PRIME IDEALS IN BIRATIONAL EXTENSIONS OF POLYNOMIAL RINGS II

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1. Introduction

Let R be a semilocal Noetherian domain of dimension one, and let x be an indeterminate over R. By a *birational extension* of the polynomial ring $R[x]$ we mean an integral domain B containing $R[x]$ and contained in the fraction field of $R[x]$. We are interested in the prime ideal structure of birational extensions of $R[x]$. A general objective, of which the present paper is a part, is to determine which partially ordered sets, or equivalently topological spaces, can occur as the prime spectrum, $\text{Spec}(B)$, of a finitely generated birational extension B of $R[x]$. A related objective is to determine necessary and sufficient conditions on two finitely generated birational extensions B_1 and B_2 of $R[x]$ in order that $\text{Spec}(B_1)$ and $\text{Spec}(B_2)$ be order-isomorphic as partially ordered sets.

Let $\mathbf{m}_1, \mathbf{m}_2, \dots, \mathbf{m}_n$ denote the maximal ideals of R. In order to obtain precise results, we restrict to extensions of the form $B = R[x, \{g_i/f_i\}_{i=1}^t]$, where t is a positive integer, each $f_i \in R[x] - \bigcup_{j=1}^n \mathbf{m}_j[x]$, and $(f_i, g_i)R[x]$ is not contained in any height-one prime ideal of $R[x]$. For ease of reference we refer to a finitely generated birational extension of this special form as a *(t-generated) sfb-extension* of $R[x]$. We often restrict to the case where R is the localization $k[y]_{(y)}$ of the polynomial ring in an indeterminate y over a countable algebraically closed field k. This article is a continuation of our work in [Heinzer et al. (1994b)] and also relies heavily on results and notation in [Heinzer and Wiegand (1989)] and [Heinzer et al. (1994a)].

Let D be a two-dimensional Noetherian domain with infinitely many height-two maximal ideals and only finitely many height-one prime ideals that are contained in infinitely many maximal ideals. We refer to the height-one primes of D that are contained in infinitely

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many maximal ideals as the *special primes* of D. Note that a height-one prime ideal P of D is special if and only if it is nonmaximal and is an intersection of maximal ideals. It is easy to see that $\mathrm{Spec}(D)$ as a partially ordered set is determined by the maximal prime ideals of D and the containments between the nonmaximal height-one primes and the height-two maximal ideals. This amounts to considering two things:

(1) The j-spectrum of D, $j\text{-}\mathrm{Spec}(D)$, (a prime $P \in \mathrm{Spec}(D)$ is a j-prime if P is an intersection of maximal ideals, and $j\text{-}\mathrm{Spec}(D)$ is the partially ordered set of j-primes in $\mathrm{Spec}(D)$) and
(2) The "inverse j-radicals" of finite sets T of height-two maximal ideals of $\mathrm{Spec}(D)$, that is, the set of all height-one prime ideals that are contained in every element of T but are in no other maximal ideals.

In [Heinzer et al. (1994b)], restricting our attention to a cyclic (i.e., one-generated) sfb-extension B of $R[x]$:

(1) we characterize $j\text{-}\mathrm{Spec}(B)$,
(2) we show that the inverse j-radical of every singleton set T of height-two maximals of B is infinite, and
(3) for certain domains R we obtain complete results on the inverse j-radicals of finite sets of height-two maximals, thus describing the entire spectrum of B.

In Section 2 of the present article, we prove that if R is of finite type over a countable field and B is a cyclic sfb-extension of R, then $\mathrm{Spec}(B)$ is uniquely determined by the type of $j\text{-}\mathrm{Spec}(B)$. This extends the result in [Heinzer et al. (1994b)] mentioned in (3). For the rest of the paper, we allow noncyclic sfb-extensions and show that the j-spectrum of a noncyclic sfb-extension can be much different from the cyclic case. A transient in an sfb-extension $B = R[x, g_1/f_1, \ldots, g_n/f_n]$ is a height-one nonmaximal j-prime that does not survive in $R[x, 1/(f_1 \ldots f_n)]$. In Section 3, we provide explicit examples to show that the j-spectra of noncyclic sfb-extensions may differ from the cyclic case, in that two transients need not be comaximal, and in fact their sum may be contained in more than one maximal ideal. In Section 4, we formulate an axiom on partially ordered sets that is satisfied by the j-spectra of sfb-extensions of $R[x]$ for R local. We describe a procedure that we believe will realize a given "feasible" partially ordered set satisfying this axiom as the j-spectrum of an sfb-extension.

All rings we consider are commutative and contain a multiplicative identity. The terms "local" and "semilocal" as used in this paper include a Noetherian hypothesis on the ring.

1.1 Notation. For U a partially ordered set of finite dimension, elements u, w of U, and T

a finite subset of U, we set

$$\begin{aligned} \mathrm{G}(u) &= \{w \in U \mid w > u\} \qquad \text{and} \\ \mathrm{L_e}(T) &= \{w \in U \mid w < t \iff t \in T\} \\ &= \{w \in U \mid \mathrm{G}(w) = T\} \quad . \end{aligned}$$

And we denote by $\mathcal{H}_i(U)$ the set of elements of U of height i.

Note. We say that a list of properties *characterizes* a given partially ordered set if every partially ordered set with these properties is order-isomorphic to the given partially ordered set.

Let R be a countable semilocal domain of dimension one with exactly n maximal ideals. As Theorem 1.2 below indicates, j- $\mathrm{Spec}(R[x])$ is then uniquely determined, and there are at most two possibilities for $\mathrm{Spec}(R[x])$. For notational purposes we introduce the *j-subset*, j-V, of a partially ordered set V of finite dimension, which we define recursively as follows: $u \in j$-V iff either (i) u is maximal in V, or (ii) there are infinitely many covers of u in j-V, where an element w of V is called a *cover* of u if $u < w$ and there is no element v of V for which $u < v < w$.

1.2 Theorem. [Heinzer et al. (1994b), Theorem 1.3] *Let R be a countable semilocal domain of dimension one with exactly n maximal ideals, x an indeterminate, and $U = \mathrm{Spec}(R[x])$. Then:*

(1) There are at most two possibilities for the partially ordered set U, and they are distinguished by the properties (P6) and (P6′) below:

In case R is not Henselian, U satisfies

(P6) For each nonempty finite subset T of $\mathcal{H}_2(U)$, $\mathrm{L_e}(T)$ is infinite.

In case R is Henselian (and then $n = 1$), U satisfies

(P6′) For each finite subset T of $\mathcal{H}_2(U)$ of cardinality greater than one, $\mathrm{L_e}(T)$ is empty. For each element t of $\mathcal{H}_2(U)$, $\mathrm{L_e}(\{t\})$ is infinite.

(2) Let V_1, V_2 be countable partially ordered sets of dimension two with unique minimal elements such that

- *(i) There exists an order-isomorphism φ: j-$V_1 \to j$-V_2.*
- *(ii) For $i = 1, 2$, the height of each element of j-V_i agrees with its height in V_i, and*
- *(iii) For each finite subset T of $\mathcal{H}_2(V_1)$, $\mathrm{L_e}(T)$ and $\mathrm{L_e}(\varphi(T))$ are either both empty or both infinite.*

Then V_1 is order-isomorphic to V_2 by an extension of φ.

(3) Let V be a countable partially ordered set of dimension two with unique minimal element such that:

(i) The subset j-V has the properties (P0)–(P4) below:

(P0) j-V is countable.

(P1) j-V has a unique minimal element u_0.

(P2) j-V has dimension 2.

(P3) j-V has infinitely many height-one maximal elements.

(P4) j-V has exactly n special elements. We denote these elements $u_1, u_2, \dots u_n$. They satisfy:

(a) $\mathrm{G}(u_1) \cup \cdots \cup \mathrm{G}(u_n) = \mathcal{H}_2(j\text{-}V)$,

(b) $\mathrm{G}(u_i) \cap \mathrm{G}(u_j) = \emptyset$ for $i \neq j$, and

(c) $\mathrm{G}(u_i)$ is infinite for each i, $1 \leq i \leq n$.

