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THE HOMOGENEOUS SPECTRUM OF A GRADED COMMUTATIVE RING

WILLIAM HEINZER AND MOSHE ROITMAN

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ABSTRACT. Suppose Γ is a torsion-free cancellative commutative monoid for which the group of quotients is finitely generated. We prove that the spectrum of a Γ-graded commutative ring is Noetherian if its homogeneous spectrum is Noetherian, thus answering a question of David Rush. Suppose A is a commutative ring having Noetherian spectrum. We determine conditions in order that the monoid ring $A[\Gamma]$ have Noetherian spectrum. If $\operatorname{rank}\Gamma \leq 2$, we show that $A[\Gamma]$ has Noetherian spectrum, while for each $n \geq 3$ we establish existence of an example where the homogeneous spectrum of $A[\Gamma]$ is not Noetherian.

0. INTRODUCTION.

All rings we consider are assumed to be nonzero, commutative and with unity. All the monoids are assumed to be torsion-free cancellative commutative monoids. Let Γ be a monoid such that the group of quotients G of Γ is finitely generated, and let $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ be a commutative Γ-graded ring. A goal of this paper is to answer in the affirmative a question mentioned to one of us by David Rush as to whether $\operatorname{Spec} R$ is necessarily Noetherian provided the homogeneous spectrum, h-$\operatorname{Spec} R$, is Noetherian.

If I is an ideal of a ring R, we let $\operatorname{rad}(I)$ denote the radical of I, that is $\operatorname{rad}(I) = \{r \in R : r^n \in I \text{ for some positive integer } n\}$. We say that I is a *radical ideal* if $\operatorname{rad}(I) = I$. A subset S of the ideal I *generates I up to radical* if $\operatorname{rad}(I) = \operatorname{rad}(SR)$. The ideal I is *radically finite* if it is generated up to radical by a finite set.

We recall that a ring R is said to have *Noetherian spectrum* if the set $\operatorname{Spec} R$ of prime ideals of R with the Zariski topology satisfies the descending chain condition on closed subsets. In ideal-theoretic terminology, R has Noetherian spectrum if and only if R satisfies the ascending chain condition (a.c.c.) on radical ideals.

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Thus a Noetherian ring has Noetherian spectrum and each ring having only finitely many prime ideals has Noetherian spectrum. As shown in [8, Prop. 2.1], $\operatorname{Spec} R$ is Noetherian if and only if each ideal of R is radically finite. It is well known that R has Noetherian spectrum if and only if R satisfies the two properties: (i) a.c.c. on prime ideals, and (ii) every ideal of R has only finitely many minimal prime ideals [6], [3, Theorem 88, page 59 and Ex. 25, page 65].

In analogy with the result of Cohen that a ring R is Noetherian if each prime ideal of R is finitely generated, it is shown in [8, Corollary 2.4] that R has Noetherian spectrum if each prime ideal of R is radically finite. It is shown in [8, Theorem 2.5] that Noetherian spectrum is preserved under polynomial extension in finitely many indeterminates. Thus finitely generated algebras over a ring with Noetherian spectrum again have Noetherian spectrum.

In Section 1 we prove that if R is a Γ-graded ring, where Γ is a monoid with finitely generated group of quotients, and if h-Spec R is Noetherian, then $\operatorname{Spec} R$ is Noetherian (Theorem 1.7). In Section 2 we deal with monoid rings. It turns out that if M is a monoid with finitely generated group of quotients and k is a field, then the homogeneous spectrum of the monoid ring $k[M]$ is not necessarily Noetherian (Example 2.9). On the positive side, h-Spec $A[M]$ is Noetherian if A is a ring with Noetherian spectrum and M is a monoid of torsion-free rank ≤ 2.

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1. The homogeneous spectrum

The *homogeneous spectrum,* h-Spec R, of a graded ring $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ is the set of homogeneous prime ideals of R. The most common choices for the commutative monoid Γ are the monoid $\mathbb{N}$ of nonnegative integers or its group of quotients $\mathbb{Z}$. A standard technique using homogeneous localization shows the following: if $R = \bigoplus_{n\in\mathbb{Z}} R_n$ is a $\mathbb{Z}$-graded integral domain, if t is a nonzero element of R_1, and if H is the multiplicative set of nonzero homogeneous elements of R, then the localization R_H of R with respect to H is the graded Laurent polynomial ring $K_0[t, t^{-1}]$, where K_0 is a field [10, page 157]. This implies the following remark.

