NON-FINITELY GENERATED PRIME IDEALS IN SUBRINGS OF POWER SERIES RINGS

William Heinzer, Christel Rotthaus and Sylvia Wiegand

Abstract. Given a power series ring R^* over a Noetherian integral domain R and an intermediate field L between R and the total quotient ring of R^*, the integral domain $A = L \cap R^*$ often (but not always) inherits nice properties from R^* such as being Noetherian. For certain fields L it is possible to approximate A using a localization B of a particular nested union of polynomial rings over R associated to A; if B is Noetherian then $B = A$. If B is not Noetherian, we can sometimes identify which prime ideals of B are not finitely generated. We have obtained in this way, for each positive integer n, a three-dimensional quasilocal unique factorization domain B such that the maximal ideal of B is two-generated, B has precisely n prime ideals of height two, each prime ideal of B of height two is not finitely generated and all the other prime ideals of B are finitely generated. We examine the structure of the map $\operatorname{Spec} A \to \operatorname{Spec} B$ for this example. We also present a generalization of this example to higher dimensions.

1. Introduction. In this paper we analyze the prime ideal structure of particular non-Noetherian integral domains arising from a general construction developed in our earlier papers [3], ... , [9]. Briefly, with this technique two types of integral domains are constructed: (1) the intersection of a power series ring with a field yields an integral domain A as in the abstract, and (2) an approximation of the domain A by a nested union B of localized polynomial rings has the second form described in the abstract. Classical examples such as those of Akizuki [1] and Nagata [12, pages 209-211] were created using the second (nested union) description of this construction. As we observe in [8], it is also possible to realize these classical examples as the intersection domains of the first description.

In [8] we observe that in certain applications of this technique flatness of a map of associated polynomial rings implies that the constructed domains are Noetherian and that $A = B$. In [8] and [9], we apply this observation to the construction of examples of both Noetherian and non-Noetherian integral domains.

The authors would like to thank the National Science Foundation and the National Security Agency for support for this research. In addition they are grateful for the hospitality and cooperation of Michigan State, Nebraska and Purdue, where several work sessions on this research were conducted.

This paper represents a continuation of this project. We present several examples constructed using this technique of building non-Noetherian integral domains inside a power series ring for which the prime ideal structure can be described explicitly.

In Section 2 we give the notation for a simplified adaptation of the construction which we use to produce these examples. This adaptation produces "insider" examples A'' and B'' having the form A and B described above and fitting inside an easier and more straightforward Noetherian domain $A' = B'$ (where the two forms of the construction are equal). We have shown in our previous work that the question of whether or not the insider domains A'' and B'' that arise in this adaptation are Noetherian or equal is related to flatness of a map of polynomial rings corresponding to the extension $B'' \hookrightarrow B'$ and having the form $\varphi : S := R[\underline{f}] \hookrightarrow T := R[\underline{x}]$, where $\underline{x} := (x_1, \dots, x_n)$ is a tuple of indeterminates over R and $\underline{f} := (f_1, \dots, f_m)$ consists of polynomials f_i in $R[\underline{x}]$ that are algebraically independent over the field of fractions $\mathcal{Q}(R)$ of R.

In Sections 3 and 4, we analyze and provide more details about an "insider" example B constructed in [9]. For each positive integer n, this construction produces an example of a 3-dimensional quasilocal unique factorization domain that is a generalized local ring in the sense of Cohen [2] and has as its completion a 2-dimensional regular local domain. Moreover, B is not catenary and has precisely n prime ideals of height two. The associated intersection domain A for this construction is a 2-dimensional regular local domain. In Section 4, we examine the prime spectrum map $\operatorname{Spec} A \to \operatorname{Spec} B$ and describe three types of height-one prime ideals of B by how they relate to primes of A.

In Section 5 we prove for each positive integer $n \geq 2$ and each positive integer t, the existence of a non-Noetherian integral domain B such that:

(a) $\dim B = n + 1$.
(b) The maximal ideal of B is generated by n elements.
(c) B has exactly t prime ideals of height n and each of these primes is not finitely generated.
(d) B is a factorial domain.
(e) The completion of B is a regular local domain of dimension n.
(f) B is a birational extension of the localized polynomial ring over a field in

$n+1$ variables.

2. Background and Notation.

We begin this section by recalling some details from Section 3 of [9] concerning the insider adaptation of the construction utilized in this article[1].