(ii) The height of each element of j-V agrees with its height in V, and

(iii) V satisfies (P6) or (P6′) of 1.2(1) (whichever U satisfies).

Then V is order-isomorphic to U.

Note. When we stipulate the j-spectrum and either (P6) or (P6′) of 1.2, we have described all possibilities for $\mathrm{Spec}(R[x])$ given that R is a countable semilocal domain of dimension one with exactly n maximal ideals. However, condition (ii) is necessary in (2) and (3), because conditions (i) and (iii) together do not uniquely determine a partially ordered set V. For example, there could be maximal elements of V of height two in V but only of height one in j-V.

(1.3) Let $(R, \mathbf{m}_1, \dots, \mathbf{m}_n)$ be a semilocal domain of dimension one, x an indeterminate, and let B be a *t-generated sfb-extension* of $R[x]$, i.e., $B = R[x, \{g_i/f_i\}_{i=1}^t]$, where t is a positive integer, each $f_i \in R[x] - \bigcup_{j=1}^n \mathbf{m}_j[x]$ and each $(f_i, g_i)R[x] = R[x]$ is not contained in any height-one prime ideal of $R[x]$. It follows from the dimension formula [M2, page 119] that for each height-two maximal ideal M of B, $M \cap R = \mathbf{m}_j$ for some j and B/M is a finite algebraic extension of $R/\mathbf{m}_j$. Let $f = \prod_{i=1}^t f_i$. If $f \in P$, a prime ideal of $R[x]$, then $f \in Q \subseteq P$ for some height-one prime Q of $R[x]$. Now $f \in Q$ implies $f_i \in Q$ for some i. But then $g_i \notin Q$ and so $g_i/f_i \notin R[x]_Q \supseteq R[x]_P$. Hence $B \nsubseteq R[x]_P$. Thus a prime ideal P of $R[x]$ does not contain f if and only if $B \subseteq R[x]_P$. Therefore the radical of $fR[x]$ is uniquely determined by B. For convenience we use the following terms from [Heinzer et al. (1994b)] for the *special primes* of B — the nonmaximal height-one j-primes.

(1) A special prime ideal P of B is called a *survivor* if it survives in $B[1/f]$, that is, if $PB[1/f] \subsetneq B[1/f]$, or, equivalently, if $f \notin P$. (There is exactly one survivor

j-prime of B contracting to each maximal ideal $\mathbf{m}_i$ of R, namely $\mathbf{m}_i R[x, 1/f] \cap B$.)

(2) A special prime ideal P of B is called a *transient* if it does not survive in $B[1/f]$, that is, if $PB[1/f] = B[1/f]$, or, equivalently, if $f \in P$.

(1.3.1) The special prime ideals of B are precisely the minimal primes of the $\mathbf{m}_i B$. To see this, we may suppose that $(R, \mathbf{m})$ is local and consider P a special prime of B; then P is the intersection of maximal ideals of B of height two. Each of these maximal ideals contains $\mathbf{m}$, so $\mathbf{m} \subseteq P$ and P is a minimal prime of $\mathbf{m}B$. On the other hand, if P is a minimal prime of $\mathbf{m}B$, then P is a minimal prime of a principal ideal, so P is of height one. Again by the dimension formula [M2, page 119], B/P is of transcendence degree one over $R/\mathbf{m}$. Since B/P is finitely generated as an $R/\mathbf{m}$-algebra, it follows from Hilbert's Nullstellensatz (cf. [M2, page 33]) that P is not maximal and is an intersection of maximal ideals of B. Thus P is a special prime of B.

(1.3.2) In [Heinzer et al. (1994a)] and [Heinzer et al. (1994b)], we consider only extensions $B = R[x, g/f]$ in which f, g is an $R[x]$-sequence. This restriction allows us to use the explicit description $(fu - g)R[x, u]$ for the kernel of the homomorphism of $R[x]$-algebras from the polynomial ring $R[x, u]$ in the indeterminates x, u onto B determined by sending u to g/f; and it rules out the case $R[x, g/f] = R[x, 1/f]$, a ring of fractions of $R[x]$, of which the spectrum is well understood. Since R and hence $R[x]$ are Cohen-Macaulay rings, to say that f, g is an $R[x]$-sequence is equivalent to saying that $(f, g)R[x]$ is not contained in any height-one prime of $R[x]$ <u>and</u> that $(f, g)R[x] \neq R[x]$. In continuing our study of sfb-extensions of $R[x]$, we now allow as generators fractions g_i/f_i for which $(f_i, g_i)R[x] = R[x]$. We have two reasons for this. First, even if $(f_1, g_1)R[x]$ and $(f_2, g_2)R[x]$ are proper ideals of $R[x]$, there is no description, analogous to the case of one generator, for the kernel of $R[x, u, v] \to R[x, g_1/f_1, g_2/f_2] : u \mapsto g_1/f_1, v \mapsto g_2/f_2$. (We use the computer program MACAULAY to compute such a kernel. See Section 3.) Second, it seems useful to us to be able to say, if $R[x] \subseteq A \subseteq B$ with B an sfb-extension and A a finitely generated extension of $R[x]$, that A is also an sfb-extension. In fact, we do not know that in general; cf. (1.3.3). But if we had required that $(f_i, g_i)R[x] \neq R[x]$ for each i, the statement would have failed even for a cyclic sfb-extension of $R[x] = k[y]_{(y)}[x]$. For example, consider $B = R[x][y/(x(yx + 1))]$ and $A = R[x][1/(yx + 1)]$.

(1.3.3) If $B = R[x, g_1/f_1, \ldots, g_t/f_t]$ is an sfb-extension of $R[x]$ and if the irreducible factors of the f_i are prime elements of $R[x]$, then every finitely generated $R[x]$-subalgebra of B is also an sfb-extension of $R[x]$. In particular this holds if R is a PID.

(1.3.4) Let $B = R[x, g_1/f_1, \ldots, g_t/f_t]$ be an sfb-extension of $R[x]$, let P_0 be a survivor prime and P_1 a transient prime of B with respect to $R[x]$. We show that $P_0 + P_1$ is contained in at most one maximal ideal of B. By considering their respective extensions to the ring of fractions $R[x, 1/(f_1 \dots f_t)]$ of B, we see that the contraction $P_0 \cap R[x]$ has the form $\mathbf{m}R[x]$ for a maximal ideal $\mathbf{m}$ of R, while $P_1 \cap R[x] = Q$ is a height-two maximal ideal. If $\mathbf{m}R[x] \nsubseteq Q$, then P_0 and P_1 are comaximal. If $\mathbf{m}R[x] \subseteq Q$, the maximal ideals of B containing both P_0 and P_1 correspond to the maximal ideals of $(R[x]-Q)^{-1}(B/P_0)$ that contain the image of P_1. But $(R[x]-Q)^{-1}(B/P_0)$ is a birational extension of $((R/\mathbf{m})[x])_Q$, a DVR, so $(R[x]-Q)^{-1}(B/P_0)$ is that DVR. Therefore, a survivor and a transient are contained in at most one (height-two) maximal ideal of B. Moreover, if P_0 is a survivor, if P_1 and P_2 are two transients with the same center on $R[x]$, and if M_1 is a maximal ideal containing P_0+P_1 and M_2 is a maximal ideal containing P_0+P_2, then $M_1 = M_2$. In other words, P_1 and P_2 cannot be contained in different maximal ideals of B that both contain a survivor.

It is noted in [Heinzer et al. (1994b)] that, when R is local and B is a cyclic sfb-extension, the survivor is often distinguished in the partially ordered set $j\text{-}\operatorname{Spec}(B)$ from other special primes, in that it is the only special prime of B which might be contained in two maximal ideals which contain other special primes. Specifically, in Diagram 1.4.1 below, for each of the transients Q_j, the survivor P and Q_j are both contained in one maximal ideal, while P is comaximal with each of the transients H_i.