Remark 1.1. Suppose $R = \bigoplus_{n\in\mathbb{Z}} R_n$ is a graded integral domain and P is a nonzero prime ideal of R. If zero is the only homogeneous element contained in P, then the localization R_P is one-dimensional and Noetherian.

If $R = \bigoplus_{n\in\mathbb{Z}} R_n$ is a graded ring with no nonzero homogeneous prime ideals, then R_0 is a field and either $R = R_0$, or R is a Laurent polynomial ring $R_0[x, x^{-1}]$ [1, page 83].

Every ring can be viewed as a graded ring with the trivial gradation that assigns degree zero to every element of the ring. Thus Nagata in [7, Section 8] develops primary decomposition for graded ideals in a graded Noetherian ring. It is not surprising that there is an interrelationship among the Noetherian properties of $\operatorname{Spec} R$, h-$\operatorname{Spec} R$, $\operatorname{Spec} R[X]$ and h-$\operatorname{Spec} R[X]$.

Proposition 1.2 is useful in considering the Noetherian property of spectra. It follows by induction from [8, Prop. 2.2 (ii)], but we prefer to prove it directly.

Proposition 1.2. *Let I be an ideal of a ring R. Let J be an ideal of R and S a subset of J such that $J = \operatorname{rad} SR$. If $I + J$ is radically finite and if for each $s \in S$, the ideal $IR[1/s]$ is radically finite, then I is radically finite.*

Proof. Since $I + J$ is radically finite and since $J = \operatorname{rad} SR$, there exist finite sets $F \subseteq I$ and $G \subseteq S$ such that $\operatorname{rad}(I + J) = \operatorname{rad}((F, G)R)$. For each $g \in G$ there exists a finite subset T_g of I such that $\operatorname{rad}(IR[1/g]) = \operatorname{rad}(T_g R[1/g])$. Let $I' = (F \cup \bigcup_{g\in G} T_g)R$, thus $I' \subseteq I$. Suppose $P \in \operatorname{Spec} R$ and $I' \subseteq P$. If $G \subseteq P$, then $I \subseteq P$ since $\operatorname{rad}(I' + GR) = \operatorname{rad}(I + J)$. Otherwise, we have $g \notin P$ for some element $g \in G$. Therefore $\operatorname{rad}(I'R[1/g]) = \operatorname{rad}(IR[1/g]) \subseteq PR[1/g]$. Since P is the preimage in R of $PR[1/g]$, we have $\operatorname{rad}(I) \subseteq P$. Therefore $\operatorname{rad}(I') = \operatorname{rad}(I)$, so I is radically finite. □

For Corollary 1.3, we use that the (homogeneous) spectrum of a graded ring R is Noetherian iff each (homogeneous) ideal of R is radically finite.

Corollary 1.3. (1) *Let S be a finite subset of a ring R. If* $\operatorname{Spec}(R/SR)$ *is Noetherian and for each $s \in S$,* $\operatorname{Spec}(R[1/s])$ *is Noetherian, then* $\operatorname{Spec} R$ *is Noetherian.*

(2) *Let S be a finite set of homogeneous elements of a graded ring R. If* h-$\operatorname{Spec}(R/SR)$ *is Noetherian and for each $s \in S$,* h-$\operatorname{Spec}(R[1/s])$ *is Noetherian, then* h-$\operatorname{Spec} R$ *is Noetherian.*

The hypotheses in Proposition 1.2 and Corollary 1.3 concerning the set S may be modified as follows and still give the same conclusion:

Proposition 1.4. *Let I be an ideal of a ring R, and let S be a finite subset of R. Let U be the multiplicatively closed subset of R generated by S.*

(1) *If $I + sR$ is radically finite for each $s \in S$ and IR_U is radically finite, then I is radically finite.*
(2) *If* $\operatorname{Spec}(R/sR)$ *is Noetherian for each $s \in S$ and* $\operatorname{Spec} R_U$ *is Noetherian, then* $\operatorname{Spec} R$ *is Noetherian.*
(3) *If R is a Γ-graded ring for some monoid Γ, each $s \in S$ is homogeneous with* h-$\operatorname{Spec}(R/sR)$ *Noetherian and if* h-$\operatorname{Spec} R_U$ *is Noetherian, then* h-$\operatorname{Spec} R$ *is Noetherian.*

The next Corollary is a special case of Proposition 1.2.

