{\mathcal{R}}(I) = 1$. Thus I is of linear type if and only if $R[It]$ is canonically isomorphic to the symmetric algebra $\mathrm{Sym}_R(I)$ of I.

The relation type $N_G(I)$ of the associated graded ring $G(I)$ is defined in a similar way, using the ideal L or the kernel of the canonical homogeneous epimorphism $\beta : \mathrm{Sym}_{R/I}(I/I^2) \to G(I)$. Likewise, the relation type $N_F(I)$ of the fiber cone $F(I)$ is defined via the ideal K or the kernel of the canonical homogeneous epimorphism $\gamma : \mathrm{Sym}_{R/\mathbf{m}}(I/\mathbf{m}\, I) \to F(I)$. The surjectivity of the maps π_1 and π_1' in diagram (2) imply that the inequalities $N_F(I) \le N_G(I) \le N_{\mathcal{R}}(I)$ hold in general.

Discussion 2.2. It is shown by Valla [Va, Thm.1.3] and by Herzog-Simis-Vasconcelos [HSV, Thm.3.1] that if $N_G(I) = 1$, then also $N_{\mathcal{R}}(I) = 1$, that is, the relation type of $G(I)$ is one if and only if I is of linear type. Planas-Vilanova shows [PV, Prop.3.3] that the equality $N_{\mathcal{R}}(I) = N_G(I)$ also holds in the case where $N_G(I) > 1$. We reprove this fact by using an 'extended symmetric algebra' analogue to the extended Rees algebra $R[It, t^{-1}]$ of the ideal I. As an R-module we define the *extended symmetric algebra* $\mathrm{Sym}_R(I, t^{-1})$ of I to be $\oplus_{i \in \mathbb{Z}} S_i(I)$, where $S_i(I) = t^i R \cong R$ for $i \le 0$ and $S_i(I)$ is the i-th symmetric power of I for $i \ge 1$. We define a multiplication on $\mathrm{Sym}_R(I, t^{-1})$ that extends the ring structure on the standard symmetric algebra $\mathrm{Sym}_R(I) = \oplus_{i \ge 0} S_i(I)$. To do this, it suffices to define multiplication by t^{-1} on $S_m(I)$ for $m \ge 1$. Consider the product $I \times \cdots \times I$ of m copies of the ideal I and the map $I \times \cdots \times I \to S_{m-1}(I)$ that takes $(a_1, \ldots, a_m) \mapsto (a_1 \cdot a_2 \cdot \ldots \cdot a_{m-1})a_m$, where the multiplication on $a_1, \ldots, a_{m-1}$ is multiplication in the symmetric algebra and where the multiplication with a_m is the scalar multiplication given by the R-module structure of $S_{m-1}(I)$. This map is m-multilinear over R and hence factors through the tensor power $I \otimes \ldots \otimes I$. Moreover, the map is symmetric in $a_1, \ldots, a_m$ and hence induces a map $S_m(I) \to S_{m-1}(I)$. To verify this symmetry, it suffices to observe that by associativity and commutativity of the two multiplications, one has $(a_1 \cdot \ldots \cdot a_{m-2} \cdot a_{m-1})a_m = (a_1 \cdot \ldots \cdot a_{m-2}) \cdot (a_{m-1}a_m) = (a_1 \cdot \ldots \cdot a_{m-2}) \cdot (a_m a_{m-1}) = (a_1 \cdot \ldots \cdot a_{m-2} \cdot a_m)a_{m-1}$. The multiplication by t^{-1} on $S_m(I)$ we just defined coincides with the downgrading homomorphism λ_{m-1} introduced by Herzog, Simis and Vasconcelos [HSV, p.471]. Assigning to the element t^{-1} degree -1, $\mathrm{Sym}_R(I, t^{-1})$ becomes a $\mathbb{Z}$-graded R-algebra. This algebra is *local [BH, (1.5.13), p.35] in the sense that the ideal of $\mathrm{Sym}_R(I, t^{-1})$ generated by $t^{-1}, \mathbf{m}$ and $\oplus_{i \ge 1} S_i(I)$ is the unique maximal homogeneous ideal of $\mathrm{Sym}_R(I, t^{-1})$.

The canonical surjective R-algebra homomorphism from the symmetric algebra $\mathrm{Sym}_R(I)$ onto the Rees algebra $R[It]$ extends to a surjective homogeneous R-algebra homomorphism $\alpha : S = \mathrm{Sym}_R(I, t^{-1}) \to R[It, t^{-1}]$, where $\alpha(t^{-1}) = t^{-1}$. Notice that the two maps have the same kernel $\mathcal{A}$. Tensoring the short exact sequence

$$0 \longrightarrow \mathcal{A} \longrightarrow S \xrightarrow{\alpha} R[It, t^{-1}] \longrightarrow 0$$

with $\otimes_S S/t^{-1}S$ gives the following isomorphisms:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{A} \otimes S/t^{-1}S & \longrightarrow & S \otimes S/t^{-1}S & \longrightarrow & R[It,t^{-1}] \otimes S/t^{-1}S & \longrightarrow & 0 \\ & & \cong \big\downarrow & & \cong \big\downarrow & & \cong \big\downarrow & & \\ 0 & \longrightarrow & \mathcal{B} := \ker(\beta) & \longrightarrow & \mathrm{Sym}_{R/I}(I/I^2) & \xrightarrow{\beta} & G(I) & \longrightarrow & 0. \end{array}$$

Here we are using that t^{-1} is a regular element on $R[It,t^{-1}]$. A graded version of Nakayama's lemma [BH, (1.5.24), p.39] now implies that $\mathcal{A}$ is generated in degrees $\leq N_G(I)$ as a module over $S = \mathrm{Sym}_R(I)[t^{-1}]$ and hence over $\mathrm{Sym}_R(I)$. Therefore the equality $N_{\mathcal{R}}(I) = N_G(I)$ holds in general.

We observe in Corollary 2.6 a sufficient condition for $N_G(I) = N_F(I)$. There exist, however, examples where $N_G(I) > N_F(I)$. If $(R, \mathbf{m})$ is a d-dimensional Noetherian local ring and $I = (a_1, \ldots, a_d)R$ is an $\mathbf{m}$-primary ideal, then $N_F(I) = 1$ [M, Thm.14.5, p.107]; while if R is not Cohen-Macaulay it may happen that $N_G(I) > 1$. For example, let k be a field and let $A = k[x,y] = k[X,Y]/(X^2Y, Y^3)$. Then $R = A_{(x,y)A}$ is a one-dimensional Noetherian local ring and $I = xR$ is primary to the maximal ideal $\mathbf{m} = (x,y)R$. Let U be an indeterminate over R/I and consider the graded homomorphism $\psi : (R/I)[U] \to G(I)$, where $\psi(U) = x + I^2 \in I/I^2 = G_1$. Let $L = \oplus_{i \geq 0} L_i = \ker(\psi)$. Since $R/I \cong k[X,Y]/(X,Y^3) \cong k[Y]/(Y^3)$, we see that $\lambda(R/I) = 3$, where λ denotes length. Similarly, $R/I^2 \cong k[X,Y]/(X^2,Y^3)$ and $\lambda(R/I^2) = 6$. Thus $\lambda(I/I^2) = 3$. It follows that $L_0 = L_1 = 0$. On the other hand, $0 \neq (y+I)U^2 \in L_2$. Therefore $N_G(I) \geq 2$, while $N_F(I) = 1$.

In the case where $N_F(I) \geq 2$, it would be interesting to have necessary and sufficient conditions for $N_G(I) = N_F(I)$. That this is not always true is shown, for example, by taking k to be a field and $A = k[x,y,z] = k[X,Y,Z]/(X^2, Y^2, XYZ^2)$. Let $R = A_{(x,y,z)A}$ and let $I = (y,z)R$. Then $F(I) \cong k[Y,Z]/(Y^2)$ has relation type 2, while if $\psi : (R/I)[U,V] \to G(I)$ with $\psi(U) = y + I^2$ and $\psi(V) = z + I^2$ is a presentation of $G(I)$, then the relation $xyz^2 = 0$ in the definition of R implies $(x+I)UV^2 \in \ker(\psi)$. Let $L = \oplus_{i \geq 0} L_i = \ker(\psi)$. Since $R/I \cong k[X]/(X^2)$, we see that $\lambda(R/I) = 2$. We have $\lambda(I/I^2) = 4$, hence $L_1 = 0$. Also $\lambda(I^2/I^3) = 4$, so $\lambda(L_2) = 2$ and L_2 is generated by U^2. Since $(x+I)UV^2 \notin (U^2 + I)(R/I)[U,V]$, we see that $N_G(I) \geq 3$.