1.4 Theorem. [Heinzer et al. (1994b), Theorems 2.6, 2.7] *(1) Let R be a local domain of dimension one, and let x be an indeterminate. Then every cyclic sfb-extension B of $R[x]$ has $j\text{-}\operatorname{Spec}(B)$ like Diagram 1.4.1 below, for some choice of positive integers m and n, not both zero.*

(2) Moreover, for every countable partially ordered set U of the form of Diagram 1.4.1, with m and n not both zero, there exists a cyclic sfb-extension B of $R[x]$ with j-spectrum order-isomorphic to U.

(3) If R and B are as in (1) except that R has r maximal ideals, where $r > 1$, then $j\text{-}\operatorname{Spec}(B)$ is a union of $j\text{-}\operatorname{Spec}((R-\mathbf{m})^{-1}B$, over all maximal ideals $\mathbf{m}$ of R—the union is disjoint except that the zeros and the height-one maximal ideals are identified, and no new inclusions are added.

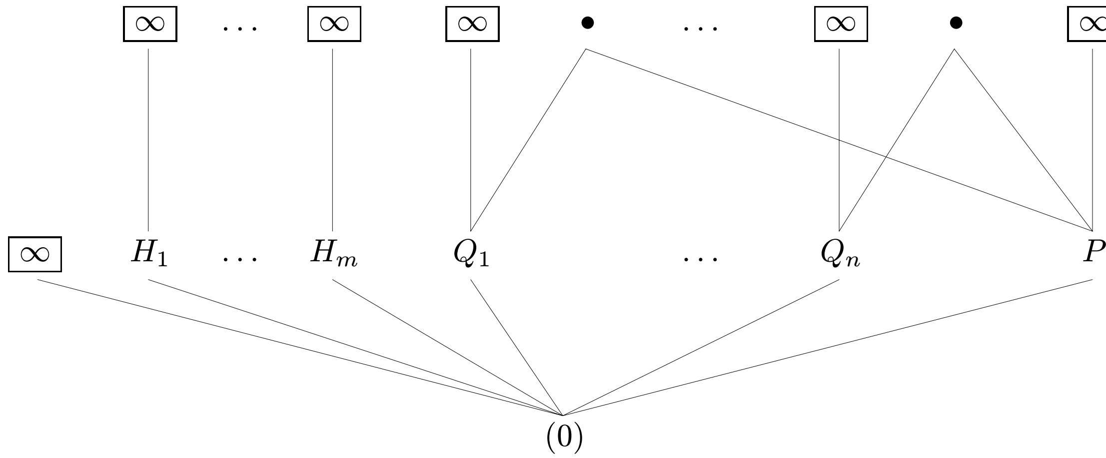


Diagram 1.4.1

The paper [Heinzer et al. (1994b)] contains the following four results on $\operatorname{Spec}(B)$ for cyclic sfb-extensions of $R[x]$:

1.5 Theorem. [Heinzer et al. (1994b), Theorem 3.1] *Suppose R is a semilocal domain of dimension one, x is an indeterminate, and B is a cyclic sfb-extension of $R[x]$. Then for each height-two maximal ideal N of B, there exist infinitely many height-one primes P such that N is the only height-two maximal ideal containing P. That is, (P6) or (P6′) of Theorem 1.2(1) holds for singleton subsets T of the set of all height-two elements of Spec(B).*

1.6 Theorem. [Heinzer et al. (1994b), Theorem 4.1] *Let R be a Henselian local domain of dimension one, and let B be a cyclic sfb-extension of $R[x]$. Then* $\operatorname{Spec}(B)$ *satisfies (P6′) of Theorem 1.2(1). If R is countable, then* $\operatorname{Spec}(B)$ *is uniquely determined by the partially ordered set* $j\text{-}\operatorname{Spec}(B)$.

1.7 Theorem. [Heinzer et al. (1994b), Theorem 4.5] *Let R be a semilocal domain of dimension one, x an indeterminate over R, and $B = R[x, g/f]$ a cyclic sfb-extension of $R[x]$. Suppose there exists a one-dimensional Noetherian domain $D \subset R$ such that R is a localization of D and such that $U = \operatorname{Spec}(D[x, g/f])$ satisfies (W):*

(W) Let S be a finite set of height-one elements of U and T a finite subset of $\mathcal{H}_2(U)$. Then there exist infinitely many height-one elements $w \in U$ such that $\operatorname{G}(s) \cap \operatorname{G}(w) \subseteq T \subseteq \operatorname{G}(w)$, for all $s \in S$.

Then $\operatorname{Spec}(B)$ *satisfies (P6) of 1.2.*

The condition (W) is similar to an axiom described by Roger Wiegand in [R. Wiegand

(1978)] and [R. Wiegand (1986)]. Condition (W) is satisfied by the prime spectrum of a polynomial ring over the integers, $\mathrm{Spec}(\mathbb{Z}[x])$ and by the spectrum of every domain of dimension two that is finitely generated as a k-algebra, if k is a field contained in the algebraic closure of a finite field [R. Wiegand (1986)], [Heinzer et al. (1994b)]. Thus the following corollary is obtained in [Heinzer et al. (1994b)].

1.8 Corollary. [Heinzer et al. (1994b), Corollary 4.6] *Suppose k is a field algebraic over a finite field. Let z and x be indeterminates over k, $\{\mathbf{p}_i \mid 1 \le i \le n\}$ a finite set of nonzero prime ideals of $k[z]$, $S = k[z] - \bigcup_{i=1}^{n} \mathbf{p}_i$, and $R = S^{-1}k[z]$. Let $B = R[x, g/f]$ be a cyclic sfb-extension of $R[x]$. Then $\mathrm{Spec}(B)$ satisfies (P6) of Theorem 1.2 and so is uniquely determined by the isomorphism type of j-$\mathrm{Spec}(B)$.*

2. Further results on cyclic sfb-extensions of $R[x]$

In Theorem 2.2 we extend Corollary 1.8 to the case where R is a one-dimensional semilocal domain which is a ring of fractions of a finitely generated algebra over an arbitrary ground field k.

2.1 Remarks. (1) Corollary 1.8 stated above holds more generally if R is a semilocal ring of fractions of a one-dimensional affine domain over a field k which is algebraic over a finite field and B is any sfb-extension of $R[x]$ (not necessarily cyclic). To prove this more general statement, modify the original proof in [Heinzer et al. (1994b)] by replacing "g/f" by "$g_1/f_1, \dots, g_t/f_t$". (The crucial tool used in the proof of (1.8) is Theorem 2 of [R. Wiegand (1986)] which holds for all two-dimensional affine domains over a field algebraic over a finite field.)

(2) If R is a one-dimensional semilocal domain that is a localization of an affine domain D over a field k, then R is also a localization of a one-dimensional affine domain over a possibly larger field. This can be seen using (14.G) and (14.F) of Matsumura (1980).

The j-spectrum of a cyclic sfb-extension B of $R[x]$ is described in Theorem 1.4. In view of Theorem 1.2(2), in order to determine the entire spectrum of B, we consider, as in [Heinzer et al. (1994b)], the exactly-less-than sets $\mathrm{L_e}(\mathcal{T})$, where $\mathcal{T}$ is a finite subset of $\mathcal{H}_2(\mathrm{Spec}(B))$.

2.2 Theorem. *Let R be a semilocal (Noetherian) domain of dimension one, and let x be an indeterminate over R. Suppose that R is a ring of fractions of an affine domain over a field and B is a cyclic sfb-extension of $R[x]$. Then $\mathrm{Spec}(B)$ satisfies (P6) of Theo-*

rem 1.2. In particular, if R is countable, then $\mathrm{Spec}(B)$ *is uniquely determined by the type of* $j\text{-}\mathrm{Spec}(B)$.

