MIXED POLYNOMIAL/POWER SERIES RINGS AND RELATIONS AMONG THEIR SPECTRA

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OVERVIEW 1

At first glance the rings

$$B := k[[y]]\,[x] \qquad \text{and} \qquad C := k[x]\,[[y]]$$

look similar. One has

$$B = k[[y]]\,[x] \hookrightarrow k[x]\,[[y]] = C,$$

but this is a strict inclusion. For example, $1 - xy$ is a nonunit of B, and

$$\frac{1}{1-xy} = \sum_{i=0}^{\infty} x^n y^n \in C,$$

so $1 - xy$ is a unit of C. Indeed, the rings B and C are not isomorphic: the intersection of the maximal ideals of B is (0), while y is in every maximal ideal of C.

OVERVIEW 2

Consider the mixed polynomial/power series rings

$$A := k[x,y] \hookrightarrow B := k[[y]]\,[x] \hookrightarrow C := k[x]\,[[y]] \hookrightarrow D := k[[x,y]],$$

where k is a field. The inclusion maps here are all flat homomorphisms. The prime ideal structure of these rings is well understood. The above inclusions induce maps

$$\operatorname{Spec} A \quad \leftarrow \quad \operatorname{Spec} B \quad \leftarrow \quad \operatorname{Spec} C \quad \leftarrow \quad \operatorname{Spec} D.$$

We are interested in describing these Spec maps.

OVERVIEW 3

Consider

$$k[x]\,[[y]] = C \hookrightarrow C[1/x] \hookrightarrow k[x, 1/x]\,[[y]] := E,$$

At first glance, it appears that E is a localization of C, but it is not.

There are elements in E that are not in the fraction field of C.

However, E is obtained from C by the localization $C[1/x]$ followed by the (y)- adic completion of $C[1/x]$. Thus E is flat over C.

The map $C \hookrightarrow E$ induces $\operatorname{Spec} C \leftarrow \operatorname{Spec} E$, and again we are interested in describing this Spec map.

OVERVIEW 4

Also consider

$$C \hookrightarrow C_1 := k[x]\,[[\frac{y}{x}]] \hookrightarrow \cdots \hookrightarrow C_n := k[x]\,[[\frac{y}{x^n}]] \hookrightarrow \cdots \hookrightarrow E.$$

The maps $C \hookrightarrow C_n$ and $C_i \hookrightarrow C_n$ for $i < n$ are not flat, but $C_n \hookrightarrow E = k[x, 1/x]\,[[y]]$ is the localization $C_n[1/x]$ followed by the (y)-adic completion of $C_n[1/x]$. Thus $C_n \hookrightarrow E$ is flat. These inclusion maps induce maps

$$\operatorname{Spec} C \leftarrow \operatorname{Spec} C_1 \leftarrow \cdots \leftarrow \operatorname{Spec} C_n \leftarrow \cdots \leftarrow \operatorname{Spec} E.$$

We are interested in describing these Spec maps.

GENERIC FIBER RINGS

Let $R \hookrightarrow S$ be an injective homomorphism of commutative rings with R an integral domain. The **generic fiber ring** of the map $R \hookrightarrow S$ is the localization $(R \setminus \{0\})^{-1}S$ of S. With

$A := k[x,y] \hookrightarrow B := k[[y]]\,[x] \hookrightarrow C := k[x]\,[[y]] \hookrightarrow D := k[[x,y]]$,

the generic fiber ring of $A \hookrightarrow R$ is one-dim. for $R \in \{B, C, D\}$, while the generic fiber ring of $R \hookrightarrow S$ is zero-dim for $R \subseteq S$ in $\{B, C, D\}$.

TRIVIAL GENERIC FIBER EXTENSIONS

Let R be a subring of an integral domain S.

Definition. $R \hookrightarrow S$ is a **trivial generic fiber** extension or a **TGF** extension if

$$(0) \neq P \in \operatorname{Spec} S \implies P \cap R \neq (0).$$

One obtains a TGF extension S of R by considering

$$R \hookrightarrow T \to T/P := S,$$

where T is an extension ring of R and $P \in \operatorname{Spec} T$ is maximal with respect to $P \cap R = (0)$.

Thus the generic fiber ring of $R \hookrightarrow T$ is relevant to constructing TGF extensions S of R.

