PARAMETRIC DECOMPOSITION OF MONOMIAL IDEALS (I)

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Abstract. Emmy Noether showed that every ideal in a Noetherian ring admits a decomposition into irreducible ideals. In this paper we explicitly calculate this decomposition in a fundamental case. Specifically, let R be a commutative ring with identity, let $x_1, \dots, x_d$ $(d > 1)$ be an R -sequence, let $X = (x_1, \dots, x_d)R$, and let I be a monomial ideal (that is, a proper ideal generated by monomials $x_1^{e_1} \cdots x_d^{e_d}$) such that $Rad(I) = Rad(X)$. Then the main result gives a canonical and unique decomposition of I as an irredundant finite intersection of ideals of the form $(x_1^{n_1}, \dots, x_d^{n_d})R$, where the exponents $n_1, \dots, n_d$ are positive integers. Specifically, if $z_1, \dots, z_m$ are the monomials in $(I : X) - I$, and if $z_j = x_1^{a_{j,1}-1} \cdots x_d^{a_{j,d}-1}$, then $I = \cap\{(x_1^{a_{j,1}}, \dots, x_d^{a_{j,d}})R;\ j = 1, \dots, m\}$. We also calculate the decomposition of the ideals $I^{[k]}$ generated by the k -th powers of the monomial generators of I. The methods we use are algebraic, but they were suggested by the geometry of lattices.

1. Introduction. *Throughout this paper*, R is a commutative ring with identity $1 \neq 0$, $x_1, \dots, x_d$ $(d > 1)$ is an R -sequence, $X = (x_1, \dots, x_d)R$, and I is a monomial ideal (that is, a proper ideal generated by monomials $x_1^{e_1} \cdots x_d^{e_d}$) such that $Rad(I)$ $= Rad(X)$.

It is known (for example, see [HRS2, (3.15)]) that in a regular local ring R of altitude two, irreducible ideals are parameter ideals. Therefore in altitude two regular local rings, Emmy Noether's fundamental decomposition theorem [N, Satz IV] shows that each open ideal in R is a finite intersection of parameter ideals (but of course the x's may vary). One consequence of our main result, (4.1), is that a similar statement holds for open monomial ideals in a Cohen-Macaulay local ring.

Monomial ideals are important in several areas of current research, and they have been studied in their own right in several papers (for example, [EH] and [T]), so many useful results are known about such ideals. In the present paper we are

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interested in giving an explicit decomposition of I as an irredundant finite intersection of parameter ideals. We do this in Section 2 for the special case when $I = X^n$ (n a positive integer), and it is shown that X^n is the irredundant intersection of the $\binom{n+d-2}{d-1}$ parameter ideals $(x_1^{a_1}, \ldots, x_d^{a_d})R$, where $a_1, \ldots a_d$ are positive integers that sum to $n + d - 1$.

To extend this result to an arbitrary monomial ideal I (such that $Rad(I) = Rad(X)$), in Section 3 we introduce and study the J -corner-elements of a monomial ideal J. We show that they are the monomials in $(J : X) - J$, that there are only finitely many of them, and that if $(x_1, \ldots, x_{d-1})R \subseteq Rad(J)$, then their J -residue classes are a minimal basis, in any order, of $(J : X)/J$. Also, if Q is an open monomial ideal in a regular local ring (R, M) of altitude two, then $v((Q : X)/Q)$ $= v(Q) - 1$ (where $v(J)$ denotes the number of elements in a minimal basis of the ideal J), and if t is an integer such that $v(Q) - 1 \leq t \leq 2v(Q) - 1$, then Q can be chosen such that $v(Q : X) = t$. Finally, we give a geometric interpretation of I -corner-elements, an algebraic construction of them, and then close this section with several examples of such elements.

In Section 4 we show that if $z_1, \ldots z_m$ are the I -corner-elements, then I is the irredundant intersection of the m parameter ideals $P(z_j) = (x_1^{a_{j,1}}, \ldots x_d^{a_{j,d}})R$, where $z_j = x_1^{a_{j,1}-1} \cdots x_d^{a_{j,d}-1}$. Three interesting corollaries are: $\cup\{Ass(R/I^n);\, n \geq 1\} \subseteq Ass(R/X)$; and, if R is a Gorenstein local ring with maximal ideal M, if X is generated by a system of parameters, and if I is open, then $v((I : M)/I)$ $= v((I : X)/I)$, and I is irreducible if and only if there exists exactly one I -corner-element, and then I is generated by a system of parameters. Also, unique factorization holds in the sense that if $I = \cap\{P(z_j);\, j = 1, \ldots, m\} = \cap\{P(w_i);\, i = 1, \ldots, n\}$, then $n = m$ and $\{z_1, \ldots z_m\} = \{w_1, \ldots, w_n\}$. Further, if k is a positive integer and $I^{[k]}$ is the ideal generated by the k-th powers of the monomial generators of I, then $I^{[k]} = \cap\{(P(z_j))^{[k]};\, j = 1, \ldots, m\}$ and the $I^{[k]}$ -corner-elements are the m monomials $z_j^{(k)} = x_1^{ka_{j,1}+k-1} \cdots x_d^{ka_{j,d}+k-1}$.