Proof. By (2.1.2) we may assume that R is a localization of a one-dimensional affine domain $D = k[d_1, \ldots, d_m]$ over a field k. Let $B = R[x, g/f]$, where f is outside all the extensions to $R[x]$ of the maximal ideals of R and either (f, g) is an $R[x]$-sequence or $(f, g)R[x] = R[x]$. We first give the proof in the case where k is an algebraically closed field.

By localizing D at a unit of R, we may assume that $f, g \in D[x]$ and $(f, g)D[x]$ is either of height two or all of $D[x]$. It follows that $A = D[x, g/f]$ is of the form a polynomial ring $D[x, y]$ modulo a principal ideal and hence is Cohen-Macaulay.

Let $\mathcal{T}$ be a finite set of height-two maximal ideals of B. The proof of Theorem 1.5 given in [Heinzer et al. (1994b), Theorem 3.1] shows that, for each height-two maximal ideal N of B, there is an element of B which is contained in N and in no other height-two maximal ideal of B; and clearly we can choose that element to be in A. So by multiplication we obtain an element h of A that is contained in all the elements of $\mathcal{T}$ and in no other height-two maximal ideal of B. If we choose from D an element r in the intersection of the maximal ideals of R and outside all the minimal primes of hA, then h, r is an A-sequence: Since h and r are both contained in the elements of $\mathcal{T}$, they do not generate the unit ideal even in B, much less in A; and our choice of r assures that $(h, r)A$ has height greater than 1.

With $A = k[d_1, \ldots, d_m, x, g/f]$ a two-dimensional affine domain over an algebraically closed field k, by Lemma 4 of [R. Wiegand (1978)], there is a nonempty open subset U of $(k)^{m+3}$ such that, for each point $\mathbf{p} = (\alpha, \beta_1, \ldots, \beta_m, \gamma, \delta)$ in U, the radical of the principal ideal in A generated by the element

$$a_{\mathbf{p}} = h + (\alpha + \beta_1 d_1 + \cdots + \beta_m d_m + \gamma x + \delta(g/f))r$$

is a (height-one) prime ideal P. Every height-two maximal ideal N of B contains a maximal ideal of R and hence r, so if N contains P, then it also contains h and hence is one of the elements of $\mathcal{T}$; and clearly the elements of $\mathcal{T}$ contain P, that is, $P \in \mathrm{L_e}(\mathcal{T})$.

We want to show that varying the elements $a_{\mathbf{p}}$ over U will give rise to infinitely many such height-one primes: Select one point $\mathbf{p}$ in U; then by leaving fixed the values of the β_i's, γ and δ, and varying the value of α, we get infinitely many other points in U. Assume by way of contradiction that two of these $\mathbf{p}$ in U, differing only in the first component α, yield elements $a_{\mathbf{p}}$ with the same prime radical P; then the difference between the two $a_{\mathbf{p}}$ is in P, and so $r \in P$. But since P is contained in only finitely many of the height-two maximals that survive in B, it intersects R in (0), the desired contradiction. This completes the proof in the case where k is algebraically closed.

Now let us consider the case where k is not algebraically closed and observe how the proof follows in this case. Again by localizing D at a unit of R we may assume that $f, g \in D[x]$. It follows that $A = D[x, g/f]$ is of the form the polynomial ring $D[x, y]$ modulo a principal ideal and hence is Cohen-Macaulay. Let k^* denote an algebraic closure of k. Let D^* denote a domain quotient of $k^* \otimes_k D$ obtained by factoring out a minimal prime, and let R^* denote the result of localizing D^* at the same multiplicative system at which D was localized to get R, $A = D[x, g/f]$, $A^* = D^*[x, g/f]$, and $B^* = R^*[x, g/f]$.

(Notes: (1) Though k^* is usually an infinite algebraic extension of k, the domain D^* is a one-dimensional Noetherian domain of finite type over k^* and an integral extension of D. Hence there are only finitely many primes of D^* lying over a prime in D — namely the minimal primes of the extension to D^* of the prime in D — so R^* is still semilocal, as well as an integral extension of R. (2) Our hypothesis on (f, g) implies A^* is still a domain.)

Let $\mathcal{T}$ be a finite set of height-two maximal ideals of B, and let $\mathcal{T}^*$ denote the set of all primes of B^* lying over elements of $\mathcal{T}$. As above, we obtain an element h of A that is contained in all the elements of $\mathcal{T}$ and in no other height-two maximal ideals of B and an element $r \in D$ in the intersection of the maximal ideals of R and outside all of the minimal primes of hA.

Let U be a nonempty open subset of $(k^*)^{m+3}$ obtained via an application of Lemma 4 of [R. Wiegand (1978)] where R^*, A^*, B^* and $\mathcal{T}^*$now play the roles previously played by R, A, B and $\mathcal{T}$. We consider the intersections with R of the extensions to R^* of all the (prime) radicals P^* of the principal ideals generated by the elements $a_{\mathbf{p}}$ as $\mathbf{p}$ varies over U. Since only finitely many primes of R^* lie over a single prime in R, there are infinitely many different such intersections, and it is clear that each is a prime contained in every element of $\mathcal{T}$. By the Going Up Theorem and the fact that each P^* is contained only in elements of $\mathcal{T}^*$, each such intersection is contained in no other height-two maximal ideal of B. Therefore, Spec(B) satisfies (P6). □

2.3 Remark. It would be interesting to know whether Theorem 2.2 also holds for noncyclic sfb-extensions. The proof given here does not extend to noncyclic sfb-extensions because it relies on Lemma 3.5 of [Heinzer et al. (1994b)] to deduce that for each height-two maximal ideal N of B there is an element of B which is contained in N and in no other height-two maximal ideal of B. We show in Example 2.4 that this need not hold for B a noncyclic sfb-extension.

2.4 Example. Let k be a field of characteristic zero, let R be the DVR $k[y]_{(y)}$, and consider the sfb-extension $B = R[x, (y/x)^2, (y/x)^3]$ of $R[x]$. Then $B \subseteq R[x, y/x] = C$ and $xC \cap B = P$ is a transient height-one prime of B such that $B/P \cong k[t^2, t^3]$, where t is an indeterminate over k. Since k is of characteristic zero, for any nonzero $a \in k$, the maximal ideal $(t - a)k[t] \cap k[t^2, t^3]$ is not the radical of a principal ideal. Hence if N is a height-two maximal ideal of B for which $N/P \cong (t - a)k[t] \cap k[t^2, t^3]$, then there exists no element of B which is contained in N and in no other height-two maximal ideal of B.

3. Examples of j-spectra of finite birational extensions

This section displays two examples showing that the j-spectra of noncyclic sfb-extensions can be more complicated that the cyclic case. In both cases, k is the field $\mathbb{Z}_{31991}$ of integers modulo the prime 31991.

3.1 Example. An example of a 2-generated sfb-extension of $k[a]_{(a)}[x]$ (where a, x are indeterminates over k) in which two transient j-primes are not comaximal. This shows that if an sfb-estension is not cyclic, a picture different from Diagram 1.4.1 can occur.