A TGF EXTENSION

Let x and y be indeterminates over a field k. Then

$$R := k[[x, y]] \hookrightarrow S := k[[x]]\,[[\frac{y}{x}]] \quad \text{is TGF.}$$

Proof. It suffices to show $P \cap R \neq (0)$ for each $P \in \operatorname{Spec} S$ with $\operatorname{ht} P = 1$. This is clear if $x \in P$, while if $x \notin P$, then $k[[x]] \cap P = (0)$, so $k[[x]] \hookrightarrow R/(P \cap R) \hookrightarrow S/P$. Now S/P is one-dim local with residue field k. Hence by Cohen's Theorem 8, S/P is finite over $k[[x]]$. Thus $\dim R/(P \cap R) = 1$, so $P \cap R \neq (0)$.

Cohen's Theorem 8

Theorem (Classical) Let I be an ideal of a ring R and let M be an R-module. Assume that R is complete in the I-adic topology and $\bigcap_{n=1}^{\infty} I^n M = (0)$. If M/I is generated over R/I by elements $\overline{w}_1, \ldots, \overline{w}_s$ and w_i is a preimage in M of $\overline{w}_i$ for $1 \leq i \leq s$, then M is generated over R by $w_1, \ldots, w_s$.

This is useful for proving that with

$$B := k[[y]]\,[x] \hookrightarrow C := k[x]\,[[y]] \hookrightarrow D := k[[x, y]],$$

then $R \hookrightarrow S$ is TGF for $R \subseteq S$ in $\{B, C, D\}$.

TGF EXTENSIONS

PROP. 1. Let $R \hookrightarrow S$ and $S \hookrightarrow T$ be injective maps, where R, S and T are integral domains.

(1) If $R \hookrightarrow S$ and $S \hookrightarrow T$ are TGF extensions, then so is $R \hookrightarrow T$.

Equivalently if $R \hookrightarrow T$ is not TGF, then at least one of the extensions $R \hookrightarrow S$ or $S \hookrightarrow T$ is not TGF.

(2) If $R \hookrightarrow T$ is TGF, then $S \hookrightarrow T$ is TGF.

(3) If the map $\operatorname{Spec} T \to \operatorname{Spec} S$ is surjective, then $R \hookrightarrow T$ is TGF implies $R \hookrightarrow S$ is TGF.

A NON-TGF EXTENSION

PROP. 2. $R = k[[x]][y, z] \hookrightarrow k[y, z][[x]] = S$ is not TGF.

Proof. There exists $\sigma \in k[y][[x]]$ that is transcendental over $k[[x]][y]$. Let $\mathbf{q} = (z - \sigma x)k[y, z][[x]]$.

Define $\pi : k[y, z][[x]] \to k[y, z][[x]]/\mathbf{q} \cong k[y][[x]]$. Thus $\pi(z) = \sigma x$. If $h \in \mathbf{q} \cap (k[[x]][y, z])$, then $\exists\ s, t \in \mathbb{N}$ so that

$h = \sum_{i=0}^{s} \sum_{j=0}^{t} (\sum_{\ell \in \mathbb{N}} a_{ij\ell} x^{\ell}) y^i z^j$, where $a_{ij\ell} \in k$.

Hence $0 = \pi(h) = \sum_{i=0}^{s} \sum_{j=0}^{t} (\sum_{\ell \in \mathbb{N}} a_{ij\ell} x^{\ell}) y^i (\sigma x)^j$.

Since σ is transcendental over $k[[x]][y]$, each $a_{ij\ell} = 0$.

Therefore $\mathbf{q} \cap (k[[x]][y, z]) = (0)$, and $R \hookrightarrow S$ is not TGF.

POWER SERIES RINGS

Lemma. Let $R[[y]]$ denote the power series ring in the variable y over the commutative ring R. Then