In Section 5 a related decomposition of I as an irredundant finite intersection of irreducible ideals is proved. Specifically, with the notation of the preceding paragraph, if R is local with maximal ideal M and if Q_j is maximal in $S_j = \{Q;$ Q is an ideal in R, $P(z_j) \subseteq Q$, and $z_j \notin Q\}$ for $j = 1, \ldots, m$, then each Q_j is irreducible, $\cap\{Q_j;\, j = 1, \ldots, m\}$ is an irredundant intersection, and $(\cap\{Q_j;\, j =$

$1, \ldots, m\}) \cap (I : X) = I + M(I : X)$. It then follows that if R is a regular local ring and $X = M$, then $Q_j = P(z_j)$ for $j = 1, \ldots m$.

Finally, in Section 6 we show that: if I is irreducible, then I is a parameter ideal; I is a parameter ideal if and only if I has exactly one corner-element; and, if R is a Gorenstein local ring of altitude d, then I is a parameter ideal if and only if I is irreducible. Also, the parameter ideals that are minimal with respect to containing I are the ideals $P(z)$, where z is an I -corner-element.

The authors have been fascinated by the historic and fundamental decomposition theorems of Emmy Noether, and this fascination gave rise to the results in [HRS1, HRS2, HRS3, HRS4] and the present paper. We are pursuing further topics in this area (in particular, in [HMRS]), and we hope this theory turns out to be fascinating and useful to others.

2. Parametric Decompositions of Powers of an R -Sequence. The main result in this section, (2.4), shows that if X is an ideal generated by an R -sequence, then X^n is the irredundant intersection of $\binom{n+d-2}{d-1}$ parameter ideals. To prove this, we need a few preliminary results, so we begin with these.

(2.1) Definition. Let R be a ring, let $x_1, \ldots, x_d$ $(d > 1)$ be an R -sequence, and let $X = (x_1, \ldots, x_d)R$. Then:

(2.1.1) A **monomial** (in $x_1, \ldots x_d$) is a power product $x_1^{e_1} \cdots x_d^{e_d}$, where $e_1, \ldots, e_d$ are nonnegative integers (so a monomial is either a nonunit or the element 1), and a **monomial ideal** is a proper ideal generated by monomials.

(2.1.2) A **parameter ideal** (in $x_1, \ldots, x_d$) is an ideal of the form $(x_1^{a_1}, \ldots, x_d^{a_d})R$, where $a_1, \ldots, a_d$ are positive integers (so the parameter ideal $(x_1^{a_1}, \ldots, x_d^{a_d})R$ is a monomial ideal generated by the R -sequence $x_1^{a_1}, \ldots, x_d^{a_d}$). If $f = x_1^{e_1} \cdots x_d^{e_d}$ is a monomial, then we let $\mathbf{P(f)}$ denote the parameter ideal $(x_1^{e_1+1}, \ldots, x_d^{e_d+1})R$. (Note that if $f = 1$, then $P(f) = X$.) And if $a_1, \ldots a_d$ are positive integers, then we let $\mathbf{P(a_1, \ldots, a_d)}$ denote the parameter ideal $(x_1^{a_1}, \ldots, x_d^{a_d})R$ (so $P(a_1, \ldots, a_d) = P(f)$, where $f = x_1^{a_1-1} \cdots x_d^{a_d-1}$).

(2.2) Remark. Let f and g be monomials. Then:

(2.2.1) If $f_1, \ldots, f_n$ are monomials then $f \in (f_1, \ldots, f_n)R$ if and only if $f \in f_iR$ for some $i = 1, \ldots, n$.

(2.2.2) If $f \in gR$, then there exists a monomial k (possibly $k = 1$) such that $f = gk$.

(2.2.3) If h is a monomial such that $fh = gh$, then $f = g$.

(2.2.4) If $fx_j = gx_i$ for some $i \neq j$ in $\{1, \dots, d\}$, then $f \in x_iR$ and $g \in x_jR$.

Proof. It is shown in [T, Theorem 1] that if $r \in R$ and $rf \in (f_1, \dots, f_n)R$, then either $f \in f_iR$ for some $i = 1, \dots, n$ or $r \in (x_1, \dots, x_d)R$. (2.2.1) readily follows from this.

(2.2.2)–(2.2.4) readily follow by the "independence" of power products in an R -sequence (that is, $x_1^{e_1} \cdots x_d^{e_d} = x_1^{a_1} \cdots x_d^{a_d}$ if and only if $a_i = e_i$ for $i = 1, \dots, d$), $\square$

(2.3) Lemma. *Let f and g be monomials. Then $g \in P(f)$ (see (2.1.2)) if and only if $f \notin gR$.*

Proof. Let $f = x_1^{e_1} \cdots x_d^{e_d}$. Then $f \notin x_i^{e_i+1}R$ for $i = 1, \dots, d$, since $e_i < e_i + 1$ for $i = 1, \dots, d$, so (2.2.1) shows that $f \notin P(f)$. Therefore if $g \in P(f)$, then $f \notin gR$.

For the converse assume that $g \notin P(f)$ and let $g = x_1^{a_1} \cdots x_d^{a_d}$. Then $a_i < e_i + 1$ for $i = 1, \dots, d$, so $e_i \geq a_i$ for $i = 1, \dots, d$, hence $f \in gR$, $\square$

(2.4), the main result in this section, extends [HRS2, (3.5)] (where it is shown that in a regular local ring $(R, M = (x, y)R)$, $M^n = \cap\{(x^{n+1-i}, y^i)R; i = 1, \dots, n\}$).

Concerning the ideals $P(a_1, \dots, a_d)$ in (2.4), see (2.1.2).