Corollary 1.5. *Suppose S is a subset of a ring R that generates R as an ideal and let I be an ideal of R. If $IR[1/s]$ is radically finite for each $s \in S$, then I is radically finite.*

If $R[1/s]$ has Noetherian spectrum for each $s \in S$, then R has Noetherian spectrum.

In analogy with Corollary 1.5, it is a standard result in commutative algebra that if $SR = R$ and $R[1/s]$ is a Noetherian ring for each $s \in S$, then R is a Noetherian ring. However, the analogue of Corollary 1.3 for the Noetherian property of a ring is false: There exists a non-Noetherian ring R and an element $s \in R$ such that R/sR and $R[1/s]$ are Noetherian. For example, let X, Y be indeterminates over a field k, let $R := k[X, \{Y/X^n\}_{n=0}^{\infty}]$ and let $s = X$. Then $P = (\{Y/X^n\}_{n=0}^{\infty})$ is a nonfinitely generated prime ideal of R, so R is not Noetherian, although both $R/XR = k$ and $R[1/X] = k[X, Y, 1/X]$ are Noetherian. Incidentally, both the ideal $(P + XR)/XR = (0)$ of R/XR and the ideal $PR[1/X]$ of $R[1/X]$ are principal.

Proposition 1.6 is the graded analogue of [8, Theorem 2.5].

Proposition 1.6. *Suppose Γ is a torsion-free cancellative commutative monoid with group of quotients G and $R = \bigoplus_{\gamma \in \Gamma} R_\gamma$ is a Γ-graded commutative ring. Fix $g \in G$, and consider the polynomial ring $R[X]$ as a graded extension ring of R uniquely determined by defining X to be a homogeneous element of degree g. If* h-$\operatorname{Spec} R$ *is Noetherian, then* h-$\operatorname{Spec} R[X]$ *is Noetherian.*

Proof. Assume that h-$\operatorname{Spec} R$ is Noetherian, but h-$\operatorname{Spec} R[X]$ is not Noetherian. Then there exists a homogeneous prime ideal P of $R[X]$ that is maximal with respect to not being radically finite. Since $P \cap R = p$ is a homogeneous prime ideal of R and h-$\operatorname{Spec} R$ is Noetherian, we may pass from $R[X]$ to $R[X]/p[X] \cong (R/p)[X]$ and assume that $P \cap R = (0)$. Then R is a graded domain and h-Spec R is Noetherian. Choose an element $f \in P$ having minimal degree d as a polynomial in $R[X]$. By replacing f by one of its nonzero homogeneous components, we may assume that $f = a_d X^d + a_{d-1}X^{d-1} + \cdots + a_0$, where the elements $a_i \in R$ are homogeneous elements of R with $a_d \neq 0$. Since $P \cap R = (0)$, we have $d > 0$ and $a_d \notin P$. The maximality of P with respect to not being radically finite implies $(P, a_d)R[X]$ is radically finite. Since $a_d^{-1} f$ is a polynomial of minimal degree in $PR[1/a_d][X]$ and since this polynomial is monic in $R[1/a_d][X]$, we see that $PR[1/a_d][X] = (f)$. But Proposition 1.4 (1) then implies that P is radically finite, a contradiction. □

We use Proposition 1.6 in the proof of Theorem 1.7.

Theorem 1.7. *Let* $R = \bigoplus_{\gamma \in \Gamma} R_\gamma$ *be a* Γ*-graded commutative ring, where* Γ *is a nonzero torsion-free cancellative commutative monoid such that its group of quotients* G *is finitely generated. If* h-Spec R *is Noetherian, then* Spec R *is also Noetherian.*