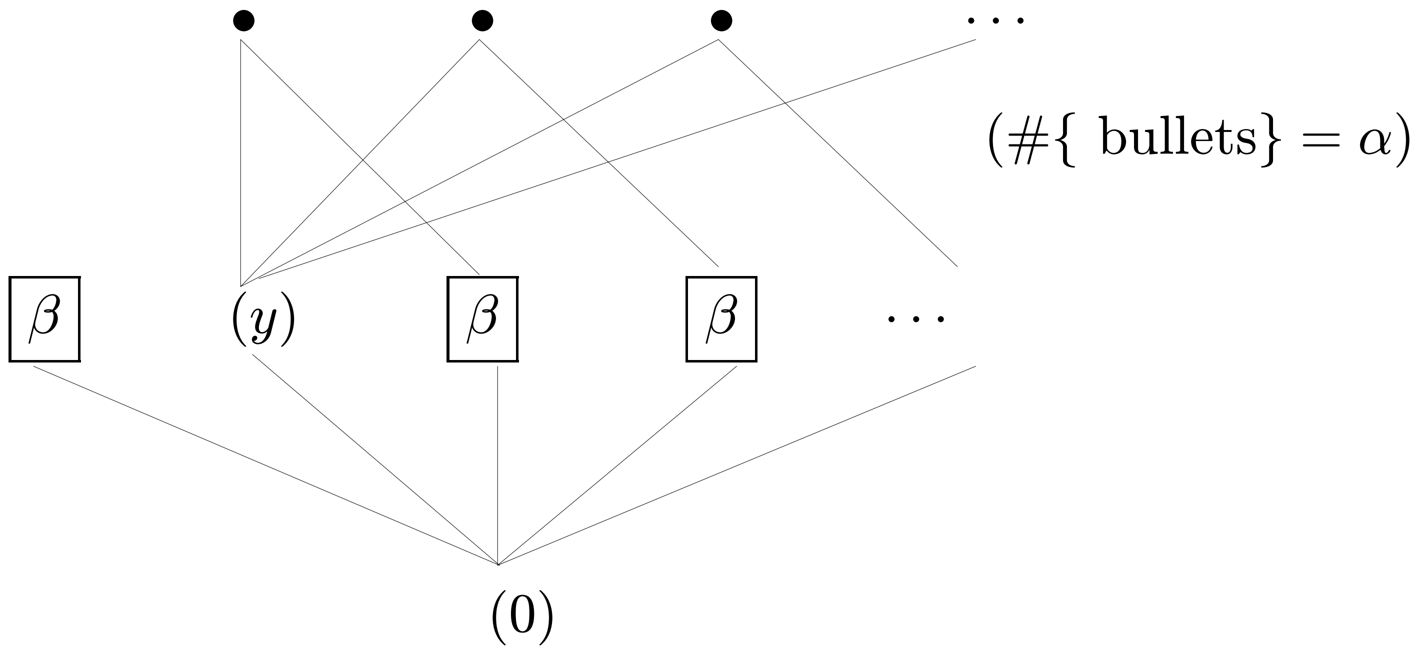
(1) Each maximal ideal of $R[[y]]$ has the form $(\mathbf{m}, y)R[[y]]$, where $\mathbf{m}$ is a maximal ideal of R. Thus y is in every maximal ideal of $R[[y]]$.

(2) If R is Noetherian with $\dim R[[y]] = n$ and $x_1, \dots, x_m$ are indeterminates over $R[[y]]$, then y is in every maximal ideal of height $n + m$ of the polynomial ring $R[[y]]\,[x_1, \dots, x_m]$.

Lemma. Let R be an n-dim. Noetherian domain, let y be an indeterminate over R, and let $\mathbf{q}$ be a prime ideal of height n in the power series ring $R[[y]]$. If $y \notin \mathbf{q}$, then $\mathbf{q}$ is contained in a unique maximal ideal of $R[[y]]$.