We use the following setting throughout the paper.

2.1 General Setting. Let k be a field and let $x, y_1, \ldots, y_s$ be indeterminates over k; for convenience, we abbreviate the y_i by $\underline{y}$. (Often we assume k to have characteristic zero. This ensures excellence of the constructed domains in the simplest applications.) Let $R := k[x, y_1, \ldots, y_s]_{(x,y_1,\ldots,y_s)} = k[x, \underline{y}]_{(x,\underline{y})}$ and let R^* be the (x)-adic completion of R, that is, $R^* := k[\underline{y}]_{(\underline{y})}[[x]]$, a power series ring in x. We write $\mathcal{Q}(R), \mathcal{Q}(R^*)$, etc, for the total rings of quotients of R, R^*, etc, respectively. Let $\tau_1, \ldots, \tau_n$ be elements of $xk[[x]]$ which are algebraically independent over $k(x)$; we abbreviate them by $\underline{\tau}$. Let $T := R[\underline{\tau}]$. We define the *intersection domain* $A_{\underline{\tau}}$ corresponding to $\underline{\tau}$ by $A_{\underline{\tau}} := k(x, \underline{y}, \underline{\tau}) \cap R^* = \mathcal{Q}(T) \cap R^*$.

Next we select elements $f_1, \ldots, f_m$ of $R[\underline{\tau}]$ which are algebraically independent over $\mathcal{Q}(R)$; we abbreviate them by $\underline{f}$. The *intersection domain* corresponding to $\underline{f}$ is $A_{\underline{f}} = \mathcal{Q}(R[\underline{\tau}]) \cap R^*$.

In order to obtain $A_{\underline{\tau}}$ as a nested union of polynomial rings over k in $s+m+1$ variables, we recall some of the details of the construction.[2]

2.2 Approximation technique. With $k, x, \underline{y}, s, R$ and R^* as in (2.1), let $\rho_1, \ldots, \rho_m$ be elements of R^* which are algebraically independent over $\mathcal{Q}(R)$; we abbreviate them by $\underline{\rho}$. (Thus the *intersection domain* $A_{\underline{\rho}}$ corresponding to $\underline{\rho}$ is $A_{\underline{\rho}} := k(x, \underline{y}, \underline{\rho}) \cap R^* = \mathcal{Q}(R[\underline{\rho}]) \cap R^*$.) Write each $\rho_i := \sum_{j=1}^{\infty} b_{ij} x^j$, with the $b_{ij} \in R$. There are natural sequences $\{\rho_{ir}\}_{r=0}^{\infty}$ of elements in R^*, called the r^{th} *endpieces* for the ρ_i, which "approximate" the ρ_i, defined by:

(2.2.1) For each $i \in \{1, \ldots, n\}$ and $r \geq 0$, $\displaystyle \rho_{ir} := \sum_{j=r+1}^{\infty} (b_{ij} x^j)/x^r.$

Now, for each $r \geq 0$, define $U_r := R[\rho_{1r}, \ldots, \rho_{nr}]$ and set B_r to be U_r localized at the multiplicative system $1 + xU_r$. Then set $U := \cup_{r=0}^{\infty} U_r$ and $B_{\underline{\rho}} = \cup_{r=0}^{\infty} B_r$. Thus U is a nested union of polynomial rings over R and $B_{\underline{\rho}}$ is a nested union of

[1] For more detail see [8] and [9]

[2] Again more details may be found in [9].

localized polynomial rings over R; clearly $U \subseteq B_{\underline{\rho}} \subseteq A_{\underline{\tau}}$. We refer to $B_{\underline{\rho}}$ as the *nested union domain* corresponding to $\underline{\rho}$. The definition of the U_r (and hence also of B_r, U and B) is independent of the representation of the ρ_i as power series with coefficients in R [4, Proposition 2.3].

For the two sequences of power series in (2.1), namely $\underline{\tau}$ and $\underline{f}$, we have the approximation sequences $\underline{\tau}_i$ and $\underline{f}_i$ and the nested union domains $B_{\underline{\tau}}$ and $B_{\underline{f}}$, which are localizations of $\cup_{r=0}^{\infty} R[\tau_{1r}, \dots, \tau_{nr}]$ and of $\cup_{r=0}^{\infty} R[f_{1r}, \dots, f_{mr}]$ respectively. Clearly $B_{\underline{f}} \subseteq B_{\underline{\tau}}$ are quasilocal domains with $B_{\underline{\tau}}$ dominating $B_{\underline{f}}$.