Proof. Up to a group isomorphism, we have $G \cong \mathbb{Z}^d$ for some positive integer d. Hence we may assume $G = \mathbb{Z}^d$. For $1 \leq i \leq d$, let g_i be the element of G having 1 as its i-th coordinate and zeros elsewhere. Consider the graded polynomial extension ring $R[\mathbf{X}] := R[X_1, \ldots, X_d]$ obtained by defining X_i to be a homogeneous element of degree g_i for $i = 1, \ldots, d$. Proposition 1.6 implies that h-Spec $R[\mathbf{X}]$ is Noetherian. We associate with each nonzero element $r \in R$ a homogeneous element $\widetilde{r} \in R[\mathbf{X}]$ such that $\deg(\widetilde{r}) = (c_1, \ldots, c_d)$, where c_i is the maximum of the i-th coordinates of the degrees of the nonzero homogeneous components of $r \in R$ as follows: let $r = r_0 + \cdots + r_k$ be the homogeneous decomposition of r; set $\widetilde{r} = \sum_{i=1}^{k} r_i \mathbf{X}^{m_i}$, where $m_i = (c_1, \ldots, c_d) - \deg r_i$ for each i and $\mathbf{X}^{(a_1, \ldots, a_d)} = \prod_{i=1}^{d} X_i^{a_i}$ for each sequence $(a_1, \ldots, a_d)$ in $\mathbb{Z}^d$. We define $\widetilde{0} = 0$. With each ideal I of R, let $\widetilde{I}$ denote the homogeneous ideal of $R[\mathbf{X}]$ generated by $\{\widetilde{r} : r \in I\}$ ($\widetilde{r}$ is the *homogenization* of r and $\widetilde{I}$ is the homogenization of I).

Let $\phi : R[\mathbf{X}] \to R$ denote the R-algebra homomorphism defined by $\phi(X_i) = 1$ for $i = 1, \ldots, n$. Since ϕ is an R-algebra homomorphism and $\phi(\widetilde{r}) = r$ for each $r \in R$, for each ideal I of R, we have $\phi(\widetilde{I}) = I$ (the meaning of ϕ is *dehomogenization*). Therefore the map $I \to \widetilde{I}$ is a one-to-one inclusion preserving correspondence of the set of ideals of R into the set of homogeneous ideals of $R[\mathbf{X}]$.

Let I be an ideal of R. Since h-Spec $R[\mathbf{X}]$ is Noetherian there exists a finite set S such that $\operatorname{rad} \widetilde{I} = \operatorname{rad}(SR[\mathbf{X}])$. We have $\operatorname{rad} I = \operatorname{rad} \phi(\widetilde{I}) = \operatorname{rad}(\phi(S))R$, thus I is radically finite. Therefore Spec R is Noetherian. □

The following corollary is immediate from Theorem 1.7.

Corollary 1.8. *Let R be an $\mathbb{N}$-graded or a $\mathbb{Z}$-graded ring. If* h-Spec R *is Noetherian, then* Spec R *is Noetherian.*

Without the assumption in Theorem 1.7 that the group of quotients of Γ is finitely generated, it is possible to have h-Spec R is Noetherian and yet Spec R is not Noetherian. For example, if K is an algebraically closed field of characteristic zero and $\Gamma = \mathbb{Q}$, then (0) is the only homogeneous prime ideal of the group ring $R := K[\mathbb{Q}]$ so h-Spec R is Noetherian, but as we note in Theorem 2.6 below, Spec R is not Noetherian.

2. The Noetherian spectra of monoid rings

Suppose A is a ring and M is a cancellative torsion-free commutative monoid. We consider the monoid ring $A[M]$ as a graded ring with its natural M-grading where the nonzero elements of A are of degree zero. The monoid M is naturally identified with a subset of $A[M]$. We write X^m for $m \in M \subseteq A[M]$. Note that $0 \in M$ is identified with $1 \in A[M]$.

A $\mathbb{Q}$-*monoid* in a $\mathbb{Q}$-vector space V is an additive submonoid of V that is closed under multiplication by positive rationals. A subset of a $\mathbb{Q}$-monoid W is called a $\mathbb{Q}$-*ideal* of the $\mathbb{Q}$-monoid W if it is an ideal of the monoid W that is closed under multiplication by positive (that is, strictly positive) rationals.

If M is a cancellative torsion-free monoid with group of quotients G, we denote by $M^{(\mathbb{Q})}$ the $\mathbb{Q}$-monoid generated by M in $G \otimes_{\mathbb{Z}} \mathbb{Q}$; thus $M = \{qm \mid q > 0 \text{ in } \mathbb{Q}, m \in M\}$.

Remark 2.1. Let S be a subset of a monoid M, let R be a ring, and let I be a homogeneous ideal of $R[M]$ containing S and generated by monomials in M. Then S generates I up to radical iff S generates the $\mathbb{Q}$-ideal $(I \cap M)^{(\mathbb{Q})}$ of $M^{(\mathbb{Q})}$.