Theorem 2.3. *Let $(R, \mathbf{m})$ be a Noetherian local ring and let I be an $\mathbf{m}$-primary ideal with $\mu(I) = \lambda(I/\mathbf{m}I) = n$. Let $A = (R/I)[X_1, \ldots, X_n]$ and let $\psi : A \to G(I)$ be defined as in diagram (2). The following are equivalent :*

(1) *I^i/I^{i+1} is free as an (R/I)-module for every $i \geq 0$.*
(2) *$G(I)$ has finite projective dimension as an A-module, i.e., $\mathrm{pd}_A G(I) < \infty$.*
(3) *If $\mathbb{F}_\bullet$ is a homogeneous minimal free resolution of $G(I)$ as an A-module and $k = R/\mathbf{m}$, then $\mathbb{F}_\bullet \otimes k$ is a homogeneous minimal free resolution of $F(I) = G(I) \otimes k$ as a module over $A \otimes k = k[X_1, \ldots, X_n]$.*

If these equivalent conditions hold, then $\operatorname{depth} G(I) = \operatorname{depth} F(I)$.

Proof. (1) $\implies$ (3): Condition (1) implies that $\operatorname{Tor}_i^{R/I}(G(I), k) = 0$ for every $i > 0$, and this implies condition (3).
(3) $\implies$ (2): Let $\mathbb{F}_\bullet$ be a homogeneous minimal free resolution of $G(I)$ over A. Condition (3) implies that $\mathbb{F}_\bullet \otimes k$ is a homogeneous minimal free resolution of $F(I)$ over the regular ring $B = k[X_1, \dots, X_n]$. Since B is regular, this homogeneous minimal free resolution of $F(I)$ is finite. Therefore $\mathbb{F}_\bullet$ is finite and $\operatorname{pd}_A G(I) < \infty$.
(2) $\implies$ (1): Condition (2) implies that $\operatorname{pd}_{R/I}(I^i/I^{i+1}) < \infty$ for every $i \geq 0$. Since $\dim(R/I) = 0$, I^i/I^{i+1} is free over R/I by the Auslander-Buchsbaum formula, see for instance [BH, Thm.1.3.3, p.17].

If these equivalent conditions hold, then we have $\operatorname{pd}_A G(I) = \operatorname{pd}_B F(I)$ by (3). Hence $\operatorname{depth} G(I) = \operatorname{depth} F(I)$ by the Auslander-Buchsbaum formula. □

Remark 2.4. If $(R, \mathbf{m})$ is a one-dimensional Cohen-Macaulay local ring, I is an $\mathbf{m}$-primary ideal and $J = xR$ is a principal reduction of I with reduction number $r = r_J(I)$, then $x^{i-r}I^r = I^i$ for $i \geq r$, so $I^i/I^{i+1} \cong I^r/I^{r+1}$. Thus in this case a fourth equivalent condition in Theorem 2.3 is that I^i/I^{i+1} is free over R/I for every $i \leq r$. On the other hand, if $I \subset \mathbf{m}$ is an ideal of a zero-dimensional Noetherian local ring $(R, \mathbf{m})$, then a condition equivalent to the conditions of Theorem 2.3 is that $\lambda(R) = \lambda(R/I) \cdot \lambda(F(I))$.

Under the equivalent conditions of Theorem 2.3, the ring R is said to be *normally flat* along the ideal I. This is a concept introduced by Hironaka in his work on resolution of singularities [M, p.188]. A well known result of Ferrand [F] and Vasconcelos [V, Cor.1] asserts that if the conormal module I/I^2 is a free module over R/I and if R/I has finite projective dimension as an R-module, then I is generated by a regular sequence. It then follows that $N_G(I) = 1$. Thus, in the case where R is a regular local ring, the equivalent conditions of Theorem 2.3 imply $\mu(I) = \dim R$. However, as we indicate in Example 2.5, there exist examples of Gorenstein local rings $(R, \mathbf{m})$ having $\mathbf{m}$-primary ideals I such that $\mu(I) > \dim R$ and the equivalent conditions of Theorem 2.3 hold.

In Examples 2.5, 2.8, 2.10, 3.6, 3.12, 4.9, and 4.10, we present examples involving an additive monoid S of the nonnegative integers that contains all sufficiently large integers and a complete one-dimensional local domain of the form $R = k[[t^s \mid s \in S]]$. The formal power series ring $k[[t]] = R[t]$ is the integral closure of R and is a finitely generated R-module. Properties of R are closely related to properties of the numerical semigroup S. For example, R is Gorenstein if and only if S is symmetric [BH, Thm.4.4.8, p.178].

Example 2.5. Let k be a field, $R = k[[t^4, t^9, t^{10}]]$ and $I = (t^8, t^9, t^{10})R$. Then R is a one-dimensional Gorenstein local domain, $\mu(I) = 3$ and $J = t^8R$ is a reduction of I. We have $JI \subsetneq I^2$ and $JI^2 = I^3$. Hence $r_J(I) = 2$. If w denotes the image of t^4

in R/I, then $R/I \cong k[w]$, where $w^2 = 0$. Thus R/I is Gorenstein with $\lambda(R/I) = 2$. Let $A = (R/I)[X, Y, Z]$ and define $\psi : A \to G(I)$ by $\psi(X) = t^8 + I^2, \psi(Y) = t^9 + I^2$ and $\psi(Z) = t^{10} + I^2$. Consider the short exact sequence

$$0 \longrightarrow L = \ker(\psi) \longrightarrow A \xrightarrow{\psi} G(I) \longrightarrow 0.$$

The ring $G(I) \cong A/L$ has multiplicity $e(I) = 8$, A is a Cohen-Macaulay ring of multiplicity 2, and the two quadrics $XZ - Y^2, wX^2 - Z^2$ form an A-regular sequence contained in L. Hence $L = (XZ - Y^2, wX^2 - Z^2)A$. Therefore the equivalent conditions of Theorem 2.3 are satisfied. It also follows that I^i/I^{i+1} is free of rank 4 over R/I for every $i \geq 2$.

Corollary 2.6. *Let $(R, \mathbf{m})$ be a Noetherian local ring and let I be an $\mathbf{m}$-primary ideal with $\mu(I) = \lambda(I/\mathbf{m}\, I) = n$. With notation as in Theorem 2.3 and diagram (2), let $A = (R/I)[X_1, \ldots, X_n]$ and $B = (R/\mathbf{m})[X_1, \ldots, X_n]$, and let $\mathcal{M}_G = (\mathbf{m}/I, G(I)_+)$ and $\mathcal{M}_F = F(I)_+$ denote the maximal homogeneous ideals of $G(I)$ and $F(I)$, respectively. Suppose the equivalent conditions of Theorem 2.3 hold. Then:*

1. *$G(I)$ is Cohen-Macaulay $\iff$ $F(I)$ is Cohen-Macaulay.*
2. *If $G(I)$, or equivalently $F(I)$, is Cohen-Macaulay, then the type of $G(I)_{\mathcal{M}_G}$ is the type of R/I times the type of $F(I)_{\mathcal{M}_F}$. In particular, $G(I)$ is Gorenstein $\iff$ both $F(I)$ and R/I are Gorenstein.*
3. *The relation type $N_G(I)$ of the associated graded ring $G(I)$ is equal to the relation type $N_F(I)$ of the fiber cone $F(I)$.*
4. *The defining ideal L of $G(I)$ is generated by a regular sequence on A if and only if the defining ideal K of $F(I)$ is generated by a regular sequence on B.*
5. *The multiplicity of $G(I)$ is $\lambda(R/I) \cdot e(F(I))$, where $e(F(I))$ denotes the multiplicity of $F(I)$.*

Proof. As $G(I)$ is flat over R/I, statements (1) and (2) follow from [BH, Prop.1.2.16, p.13]. Since the relation types of $F(I)$ and $G(I)$ are determined by the degrees of minimal generators of first syzygies, statement (3) follows from part (3) of Theorem 2.3. Indeed, with the notation of diagram (2), the relation type of $G(I)$ is $\max\{1, w\}$, where w is the maximal degree of a homogeneous minimal generator of $\ker(\psi)$. Since $\mathbb{F}_\bullet \otimes k$ is a minimal free resolution of $F(I)$, π'_1 maps a set of homogeneous minimal generators of $L = \ker(\psi)$ onto a set of homogeneous minimal generators of $K = \ker(\phi)$. Therefore $N_G(I) = N_F(I)$. Since A and B are Cohen-Macaulay and $\operatorname{ht}(L) = \operatorname{ht}(K)$, it also follows that L is generated by a regular sequence on A if and only if K is generated by a regular sequence on B, which is part (4). Statement (5) is clear in view of the freeness of the I^i/I^{i+1} over R/I. $\square$

We observe in part (1) of Remark 2.9 that L is generated by a regular sequence implies K is generated by a regular sequence holds even without the equivalent conditions of Theorem 2.3.