2.3 Proposition. [5, Proposition 4.1] *Assume the set-up as in (2.1). Then $T \to R^*[1/x]$ is flat, $A_{\underline{\tau}} = B_{\underline{\tau}}$ and $A_{\underline{\tau}}$ is a Noetherian regular local ring; moreover, if* $\operatorname{char} k = 0$*, then $A_{\underline{\tau}}$ is excellent.*

2.4 Insider construction details. Let $\underline{\tau} = \tau_1, \dots, \tau_n$ and $\underline{f} = f_1, \dots, f_m$ be as in (2.1) and (2.2). We assume the constant terms in $R = k[x, \underline{y}]$ of the f_i are zero. Let $S := R[\underline{f}]$. The inclusion map $S \hookrightarrow T$ is an injective R-algebra homomorphism, and $m \leq n$.

Let A be the intersection domain for $\underline{f}$, i.e. $A := A_{\underline{f}} = \mathcal{Q}(S) \cap R^*$. Let $B := B_{\underline{f}}$ be the nested union domain associated to the $f_1, \dots, f_m$, as in (2.2). We consider the inclusion maps $\phi : S \hookrightarrow T$ and $\psi : T \hookrightarrow R^*[1/x]$ as shown in the following diagram.

(2.4.1)
$$
\begin{tikzcd}
 & & R^*[1/x] \\
R \arrow[r, phantom, "\subseteq"] & S := R[\underline{f}] \arrow[ur, "\alpha := \psi\varphi"] \arrow[r, "\varphi"] & T := R[\underline{\tau}] \arrow[u, "\psi"']
\end{tikzcd}
$$

2.5 Theorem. [3, Theorem 2.2], [9, Theorem 3.2]

(1) *B is Noetherian and $B = A$ if and only if the map $\alpha : S \to R^*[1/x]$ in (2.4.1) is flat.*

(2) *For $Q^* \in \operatorname{Spec}(R^*[1/x])$, the localization $\alpha_{Q^*} : S \to (R^*[1/x])_{Q^*}$ of the map α in (2.4.1) is flat if and only if the localization $\varphi_{Q^* \cap T}$ of the map φ in (2.4.1) is flat.*

(3) *The following are equivalent:*

(i) B is Noetherian and $A = B$.

(ii) B is Noetherian.

(iii) The localized map $\varphi_{Q^*\cap T}$ from (2.4.1) is flat for every maximal ideal $Q^* \in \operatorname{Spec}(R^*[1/x])$.

Proposition 2.6 is helpful for testing whether the map ϕ of (2.4.1) is flat.

2.6 Proposition. [9, Proposition 2.4] *Let R be a Noetherian ring and let $x_1, \dots, x_n$ be indeterminates over R. Assume that $f_1, \dots, f_m \in R[x_1, \dots, x_n]$ are algebraically independent over R. Then*

(1) *$\varphi : S := R[f_1, \dots, f_m] \hookrightarrow T := R[x_1, \dots, x_n]$ is flat if and only if, for each prime ideal P of T, we have $ht(P) \geq ht(P \cap S)$.*
(2) *For $Q \in \operatorname{Spec} T$, $\varphi_Q : S \to T_Q$ is flat if and only if for each prime ideal $P \subseteq Q$ of T, we have $ht(P) \geq ht(P \cap S)$.*
(3) *If $\varphi_x : S \to T[1/x]$ is flat, then B is Noetherian and $B = A$.*

3. Non-Noetherian Examples. We review the construction of a series of examples of non-Noetherian integral domains inside power series rings given in [9, Examples 5.1].

3.1 Specific construction details for the examples of [9]**.** This construction is a localized version of (2.4), with $s = 1$. Thus k is a field, $R = k[x, y]_{(x,y)}$ is a two-dimensional regular local ring and $R^* = k[y]_{(y)}[[x]]$ is the (x)-adic completion of R. Let $\tau = \sum_{j=1}^{\infty} c_j x^j \in xk[[x]]$ be algebraically independent over $k(x)$. (It is easy to see that (2.3) still holds; that is, the modified $A_\tau := \mathcal{Q}(R[\tau]) \cap R^* = \bigcup_{r=0}^{\infty} R[\tau_r]_{(-)} = B_\tau$.)