The example is the ring

$$B = k[a]_{(a)}[x, x/(x - a), (x - a)^2/x] \ .$$

To compute the kernel P of the $k[a]_{(a)}[x]$-homomorphism from the polynomial ring $k[a]_{(a)}[x, u, v]$ onto B determined by sending u to $x/(x-a)$ and v to $(x-a)^2/x$, we used the computer algebra program MACAULAY, written by David Bayer and Michael Stillman. But since MACAULAY is written for computations with quasihomogeneous polynomial rings, we added a homogenizing indeterminate d; and to be sure that the program found the entire kernel, and not just the piece generated by $(x - a)u - x$ and $xv - (x - a)^2$, we added a indeterminate w to map to $1/x(x-a)$. Thus, we really began with the polynomial ring $C = k[a, x, u, v, d][w]$ (the isolation of w is accomplished by placing it first and specifying the monomial order 1 5) and its ideal I generated by $(x - a)u - x$, $xv - (x - a)^2$ and $x(x-a)w - d^3$. The ideal is described by its generators, with the MACAULAY convention that an integer after a variable is interpreted as an exponent. (The example is a bit unusual in that we did not have to multiply some terms of the numerators and denominators of the generators of our sfb-extension by powers of d to "homogenize" them, and that after we had given weight 1 to each of w, a, x, d, the weights we had to assign to u, v to make the generators into quasihomogeneous polynomials were also 1.) Then we replaced the given generating set for I with a "standard" (i.e., Gröbner) basis, "eliminated" w (i.e, found the intersection J of I with the coefficient ring $k[a,x,u,v,d]$), and found a standard basis for J, of which we then displayed a minimal generating set. Here is the slightly edited "monitor file" of our computer run; the characters we entered are in "typewriter" typeface:

% `ring C`
! characteristic (if not 31991) ?
! number of variables ? `6`
! 6 variables, please ? `waxuvd`
! variable weights (if not all 1) ? `1 1 1 1 1 1`
! monomial order (if not rev. lex.) ? `1 5`
; largest degree of a monomial : 512 217
% `ideal I`
! number of generators ? `3`
! (1,1) ? `(x-a)u-xd`
! (1,2) ? `xv-(x-a)2`
! (1,3) ? `x(x-a)w-d3`
% `std I I`
% `elim I J`
% `std J J`
% `type J`
; au-xu+xd a2-2ax+x2-xv uvd3+ad4-xd4

Setting $d=1$, we obtain a generating set for the desired kernel P: $(x-a)u-x$, $xv-(x-a)^2$, and $uv-(x-a)$. (The first two, of course, are not new, and the third is not surprising but is not in the ideal generated by the first two.) Now since all the height-two and special primes in B contain the extension of the maximal ideal $\mathbf{m}$ of $k[a]_{(a)}$, we can take their preimages in $k[a]_{(a)}[x,u,v]$ and factor out the extension of $\mathbf{m}$ to this ring, i.e., set $a=0$ to get $k[x,u,v]$. The resulting generators of the image of P are $x(u-1)$, $x(v-x)$ and $uv-x$. The image in $k[x,u,v]$ of the preimage in $k[a]_{(a)}[x,u,v]$ of the survivor prime in B is the minimal prime of the image of P that does not contain x and hence is generated by $u-1$ and $v-x$. So the preimage of the survivor in $k[a]_{(a)}[x,u,v]$ is $(a,u-1,v-x)$. The image of the preimage of a transient prime contains x and hence contains either u or v; so the preimages of the transients are (a,x,u) and (a,x,v). These primes are clearly contained in the maximal ideal (a,x,u,v), so their images in B, the transients, are not comaximal. Moreover, the preimages of the second transient and the survivor are both contained in the maximal ideal $(a,x,u-1,v)$. Here is a diagram of the poset j-Spec(B):

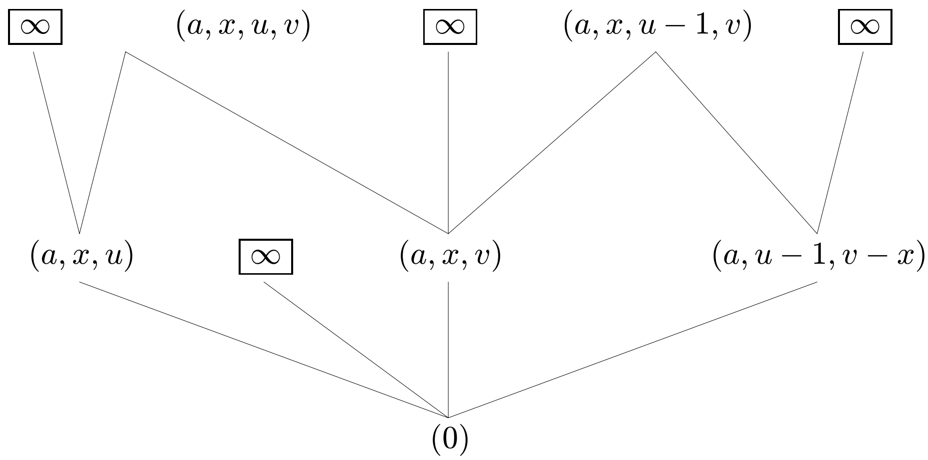


Diagram 3.1.1

3.2 Example. An example of a 2-generated sfb-extension of $k[x-y]_{(x-y)}[x]$, where x, y are indeterminates over the field k, in which two transients are contained in two maximal ideals. This cannot happen in a cyclic sfb-extension.

Let

$$B = k[x-y]_{(x-y)}[x, (x^2+2y^3)/xy, (x^2+2y^3-2xy)/y^2] .$$

By using MACAULAY, we find that the kernel of the map from $k[x-y]_{(x-y)}[x,u,v]$ onto B determined by sending u to $(x^2+2y^3)/xy$ and v to $(x^2+2y^3-2xy)/y^2$ is generated by

$$x^2(v+1-2x) , \qquad x^2(u-1-2x) , \qquad x(u-2-v) ,$$
$$x(u(v-2x)-v+4x) , \qquad v(u^2-2u-v)-2x(u-2)^2 .$$

In this case, factoring out the extension of $\mathbf{m}$ means replacing y with x in the generators, and we find that the preimage of the survivor is $(x-y, u-v-2, 2x-u+1)$ and those of the transients are (x,y,v) and (x,y,u^2-2u-v). Thus, the preimages of the transients are contained in both (x,y,u,v) and $(x,y,u-2,v)$, while the preimages of the survivor and the first transient are contained in $(x,y,u-1,v+1)$. The survivor and the other transient are comaximal.

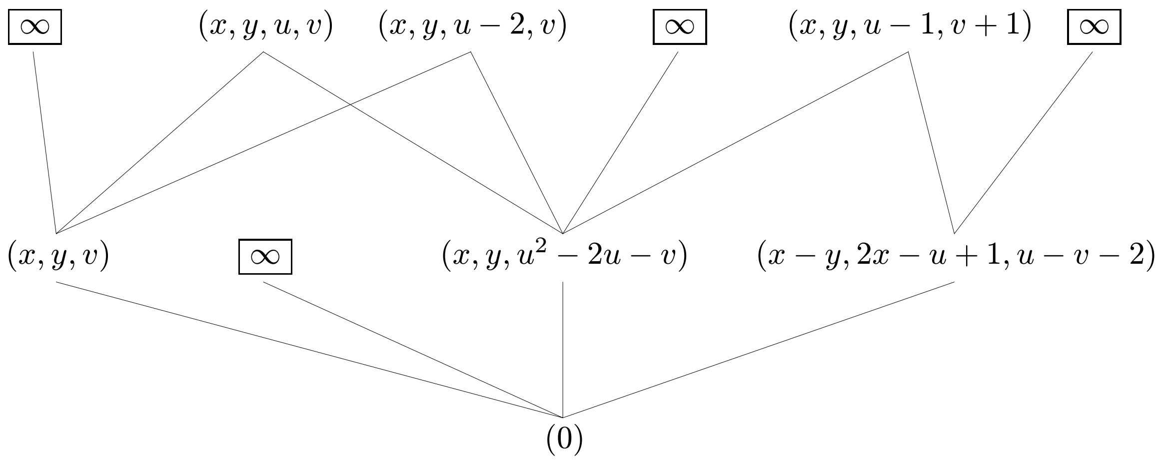


Diagram 3.2.1

4. j-Spectra of birational extensions

Let B be an sfb-extension of $R[x]$ where R is local, so that there is a unique survivor P_0 in $j\text{-}\operatorname{Spec}(B)$. Recall that for P a transient prime of B, $P \cap R[x]$ is a maximal ideal (from the discussion in (1.3.4)). For P, Q transient primes of B, we say $P \sim Q$ if and only if $P \cap R[x] = Q \cap R[x]$; and for maximal ideals M, N of B, we define $M \sim N$ if and only if $M \cap R[x] = N \cap R[x]$. There are finitely many transients in $j\text{-}\operatorname{Spec}(B)$ and therefore a finite number s of equivalence classes, $\mathcal{T}_1, \ldots \mathcal{T}_s$, of transient primes under this relation. Considering just the maximal ideals of B which contain transients, they also fit into s equivalence classes, $\mathcal{M}_1, \ldots, \mathcal{M}_s$, where

$$\mathcal{M}_i = \{\ M \in \mathcal{H}_2(j\text{-}\operatorname{Spec}(B)) : M \cap R[x] = P \cap R[x],\ P \in \mathcal{T}_i\ \}\ .$$