Remark 2.2. Suppose M is a cancellative torsion-free commutative monoid and k is a field. There is a natural one-to-one inclusion preserving correspondence between the homogeneous radical ideals of the monoid domain $k[M]$ and the $\mathbb{Q}$-ideals of the

$\mathbb{Q}$-monoid $M^{(\mathbb{Q})}$. Indeed, to each $\mathbb{Q}$-ideal L of $M^{(\mathbb{Q})}$ (which is generated by $L \cap M$) we make correspond the ideal of $k[M]$ generated by $L \cap M$.

Lemma 2.3. *Suppose M is a torsion-free cancellative commutative monoid, A is a ring with Noetherian spectrum, and P is a homogeneous prime ideal of the monoid ring $A[M]$. Then the following two conditions are equivalent:*

(1) *The prime ideal P is radically finite in $A[M]$.*
(2) *The $\mathbb{Q}$-ideal $(P \cap M)^{(\mathbb{Q})}$ of $M^{(\mathbb{Q})}$ is finitely generated.*

Proof. Since P is prime and homogeneous, P is generated by $(P \cap A) \cup (P \cap M)$. Since $\operatorname{Spec} A$ is Noetherian, we see that P is radically finite iff the ideal in $A[M]$ generated by $P \cap M$ is radically finite iff the $\mathbb{Q}$-ideal $(P \cap M)^{(\mathbb{Q})}$ of $M^{(\mathbb{Q})}$ is finitely generated (Remark 2.1). This proves Lemma 2.3. □

The following is an immediate corollary to Lemma 2.3.

Corollary 2.4. *Let M be a torsion-free cancellative commutative monoid and let A be a ring with Noetherian spectrum. Then the following two conditions are equivalent:*

(1) *The monoid ring $A[M]$ has Noetherian homogeneous spectrum.*
(2) *Each $\mathbb{Q}$-ideal in the $\mathbb{Q}$-monoid $M^{(\mathbb{Q})}$ is finitely generated.*

We denote the torsion-free rank of a monoid M by $\operatorname{rank} M$.

Proposition 2.5. *Suppose A is a ring and M is a cancellative torsion-free commutative monoid.*

(1) *If* $\operatorname{Spec} A[M]$ *is Noetherian, then* $\operatorname{Spec} A$ *is Noetherian and* $\operatorname{rank} M$ *is finite.*
(2) *If* $\operatorname{Spec} A$ *is Noetherian and if* $\operatorname{rank} M \leq 2$*, then* h-$\operatorname{Spec} A[M]$ *is Noetherian.*

Proof. (1) $\operatorname{Spec} A$ is Noetherian since every ideal I of A satisfies $I = IA[M] \cap A$, and if I is a radical ideal of A, then $IA[M]$ is a radical ideal in $A[M]$. If $\operatorname{rank} M$ is infinite, let B be an infinite set of elements in M which are linearly independent over $\mathbb{Q}$ in the $\mathbb{Q}$-vector space $G \otimes_{\mathbb{Z}} \mathbb{Q}$, where G is the group of quotients of M. Then the ideal of $A[M]$ generated by $\{X^b - X^c : b, c \in B\}$ is not radically finite. Therefore $\operatorname{Spec} A[M]$ is not Noetherian.

(2) By Lemma 2.3 it suffices to show each $\mathbb{Q}$-ideal in $M^{(\mathbb{Q})}$ is finitely generated. We may assume that $M \subseteq \mathbb{Q}^2$. Let W be a nonempty $\mathbb{Q}$-ideal of $M^{(\mathbb{Q})}$. We show that W is a finitely generated ideal of $\widetilde{W} := W \cup \{\mathbf{0}\}$. If W

spans a one-dimensional subspace and $\mathbf{v}$ is a nonzero element of W, then the $\mathbb{Q}$-ideal W is generated by $\mathbf{0}$ if $-\mathbf{v} \in W$, and by $\mathbf{v}$ otherwise. If W spans $\mathbb{Q}^2$, then choose two linearly independent vectors in W. By changing coordinates, we may assume that these vectors are $(1,0)$ and $(0,1)$. If W contains a vector $\mathbf{v}$ with both coordinates strictly negative, then W is generated by $\mathbf{0}$ as a $\mathbb{Q}$-ideal. Otherwise, define vectors $\mathbf{u}$ and $\mathbf{v}$ as follows: if $a = \min\{y \mid (1,y) \in W\}$ exists, let $\mathbf{u} = (1,a)$; if the minimum does not exist, let $\mathbf{u} = (1,0)$. Similarly, define a vector $\mathbf{v}$ with second coordinate 1. Then $\mathbf{u}$ and $\mathbf{v}$ generate W as a $\mathbb{Q}$-ideal of $\widetilde{W}$.