For the insider domains, let $p_i \in R \setminus xR$ be such that $p_1R^*, \dots, p_nR^*$ are n distinct prime ideals. For example, we could take $p_i = y - x^i$. Let $q = p_1 \cdots p_n$. We set $f := q\tau$ (i.e., $m = 1$).

Let $B := B_f$ be the nested union domain associated to f as in (2.4).

If $\tau_r = \sum_{j=r+1}^{\infty} \frac{c_j x^j}{x^r}$ is the r^{th} endpiece of τ, then $f_r := q\tau_r$ is the r^{th} endpiece of f. For each $r \geq 0$, let $B_r = R[f_r]_{(x,y,f_r)}$. Then each B_r is a 3-dimensional regular local ring and $B = \bigcup_{r=0}^{\infty} B_r$.

In this example, the intersection domain A_f associated to f is the same as that associated to τ; that is, $A = A_\tau = \mathcal{Q}(R[\tau]) \cap R^* = \mathcal{Q}(R[f]) \cap R^* = A_f$, since $\mathcal{Q}(R[\tau]) = \mathcal{Q}(R[f])$.

3.2 Proposition. [9] *For each positive integer n, the nested union domain B constructed in (3.1) is a three-dimensional quasilocal unique factorization domain such that*

(1) *B is not catenary.*
(2) *The maximal ideal of B is two generated.*
(3) *B has precisely n prime ideals of height two.*
(4) *Each prime ideal of B of height two is not finitely generated.*
(5) *Each height-one prime ideal of B is principal and each nonzero prime ideal of B is the union of the prime ideals of height one that it contains, so B has infinitely many prime ideals of height one.*
(6) *For every non-maximal prime P of B, the ring B_P is Noetherian.*

3.3 Notes. (1) The nested union domain B is not Noetherian by (2.5.1) and (2.6.2); each $p_iR^*[1/x]$ is a height-one prime of $R^*[1/x]$, but $p_iR^*[1/x] \cap S = (p_i, f)S$ has height two, thus the map $\alpha : S \to R^*[1/x]$ is not flat.

(2) The maximal ideal of B is $(x, y)B$, because $B/xB = R/xR$.

(3) As noted above, $B_\tau = A_\tau = A = R^* \cap \mathcal{Q}(R(\tau)) = R^* \cap \mathcal{Q}(R(f)) = A_f$ in the notation of (2.1) and (2.2), so that A_f is a nested union of three-dimensional regular local domains (although we will see that it is not equal to the nested union domain B_f).

We consider the inclusion map $B \hookrightarrow A$ and the map $\operatorname{Spec} A \to \operatorname{Spec} B$. The following Proposition is proved in [9].

3.4 Proposition. *With the notation of (3.1) and $A = R^* \cap \mathcal{Q}(R(f))$, we have*

(1) *A is a two-dimensional regular local domain with maximal ideal $\mathbf{m}_A = (x, y)A$.*
(2) *$\mathbf{m}_A$ is the unique prime of A lying over $\mathbf{m}_B = (x, y)B$, the maximal ideal of B.*
(3) *If $P \in \operatorname{Spec} B$ is nonmaximal, then $\operatorname{ht}(PR^*) \leq 1$ and $\operatorname{ht}(PA) \leq 1$. Thus every nonmaximal prime of B is contained in a nonmaximal prime of A.*
(4) *If $P \in \operatorname{Spec} B$ and $xq \notin P$, then $\operatorname{ht} P \leq 1$.*
(5) *If $P \in \operatorname{Spec} B$, $\operatorname{ht} P = 1$ and $P \cap R \neq 0$, then $P = (P \cap R)B$.*
(6) *If pA is a height-one prime of A with $pA \notin \{p_1A, \dots p_nA\}$, then $A_{pA} = B_{pA\cap B}$ and $\operatorname{ht}(pA \cap B) = 1$. However, $p_iA \cap B$ has height two and is not*

finitely generated.