In Proposition 2.7, we give a partial converse to part (5) of Corollary 2.6. Proposition 2.7 is closely related to results of Shah [Sh, Lemma 8 and Thm.8].

Proposition 2.7. *Let $(R, \mathbf{m})$ be a Noetherian local ring and let I be an $\mathbf{m}$-primary ideal. With notation as in Theorem 2.3 assume that $e(G(I)) = \lambda(R/I) \cdot e(F(I))$.*

(1) *If all associated primes of $F(I)$ have the same dimension, then I^i/I^{i+1} is a free (R/I)-module for every $i \geq 0$.*
(2) *If all relevant associated primes of $F(I)$ have the same dimension, then I^i/I^{i+1} is a free (R/I)-module for all sufficiently large i.*

Proof. A composition series of R/I induces a filtration on $G(I)$ whose factors are homogeneous $F(I)$-modules of the form $F(I)/\mathbf{a}_j$ with $1 \leq j \leq \lambda(R/I)$. Since $e(G(I)) = \lambda(R/I) \cdot e(F(I))$, these factors all have the same dimension and the same multiplicity as $F(I)$. Thus in the setting of part (1), $\mathbf{a}_j = 0$ for every j, whereas in part (2), $\mathbf{a}_j$ is concentrated in finitely many degrees for every j. Computing the Hilbert function of $G(I)$ one sees that $\lambda(I^i/I^{i+1}) = \lambda(R/I) \cdot \lambda(I^i/\mathbf{m} I^i) = \lambda(R/I) \cdot \mu(I^i)$ for every $i \geq 0$ in the setting of part (1), and for every $i >> 0$ in the setting of part (2). $\square$

In Example 2.8, we exhibit an $\mathbf{m}$-primary ideal I of a one-dimensional Cohen-Macaulay local domain $(R, \mathbf{m})$ such that I^i/I^{i+1} is free over R/I for every $i \geq 2$, while I/I^2 is not free over R/I. This example illustrates that the equality $e(G(I)) = \lambda(R/I) \cdot e(F(I))$ does not imply the equivalent conditions of Theorem 2.3, even if the assumption of part (2) of Proposition 2.7 is satisfied.

Example 2.8. Let k be a field, $R = k[[t^3, t^7, t^{11}]]$ and $I = (t^6, t^7, t^{11})R$ as in [DV, Ex.6.4]. Then $\mu(I) = 3$ and $J = t^6 R$ is a principal reduction of I. Also $JI \subsetneq I^2$ and $JI^2 = I^3$, so $r_J(I) = 2$. We have $\lambda(R/I) = 2$ and $\lambda(R/I^2) = 7$. Hence $\lambda(I/I^2) = 5$ and I/I^2 is not free over R/I. On the other hand, $I^2 = (t^{12}, t^{13}, t^{14})R$. Therefore I^i/I^{i+1} is generated by 3 elements and $\lambda(I^i/I^{i+1}) = 6$, so I^i/I^{i+1} is free over R/I of rank 3 for every $i \geq 2$.

By Proposition 2.7, the fiber cone $F(I)$ in Example 2.8 is not Cohen-Macaulay. To see this explicitly, let w denote the image of t^3 in R/I and let $A = (R/I)[X, Y, Z]$. Define $\psi : A \to G(I)$ by $\psi(X) = t^6 + I^2, \psi(Y) = t^7 + I^2$ and $\psi(Z) = t^{11} + I^2$, and consider the exact sequence

$$0 \longrightarrow L = \ker(\psi) \longrightarrow A \xrightarrow{\psi} G(I) \longrightarrow 0.$$

Then $L = (wZ, XZ - wY^2, YZ, Z^2, Y^3 - wX^3)A$. Hence

$$e(G(I)) = e(I) = 6 = \lambda(G(I)/\psi(X)G(I)),$$

showing that $G(I)$ is Cohen-Macaulay, see for instance [M, Thm.17.11, p.138]. However, the fiber cone $F(I)$ is isomorphic to $(R/\mathbf{m})[X, Y, Z]/(XZ, YZ, Z^2, Y^3)$ and is not Cohen-Macaulay.

Remark 2.9. With notation as in Theorem 2.3 and diagram (2), we have:

(1) If L is generated by a regular sequence in A, then $\operatorname{pd}_A G(I)$ is finite and the equivalent conditions of Theorem 2.3 hold. Hence by part (4) of Corollary 2.6, K is also generated by a regular sequence. However, as we demonstrate in Example 2.10, the converse fails in general, that is, without the hypothesis that $\operatorname{pd}_A G(I) < \infty$, it can happen that K is generated by a regular sequence, while L is not generated by a regular sequence.

(2) Suppose R/I is Gorenstein and $\mu(I) = \dim R + 2$. If $G(I)$ is Gorenstein and $\operatorname{pd}_A G(I) < \infty$, then L is generated by an A-regular sequence. For $\mu(I) = \dim R + 2$ implies $\operatorname{ht}(L) = 2$; then $G(I) = A/L$ Cohen-Macaulay and $\operatorname{pd}_A(A/L) < \infty$ imply $\operatorname{pd}_A(A/L) = 2$ by the Auslander-Buchsbaum formula. Since A and A/L are Gorenstein rings, the homogeneous minimal free resolution of A/L has the form

$$0 \longrightarrow A \longrightarrow A^2 \longrightarrow A \longrightarrow A/L \longrightarrow 0.$$

Hence $\mu(L) = 2 = \operatorname{ht}(L)$, and therefore L is generated by a regular sequence.

(3) Assume that R is Gorenstein and J is a reduction of I with $\mu(J) = \dim R$. If $r_J(I) \leq 1$ and $\dim R \geq 2$, then it follows from [CP, Cor.3.2] that $G(I)$ is Gorenstein if and only if $I = J : I$. If R/I is Gorenstein and $r_J(I) \leq 1$, we prove by induction on $\dim R$ that $G(I)$ is Gorenstein implies $\operatorname{pd}_A G(I) < \infty$. Suppose $\dim R = 0$, in which case $G(I) = R/I \oplus I$. If $G(I)$ is Gorenstein and $I \neq 0$, then $0 : I = I$ by Theorem 3.1. Thus I is principal since R and R/I are Gorenstein, see for instance [BH, Prop.3.3.11(b), p.114]. Therefore $I/I^2 = I \cong R/I$ and hence $\operatorname{pd}_A G(I) < \infty$. Now suppose $\dim R > 0$. With the notation of diagram (2), we may assume that $\psi(X_1) = a^* \in G(I)$ is a regular element of $G(I)$, where a^* is the leading form of some $a \in J \backslash \mathbf{m}J$. Therefore $G(I)/a^*G(I) \cong G(I/aR)$ and $\operatorname{pd}_A G(I) = \operatorname{pd}_{A/X_1A} G(I)/a^*G(I)$. We conclude by induction that $\operatorname{pd}_A G(I) < \infty$. Thus in the case where R/I is Gorenstein and $r_J(I) \leq 1$, if $G(I)$ is Gorenstein, then the equivalent conditions of Theorem 2.3 hold.