For each i, the only maximal ideals of B containing elements of $\mathcal{T}_i$ are elements of $\mathcal{M}_i$. If two transients P and Q of B are not equivalent, they must be comaximal in B. By (1.3.4) there is at most one connection of the survivor prime P_0 to each $\mathcal{T}_i$ via $\mathcal{M}_i$; that is, if there is a maximal ideal M containing P_0 and some $P \in \mathcal{T}_i$, then $M \in \mathcal{M}_i$ and for each $Q \in \mathcal{T}_i$ and $N \in \mathcal{M}_i$ with $P_0 + Q \subseteq N$, we have $N = M$. Consider the poset diagram of $j\text{-}\operatorname{Spec}(B)$; there is at most one line upward joining P_0 to each set $\mathcal{M}_i$ and thus at most one connection from P_0 to each set $\mathcal{T}_i$. In other words, suppose we remove from the poset diagram of $j\text{-}\operatorname{Spec}(B)$ the prime (0) and regard the rest as an undirected graph. Then P_0 is adjacent to at most one element of each $\mathcal{M}_i$; thus a path through P_0 cannot be closed (without passing back through P_0). Thus $U = j\text{-}\operatorname{Spec}(B)$ satisfies

(P7) Let V be the graph resulting from taking the poset diagram of U and removing the unique minimal element. Among the special elements there is one, u_0, for which there is no closed path containing u_0 (without repeated edges) in V.

In Theorem 4.1, we describe a systematic procedure for constructing an sfb-extension with j-spectrum a given countable partially ordered set U that is otherwise reasonable (dimension two, unique minimal element, infinitely many height-one maximals, finitely many height-two points that are greater than two given height-one points, etc.) and satisfies a strong version of (P7): the special element u_0 has degree one in the graph V. In (4.4), we construct an example in which u_0 has degree two. We believe that this construction can be generalized to yield an sfb-extension with j-spectrum isomorphic to a given feasible U satisfying (P7). The local ring R used in our constructions is the DVR $k[y]_{(y)}$ where k is a countable algebraically closed field and y is an indeterminate over k.

4.1 Theorem. *Let U be a countable two-dimensional partially ordered set with a unique minimal element and precisely $n+1$ height-one elements $u_0, u_1, \ldots, u_n$, which are not maximal. Assume that*

(1) *The set of maximal height-one elements of U is infinite,*
(2) *For each i, $0 \le i \le n$, $\mathrm{G}(u_i)$ is infinite,*
(3) $|G(u_0) \cap (\cup_{i>0} G(u_i))| \le 1$,
(4) *For every $i \ne j$, $1 \le i, j \le n$, $\mathrm{G}(u_i) \cap \mathrm{G}(u_j)$ is finite,*

Then, for k a countable algebraically closed field, y, x indeterminates over k, and $R = k[y]_{(y)}$, there exists an sfb-extension B of $R[x]$ for which $U \cong j\text{-}\mathrm{Spec}(B)$. Furthermore the order-isomorphism $\varphi : U \to j\text{-}\mathrm{Spec}(B)$ can be chosen so that $\varphi(u_i) \cap R[x] = (x, y)$, for each $i > 0$, and so that $\varphi(u_0)$ is the survivor prime of the sfb-extension B of $R[x]$.

We first prove a version for which the partially ordered set is closer to a tree.

4.2 Lemma. *Let U be a countable two-dimensional partially ordered set with a unique minimal element and precisely $n+1$ height-one elements $u_0, u_1, \ldots, u_n$, which are not maximal. Assume that*

(1) *The set of maximal height-one elements of U is infinite,*
(2) *For each i, $0 \le i \le n$, $\mathrm{G}(u_i)$ is infinite,*
(3) *For the pair u_0, u_1, $|G(u_0) \cap G(u_1)| \le 1$, while for every $i \ne j, \{i, j\} \ne \{0, 1\}$, $0 \le i, j \le n$, $\mathrm{G}(u_i) \cap \mathrm{G}(u_j) = \emptyset$.*

Then, for k a countable algebraically closed field, y, x indeterminates over k, and $R = k[y]_{(y)}$, there exists an sfb-extension A of $R[x]$ so that $U \cong j\text{-}\mathrm{Spec}(A)$. Furthermore the order-isomorphism $\varphi : U \to j\text{-}\mathrm{Spec}(A)$ can be chosen so that $\varphi(u_i) \cap R[x] = (x, y)$, for *each $i > 0$, and so that $\varphi(u_0)$ is the survivor prime of the sfb-extension A of $R[x]$. Also if M is a maximal ideal of A, then $A/M \cong k$.*

Proof of Lemma 4.2. Choose distinct elements $a_1, \ldots, a_n$ of k. If $|G(u_0) \cap G(u_1)| = 1$, take $a_1 = 0$, while for $|G(u_0) \cap G(u_1)| = 0$, take all $a_i \neq 0$. For $1 \leq i \leq n$, let $V_i \supset R[x][y/x]_{(x,(y/x)-a_i)}$ be the valuation domain of the ord valuation v_i (determined by the powers of the ideal $(x, (y/x) - a_i)$); let $D = R[x, 1/x]$. The center of V_i on the regular ring $R[x, y/x]$ is $(x, (y/x) - a_i)R[x, y/x]$ and the residue field of V_i is a simple transcendental extension of k with the image of $((y/x) - a_i)/x$ as a field generator for this residue field over k. Set $A = D \cap V_1 \cdots \cap V_n$.

Let $w = (\prod_{i=1}^{n}((y/x) - a_i))/x$; we show that $A = R[x, y/x, w]$. It is easily seen that $R[x, y/x, w] \subseteq A$, so we prove the reverse inclusion. Since $R[x, y/x]$ is regular, the extension $R[x, y/x][w]$ is locally a complete intersection and hence Cohen-Macaulay. Thus to show $A \subseteq R[x, y/x, w]$, it suffices to show for each height-one prime Q of $R[x, y/x, w]$ that $A \subseteq R[x, y/x, w]_Q$. If $x \notin Q$, then $R[x, y/x, w]_Q$ contains D and hence A. Suppose $x \in Q$. Then $\prod_{i=1}^{n}((y/x) - a_i) = xw \in Q$, so one factor $(y/x) - a_i$ is in Q and the rest are not because the pairwise differences of the factors are units. Thus, one element of $R[x, y/x, w]_Q$ is

$$\frac{w}{\prod_{j \neq i}((y/x) - a_j)} = \frac{(y/x) - a_i}{x} \ .$$

It follows that $R[x, y/x, w]_Q$ contains $R[x, y/x]_{(x,(y/x)-a_i)}[((y/x) - a_i)/x]$, of which V_i is a localization, i.e., $R[x, y/x, w]_Q = V_i$. This completes the proof that $A = R[x, y/x, w]$. In particular, A is an sfb-extension of $R[x]$.

Setting $P_0 = yD \cap A$, we have $x \notin P_0$, and P_0 is the survivor prime with respect to the extension $R[x] \subset A \subset R[x, 1/x] = D$, i.e., we have $P_0 = P_0 D \cap A$, a height-one prime of A. For $i \geq 1$, let P_i be the center of V_i on A; then the P_i are height-one primes of A, and $A_{P_i} = V_i$. Moreover the P_i have the property that $P_i \cap R[x] = (x, y)R[x]$.