(7) *Each p_iB is prime in B.*

(8) *p_iB and $Q_i := (p_i, f_1, f_2, \dots)B = p_iA \cap B$ are the only primes of B lying over p_iR in R.*

(9) *Q_i has height two and is not finitely generated.*

3.5 Notes. (1) With regard to the birational inclusion $B \hookrightarrow A$ and the map $\operatorname{Spec} A \to \operatorname{Spec} B$, we remark that the following hold: Each Q_i contains infinitely many height-one primes of B that are the contraction of primes of A and infinitely many that are not. Among the primes that are not contracted from A are the p_iB. In the terminology of [14, page 325], P is *not lost* in A if $PA \cap B = P$. Since $p_iA \cap B = Q_i$ properly contains p_iB, p_iB is lost in A. Since $(x, y)B$ is the maximal ideal of B and $(x, y)A$ is the maximal ideal of A and B is integrally closed, a version of Zariski's Main Theorem [13] implies that A is not essentially finitely generated as a B-algebra.

(2) The quasilocal domains B constructed in Example 3.1 are *generalized local rings* in the sense of Cohen [2, page 56], that is, the maximal ideal $\mathbf{m}$ of B is finitely generated and the intersection of the powers of the maximal ideal is zero. Cohen proves in [2] that the completion of a generalized local ring is Noetherian. In our situation, the $\mathbf{m}_B$-adic completion of B is equal to the $\mathbf{m}_A$-adic completion of A and is a 2-dimensional regular local domain.

4. Further analysis of B for $n = 1$.

In this section we consider the ring $B = \bigcup_{r=0}^{\infty} B_r$ of the previous section in the case where $n = 1$ and $q = p_1 = y$. Thus $f = y\tau$, $R = k[x, y]_{(x,y)}$, $B_r = R[y\tau_r]_{(x,y,y\tau_r)}$ and $B = \bigcup_{r=0}^{\infty} R[y\tau_r]_{(x,y,y\tau_r)}$. This ring B has exactly one prime ideal $Q = (y, \{y\tau_r\}_{r=0}^{\infty})B$ of height 2. Moreover, Q is not finitely generated and is the only prime ideal of B that is not finitely generated. We also have $Q = yA \cap B$, and $Q \cap B_r = (y, y\tau_r)B_r$ for each $r \geq 0$.

If q is a height-one prime of B, then B/q is Noetherian if and only if q is not contained in Q. This is clear since Q is the unique prime of B that is not finitely generated and a ring is Noetherian if each prime ideal of the ring is finitely generated.

The height-one primes q of B may be separated into three types as follows:

Type I. The primes $q \not\subseteq Q$, such as xB. As mentioned above, B/q is Noetherian. These primes are contracted from A. To see this, consider $q = gB$ where $g \notin Q$. Then gA is contained in a height one prime P of A. Then $g \in (P \cap B) \setminus Q$ so $P \cap B \neq Q$. Since $\mathbf{m}_B A = \mathbf{m}_A$, we have $P \cap B \neq \mathbf{m}_B$. Therefore $P \cap B$ is a height-one prime containing q, so $q = P \cap B$ and $B_q = A_P$.

There are infinitely many primes q of type I, because every element of $\mathbf{m}_B \setminus Q$ is contained in a prime q of type I. Thus $\mathbf{m}_B \subseteq Q \cup \bigcup\{q \text{ of type I}\}$. Since $\mathbf{m}_B$ is not the union of finitely many strictly smaller prime ideals, there are infinitely many primes q of type I.

If q is a height-one prime of B not of type I, then $\overline{B} = B/q$ has precisely three prime ideals. These prime ideals form a chain: $(\overline{0}) \subset \overline{Q} \subset \overline{(x,y)B} = \overline{\mathbf{m}_B}$.

Type II. The primes $q \subset Q$, where q has height one and is contracted from a prime p of $A = k(x,y) \cap R^*$, for example, the prime $y(y+\tau)B$. For q of this type, B/q is dominated by the one-dimensional Noetherian local domain A/p. Thus B/q is a non-Noetherian generalized local ring in the sense of Cohen.

For q of Type II, the maximal ideal of B/q is not principal. This follows because a quasilocal generalized local domain having a principal maximal ideal is a DVR [12, (31.5)].

There are infinitely many height-one primes of type II, for example, $y(y+x^n\tau)B$ for each $n \in \mathbb{N}$. For q of type II, the DVR B_q is birationally dominated by A_p. Hence $B_q = A_p$ and the ideal $\sqrt{qA} = p \cap yA$.