(4) There exist examples where R is Gorenstein of dimension zero, R/I is Gorenstein, $I^3 = 0$ and $G(I)$ is Gorenstein, but $\operatorname{pd}_A G(I) = \infty$. To obtain an example illustrating this consider the submaximal Pfaffians $Y^3, XZ, XY^2 + Z^2, YZ, X^2$ of the matrix

$$\begin{bmatrix} 0 & X & 0 & 0 & Z \\ -X & 0 & Y & Z & 0 \\ 0 & -Y & 0 & X & 0 \\ 0 & -Z & -X & 0 & Y^2 \\ -Z & 0 & 0 & -Y^2 & 0 \end{bmatrix}.$$

Let k be a field and let H denote the ideal of the polynomial ring $k[X, Y, Z]$ generated by these Pfaffians. Notice that H is homogeneous with respect to the grading that assigns $\deg X = 0$ and $\deg Y = \deg Z = 1$. Let $R = k[X,Y,Z]/H$, write x, y, z for the images of X, Y, Z in R and let $I = (y,z)R$. Then R is an Artinian Gorenstein local ring by [BE, Thm.2.1]. Furthermore $R/I = k[X]/(X^2)$ is Gorenstein and $I^3 = 0$. Finally $G(I) \cong R$ by our choice of the grading. With $A = (R/I)[U,V]$ and $\psi : A \to G(I) \cong R$ defined by $\psi(U) = y$ and $\psi(V) = z$, we have $L = \ker(\psi) = (xV, UV, V^2 + xU^2, U^3)A$. Thus I/I^2 is not free over R/I and then $\operatorname{pd}_A G(I) = \infty$ by Theorem 2.3. This even provides an example where the associated graded ring $G(I)$ is Gorenstein and the fiber cone $F(I) \cong k[U,V]/(UV, V^2, U^3)$ is not Gorenstein.

(5) Suppose R is Gorenstein. If R/I is Gorenstein and of finite projective dimension over R, it is shown in [NV, Thm.2.6] that $G(I)$ is Cohen-Macaulay implies I is generated by a regular sequence.

Example 2.10. Let k be a field, $R = k[[t^3,t^4,t^5]]$ and $I = (t^3,t^4)R$. Then R is a one-dimensional Cohen-Macaulay local domain, $\mu(I) = 2$ and $J = t^3R$ is a reduction of I. We have $JI \subsetneq I^2$ and $JI^2 = I^3$. Hence $r_J(I) = 2$. If w denotes the image of t^5 in R/I, then $R/I \cong k[w]$, where $w^2 = 0$. Thus R/I is Gorenstein with $\lambda(R/I) = 2$. Consider the commutative diagram with exact rows and surjective column maps

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L := \ker(\psi) & \longrightarrow & (R/I)[X,Y] & \xrightarrow{\psi} & G(I) & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \pi_1'} & & \Big\downarrow{\scriptstyle \pi_2'} & & \Big\downarrow{\scriptstyle \pi_3'} & & \\ 0 & \longrightarrow & K := \ker(\phi) & \longrightarrow & (R/\mathbf{m})[X,Y] & \xrightarrow{\phi} & F(I) & \longrightarrow & 0, \end{array}$$

where $\psi(X) = t^3 + I^2$ and $\psi(Y) = t^4 + I^2$. It is readily checked that wX, wY and Y^3 generate $L = \ker(\psi)$. In particular $K = \ker(\phi)$ is generated by Y^3, so K is generated by a regular sequence, while L is not generated by a regular sequence. Also notice that $G(I)$ and $F(I)$ both have multiplicity 3, and I/I^2 has length 2 and is not a free module over R/I.

3. The Gorenstein property for $G(I)$

In this section, we establish a necessary and sufficient condition for $G(I)$ to be Gorenstein in terms of residuation of powers of I with respect to a reduction J of I for which $\mu(J) = \dim R$. We first state this in dimension zero. Among the equivalences in Theorem 3.1, the equivalence of (1), (3) and (5) are due to Ooishi [O, Thm.1.5]. We include elementary direct arguments in the proof. We use the floor function $\lfloor x \rfloor$ to denote the largest integer which is less than or equal to x.

Theorem 3.1. *Let $(R,\mathbf{m})$ be a zero-dimensional Gorenstein local ring and let $I \subseteq \mathbf{m}$ be an ideal of R. Assume that $I^r \neq 0$ and $I^{r+1} = 0$. Let $G := G(I) = \oplus_{i=0}^r G_i$ be* *the associated graded ring of I, and let $S := \mathrm{Soc}(G) = \oplus_{i=0}^{r} S_i$ denote the socle of G. Then the following are equivalent :*

(1) *G is a Gorenstein ring.*
(2) *$S_i = 0 \quad for\ 0 \le i \le r-1$.*
(3) *$0 :_R I^{r-i} = I^{i+1} \quad for\ 0 \le i \le r-1$.*
(4) *$0 :_R I^{r-i} = I^{i+1} \quad for\ 0 \le i \le \lfloor \frac{r-1}{2} \rfloor$.*
(5) *$\lambda(G_i) = \lambda(G_{r-i}) \quad for\ 0 \le i \le \lfloor \frac{r-1}{2} \rfloor$.*
(6) *$I^r :_R I^{r-i} = I^i$ for $1 \le i \le r-1$, and $0 :_R I = I^r$.*
(7) *I^{r-i}/I^r is a faithful module over R/I^i for $1 \le i \le r-1$, and I is faithful over R/I^r.*

Proof. We may assume $r > 0$. Write $k = R/\mathbf{m}$ and let $\mathfrak{M}$ denote the unique maximal ideal of G. We first compute $S_i = 0 :_{G_i} \mathfrak{M}$ for $0 \le i \le r$. Since $\mathfrak{M}$ is generated by $\mathbf{m}/I$ and I/I^2 it follows that

$$S_i = (0 :_{I^i/I^{i+1}} \mathbf{m}/I) \cap (0 :_{I^i/I^{i+1}} I/I^2) = \frac{I^i}{I^{i+1}} \cap \frac{(I^{i+1} :_R \mathbf{m})}{I^{i+1}} \cap \frac{(I^{i+2} :_R I)}{I^{i+1}}.$$

Therefore

$$S_i = \frac{I^i \cap (I^{i+1} : \mathbf{m}) \cap (I^{i+2} : I)}{I^{i+1}} \quad \text{for} \quad 0 \le i \le r. \tag{3.2}$$

In particular $S_r = 0 :_{I^r} \mathbf{m}$ because $I^{r+1} = 0$. Note that $S_r \neq 0$. Since $(R, \mathbf{m})$ is a zero-dimensional Gorenstein local ring, we have

$$S_r \cong k. \tag{3.3}$$

(1) $\Longleftrightarrow$ (2) : The ring G is Gorenstein if and only if $\dim_k S = 1$ if and only if $S_i = 0$ for $0 \le i \le r-1$, by (3.3).

(2) $\Longrightarrow$ (3) : Condition (2) implies that $S = S_r \cong k$ by (3.3). Hence $S = s^* k$ for some $0 \neq s^* \in S_r$.

Let $0 \le i \le r-1$. It is clear that $0 : I^{r-i} \supseteq I^{i+1}$ because $I^{r+1} = 0$. To show the reverse inclusion suppose that $0 : I^{r-i} \not\subseteq I^{i+1}$. In this case there exists an element $z \in 0 : I^{r-i}$ with $z \in I^j \backslash I^{j+1}$ for some $0 \le j \le i$. Since $0 \neq z^* \in I^j/I^{j+1}$ and $S = S_r = s^* k$, we can express $s^* = z^* w^*$ for some $w^* \in I^{r-j}/I^{r-j+1}$. As $s^* \neq 0$ it follows that $zw \neq 0$. This is impossible since $z \in 0 : I^{r-i}$ and $w \in I^{r-j} \subseteq I^{r-i}$.

(3) $\Longrightarrow$ (4) : This is obvious.