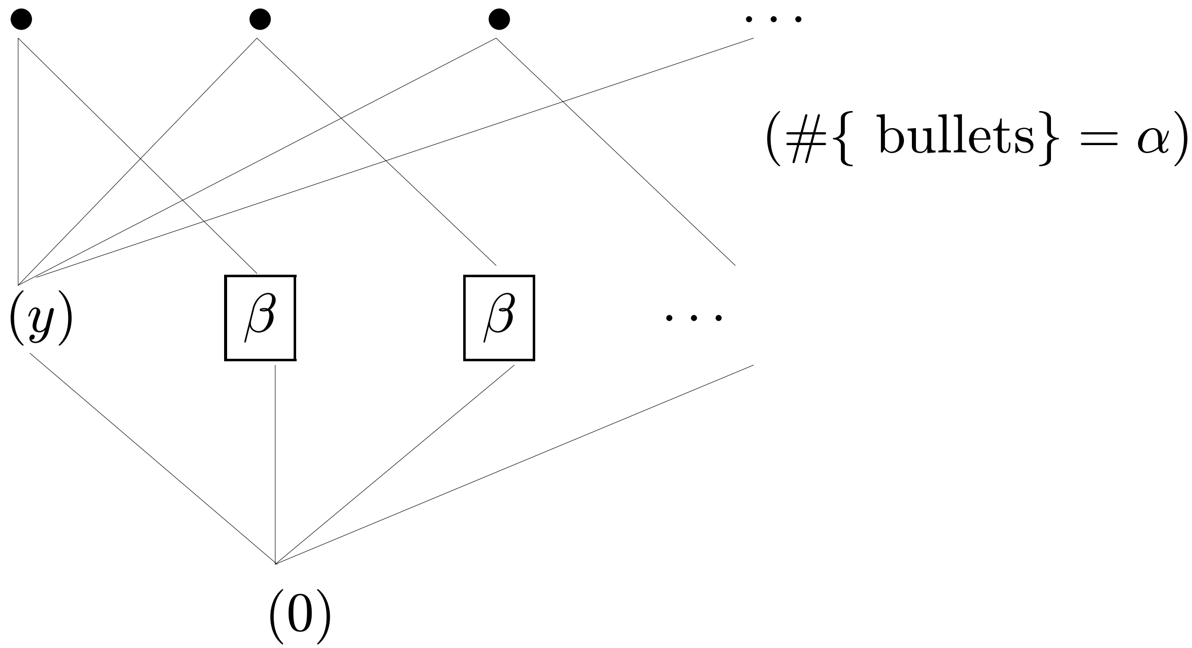
Proof. Let $S := R[[y]]/\mathbf{q}$. The assertion is clear if $\mathbf{q}$ is maximal. Otherwise, $\dim S = 1$. Moreover, S is complete in its yS-adic topology and every maximal ideal of S is a minimal prime of the principal ideal yS. Hence S is a complete semilocal ring. Since S is also an integral domain, it is local by [Mat., Theorem 8.15]. Thus $\mathbf{q}$ is contained in a unique maximal ideal of $R[[y]]$.

The picture of $\operatorname{Spec} k[[y]]\,[x]$ is shown below:



In the diagram, β is the cardinality of $k[[y]]$, and α is the cardinality of the set of maximal ideals of $k[x]$; the boxed β means there are cardinality β height-one primes in that position with respect to the partial ordering.

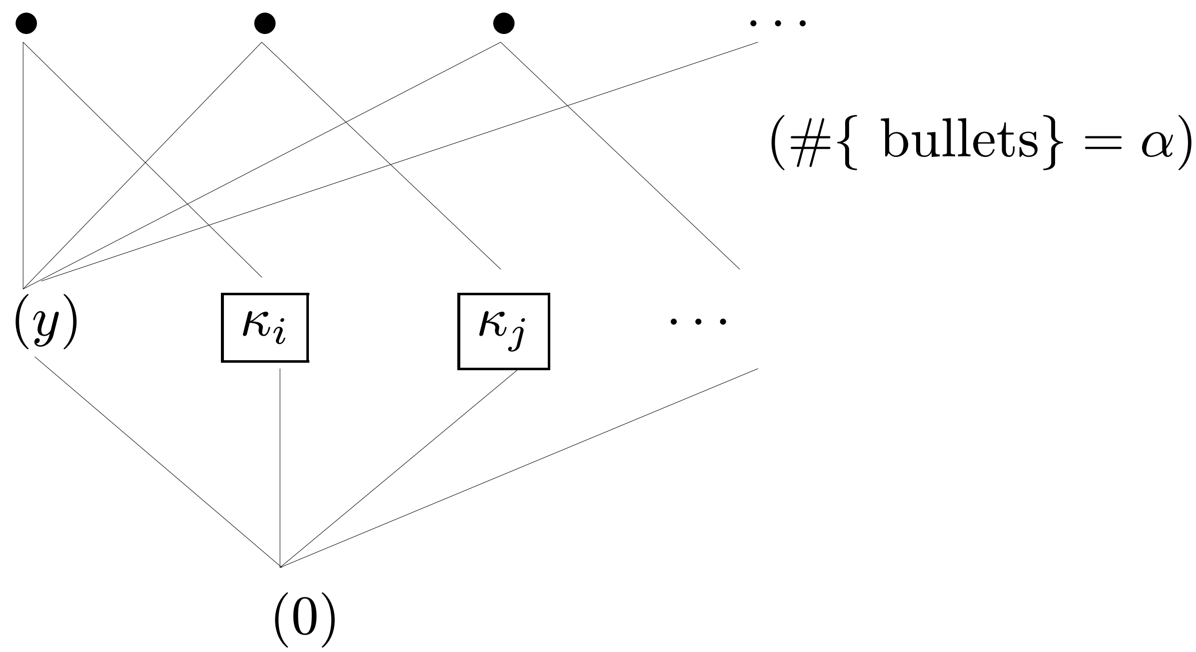
The picture of $\operatorname{Spec} k[x]\,[[y]]$ is shown below:



Here α is the cardinality of the set of maximal ideals of $k[x]$, and

β is the uncountable cardinal equal to the cardinality of $k[[y]]$.

The picture of $\operatorname{Spec} R[[y]]$ for a one-dim Noetherian domain R:



$\operatorname{Spec} R[[y]]$

Here κ_i and κ_j are uncountable cardinals.

ISOMORPHIC SPECTRA

REMARK. Let F be a field that is algebraic over a finite field.

Roger Wiegand proved that as partially ordered sets or

topological spaces

$$\operatorname{Spec} \mathbb{Q}[x, y] \ncong \operatorname{Spec} F[x, y] \cong \operatorname{Spec} \mathbb{Z}[y].$$

The spectra of power series extensions in y behave differently:

We have

$$\operatorname{Spec} \mathbb{Z}[[y]] \cong \operatorname{Spec} \mathbb{Q}[x]\,[[y]] \cong \operatorname{Spec} F[x]\,[[y]].$$

Higher dimensional mixed power series/polynomial rings

We display several extensions involving three variables:

$$k[x,y,z] \overset{\alpha}{\hookrightarrow} k[[z]]\,[x,y] \overset{\beta}{\hookrightarrow} k[x]\,[[z]]\,[y] \overset{\gamma}{\hookrightarrow} k[x,y]\,[[z]] \overset{\delta}{\hookrightarrow} k[x]\,[[y,z]],$$

$$k[[z]]\,[x,y] \overset{\epsilon}{\hookrightarrow} k[[y,z]]\,[x] \overset{\zeta}{\hookrightarrow} k[x]\,[[y,z]] \overset{\eta}{\hookrightarrow} k[[x,y,z]],$$

We have been able to show most of these extensions are not TGF.

PROP. 3. $k[[z]]\,[x,y] \overset{\beta}{\hookrightarrow} k[x]\,[[z]]\,[y]$ is not TGF.

Proof. Fix $\sigma \in k[x]\,[[z]]$ that is transcendental over $k[[z]]\,[x]$.

Define $\pi : k[x]\,[[z]]\,[y] \to k[x]\,[[z]]$ to be the identity map on $k[x]\,[[z]]$ and $\pi(y) = \sigma z$. Let $\mathbf{q} = \ker \pi$. Then $y - \sigma z \in \mathbf{q}$. If $h \in \mathbf{q} \cap (k[[z]]\,[x,y])$, then

$$h = \sum_{j=0}^{s} \sum_{i=0}^{t} (\sum_{\ell\in\mathbb{N}} a_{ij\ell} z^{\ell}) x^i y^j, \text{ for some } s,t \in \mathbb{N} \text{ and } a_{ij\ell} \in k, \text{ and so}$$

$$0 = \pi(h) = \sum_{j=0}^{s} \sum_{i=0}^{t} (\sum_{\ell\in\mathbb{N}} a_{ij\ell} z^{\ell}) x^i (\sigma z)^j = \sum_{j=0}^{s} \sum_{i=0}^{t} (\sum_{\ell\in\mathbb{N}} a_{ij\ell} z^{\ell+j}) x^i \sigma^j.$$

Since σ is trans. over $k[[z]]\,[x]$, x and σ are alg. indep. over $k((z))$. Thus each $a_{ij\ell} = 0$. Therefore $\mathbf{q} \cap (k[[z]][x,y]) = (0)$, and the embedding β is not TGF.

QUESTION. Is $k[x,y]\,[[z]] \overset{\theta}{\hookrightarrow} k[x,y,1/x]\,[[z]]$ TGF?

REMARK. For k a field and x, y, u and z indeterminates over k, the extension $k[x,y,u]\,[[z]] \hookrightarrow k[x,y,u,1/x,]\,[[z]]$ is not TGF.