Type III. The primes $q \subset Q$, where q has height one and is not contracted from A, for example, the prime yB and the prime $(y + x^n y\tau)B$ for $n \in \mathbb{N}$. Since the elements y and $y + x^n y\tau$ are in $\mathbf{m}_B$ and are not in $\mathbf{m}_B^2$ and since B is a UFD, these elements are necessarily prime. Also the prime ideals $(y+x^n y\tau)B$ and $(y+x^m y\tau)B$ are distinct for n and m distinct positive integers, for if W is a prime ideal of B that contains them both and if $m > n$, then $x^{m-n}y\tau \in W$. But $x \notin Q$ and Q and $\mathbf{m}_B$ are the only primes of B containing $(y, y\tau)B$. For q of type III, we have $\sqrt{qA} = yA$.

If $q = yB$ or $q = (y + x^n y\tau)B$, then the image $\overline{\mathbf{m}_B}$ of $\mathbf{m}_B$ in B/q is principal. It follows that the intersection of the powers of $\overline{\mathbf{m}_B}$ is Q/q and B/q is not a generalized local ring. For if P is a principal prime ideal of a ring and P' is a prime ideal properly contained in P, then P' is contained in the intersection of the powers of P. [11, page 7, ex. 5].

Another way to examine the height-one primes of B.

Observe that $B[1/x]$ is a localization of the polynomial ring $k[x, y, f]$ while $A[1/x]$ is a localization of $k[x, y, \tau]$. Thus the embedding $B[1/x] \hookrightarrow A[1/x]$ is a localization of the map:

$$\phi : k[x, y, f] \longrightarrow k[x, y, \tau]$$

which is defined by $\phi(f) = y\tau$. Obviously the nonflat locus of ϕ is defined by the ideal (y) of $k[x, y, \tau]$.

If $q \in B$ is a height one prime ideal which is not contained in Q, then the extension qA is not contained in yA and for every minimal prime $p \subseteq A$ over qA the induced map $B_q \hookrightarrow A_p$ is flat. In particular, q is not lost in A. If q is a height-one prime of B which is contained in Q then q is of type II or III according to the minimal prime divisors of qA: If there is a minimal prime $p \in \operatorname{Spec} A$ of qA which is different from (y), then the map: $B_q \hookrightarrow A_p$ is flat, and q is not lost in A. In this case, q is of type II. Note that yA is also a minimal prime divisor of qA. If yA is the only prime divisor of qA, then q is lost in A. In this case the map $B_Q \hookrightarrow A_{(y)}$ is a localization.

We remark that B/yB is a rank 2 valuation ring. This can be seen directly or else one may apply [10, Prop. 3.5(iv)]. If we do a similar construction with a prime contained in $\mathbf{m}^2$ (instead of y), for example, $f = (x^2 + y^2)\tau$ (over $\mathbb{Q}$), then $B/(x^2 + y^2)$ has a two-generated maximal ideal and cannot be a valuation ring.

5. A More General Construction.

In Theorem 5.1, we obtain a generalization of the construction that gives the examples considered in Sections 3 and 4.

5.1 Theorem. *Let $n \geq 2$ be a positive integer. For each positive integer t, there exists a non-Noetherian integral domain B such that:*

(a) $\dim B = n + 1$.

(b) *The maximal ideal of B is generated by n elements.*

(c) *B has exactly t prime ideals of height n and each of these primes is not finitely generated.*

(d) *B is a factorial domain.*

(e) *The completion of B is a regular local domain of dimension n.*

(f) *B is a birational extension of the localized polynomial ring over a field in $n+1$ variables.*

Proof. Let k be a field and let $x, y_1, \dots, y_{n-1}$ be variables over k. Define:

$$R = k[x, y_1, \dots, y_{n-1}]_{(x, y_1, \dots, y_{n-1})}$$

and let $\tau_1, \dots, \tau_{n-1} \in xk[[x]]$ be algebraically independent over $\mathcal{Q}(R)$. For $1 \le i \le t$, let $p_i = y_1 - x^i$. Set $q = p_1 p_2 \dots p_t$, and consider the element

$$f = q\tau_1 + y_2\tau_2 + \dots y_{n-1}\tau_{n-1}.$$

The map:

$$R[f] \longrightarrow R[\tau_1, \dots, \tau_{n-1}]$$

has as its Jacobian ideal: $J = (q, y_2, \dots, y_{n-1})$ which is an ideal of height $n-1$ that is the intersection of t prime ideals of height $n-1$: $J = Q_1 \cap \dots \cap Q_t$, where $Q_i = (p_i, y_2, \dots, y_{n-1})$.