(4) $\Longrightarrow$ (5) : For $0 \le i \le \lfloor \frac{r-1}{2} \rfloor$ we have

$$\begin{aligned}
\lambda(G_{r-i}) &= \lambda(I^{r-i}/I^{r-i+1}) \\
&= \lambda(R/I^{r-i+1}) - \lambda(R/I^{r-i}) \\
&= \lambda(0 : I^{r-(i-1)}) - \lambda(0 : I^{r-i})) \quad \text{by [BH, Prop.3.2.12(b), p.103]} \\
&= \lambda(I^i) - \lambda(I^{i+1}) \quad \text{by condition (4)} \\
&= \lambda(I^i/I^{i+1}) = \lambda(G_i).
\end{aligned}$$

(5) $\implies$ (3) : If condition (5) holds for $0 \le i \le \lfloor \frac{r-1}{2} \rfloor$, then it obviously holds for every i. For $0 \le i \le r-1$ we have

$$\begin{aligned}\lambda(I^{i+1}) &= \lambda(G_{i+1}) + \cdots + \lambda(G_r)\\ &= \lambda(G_{r-(i+1)}) + \cdots + \lambda(G_0) \quad \text{by condition (5)}\\ &= \lambda(R/I^{r-i})\\ &= \lambda(0 : I^{r-i}).\end{aligned}$$

As $I^{r+1} = 0$, we have $I^{i+1} \subseteq 0 : I^{r-i}$. Since these two ideals have the same length, we conclude that $I^{i+1} = 0 : I^{r-i}$.

(3) $\implies$ (6) : Let $1 \le i \le r-1$. The inclusion $I^r : I^{r-i} \supseteq I^i$ is clear. To show "$\subseteq$", observe that

$$I^r : I^{r-i} \subseteq I^{r+1} : I^{r-i+1} = 0 : I^{r-(i-1)} = I^i,$$

where the last equality holds by condition (3).

(6) $\implies$ (2) : From (3.2) we have for $0 \le i \le r-2$,

$$S_i \subseteq \frac{I^{i+2} : I}{I^{i+1}} \subseteq \frac{I^{i+2+(r-i-2)} : I^{1+(r-i-2)}}{I^{i+1}} = \frac{I^r : I^{r-(i+1)}}{I^{i+1}} = \frac{I^{i+1}}{I^{i+1}},$$

where the last equality follows from condition (6). Again by (3.2) and condition (6),

$$S_{r-1} \subseteq \frac{0 : I}{I^r} = \frac{I^r}{I^r}.$$

Hence $S_i = 0$ for $0 \le i \le r-1$.

(6) $\iff$ (7) : This is clear. □

Corollary 3.4. *Let $(R, \mathbf{m})$ be a zero-dimensional Gorenstein local ring and let $I \subseteq \mathbf{m}$ be an ideal of R. Assume that $I^r \neq 0$ and $I^{r+1} = 0$. If $G(I)$ is Gorenstein, then I^i/I^{i+1} is a faithful module over R/I for $0 \le i \le r$.*

Proof. It suffices to show that $I^{i+1} : I^i = I$ for $0 \le i \le r$. The inclusion "$\supseteq$" is clear. To show the inclusion "$\subseteq$", observe that we have

$$I^{i+1} : I^i \subseteq I^{r+1} : I^r = 0 : I^r = I,$$

where the last equality follows from condition (3) of Theorem 3.1. □

Remark 3.5. (1) The existence of a zero-dimensional Gorenstein local ring $(R, \mathbf{m})$ for which the associated graded ring $G(\mathbf{m})$ is not Gorenstein shows that the converse of Corollary 3.4 is not true. A specific example illustrating this is given in Example 3.6.

(2) In general, suppose $I \subseteq \mathbf{m}$ is an ideal of a Noetherian local ring $(R, \mathbf{m})$. If the ideal $G(I)_+ = \oplus_{i \ge 1}(I^i/I^{i+1})$ of $G(I)$ has positive grade, then I^i/I^{i+1} is a faithful module over R/I for every $i \ge 0$. For if I^i/I^{i+1} is not faithful over R/I, then there exists $a \in R \setminus I$ such that $aI^i \subseteq I^{i+1}$. Hence $aI^{i+j} \subseteq I^{i+j+1}$ and I^{i+j}/I^{i+j+1} is not faithful over R/I for every integer $j \ge 0$. On the other

hand, if $G(I)_+$ has positive grade, then there exists a homogeneous $G(I)$-regular element $b \in I^k/I^{k+1}$ for some positive integer k. Then $b^\ell \in I^{k\ell}/I^{k\ell+1}$ is $G(I)$-regular, and therefore $I^{k\ell}/I^{k\ell+1}$ is a faithful module over R/I for every positive integer ℓ.

Example 3.6. Let k be a field and $R = k[[t^5, t^6, t^9]]$. As observed by Sally in [S, Ex.3.6],

$$R \cong k[[X,Y,Z]]/(YZ - X^3, Z^2 - Y^3)$$

and

$$G(\mathbf{m}) \cong k[X,Y,Z]/(YZ, Z^2, Y^4 - ZX^3).$$

Thus $(R, \mathbf{m})$ is a one-dimensional Gorenstein local domain and $G(\mathbf{m})$ is Cohen-Macaulay, but not Gorenstein. It is readily seen that $J = t^5R$ is a minimal reduction of $\mathbf{m}$ with $r_J(\mathbf{m}) = 3$. Sally also observes that $\overline{R} = R/J$ is a zero-dimensional Gorenstein local ring with maximal ideal $\mathbf{n} = \mathbf{m}\overline{R}$ such that $G(\mathbf{n})$ is not Gorenstein. Indeed, the leading form $(t^5)^*$ of t^5 in $\mathbf{m}/\mathbf{m}^2$ is a regular element of $G(\mathbf{m})$. Therefore $G(\mathbf{n}) \cong G(\mathbf{m})/(t^5)^*G(\mathbf{m})$. It follows that $G(\mathbf{n})$ is not Gorenstein, $\mathbf{n}^3 \neq 0$ and $\mathbf{n}^4 = 0$. Since $\mathbf{n}^i/\mathbf{n}^{i+1}$ is a vector space over $\overline{R}/\mathbf{n}$, $\mathbf{n}^i/\mathbf{n}^{i+1}$ is a nonzero free $(\overline{R}/\mathbf{n})$-module for $0 \le i \le 3$ and hence in particular a faithful $(\overline{R}/\mathbf{n})$-module. The dimensions of the components of $G(\mathbf{n}) = R/\mathbf{n} \oplus \mathbf{n}/\mathbf{n}^2 \oplus \mathbf{n}^2/\mathbf{n}^3 \oplus \mathbf{n}^3$ are $1,2,1,1$. This nonsymmetry reflects the fact that $G(\mathbf{n})$ is not Gorenstein, see Theorem 3.1.

We now turn to rings of arbitrary dimension. For this the following lemma is needed.

Lemma 3.7. *Let $(R, \mathbf{m})$ be a d-dimensional Gorenstein local ring and let I be an $\mathbf{m}$-primary ideal. Assume that $J \subseteq I$ is a reduction of I with $\mu(J) = d$, let $r_J(I)$ denote the reduction number of I with respect to J and let r be an integer with $r \ge r_J(I)$. If $J :_R I^{r-i} = J + I^{i+1}$ for every i with $0 \le i \le r-1$, then $G(I)$ is Cohen-Macaulay.*

Proof. The assertion is clear for $d = 0$. Next we assume that $d = 1$. By Valabrega-Valla [VV, Cor.2.7], it suffices to verify that $J \cap I^i = JI^{i-1}$ for $1 \le i \le r$. We first prove that

$$(3.8) \qquad I^r : I^{r-i} = I^i \qquad \text{for} \qquad 1 \le i \le r\,.$$

To show (3.8) we proceed by induction on i. One has

$$I^i \subseteq I^r : I^{r-i} \subseteq I^{r+1} : I^{r-i+1} = JI^r : I^{r-(i-1)} \subseteq J : I^{r-(i-1)} = J + I^{(i-1)+1} = J + I^i.$$

This proves the case $i = 1$. Now assume $i \geq 2$. The containment $I^r : I^{r-i} \subseteq J + I^i$ gives

$$\begin{aligned}
I^r : I^{r-i} &= (J + I^i) \cap (I^r : I^{r-i}) \\
&= [J \cap (I^r : I^{r-i})] + I^i \quad \text{since } I^i \subseteq I^r : I^{r-i} \\
&= J[(I^r : I^{r-i}) : J] + I^i \quad \text{since } J \text{ is principal} \\
&= J(I^r : JI^{r-i}) + I^i \\
&\subseteq J(I^{r+1} : JI^{r-i+1}) + I^i \\
&= J(JI^r : JI^{r-(i-1)}) + I^i \\
&= J(I^r : I^{r-(i-1)}) + I^i \quad \text{since } J \text{ is principal and regular} \\
&= JI^{i-1} + I^i \qquad \text{by the induction hypothesis} \\
&= I^i.
\end{aligned}$$

This completes the proof of (3.8).