(2.4) Theorem. *Let X be an ideal that is generated by an R -sequence $x_1, \dots, x_d$ $(d > 1)$ and let n be a positive integer. Then $X^n = \cap\{P(a_1, \dots, a_d); a_1 + \cdots + a_d = n + d - 1\}$ and this intersection is irredundant. Therefore X^n is the irredundant intersection of $\binom{n+d-2}{d-1}$ parameter ideals.*

Proof. If $n = 1$, then this is clear, so it will be assumed that $n > 1$.

Let $J = \cap\{P(a_1, \dots, a_d); a_1 + \cdots + a_d = n + d - 1\}$. Then since X^n is generated by the monomials $x_1^{e_1} \cdots x_d^{e_d}$, where $e_1, \dots, e_d$ are nonnegative integers that sum to n, to show that $X^n \subseteq J$ it suffices to show that each such monomial is in J. For this, fix $f = x_1^{e_1} \cdots x_d^{e_d}$ and consider any of the ideals $P(a_1, \dots, a_d)$. Then $e_i \geq a_i$ for some $i = 1, \dots, d$ (since otherwise $n = e_1 + \cdots + e_d < n + d = (e_1 + 1) + \cdots + (e_d + 1) \leq a_1 + \cdots + a_d = n + d - 1$, and this is a contradiction), so $f \in P(a_1, \dots, a_d)$. Therefore $f \in J$, so it follows that $X^n \subseteq J$.

For the opposite inclusion, since $Rad(P(a_1,\ldots,a_d)) = Rad(X)$ for all positive integers $a_1,\ldots,a_d$ that sum to $n+d-1$, [T, Lemma 6] shows that J is generated by monomials, so it suffices to show that if f is a monomial in $X - X^n$, then f is not in J. For this let $f = x_1^{e_1}\cdots x_d^{e_d} \in X^k - X^{k+1}$, where $e_1+\cdots+e_d = k$ and $1 \le k < n$. Then since $k < n$ there exists a nonnegative integer h such that $(e_1+1)+\cdots+(e_{d-1}+1)+(e_d+1+h) = n+d-1$, and since $x_1,\ldots,x_d$ is an R -sequence, it follows from (2.2.1) that $f \notin P((e_1+1),\ldots,(e_{d-1}+1),(e_d+1+h))$. Therefore it follows that $J \subseteq X^n$, so $J = X^n$.

Also, this intersection is irredundant, since if $\{a_1,\ldots,a_d\}$ and $\{b_1,\ldots,b_d\}$ are distinct sets of positive integers that sum to $n+d-1$, then $b_i > a_i$ for some $i = 1,\ldots,d$, so $x_1^{b_1-1}\cdots x_d^{b_d-1} \in P(a_1,\ldots,a_d)$, hence it follows that $x_1^{b_1-1}\cdots x_d^{b_d-1} \in \cap\{P(a_1,\ldots,a_d); a_1+\cdots+a_d = n+d-1 \text{ and } a_i \ne b_i \text{ for some } i\}$, and $x_1^{b_1-1}\cdots x_d^{b_d-1} \notin P(b_1,\ldots,b_d)$, by (2.3), so $x_1^{b_1-1}\cdots x_d^{b_d-1} \notin \cap\{P(a_1,\ldots,a_d); a_1+\cdots+a_d = n+d-1\}$.

For the final statement, each ideal $P(a_1,\ldots,a_d)$ is a parameter ideal, by (2.1.2). And the preceding paragraph shows that they are distinct for distinct d-tuples $(a_1,\ldots,a_d)$ of positive integers. To compute the number of these ideals, since we are only interested in the number of ideals, it may be assumed that $X = (x_1,\ldots,x_d)R$ is the maximal ideal M in a regular local ring (R,M). Then by [HRS2, (2.3.2) and (2.4)] the number of ideals is $d(X^n) = \dim_{R/M}((X^n : X)/X^n) = \dim_{R/M}(X^{n-1}/X^n) = v(X^{n-1}) = \binom{n+d-2}{d-1}$, $\square$

(2.5) Corollary. *If R is a Gorenstein local ring and altitude $(R) = d$, then $X^n = \cap\ \{P(a_1,\ldots,a_d);\ a_1+\cdots+a_d = n+d-1\}$ is an irredundant intersection of $\binom{n+d-2}{d-1}$ irreducible ideals.*

Proof. If R is Gorenstein, then each open parameter ideal is irreducible, so the conclusion follows from (2.4), $\square$

(2.6) Remark. It follows from (2.4) that the cardinality of $\{x_1^{e_1}\cdots x_d^{e_d}; e_1,\ldots,e_d$ are positive integers that sum to $n+d-1\}$ is $\binom{n+d-2}{d-1}$.

3. J-Corner-Elements. We now want to extend (2.4) to an arbitrary monomial ideal I such that $Rad(I) = Rad(X)$. (It should be noted that $Rad(I) = Rad(X)$ is a necessary condition to extend (2.4), since the radical of each parameter ideal is the radical of X, and in (4.1) we show that this condition is also sufficient.) To accomplish this extension, we have found it useful to use "corner-elements". So in this section we introduce such elements and derive some of their basic properties, and then use some of these properties in the proof of (4.1) to give the desired extension of (2.4).