If we construct B as usual as the union of localized polynomial rings of dimension $n+1$, then by [9, Theorem 3.9], B is not Noetherian (or use (2.5.1) and an argument similar to (3.3.1)). If $n > 2$ then J has height > 1 and it follows from [6, Theorem 5.5] or [8, Theorem 6.3] that $B = A = \mathcal{Q}(R[f]) \cap R^*$. It follows from [6, Theorem 4.5] that B is factorial.

Moreover, if we consider our usual set up:

$$R[f] \longrightarrow R[\tau_1, \dots, \tau_{n-1}][1/x] \longrightarrow R^*[1/x]$$

the preimage in $R[f]$ of $JR^*[1/x]$ is the ideal $L = (J, f)$ which has height n. The ideal L corresponds to the intersection of t prime ideals in B of height n, these are the only primes in B of height n and they are not finitely generated.

5.2 Remark. It would be interesting to know whether for B as in Theorem 5.1, the prime ideals of B of height n are the only prime ideals of B that are not finitely generated.

5.3 Remark. With the insider construction given in (2.1)-(2.4), if the dimension of B_f is greater than that of R^*, then B_f is not catenary. One way to see this is that in $\operatorname{Spec}(B_f)$ there is always a saturated chain of prime ideals that includes (x) and this chain has length equal to $\dim R^*$, while if $\dim B_f > \dim R^*$, then there exists a saturated chain of prime ideals in B_f of length greater than $\dim R^*$.

References

[1] Y. Akizuki, *Einige Bemerkungen über primäre Integritatsbereiche mit Teilerkettensatz*, Proc. Phys-Math. Soc. Japan **17** (1935), 327-336.

[2] I. S. Cohen, *On the structure and ideal theory of complete local rings*, Trans. Amer. Math. Soc. **59** (1946), 54-106.

[3] W. Heinzer, C. Rotthaus and S. Wiegand, *Building Noetherian domains inside an ideal-adic completion*, Abelian Groups, Module Theory and Topology Proceedings in honor of Adalberto Orsatti's 60th Birthday (Dikranjan and Salce, ed.), Marcel Dekker, Inc., New York, 1998, pp. 279-287.

[4] ———, *Noetherian rings between a semilocal domain and its completion*, J. of Algebra **198** (1997), 627-655.

[5] ———, *Noetherian domains inside a homomorphic image of a completion*, J. of Algebra **215** (1999), 666-681.

[6] ———, *Intermediate rings between a local domain and its completion*, Illinois J. Math. **43** (1999), 19-46.

[7] ———, *Intermediate rings between a local domain and its completion II*, Illinois J. Math. **45** (2001), 965-979.

[8] ———, *Building Noetherian and non-Noetherian integral domains using power series*, Ideal theoretic methods in commutative algebra in honor of James A. Huckaba's retirement (Anderson and Papick, ed.), Marcel Dekker, Inc., New York, 2001, pp. 251-264.

[9] ———, *Examples of integral domains inside power series rings*, Commutative ring theory and applications (Fontana, Kabbaj and Wiegand, ed.), Marcel Dekker, Inc., New York, 2002, pp. 233-254.

[10] W. Heinzer and J. Sally, *Extensions of valuations to the completion of a local domain*, J. Pure Applied Algebra **71** (1991), 175-185.

[11] I. Kaplansky, *Commutative Rings*, Allyn Bacon, Boston, 1970.

[12] M. Nagata, *Local Rings*, John Wiley, New York, 1962.

[13] C. Peskine, *Une généralsation du "main theorem" de Zariski*, Bull. Sci. Math. **90** (1966), 377-408.

[14] O. Zariski and P. Samuel, *Commutative Algebra volume II*, D. Van Nostrand, New York, 1960.

Department of Mathematics, Purdue University, West Lafayette, IN 47907-1395

Department of Mathematics, Michigan State University, East Lansing, MI 48824-1027

Department of Mathematics and Statistics, University of Nebraska, Lincoln, NE 68588-0323