Now we have for $1 \leq i \leq r$,

$$\begin{aligned}
J \cap I^i &= J(I^i : J) \quad \text{since } J \text{ is principal} \\
&\subseteq J(I^{r+1} : JI^{r+1-i}) \\
&= J(JI^r : JI^{r-(i-1)}) \\
&= J(I^r : I^{r-(i-1)}) \quad \text{since } J \text{ is principal and regular} \\
&= JI^{i-1} \qquad \text{by (3.8).}
\end{aligned}$$

Hence by the criterion of Valabrega-Valla, $G(I)$ is Cohen-Macaulay. This completes the proof of Lemma 3.7 in the case where $d = 1$.

Finally let $d \geq 2$. We may assume that the residue field of R is infinite. There exist elements $a_1, \ldots, a_{d-1}$ that form part of a minimal generating set of J and a superficial sequence for I. Write $\overline{R} = R/(a_1, \ldots, a_{d-1})R$ and $\overline{I} = I\overline{R}$. Since $\dim \overline{R} = 1$, we have that $G(\overline{I})$ is a Cohen-Macaulay ring, necessarily of dimension one. Hence by [HM, Lemma 2.2], $G(I)$ is Cohen-Macaulay. □

We are now ready to prove the main result of this section. Notice that we do not require $G(I)$ to be Cohen-Macaulay in this theorem; instead, the Cohen-Macaulayness is a consequence of the colon conditions (2) or (3) of the theorem.

Theorem 3.9. *Let $(R, \mathbf{m})$ be a d-dimensional Gorenstein local ring and let I be an $\mathbf{m}$-primary ideal. Assume that $J \subseteq I$ is a reduction of I with $\mu(J) = d$, and let $r = r_J(I)$ be the reduction number of I with respect to J. Then the following are equivalent :*

(1) *$G(I)$ is Gorenstein.*
(2) *$J :_R I^{r-i} = J + I^{i+1}$ for $0 \leq i \leq r-1$.*
(3) *$J :_R I^{r-i} = J + I^{i+1}$ for $0 \leq i \leq \lfloor \frac{r-1}{2} \rfloor$.*

Proof. The equivalence of items (2) and (3) follows from the double annihilator property in the zero-dimensional Gorenstein local ring R/J, see for instance [BH, (3.2.15), p.107]. To prove the equivalence of (1) and (2), by Lemma 3.7, we may assume that $G(I)$ is a Cohen-Macaulay ring. Write $J = (a_1, \dots, a_d)R$ and set $\overline{R} = R/J$, $\overline{I} = I/J$. Since $G(I)$ is Cohen-Macaulay and J is a minimal reduction of I, it follows that $a_1^*, \dots, a_d^*$ form a regular sequence on $G(I)$ and therefore

$$G(I)/(a_1^*, \dots, a_d^*)G(I) \cong G(I/J).$$

In particular $\overline{I}^r \neq 0$. Hence $(\overline{R}, \overline{\mathbf{m}})$ is a zero-dimensional Gorenstein local ring with $\overline{I}^r \neq 0$ and $\overline{I}^{r+1} = 0$. Now

$$\begin{aligned} G(I) \quad \text{is Gorenstein} &\Longleftrightarrow G(\overline{I}) \quad \text{is Gorenstein} \\ &\Longleftrightarrow 0 : \overline{I}^{r-i} = \overline{I}^{i+1} \quad \text{for } 0 \leq i \leq r-1 \text{ by Theorem 3.1} \\ &\Longleftrightarrow J : I^{r-i} = J + I^{i+1} \quad \text{for } 0 \leq i \leq r-1. \end{aligned}$$

This completes the proof of Theorem 3.9. □

We record the following corollary to Theorem 3.9 for the case of reduction number two.

Corollary 3.10. *Let $(R, \mathbf{m})$ be a d-dimensional Gorenstein local ring and let I be an $\mathbf{m}$-primary ideal. Assume that $J \subseteq I$ is a reduction of I with $\mu(J) = d$ and that $r_J(I) = 2$, i.e., $JI \neq I^2$ and $JI^2 = I^3$. Then the following are equivalent :*

(1) *$G(I)$ is Gorenstein.*
(2) *$J :_R I^2 = I$.*

The next corollary to Theorem 3.9 deals with the problem of lifting the Gorenstein property of associated graded rings. Notice we are not assuming that $G(I)$ is Cohen-Macaulay.

Corollary 3.11. *Let $(R, \mathbf{m})$ be a d-dimensional Cohen-Macaulay local ring and let I be an $\mathbf{m}$-primary ideal. Assume that $J \subseteq I$ is a reduction of I with $\mu(J) = d$ and that $I^r \not\subseteq J$ for $r = r_J(I)$ the reduction number of I with respect to J. Set $\overline{I} := I/J \subseteq \overline{R} := R/J$. If $G(\overline{I})$ is Gorenstein, then $G(I)$ is Gorenstein.*

Proof. If $G(\overline{I})$ is Gorenstein, then $\overline{R}$ and hence R are Gorenstein [M, p.121]. Since $I^r \not\subseteq J$, we have that $r = r_J(I)$ is also the reduction number of $\overline{I}$ with respect to the zero ideal. Now the assertion follows from Theorem 3.9. □

In Example 3.12 we exhibit a one-dimensional Gorenstein local domain $(R, \mathbf{m})$, an $\mathbf{m}$-primary ideal I and a principal reduction J of I such that for $\overline{I} := I/J \subseteq \overline{R} := R/J$, the associated graded ring $G(\overline{I})$ is Gorenstein, while $G(I)$ is not Gorenstein.

Example 3.12. Let k be a field, $R = k[[t^4, t^5, t^6]]$ and $I = (t^4, t^5)R$. Then R is a one-dimensional Gorenstein local domain, $\mu(I) = 2$ and $J = t^4R$ is a principal reduction of I. An easy computation shows that the reduction number $r_J(I) =$

3. On the other hand, $I^2 \subseteq J$. Hence the associated graded ring $G(I)$ is not Cohen-Macaulay. We have $\lambda(R/I) = \lambda(\overline{R}/\overline{I}) = 2$ and $\lambda(\overline{R}) = 4$. Therefore, by Theorem 3.1, $G(\overline{I}) = \overline{R}/\overline{I} \oplus \overline{I}$ is Gorenstein. On the other hand, the principal reduction $J' = (t^4 - t^5)R$ of I has the property that I^2 is not contained in J'. Therefore the associated graded ring $G(I/J')$ has Hilbert function $2, 1, 1$ and thus is not Gorenstein.

4. The Quasi-Gorenstein property of the extended Rees algebra

In Theorem 4.1 we give a general characterization for when the extended Rees algebra $R[It, t^{-1}]$ is quasi-Gorenstein. In case $G(I)$ is Cohen-Macaulay, this characterization would also follow from [GI, Thm.5.3]. In fact, if $G(I)$ is Cohen-Macaulay, then the quasi-Gorensteinness of $R[It, t^{-1}]$ is equivalent to the Gorensteinness of $G(I)$, since a Cohen-Macaulay quasi-Gorenstein ring is Gorenstein and $G(I) = R[It, t^{-1}]/(t^{-1})$ is Cohen-Macaulay (resp. Gorenstein) if and only if $R[It, t^{-1}]$ is Cohen-Macaulay (resp. Gorenstein).

For an ideal I of a ring R and an integer i, we define $I^i = R$ if $i \leq 0$.