□

Theorem 2.6. *Let A be a ring with Noetherian spectrum and M be a cancellative torsion-free commutative monoid. If the group of quotients of M is finitely generated and if* $\operatorname{rank} M \leq 2$*, then the monoid ring $A[M]$ has Noetherian spectrum.*

On the other hand, if $A[M]$ has Noetherian spectrum and if A contains an algebraically closed field of zero characteristic, then the group of quotients of M is finitely generated.

Proof. Assume that the group of quotients of M is finitely generated and that $\operatorname{rank} M \leq 2$. By Proposition 2.5 (2), $A[M]$ has Noetherian homogeneous spectrum. By Theorem 1.7, $\operatorname{Spec} A[M]$ is Noetherian.

For the second statement, assume that the group of quotients of M is not finitely generated. By Proposition 2.5 (1), we may assume that M has finite rank. It follows that there exists an element $s \in M$ that is divisible by infinitely many positive integers. Since A contains all roots of unity and they are distinct, we obtain that over the element $X^s - 1$ of $A[M]$ there are infinitely many minimal primes. Therefore $\operatorname{Spec} A[M]$ is not Noetherian. □

With regard to 2.6, if the monoid M is finitely generated, then it follows from [8, Theorem 2.5], that $\operatorname{Spec} A[M]$ is Noetherian if $\operatorname{Spec} A$ is Noetherian.

Example 2.7. Over a field k of characteristic $p > 0$, there exists a monoid M for which the group of quotients is not finitely generated and yet the monoid domain $k[M]$ has Noetherian spectrum. For example, if $M := \mathbb{Z}[\{1/p^n\}_{n=1}^{\infty}]$, then $k[M]$ is an integral purely inseparable extension of $k[\mathbb{Z}]$ and $\operatorname{Spec}(k[M])$ is Noetherian.

A *prime* $\mathbb{Q}$-ideal of a $\mathbb{Q}$-monoid M is a $\mathbb{Q}$-ideal Q of M that is a prime ideal, that is, if $a + b \in Q$, then either $a \in Q$ or $b \in Q$.

Let S be a subset of a vector space over $\mathbb{Q}$. S is $\mathbb{Q}$-*convex* if for any points p, q in S and rational $0 \le t \le 1$ we have $tp + (1-t)q \in S$.

Remark 2.8. Let M be a $\mathbb{Q}$-monoid in a vector space over $\mathbb{Q}$, and let I be a subset of M that is closed under addition and under multiplication by positive rationals; thus I is a $\mathbb{Q}$-convex set. Then I is an ideal of M iff for any two points $p \in I$ and $q \in M$ and any rational $0 < t < 1$, we have $tp + (1-t)q \in I$. Moreover, for I as above, if I is an ideal, then I is prime iff the set $M \setminus I$ is $\mathbb{Q}$-convex.

We denote by C the unit circle in $\mathbb{R}^2$, that is, $C = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. We let $C_{\mathbb{Q}} = C \cap \mathbb{Q}^2$. For any subset S of $\mathbb{R}^n$ we denote by S^+ the set of points in S with nonnegative coordinates.

Example 2.9. Let $n \ge 3$. Then there exists a cancellative torsion-free commutative monoid M of rank n such that the group of quotients of M is finitely generated, but for any ring A the homogeneous spectrum of $A[M]$ is not Noetherian. Furthermore, the monoid M is completely integrally closed. Hence, if A is an integrally closed (completely integrally closed) domain, then $A[M]$ is an integrally closed (completely integrally closed) domain.

First let $n = 3$. Let W be the $\mathbb{Q}$-submonoid of $\mathbb{Q}^3$ generated by the set $\{(x, y, 1) : (x, y) \in C_{\mathbb{Q}}\}$. We claim that the $\mathbb{Q}$-ideal $W \setminus \{\mathbf{0}\}$ of W is not finitely generated; moreover, if $(p, 1) \in C_{\mathbb{Q}} \times \{1\}$, then $(p, 1)$ does not belong to the $\mathbb{Q}$-ideal of W generated by $C_{\mathbb{Q}} \times \{1\} \setminus \{(p, 1)\}$. Indeed, by Remark 2.8, the set of points $(x, y, z) \in W$ such that $\dfrac{1}{z}(x, y) \ne p$ is a $\mathbb{Q}$-ideal of W which does not contain $(p, 1)$. Set $M = W \cap \mathbb{Z}^3$. More explicitly, since the convex hull of $C_{\mathbb{Q}}$ equals the rational unit disk, we see that $M = \{X^a Y^b Z^c \mid (a, b, c) \in \mathbb{Z}^3, c \ge 0 \text{ and } a^2 + b^2 \le c^2\}$.