We think "corner-elements will be of interest and use in other problems, so in this section we prove several results concerning them. Specifically, we show in (3.2) and (3.7) that if J is a monomial ideal, then there exist only finitely many J -corner-elements, that they are the monomials in $(J : X) - J$, and that if $(x_1, \ldots, x_{d-1})R \subseteq Rad(J)$, then the J -residue classes of these corner-elements are a minimal basis, in any order, of $(J : X)/J$. We then apply these results to the case when J is an open monomial ideal in a regular local ring (R, M) of altitude two, give a geometric interpretation of I -corner-elements and an algebraic construction of them, and then close this section with several examples of such elements.

We begin with the definition.

(3.1) Definition. Let J be a monomial ideal. Then a J **-corner-element** is a monomial z such that $z \notin J$ and $zx_i \in J$ for $i = 1, \ldots, d$.

(The name "corner-element" is suggested by the geometric interpretation in (3.13), where a corner-element is an element $z = x^a y^b$ with coordinates (a, b) such that $(a, b+1), (a+1, b)$, and $(a+1, b+1)$ are the coordinates of points in I and $z \notin I$.)

Concerning (3.1), note that 1 is the unique X -corner-element (since each nonunit monomial is in X). Also, if J is a monomial ideal and 1 is a J -corner-element, then $1x_i \in J$ for $i = 1, \ldots, d$, so $J = X$.

In (3.2) we characterize the J -corner-elements and show that there are only finitely many of them. (It follows from (3.2) that J uniquely determines its corner-elements. The converse of this is proved in (4.2) when $Rad(J) = Rad(X)$.)

(3.2) Proposition. *If J is a monomial ideal, then the J -corner-elements are the monomials in $(J : X) - J$. Also, if z, z' are distinct J -corner-elements, then $zR \not\subseteq z'R$ and $z'R \not\subseteq zR$, so there exist only finitely many J -corner-elements.*

Proof. Let $\mathbf{C}$ be the set of J -corner-elements (so each element in $\mathbf{C}$ is a monomial). Then it is clear from (3.1) that $\mathbf{C} \subseteq (J : X) - J$. And if z is a monomial in

$(J : X) - J$, then $z \notin J$ and $zx_i \in J$ for $i = 1, \ldots, d$, so z is a J -corner-element, hence $z \in \mathbf{C}$. Therefore $\mathbf{C}$ is the set of monomials in $(J : X) - J$.

Now let z and z' be distinct J -corner-elements and suppose that $zR \subseteq z'R$. Then (2.2.2) shows that $z = z'f$ for some monomial f (and $f \neq 1$, since $z \neq z'$). But this implies that $z = z'f \in J$ (since z' is a J -corner-element), and this is a contradiction. Therefore $zR \not\subseteq z'R$ and, similarly, $z'R \not\subseteq zR$.

Finally, the ideal generated by the J -corner-elements (viewed as elements in $Z_k[x_1, \ldots, x_d]$, where k is the characteristic of R and where Z_k is the ring generated by the identity of R) is finitely generated, so since there are no inclusion relations among the ideals they generate, (2.2.1) shows that there are only finitely many of them, □

(3.3) Corollary. *Let J be a monomial ideal and let $z_1, \ldots, z_m$ be the J -corner-elements. Then for $j = 1, \ldots, m$ it holds that $(z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_m)R \subseteq P(z_j)$ and $z_j \notin P(z_j)$. Therefore $\cap\{P(z_j); j = 1, \ldots, m\}$ is an irredundant intersection of parameter ideals.*

Proof. (It follows from (3.2) that there are only finitely many J -corner-elements. Also, if $m = 1$ and $z_1 = 1$, then $P(z_1) = X$, (0) (the ideal generated by the empty set) is contained in X, and $1 \notin P(1) = X$, so the conclusion holds in this case.)

Fix $j \in \{1, \ldots, m\}$. Then it follows from (3.2) that if $i \in \{1, \ldots, j-1, j+1, \ldots, m\}$, then $z_j \notin z_iR$, so (2.3) shows that $z_i \in P(z_j)$ (hence $(z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_m)R \subseteq P(z_j)$) and $z_j \notin P(z_j)$. This shows that $\cap\{P(z_j); j = 1, \ldots, m\}$ is an irredundant intersection, and (2.1.2) shows that the ideals $P(z_j)$ are parameter ideals, □

In (3.4) we specify the X^n -corner-elements.

(3.4) Corollary. *If $n > 1$ is a positive integer, then the X^n -corner-elements are the $\binom{n+d-2}{d-1}$ generators $x_1^{e_1} \cdots x_d^{e_d}$ of X^{n-1} (so $e_1, \ldots, e_d$ are nonnegative integers such that $e_1 + \cdots + e_d = n - 1$).*

Proof. By (3.2) the X^n -corner-elements are the monomials in $X^{n-1} - X^n$ (since $X^n : X = X^{n-1}$), and since X is generated by an R -sequence of length d it follows that there are $\binom{n+d-2}{d-1}$ distinct such elements, □

It follows from (3.5) that if z is a J -corner-element, then the d elements $zx_1, \dots, zx_d$ are members of distinct principal ideals generated by monomials in J.