Theorem 4.1. *Let $(R, \mathbf{m})$ be a Gorenstein local ring and let $I \subseteq \mathbf{m}$ be an ideal with $\mathrm{ht}(I) = g > 0$. Assume that $J \subseteq I$ is a reduction of I with $\mu(J) = g$. Let k be an integer with $k \geq r := r_J(I)$, the reduction number of I with respect to J, and let $B = R[It, t^{-1}]$ be the extended Rees algebra of the ideal I. Then the graded canonical module ω_B of B has the form*

$$\omega_B = \bigoplus_{i \in \mathbb{Z}} (J^{i+k} :_R I^k) t^{i+g-1}.$$

In particular, for $a \in \mathbb{Z}$, the following are equivalent :

(1) *$R[It, t^{-1}]$ is quasi-Gorenstein with $\mathbf{a}$-invariant a.*
(2) *$J^i :_R I^k = I^{i+a-k+g-1}$ for every $i \in \mathbb{Z}$.*

Proof. Let $K = \mathrm{Quot}(R)$ denote the total ring of quotients of R and let $A = R[Jt, t^{-1}] \subseteq B = R[It, t^{-1}]$. Notice that $A/(t^{-1}) \cong G(J)$, where t^{-1} is a homogeneous A-regular element of degree -1. Moreover, since J is generated by a regular sequence, $G(J)$ is a standard graded polynomial ring in g variables over the Gorenstein local ring R/J. Thus A is Cohen-Macaulay and $\omega_A \cong A(-g+1) \cong At^{g-1}$.

The extension $A \subseteq B$ is finite and $\mathrm{Quot}(A) = \mathrm{Quot}(B) = K(t)$ since $g > 0$. Therefore

$$\omega_B \cong \mathrm{Hom}_A(B, \omega_A) \cong At^{g-1} :_{K(t)} B = (A :_{K(t)} B)t^{g-1} = (A :_{R[t,t^{-1}]} B)t^{g-1},$$

where the last equality holds because

$$A :_{K(t)} B \subseteq A :_{K(t)} 1 \subseteq A \subseteq R[t, t^{-1}].$$

We may now make the identification $\omega_B = (A :_{R[t,t^{-1}]} B)t^{g-1}$.

Let i and j be any integers. Since J is a complete intersection, it follows that $J^{i+j+1} :_R J = J^{i+j}$. Hence

$$J^{i+j} :_R I^j = (J^{i+j+1} :_R J) :_R I^j = J^{i+j+1} :_R JI^j \supseteq J^{i+j+1} :_R I^{j+1},$$

where the last inclusion is an equality whenever $j \geq k \geq r_J(I)$. Therefore

$$\cap_{j\in\mathbb{Z}}(J^{i+j} :_R I^j) = J^{i+k} :_R I^k.$$

We conclude that

$$[A :_{R[t,t^{-1}]} B]_i = (\cap_{j\in\mathbb{Z}}(J^{i+j} :_R I^j))t^i = (J^{i+k} :_R I^k)t^i,$$

which gives

$$\omega_B = \bigoplus_{i\in\mathbb{Z}}(J^{i+k} :_R I^k)t^{i+g-1}.$$

This description shows in particular that $[\omega_B]_i = Rt^i$ for $i << 0$.

Now B is quasi-Gorenstein with **a**-invariant a if and only if $\omega_B \cong B(a)$, or equivalently, $\omega_B = \alpha B t^{-a}$ for some unit α of K. As $[\omega_B]_i = Rt^i$ for $i << 0$, it follows that α is necessarily a unit in R. Thus $\omega_B \cong B(a)$ if and only if

$$\oplus_{i\in\mathbb{Z}}(J^{i+k} :_R I^k)t^{i+g-1} = \oplus_{i\in\mathbb{Z}} I^i t^{i-a},$$

which means that $J^i :_R I^k = I^{i+a-k+g-1}$ whenever $i \in \mathbb{Z}$. □

Corollary 4.2. *With notation as in Theorem* 4.1*, the following are equivalent* :

(1) *The extended Rees algebra* $R[It, t^{-1}]$ *is quasi-Gorenstein.*

(2) *There exists an integer* u *such that* $J^i :_R I^r = I^{i-u}$ *for every* $i \in \mathbb{Z}$.

If these equivalent conditions hold, then $u = r - g + 1 - a \geq 0$ *with* a *denoting the* **a**-*invariant of* $R[It, t^{-1}]$.

Proof. To prove the last assertion notice that u is uniqely determined since I is not nilpotent. Thus part (2) of Theorem 4.1 gives $u = r - g + 1 - a$. The inequality $u \geq 0$ can be seen by setting $i = 0$ in part (2) of the present corollary. □

Corollary 4.3. *With notation as in Theorem* 4.1, $I^n(J^i :_R I^r) \subseteq J^{n+i} :_R I^r$ *for all integers* n, i, *and* $J^n(J^i :_R I^r) = J^{n+i} :_R I^r$ *for every* $n \geq 0$ *and* $i >> 0$.

Proof. This is clear since $\omega_B = \oplus_{i\in\mathbb{Z}}(J^{i+r} :_R I^r)t^{i+g-1}$ is a graded module over $B = R[It, t^{-1}]$ that is finitely generated over $A = R[Jt, t^{-1}]$. □

Remark 4.4. In the setting of Theorem 4.1 we define the *index of nilpotency* of I with respect to J to be $s_J(I) = \min\{i \,|\, I^{i+1} \subseteq J\}$.

Suppose $R[It, t^{-1}]$ is quasi-Gorenstein and let $u = r - g + 1 - a \geq 0$ be as in Corollary 4.2. The corollary implies $J : I^r = I^{1-u}$ and hence $I^{1-u+r} \subseteq J$. Thus $1 - u + r \geq s_J(I) + 1$, so $r_J(I) - s_J(I) \geq u \geq 0$. Therefore the **a**-invariant a of $R[It, t^{-1}]$ satisfies $s_J(I) - g + 1 \leq a \leq r_J(I) - g + 1$. Let $w = \max\{n \,|\, I^r \subseteq J^n\}$.

Then $w = u$; for $I^{w-u} = J^w : I^r = R$ implies $u \geq w$, while $I^r \not\subseteq J^{w+1}$ implies $I^{w+1-u} = J^{w+1} : I^r \subsetneq R$, so $w \geq u$. It follows that $s_J(I) = r_J(I)$ if and only if $u = 0$ if and only if $a = r_J(I) - g + 1$. These equalities hold in case $G(I)$, or equivalently $R[It, t^{-1}]$, is Cohen-Macaulay.

Since in dimension two, quasi-Gorenstein is equivalent to Gorenstein, we have the following corollary:

Corollary 4.5. *Let $(R, \mathbf{m})$ be a one-dimensional Gorenstein local ring and let I be an $\mathbf{m}$-primary ideal. Assume that J is a principal reduction of I and that $r = r_J(I)$ is the reduction number of I with respect to J. Then the following are equivalent :*

(1) *$G(I)$ is Gorenstein.*
(2) *$R[It, t^{-1}]$ is Gorenstein.*
(3) *There exists an integer u such that $J^i :_R I^r = I^{i-u}$ for every $i \in \mathbb{Z}$.*
(4) *$J^i :_R I^r = I^i$ for every $i \in \mathbb{Z}$.*
(5) *$J^i :_R I^r = I^i$ for $1 \leq i \leq r$.*

Proof. It remains to prove that (5) implies (4). This follows since for $i > r$ we have $I^i = J^{i-r} I^r = J^{i-r}(J^r : I^r) = J^i : I^r$, where the last equality holds because J is principal generated by a regular element. □

Remark 4.6. With notation as in Corollary 4.5, we have:

(1) As is well-known in the one-dimensional case, the reduction number $r = r_J(I)$ is independent of the principal reduction $J = xR$. Furthermore the ideal $J^r :_R I^r$ is also independent of J. Indeed, the subring $R[I/x] = I^r/x^r = \bigcup_{i=1}^{\infty}(I^i :_K I^i)$ of $K := \mathrm{Quot}(R)$ is the blowup of I, so is independent of J, and $J^r :_R I^r = J^r :_K I^r$ is the conductor of $R[I/x]$ into R.
(2) If $I^r \subseteq J$, then $I^r \subseteq J'$ for every reduction J' of I, that is, I^r is contained in the core of I. Indeed, we have $r > 0$ and then $I^r \subseteq J \iff J^{r-1} I^r \subseteq J^r \iff I^{2r-1} \subseteq J^r \iff I^{r-1} \subseteq J^r : I^r$. Notice that the latter ideal is independent of J by item (1). Therefore if the index of nilpotency $s_J(I) = r$ for one principal reduction J, then $s_{J'}(I) = r$ for every principal reduction J' of I. On the other hand, if $s_J(I) < r$ there may exist a principal reduction J' of I such that $s_J(I) \neq s_{J'}(I)$ as is illustrated in Example 3.12.
(3) If $J^r : I^r = I^r$, then $I^r \not\subseteq J$ and $s_J(I) = r$. Indeed, we may assume $r > 0$, hence $I^{r-1} \not\subseteq I^r$ and by item (2), $I^{r-1} \not\subseteq J^r : I^r \iff I^r \not\subseteq J$.