By (1.3.1), $P_0, P_1, \ldots, P_n$ are the special elements of A. Define φ on the u_i by setting $\varphi(u_i) = P_i$ for $i \geq 0$. Then $\mathrm{G}(\varphi(u_i))$ is infinite. Note that j-Spec(A) has a unique minimal element and infinitely many height-one maximal elements (since $R[x, 1/x]$ does). Thus to see that φ can be extended to an order-isomorphism on all of U, it suffices to show that (3) holds with the u_i replaced by P_i. This is done in the following claim:

Claim. *$P_0 + P_1 \neq A$ iff $a_1 = 0$. If $a_1 = 0$, then $P_0 + P_1$ is contained in a unique maximal ideal of A. All pairs P_i, P_j with $i \neq j, \{i, j\} \neq \{0, 1\}$ are comaximal.*

Proof of claim. We have P_0 contains y/x, P_i contains $y/x - a_i$ for $i > 0$, and the a_i are distinct. Thus the last sentence of the claim is clear, and if $a_1 \neq 0$, then $P_0 + P_1 = A$.

If we take $a_1 = 0$ and $n = 1$, so that $A = D \cap V_1$, then $A = R[x, y/x^2]$ by the argument above with $w = y/x^2$. It follows that $R[x, y/x^2] = D \cap V_1 = A$, $P_0 = (y/x^2)A$ and $P_1 = xA$. Thus $P_0 + P_1 = (x, y/x^2)A$, a maximal ideal of A with residue field k.

In the situation where $A = D \cap V_1 \cap \cdots \cap V_n$, $n > 1$, and $a_1 = 0$, let $A_1 = D \cap V_1$. Then $A_1 = R[x, y/x^2]$, $y/x \in P_0 \cap P_1$, and $(y/x) - a_i \in P_i$ for $i > 1$. Hence $P_0 \cap P_1$ and P_i are comaximal for each $i > 1$. If M is a maximal ideal of A containing $P_0 + P_1$ and $S = A - M$, then S meets each P_i for $i > 1$, so $S^{-1}A = S^{-1}A_1$. It follows that $P_0 + P_1$ is contained in a unique maximal ideal of A which is $(x, y/x^2)A_1 \cap A$. This completes the proof of the claim.

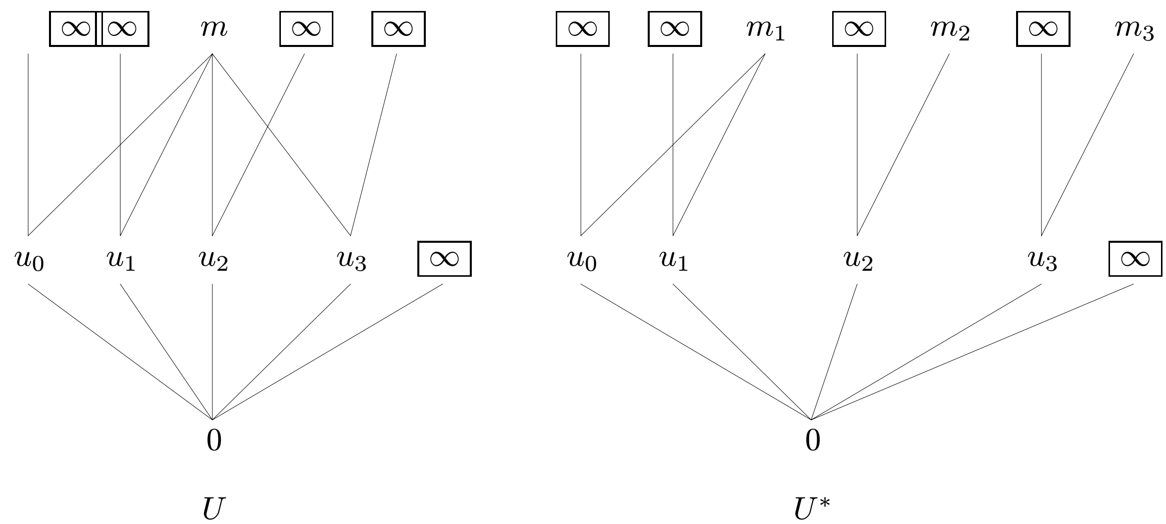
As noted above, for $i > 0$, $P_i \cap R[x] = (y, x)R[x]$ and P_0 is the survivor prime of the sfb-extension A. If M is a height-two maximal ideal of A, then A/M is an algebraic extension of k. Hence if k is algebraically closed we have $A/M \cong k$, for each height-two maximal ideal M of A. Thus we have completed the proof of the lemma. $\square$

Proof of Theorem 4.1. We adapt the proof of [S. Wiegand (1983), Theorem 1]: Suppose

$$H = \bigcup \{\mathrm{G}(u_i) \cap \mathrm{G}(u_j) : i, j = 1, \ldots, n,\ i \neq j\}$$

has t elements; use induction on t. If $t = 0$, then the theorem holds by the lemma (possibly by renumbering). Thus we may assume that $t > 0$ and that the theorem holds for every smaller number.

Let $m \in H$. For convenience rearrange the u_i's for $i \geq 1$ so that $u_i < m$ if $1 \leq i \leq r$ ($r \geq 2$), and $u_i \not< m$ if $r < i \leq n$. Now let $W = \{m_1, m_2, \ldots, m_r\}$ be a set of r elements disjoint from U, and let U^* be the disjoint union $(U - \{m\}) \cup W$, with the additional relations: $u_i < m_i$ in U^* for every $1 \leq i \leq r$; $u_0 < m_1$ in $U^* \iff u_0 < m$ in U; and no other new relations. As an illustration of the process in the case $u_0 < m$, see the diagram below:



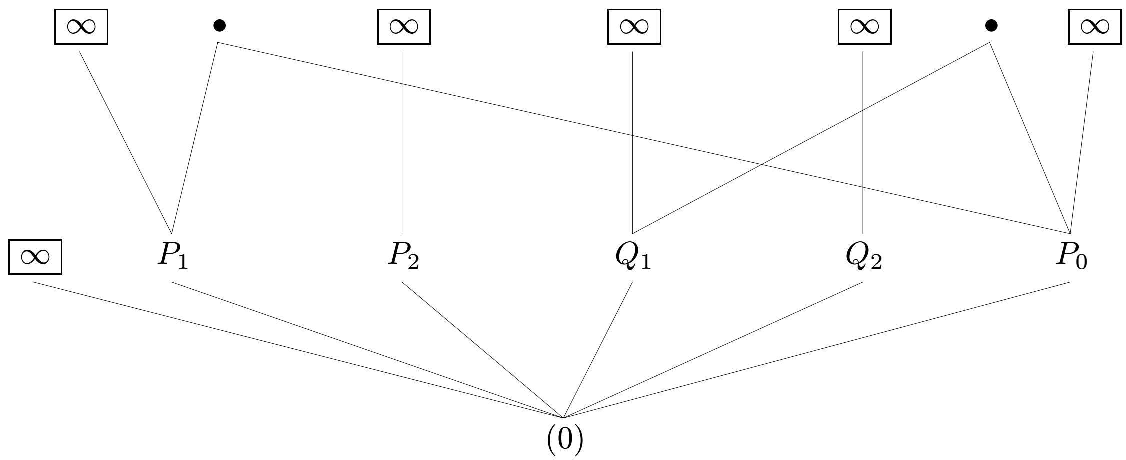
(If $u_0 \not< m$, delete the lines joining u_0 to m and u_0 to m_1.)

By the inductive hypothesis, there is an sfb-extension B^* of $R[x]$ and an order-isomorphism $\varphi : U^* \to j\text{-}\operatorname{Spec}(B^*)$ so that $\varphi(u_i) \cap R[x] = (x, y)R[x]$ for each $i > 0$ and $\varphi(u_0)$ is the survivor in B^*.