(3.5) Proposition. *Let f and g be monomials and let $i \neq j \in \{1, \dots, d\}$. If $fx_i \in gR$ and $fx_j \in gR$, then $f \in gR$.*

Proof. If $fx_i \in gR$ and $fx_j \in gR$, then (2.2.2) shows that there exist monomials h_i, h_j such that $fx_i = gh_i$ and $fx_j = gh_j$. Then $fx_ix_j = gh_ix_j = gh_jx_i$, so $h_ix_j = h_jx_i$ by (2.2.3). (2.2.4) then shows that $h_i \in x_iR$, so $h_i = kx_i$ for some monomial k by (2.2.2). Therefore $fx_ix_j = gh_ix_j = g(kx_i)x_j$, so $f = gk \in gR$ by (2.2.3), □

(3.6) Corollary. *If J is a monomial ideal that has a corner-element, and if $f_1, \dots, f_n$ are monomials that generate J, then $n \geq d$.*

Proof. This follows immediately from (3.5), □

In (3.7) it is shown that if $(x_1, \dots, x_{d-1})R \subseteq Rad(J)$, then the J -residue classes of the J -corner-elements are a minimal basis, in any order, of $(J : X)/J$. (In this regard, note that if $J : X = J$, then there are no J -corner-elements, and the empty set does generate the ideal $(J : X)/J = J/J$. On the other hand, if $Rad(J)$ $= Rad(X)$ then $J : X \neq J$.)

(3.7) Theorem. *Let $J \neq X$ be a monomial ideal such that $(x_1, \dots, x_{d-1})R \subseteq$ $Rad(J)$. Then the J -residue classes of the J -corner-elements are a minimal basis, in any order, of $(J : X)/J$.*

Proof. (By "minimal basis", we mean a basis such that no proper subset is a generating set of the ideal.) Since $Rad((x_1, \dots, x_{d-1})R) \subseteq Rad(J)$, [T, Theorem 6] shows that $J : X$ is a monomial ideal, so it follows from (3.2) that the J -residue classes of the J -corner-elements generate $(J : X)/J$.

Let $\mathbf{C} = \{z_1, \dots, z_m\}$ be the set of J -corner-elements ($\mathbf{C}$ is a finite set by (3.2)) and suppose that there exists a permutation $\pi 1, \dots, \pi m$ of $1, \dots, m$ such that $z_{\pi 1} \in (z_{\pi 2}, \dots, z_{\pi m})R$. Then $z_{\pi 1} \in z_{\pi k}R$ for some $k = 2, \dots, m$ by (2.2.1), so (2.2.2) shows that $z_{\pi 1} = z_{\pi k}f$ for some monomial f ($f \neq 1$, since $z_1, \dots, z_m$ are distinct). However, this implies that $z_{\pi 1} = z_{\pi k}f \in J$ (since $z_{\pi k}$ is a J

-corner-element), and this is a contradiction. Therefore $z_{\pi 1} \notin (z_{\pi 2}, \dots, z_{\pi m})R$ for all permutations $\pi 1, \dots, \pi m$ of $1, \dots, m$. And no z_j in in J, so it follows from (2.2.1) that the J -residue classes of $z_1, \dots, z_m$ are a minimal basis, in any order, of $(J : X)/J$, $\square$

In (3.8) we consider the case when $J = Q$ is open in a regular local ring R of altitude two. ((3.8) was noted in [HRS2, (3.3)] for the case $M = (x, y)R$, and therein a homological proof using [HS, (2.1)] was sketched for an arbitrary open ideal (in an altitude two regular local ring). (3.8) gives a non-homological proof for an arbitrary R -sequence of length two, but only for the case of an open monomial ideal.)

(3.8) Corollary. *Let (R, M) be a regular local ring of altitude two, let x, y be an R -sequence, and let $Q \neq (x, y)R$ be an open monomial ideal in x and y, say $v(Q) = n$. Then $v((Q : X)/Q) = n - 1$.*

Proof. Let $Q = (f_1, \dots, f_n)R$ and lexicographically order the f_i by saying that $f_i < f_j$ (for $f_i = x^a y^b$ and $f_j = x^c y^e$) if either $a < c$ or $a = c$ and $b < e$. Then it may be assumed that $f_1 < f_2 < \cdots < f_n$. Therefore, since $v(Q) = n$ and Q is open, it follows that there exist positive integers $h, k, h_2 < \cdots < h_{n-1}$ ($h_{n-1} \leq h - 1$), and $k_{n-1} < \cdots < k_2$ ($k_2 \leq k - 1$) such that $f_1 = y^h$, $f_n = x^k$, and $f_i = x^{k-k_i} y^{h-h_i}$ for $i = 2, \dots, n-1$. For $j = 1, \dots, n-1$ let $z_j = x^{k-k_{j+1}-1} y^{h-h_j-1}$ (with $h_1 = 0 = k_n$). Then $z_j \notin Q$, $z_j x = x^{k-k_{j+1}} y^{h-h_j-1} \in f_{j+1}R \subseteq Q$, and $z_j y = x^{k-k_{j+1}-1} y^{h-h_j} \in f_j R \subseteq Q$, so $z_j \in (Q : X) - Q$ for $j = 1, \dots, n-1$. Thus each z_j is a Q -corner-element, and the geometric interpretation in (3.13) shows that every Q -corner-element is one of these $z_1, \dots, z_{n-1}$. Therefore the conclusion follows from (3.7), $\square$

For the next corollary of (3.7) we need the following definition.

(3.9) Definition. If J is a monomial ideal, then $\mathbf{c(J)}$ denotes the number of J -corner-elements.