Let R be a Noetherian ring and I an R-ideal containing a regular element. The ideal $\widetilde{I} = \bigcup_{i=1}^{\infty}(I^{i+1} :_R I^i)$ first studied by Ratliff and Rush in [RR] is called the *Ratliff-Rush ideal associated to* I, and I is said to be a *Ratliff-Rush ideal* if $I = \widetilde{I}$. It turns out that the ideal $G(I)_+$ of $G(I)$ has positive grade if and only if all powers of I are Ratliff-Rush ideals [HLS, (1.2)]. Thus if $\dim R = 1$, then $G(I)$ is Cohen-Macaulay if and only if all powers of I are Ratliff-Rush ideals.

Remark 4.7. Let $(R, \mathbf{m})$ be a Gorenstein local ring and $I \subseteq \mathbf{m}$ an ideal with $\mathrm{ht}(I) = g > 0$. As in Theorem 4.1, assume there exists a reduction J of I with $\mu(J) = g$. Let $C = \bigoplus_{i \in \mathbb{Z}} \widetilde{I^i} t^i$ denote the extended Rees algebra of the Ratliff-Rush filtration associated to I, and let $k \geq 0$ be an integer such that $J^j \widetilde{I^k} = \widetilde{I^{j+k}}$ for every $j \geq 0$. The following are equivalent :

(1) C is quasi-Gorenstein with $\mathbf{a}$-invariant b.

(2) $J^i :_R \widetilde{I^k} = \widetilde{I^{i+b-k+g-1}}$ for every $i \in \mathbb{Z}$.

To show this equivalence one proceeds as in the proof of Theorem 4.1

Corollary 4.8. *With notation as in Corollary 4.5, if $J^r :_R I^r = I^{r-u}$ for some integer $u \geq 0$, then $u = 0$. If, in addition, all powers of I are Ratliff-Rush ideals, then $G(I)$ is Gorenstein.*

Proof. The equality $J^r : I^r = I^{r-u}$ implies $I^{r+1-u} = I(J^r : I^r)$ and by Corollary 4.3, $I(J^r : I^r) \subseteq J^{r+1} : I^r$. On the other hand, as J is generated by a regular element, $J^{r+1} : I^r = J(J^r : I^r) = JI^{r-u}$. Therefore $I^{r+1-u} = JI^{r-u}$, and $u = 0$ since r is minimal such that $I^{r+1} = JI^r$. By Corollary 4.5, to show $G(I)$ is Gorenstein it suffices to show $J^i : I^r = I^i$ for $1 \leq i \leq r$. Again according to Corollary 4.3, $I^{r-i}(J^i : I^r) \subseteq J^r : I^r$. Since $J^r : I^r = I^r$, it follows that $J^i : I^r \subseteq I^r : I^{r-i}$. We always have $I^r : I^{r-i} \subseteq \widetilde{I^i}$. Therefore, if in addition, $\widetilde{I^i} = I^i$, then $G(I)$ is Gorenstein. $\square$

With notation as in Corollary 4.5, it can happen that $J^r :_R I^r = I^r$ and yet $G(I)$ is not Cohen-Macaulay. We illustrate this in Example 4.9.

Example 4.9. Let k be a field, $R = k[[t^5, t^6, t^7, t^8]]$ and $I = (t^5, t^6, t^7)R$. Then R is a one-dimensional Gorenstein local domain with integral closure $\overline{R} = k[[t]]$ and $J = t^5 R$ is a reduction of I. We have $JI \subsetneq I^2 = \mathbf{m}^2 = t^{10}\overline{R}$ and $JI^2 = I^3$. Hence $r_J(I) = 2$. Since $t^8 \notin I$, but $t^8 I \subseteq I^2$, $G(I)$ is not Cohen-Macaulay. To see that $J^2 : I^2 = I^2$, observe that $t^{19} \notin J^2$ and that for each integer i with $5 \leq i \leq 8$ we have $t^{19-i} \in I^2$ and $t^{19} = t^{19-i}t^i$. Notice also that the Ratliff-Rush ideal $\widetilde{I}$ associated to I is $\mathbf{m} = (t^5, t^6, t^7, t^8)R$, $r_J(\mathbf{m}) = r_J(I) = 2$, $J^i : \mathbf{m}^2 = \mathbf{m}^i$ for every i (equivalently, for $1 \leq i \leq 2$), and hence $G(\mathbf{m})$ is Gorenstein by Corollary 4.5.

Example 4.10. Let $R = k[[t^4, t^5, t^6]]$, $I = (t^4, t^5)R$ and $J = t^4 R$ be as in Example 3.12. The Ratliff-Rush ideal $\widetilde{I}$ associated to I is $\mathbf{m} = (t^4, t^5, t^6)R$, and in fact $\widetilde{I^i} = \mathbf{m}^i$ for every $i \in \mathbb{Z}$. We have $r_J(I) = 3$ while $r_J(\mathbf{m}) = 2$. Also $I^3 = \mathbf{m}^3$. We have $J^i : I^3 = \mathbf{m}^{i-1}$ while $J^i : \mathbf{m}^2 = \mathbf{m}^i$ for every $i \in \mathbb{Z}$. In particular $G(\mathbf{m})$ is Gorenstein.

Question 4.11. With notation as in Theorem 4.1, is the extended Rees algebra $R[It, t^{-1}]$ Gorenstein if it is quasi-Gorenstein? Equivalently, is the associated graded ring $G(I)$ then Gorenstein?

If $\dim R = 2$ and R is pseudo-rational in the sense of [LT, p.102], we observe an affirmative answer to Question 4.11.

Corollary 4.12. *Let $(R, \mathbf{m})$ be a 2-dimensional pseudo-rational Gorenstein local ring and let I be an $\mathbf{m}$-primary ideal. If $B = R[It, t^{-1}]$ is quasi-Gorenstein, then B is Gorenstein. In particular, if R is a 2-dimensional regular local ring, then every extended Rees algebra $R[It, t^{-1}]$ that is quasi-Gorenstein is Gorenstein.*

Proof. We may assume that $R/\mathbf{m}$ is infinite. Let $J \subseteq I$ be a minimal reduction of I and let $r = r_J(I)$. If $J = I$, then I is generated by a regular sequence and $R[It, t^{-1}]$ is Gorenstein. Thus we may assume $J \subsetneq I$. By [LT, Cor.5.4], $\overline{I^{r+1}} \subseteq J^r$, where $\overline{I^{r+1}}$ denotes the integral closure of I^{r+1}. In particular $\overline{I} I^r \subseteq J^r$, which gives $\overline{I} \subseteq J^r : I^r$. By Corollary 4.2, there exists an integer u such that $J^i : I^r = I^{i-u}$ for every $i \in \mathbb{Z}$. We have $u < r$ according to Remark 4.4 since $J \neq I$. Thus $J^r : I^r \subseteq J^{u+1} : I^r$. Therefore $\overline{I} \subseteq J^r : I^r \subseteq J^{u+1} : I^r = I$, which shows that $I = \overline{I}$ is integrally closed. Now [LT, Cor.5.4] implies $r \leq 1$. Therefore B is Cohen-Macaulay by [VV, Prop.3.1] and hence Gorenstein. $\square$

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