Now let A be any ring. Since the $\mathbb{Q}$-ideal generated by $W \setminus \{\mathbf{0}\}$ in W is not finitely generated, we obtain by Lemma 2.3 that the ideal in $A[M]$ generated by the nonzero elements of M is not radically finite; thus h-Spec $A[M]$ is not Noetherian.

If $n > 3$ let $\widetilde{M} = M \oplus \mathbb{Z}^{n-3}$, where M is the monoid defined above. Then rank $\widetilde{M} = n$ and $\widetilde{M}$ satisfies our requirements.

Clearly, M is a completely integrally closed monoid. Thus the assertions on $A[M]$ follow from [2, Corollary 12.7 (2) and Corollary 12.11 (2)]. □

We now elaborate on Example 2.9, but with W replaced by W^+. As seen in Example 2.9, R is a completely integrally closed domain, and h-Spec R is not Noetherian. Moreover, $R = k[M]$ is a subring of the polynomial ring $k[X, Y, Z]$ and has fraction field $k(X, Y, Z)$. By [2, Theorem 21.4], $\dim R = 3$. It is interesting that the maximal homogeneous ideal N of R has height 3, but its homogeneous height (defined using just homogeneous prime ideals) is 2. Indeed, let $P \neq N$ be a nonzero prime homogeneous ideal of R. Let Q be the $\mathbb{Q}$-ideal of W generated by the points (a, b, c) in $\mathbb{Q}^3$ such that $X^a Y^b Z^c \in P$. Since Q is a prime $\mathbb{Q}$-ideal of W and since $C^+_{\mathbb{Q}}$ is dense in C^+, by Remark 2.8 we easily obtain that Q contains $C^+_{\mathbb{Q}} \times \{1\}$ except one point. Thus the homogeneous height of N is at most 2. Since the $\mathbb{Q}$-ideal of W generated by $C^+_{\mathbb{Q}} \times \{1\}$ with one point removed is prime, we see that the homogeneous height of N is 2.

On the other hand, $\operatorname{ht} N = 3$. More generally, if R is a k-subalgebra of a polynomial ring $k[\mathbf{X}] := k[X_1, \ldots, X_n]$ over a field k with the same fraction field $k(\mathbf{X})$, then $\operatorname{ht}((\mathbf{X})k[\mathbf{X}] \cap R) = n$. Indeed, this prime ideal has height at most n since $k(\mathbf{X})$ has transcendence degree n over k. Moreover, each nonzero ideal of $k[\mathbf{X}]$ has a nonzero intersection with R. Since the primes of height $n-1$ of $k[\mathbf{X}]$ contained in $(\mathbf{X})k[\mathbf{X}]$ intersect in zero, there exists such a prime ideal P_{n-1} of $k[\mathbf{X}]$ such that $P_{n-1} \cap R \subsetneq (\mathbf{X})k[\mathbf{X}] \cap R$. Repeating this argument, we find a strictly descending chain of prime ideals contained in R: $(\mathbf{X})k[\mathbf{X}] \cap R \supsetneq (P_{n-1} \cap R) \supsetneq \cdots \supsetneq P_0 = (0)$.

This behavior where the dimension of the homogeneous spectrum of a graded integral domain R is less than $\dim R$ also occurs in the case where R is an $\mathbb{N}$-graded integral domain. For example, if A is a one-dimensional quasilocal integral domain such that the polynomial ring $A[X]$ has dimension three [9], then the homogeneous spectrum of $A[X]$ in its natural $\mathbb{N}$-grading has dimension two.

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Department of Mathematics, Purdue University, West Lafayette, Indiana 47907-1395
E-mail address: heinzer@math.purdue.edu

Department of Mathematics, University of Haifa, Mount Carmel, Haifa 31905, Israel
E-mail address: mroitman@math.haifa.ac.il