(3.10) Corollary. *With the notation of (3.8), there exists a polynomial $p(x)$ of degree two such that $p(n) = c(Q^n)$ for large n.*

Proof. It is well known that there exists a polynomial $q(x)$ of degree two such that

$q(n) = v(Q^n)$ for large n. But Q^n is a monomial ideal, so the conclusion follows immediately from (3.7) and (3.8) with $p(x) = q(x) - 1$, □

In (3.11) it is shown that if $v(Q) = n$, where Q is as in (3.8), then $n - 1 \leq v(Q : X) \leq 2n - 1$ and for each integer t between $n - 1$ and $2n - 1$ the ideal Q can be chosen so that $v(Q : X) = t$.

(3.11) Proposition. *With the notation of (3.8) assume that $v(Q) = n$. Then $n - 1 \leq v(Q : X) \leq 2n - 1$, and for each intermediate integer t there exists an ideal Q such that $v(Q) = n$ and $v(Q : X) = t$.*

Proof. (3.8) shows that $(Q : X)/Q$ is generated by $v(Q) - 1 = n - 1$ elements, so it follows that $Q : X$ can be generated by the preimages of these $n - 1$ elements together with the n generators of Q. Therefore $n - 1 \leq v(Q : X) \leq 2n - 1$.

Now let t be a given positive integer such that $n - 1 \leq t \leq 2n - 1$ and let s be the integer such that $t = (n - 1) + s$, so $0 \leq s \leq n$. For $i = 1, \ldots, s$ let $f_i = x^{2(i-1)}y^{n+s-2i}$, for $i = s + 1, \ldots, n$ let $f_i = x^{s+i-1}y^{n-i}$, and let $Q = (f_1, \ldots, f_n)R$. Then the Q -corner-elements are the elements $z_j = x^{2j-1}y^{n+s-2j-1}$ (for $j = 1, \ldots, s$) and the elements $z_j = x^{s+j-1}y^{n-1-j}$ (for $j = s+1, \ldots, n-1$). If $s = 0$, then $f_1 \in z_1R$, $f_i \in z_{i-1}R \cap z_iR$ for $i = 2, \ldots, n-1$, and $f_n \in z_{n-1}R$, and if $s > 0$, then $f_i \in z_{i-1}R$ for $i = s+1, \ldots, n$ and $f_i \notin (z_1, \ldots, z_{n-1})R$ for $i = 1, \ldots, s$, so by (2.2.1) (and (3.8)) it readily follows that $f_1, \ldots, f_s, z_1, \ldots z_{n-1}$ is a minimal basis of $Q : X$ so $v(Q : X) = s + n - 1 = t$, □

(3.12) Corollary. *Let $(R, M = (x, y)R)$ be a regular local ring of altitude two, let $Q \neq M$ be an open monomial ideal in x and y, and let $n = v(Q)$. Then $v((Q : M)/Q) = n - 1$, $n - 1 \leq v(Q : M) \leq 2n - 1$, and for each intermediate integer t there exists an ideal Q such that $v(Q) = n$ and $v(Q : M) = t$.*

Proof. This follows immediately from (3.8) and (3.11), since $M = X$, □

In (3.13) we give a geometric interpretation of the I -corner-elements for a monomial ideal I in an R -sequence x, y of length two such that $Rad(I) = Rad((x, y)R)$.

(3.13) Geometric Interpretation. Assume that $d = 2$, let $x = x_1$ and $y = x_2$, let $f_1, \ldots, f_n$ be a minimal basis of I (where the f_l are monomials in x and y, say $f_l = x^{i_l}y^{j_l}$), and assume that $Rad(I) = Rad((x, y)R)$. Lexicographically order the f_l (as in the proof of (3.8)) and assume that $f_1 < \cdots < f_n$. Plot the n points (i_l, j_l) (corresponding to the f_l) in the first quadrant of the xy-plane. Then for each of these n points draw the horizontal line segment connecting (i_l, j_l), $(i_l + 1, j_l)$, $(i_l + 2, j_l), \ldots,$ and draw the vertical line segment connecting (i_l, j_l), $(i_l, j_l + 1)$, $(i_l, j_l + 2), \ldots.$ (Then it is clear that there is a one-to-one correspondence from the set $\mathbf{D} = \{(a, b); a \geq i_l \text{ and } b \geq j_l \text{ for some } l = 1, \ldots, n\}$ to a subset $\mathbf{M}$ of the set of monomials in Q, and it follows from (2.2.1) that, in fact, every monomial in Q is in $\mathbf{M}$.) Since $(i_l, j_l) < (i_{l+1}, j_{l+1}), (i_{l+1}, j_l)$ are the coordinates of the intersection of the rightward extending horizontal line segment thru (i_l, j_l) with the ascending vertical line segment thru (i_{l+1}, j_{l+1}). Then $z_l = x^{i_{l+1}-1}y^{j_l-1} \notin Q$, $z_l y$ has coordinates on the rightward extending horizontal line segment thru (i_l, j_l) (so $z_l y \in Q$), and $z_l x$ has coordinates on the ascending vertical line segment thru (i_{l+1}, j_{l+1}) (so $z_l x \in Q$), hence z_l is a Q -corner-element. And since a Q -corner-element must correspond to some (a, b) with $0 \leq a < i_n$ and $0 \leq b < j_1$, it is readily checked that all Q -corner-elements are obtained in this way, so there are exactly $n - 1$ of them, where $n = v(Q)$.