Now for each i, $M_i = \varphi(m_i)$ is a maximal ideal of B^*. Since k is algebraically closed, $B^*/\varphi(m_i) \cong k$ for each i, $1 \le i \le t$; let $\varepsilon_i : B^* \to k$ be the natural projection with kernel M_i. Let $B = \{b \in B^* : \varepsilon_1(b) = \cdots = \varepsilon_t(b)\}$. Then by [S. Wiegand (1983), Proposition 2], the conductor of B^* in B is the maximal ideal $M = \bigcap_{i=1}^{r} M_i$ of B, and the spectrum of B differs from the spectrum of B^* only in that M replaces $M_1, \ldots, M_t$ which are the primes of B^* lying over M. Furthermore, B^* is a finitely generated B-module; therefore by the Artin-Tate Lemma [Kunz (1985), Lemma 3.3, p. 16] and (1.3.3), B is an sfb-extension of $R[x]$. (Note that in order to insure that $R[x] \subseteq B$, the M_i's must intersect to the *same* maximal ideal of $R[x]$.) It is clear that $\varphi : U^* \to j\text{-}\operatorname{Spec}(B^*)$ can be adjusted to an order-isomorphism $\varphi : U \to j\text{-}\operatorname{Spec}(B)$ which has the desired properties regarding maximal elements. □

4.3 Remark. The proofs of Theorem 4.1 and Lemma 4.2 also apply under the assumption that R is a countable excellent DVR with algebraically closed residue field.

4.4 Generalization. The construction in Theorem 4.1 and Lemma 4.2 can achieve only those j-spectra with the survivor in at most one maximal ideal containing another special prime. In order to produce the j-spectrum shown below, we outline a modification of this construction.



Let $R = k[y]_{(y)}$, where k is a countable algebraically closed field, and let $D = R[x, 1/x(x-1)]$. Set $z = y/(x(x-1))$ and let $C = R[x,z]$. Then C is a regular ring. For a, b nonzero elements of k, consider the following valuations defined by ideals of C:

v_1 the ord valuation defined by powers of $(x,z)C$

v_2 the ord valuation defined by powers of $(x, z-a)C$

w_1 the ord valuation defined by powers of $(x-1, z)C$ and

w_2 the ord valuation defined by powers of $(x-1, z-b)C$;

let V_1, V_2, W_1, W_2 be the associated valuation rings.

Let $A = D \cap V_1 \cap V_2 \cap W_1 \cap W_2$ and let $A_0 = C[(z(z-a))/x, (z(z-b))/(x-1)]$. We show that $A = A_0$. Let us first note that A_0 is Cohen-Macaulay. To see this, observe that at each maximal ideal Q of C, $(C-Q)^{-1}A_0$ is a domain generated by one element over the regular local ring C_Q and hence the quotient of a regular ring modulo a principal ideal. Thus A_0 is locally a complete intersection and hence is Cohen-Macaulay. Since $A_0 \subseteq A$, to show $A_0 = A$ it suffices to show that $A \subseteq (A_0)_P$ for each height-one prime P of A_0. If $x(x-1) \notin P$, then $A \subseteq D \subseteq (A_0)_P$, so assume that $x(x-1) \in P$. Suppose $x \in P$. Then z or $z-a$ is in P; say, $z \in P$. Set $Q = P \cap C$; then since $z-a$ and $x-1$ are units in C_Q, we have $(C-Q)^{-1}A_0 = C_Q[z/x]$, so

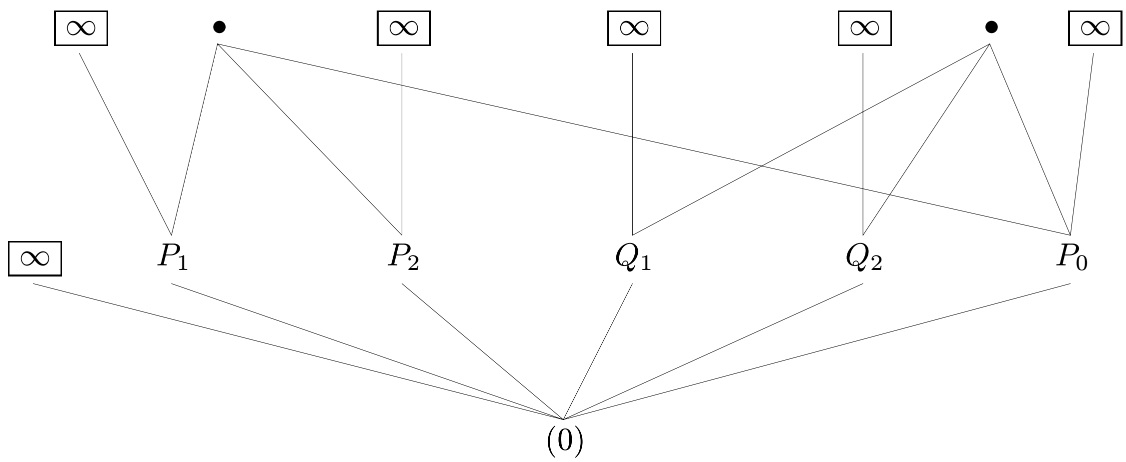
$$PC_Q[z/x] = xC_Q[z/x] \qquad \text{and} \qquad (A_0)_P = (C_Q[z/x])_{xC_Q[z/x]} = V_1.$$

Similarly, if $x, z-a \in P$, then $(A_0)_P = V_2$; if $x-1, z \in P$, then $(A_0)_P = W_1$; and if $x-1, z-b \in P$, then $(A_0)_P = W_2$. It follows that $A_0 = A$ is an sfb-extension of $k[y]_{(y)}[x]$ with j-Spec(A) as in the picture above.

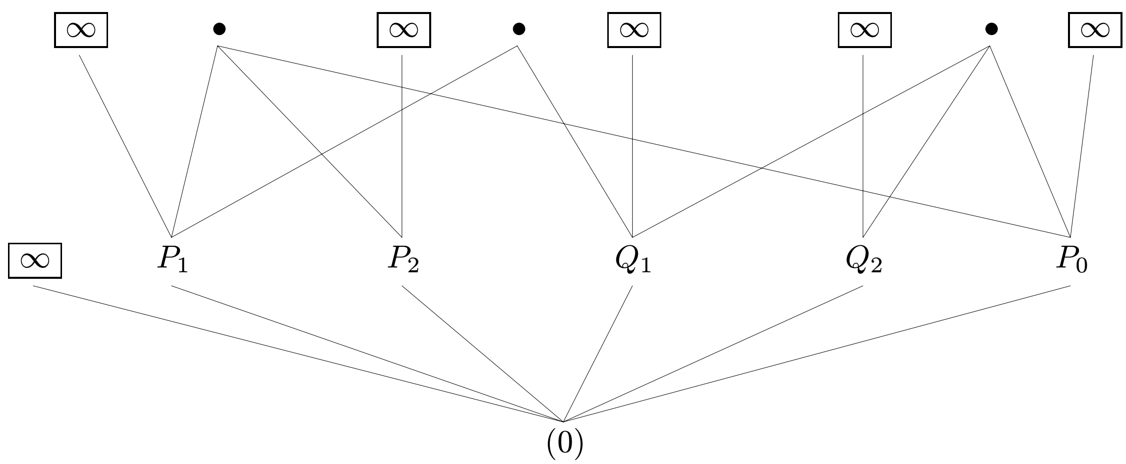
By an extension of this argument we can build examples that include the examples in [Heinzer et al. (1994b)] of cyclic sfb-extensions.

Using the gluing techniques from [S. Wiegand (1983)] as in the proof of Theorem 4.1 above, any maximal ideals above $P_1 \cap P_2$ can be glued and any maximal ideals above $Q_1 \cap Q_2$ can be glued together. However the technique does not permit gluing of pairs of maximals one above P_i and one above Q_j since the $P_i \cap R[x]$ and $Q_j \cap R[x]$ are comaximal.

Thus we can get this picture:



But we can not get this picture:



That is, for the examples that arise from this construction, if the transients T_1 and T_2 are contained in distinct maximals containing the survivor, then T_1 and T_2 are comaximal in the glued ring.

References

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