(3.14) Algebraic Construction. Let $x_1, \ldots, x_d$ be an R -sequence and let I be a monomial ideal such that $Rad(I) = Rad((x_1, \ldots, x_d)R)$. Then the following is an algebraic construction of the I -corner-elements. (For ease of description it will be said that $\deg(f) = n$ if $f = x_1^{e_1} \cdots x_d^{e_d}$ and $e_1 + \cdots + e_d = n$.) Let S be the set of monomials (in $x_1, \ldots, x_d$) that are not in I (so S is a finite set, since for $i = 1, \ldots, d$ there exists a positive integer n_i such that $x_i^{n_i} \in I$). Let $w = \max\{n; n = \deg(f) \text{ for some } f \in S\}$. For $j = 1, \ldots, w$ let $D_j = \{f \in S; \deg(f) = j\}$, let $C_w = D_w$, and for $j = 1, \ldots, w - 1$ let $C_j = \{f \in D_j; fx_i \notin D_{j+1} \text{ for } i = 1 \ldots, d\}$ (possibly some of the sets C_j are empty for $j < w$). Then $C_l \cup \cdots \cup C_w$ is the set of I -corner-elements (and this union is disjoint).

Proof. Let $f \in C_j$ for some $j = 1, \ldots, w$. Then $f \notin I$ (since $f \in C_j \subseteq S$) and for $i = 1, \ldots, d$ it holds that $\deg(fx_i) = j+1$. If $j = w$, then $fx_i \notin S$ (for no element in S has degree greater than $w = j$), and if $j < w$, then $fx_i \notin D_{j+1} = \{g \in S; \deg(g) = j + 1\}$ (by the definition of C_j). Therefore in either case $(j = w \text{ or } j < w) fx_i \notin S$ for $i = 1, \ldots, d$, so $fx_i \in I$, hence f is an I -corner-element. Therefore $C_1 \cup \cdots \cup C_w$ $\subseteq \mathbf{C} = \{f; f \text{ is an } I \text{ -corner-element}\}$.

And if $g \in \mathbf{C}$, then $g \notin I$, so $g \in S$, so $g \in D_j$, where $j = \deg(g)$. Also, $\deg(gx_i) = j+1$ and $gx_i \in I$ for $i = 1, \dots, d$, so $gx_i \notin D_{j+1}$. Therefore $g \in C_j$, so $\mathbf{C} \subseteq C_1 \cup \cdots \cup C_w$, $\square$

(3.15) Remark. With the notation of (3.14) let f be a monomial that is not in I. Then there exists a monomial g (possibly $g = 1$) such that fg is an I -corner-element.

Proof. It may be assumed that f is not an I -corner-element, so $fx_i \notin I$ for some $i = 1, \dots, d$. Let $T = \{g; g$ is a monomial in $x_1, \dots, x_d$ and $fg \notin I\}$. Then T is a finite set (since T is contained in the finite set S of (3.14)), so let $g \in T$ such that the sum of its exponents is greater than or equal to the sum of the exponents of the other monomials in T. Then $fgx_i \in I$ for $i = 1, \dots, d$, by the maximality of the sum of the exponents of g, so fg is an I -corner-element, $\square$

Before giving some examples of I -corner-elements, we first prove one more result concerning them. (Some additional properties are given in (4.11)-(4.12).)

(3.16) Proposition. *Let $I \subset J$ be monomial ideals such that $Rad(I) = Rad(X)$. Then some I -corner-element is in J.*

Proof. There exists a monomial $f \in J - I$, by hypothesis. Then (3.15) shows that there exists a monomial g (possibly $g = 1$) such that fg is an I -corner-element, and it is clear that $fg \in J$, $\square$

This section will be closed with several examples of Q -corner-elements for an open monomial ideal Q in a regular local ring.

(3.17) Example. Let $(R, M = (x,y)R)$ be a regular local ring of altitude two, let $x_1 = x$ and $x_2 = y$, and let $Q = (y^9, xy^7, x^3y^4, x^5y^2, x^{11})R$, so $f_1 = y^9$, $f_2 = xy^7$, $f_3 = x^3y^4$, $f_4 = x^5y^2$, $f_5 = x^{11}$. Then the Q -corner-elements are $z_1 = y^8$, $z_2 = x^2y^6$, $z_3 = x^4y^3$, and $z_4 = x^{10}y$. (This can be checked by using either (3.13) or (3.14).) Therefore $(Q : M)/Q = (y^8, x^2y^6, x^4y^3, x^{10}y)R/Q$ by (3.7).

(3.18) Example. Let $(R, M = (x,y,z)R)$ be a regular local ring of altitude three, let $x_1 = x$, $x_2 = y$, and $x_3 = z$, and let $Q = (z^4, y^2z^3, y^3, xyz, xy^2, x^2)R$. Then the Q -corner-elements are yz^3, y^2z^2, xz^3, and xy. (This can be checked by writing down the sets S, D_j, and C_j (for $j = 1, \ldots, 4$) of (3.14). Thus $S = \{z, z^2, z^3, y, yz, yz^2, yz^3, y^2, y^2z, y^2z^2, x, xz, xz^2, xz^3, xy\}$ (in lexicographic order), $D_1 = \{z, y, x\}$, $D_2 = \{z^2, yz, y^2, xz, xy\}$, $D_3 = \{z^3, yz^2, y^2z, xz^2\}$, and $D_4 = \{yz^3, y^2z^2, xz^3\}$. Then $C_4 = D_4$, $C_3 = \emptyset$ (since at least one of fx, fy, fz is in D_4 for each $f \in D_3$), $C_2 = \{xy\}$ (since at least one of fx, fy, fz is in D_3 for $f \in \{z^2, yz, y^2, xz\}$ and none of xyx, xy^2, xyz is in D_3) and $C_1 = \emptyset$ (since at least one of fx, fy, fz is in D_2 for each $f \in D_1$).)

(3.19) Example. Let $(R, M = (w, x, y, z)R)$ be a regular local ring of altitude four, let $x_1 = w$, $x_2 = x$, $x_3 = y$, and $x_4 = z$, and let $Q = (z^5, yz^4, y^2z^2, y^3, xz^2, xyz, x^3z, x^3y^2, x^4, w)R$. Then the Q -corner-elements are $z^4, yz^3, y^2z, x^2z, x^2y^2$, and x^3y. (This can be checked by using (3.14).)

(3.20) Example. Let $(R, M = (w, x, y, z)R)$ be a regular local ring of altitude four, let $x_1 = w$, $x_2 = x$, $x_3 = y$, and $x_4 = z$, and let $Q = (x^d, y^c, x^b, wxyz, w^a)R$, where $a > 1$, $b > 1$, $c > 1$, $d > 1$ are integers. Then the Q -corner-elements are $x^{b-1}y^{c-1}z^{d-1}$, $w^{a-1}y^{c-1}z^{d-1}$, $w^{a-1}x^{b-1}z^{d-1}$, and $w^{a-1}x^{b-1}y^{c-1}$. (This can be checked by using (3.14).)

(3.21) Example. Let $(R, M = (x_1, \ldots, x_d)R)$ be a regular local ring of altitude d and let $Q = (x_1^{a_1}, \ldots, x_d^{a_d})R$, where the a_i are positive integers. Then Q is irreducible, so by (4.3) there is only one Q -corner-element, namely $z = x_1^{a_1-1} \cdots x_d^{a_d-1}$.

4. Parametric Decompositions of Monomial Ideals. (2.4) (together with (3.4)) shows that X^n is the irredundant finite intersection of the parameter ideals $P(z)$, where z is an X^n -corner-element. The main result in this section, (4.1), generalizes this to an arbitrary monomial ideal I such that $Rad(I) = Rad(X)$. And in (4.10) we show that such a decomposition is unique.

(4.1) Theorem. *Let I be a monomial ideal such that $Rad(I) = Rad(X)$ and let $z_1, \ldots, z_m$ be the I -corner-elements. Then $I = \cap\{P(z_j); j = 1, \ldots, m\}$ is a decomposition of I as an irredundant intersection of parameter ideals.*

Proof. Let $J = \cap\{P(z_j); j = 1, \ldots, m\}$. Then (3.3) shows that J is the irredundant intersection of the m parameter ideals $P(z_j)$.

Now let f be a monomial in I and suppose that $f \notin P(z_j)$ for some $j = 1, \ldots, m$. Then $z_j \in fR \subseteq I$, by (2.3), and this contradicts the fact that $z_j \notin I$ (since z_j is an I -corner-element). Therefore $I \subseteq J$.

Finally, [T, Lemma 6] shows that J is a monomial ideal (since $Rad(P(z_j)) = Rad(X)$ for $j = 1, \dots, m$), so it suffices to show that each monomial that is not in I is not in J. For this, let f be a monomial that is not in I. Then (3.15) shows that there exists a monomial g (possibly $g = 1$) such that fg is an I -corner-element, so $fg = z_j$ for some $j = 1, \dots, m$ (since (3.2) shows that the I -corner-elements are finite in number and uniquely determined by I). Then $f \notin P(z_j)$, by (2.3), so it follows that $I \supseteq J$, hence $I = J$ by the preceding paragraph, □

In (4.2) it is shown that the corner-elements of a monomial ideal I determine I when $Rad(I) = Rad(X)$.

(4.2) Corollary. *If I and J are monomial ideals such that $Rad(I) = Rad(X) = Rad(J)$ and if $(I : X) - I = (J : X) - J$, then $I = J$.*

Proof. If $(I : X) - I = (J : X) - J$, then I and J have the same corner-elements, by (3.2), so this follows immediately from (4.1), □

(4.3) Corollary. *If Q is an open monomial ideal in a Gorenstein local ring R of altitude $d > 1$, then Q is irreducible if and only if there exists exactly one Q -corner-element, and then Q is generated by a system of parameters.*

Proof. Let m be the number of Q -corner-elements. Then Q is the irredundant intersection of m (open) parameter ideals, by (4.1). Since R is Gorenstein, an open parameter ideal is irreducible, so Q is the irredundant intersection of m (open) irreducible ideals. Since each such decomposition of Q has the same number of factors, $m = 1$ if and only if Q is irreducible.

For the final statement, if Q is irreducible, then $Q = P(z)$ is generated by a system of parameters, where z is the Q -corner-element, □

The next corollary is closely related to (2.5) and (4.3).

(4.4) Corollary. *Let I and $z_1, \dots, z_m$ be as in (4.1) and assume that R is a Gorenstein local ring of altitude d. Then $I = \cap\{P(z_j); j = 1, \dots, m\}$ is an irredundant intersection of m irreducible ideals.*

Proof. If R is Gorenstein, then each open parameter ideal is irreducible, so the conclusion follows from (4.1), □

In [T, Theorem 8] it is shown (among other things) that if $Rad(I) = Rad(X)$, then $\cup\{P; P \in Ass(R/I)\} = \cup\{Q; Q \in Ass(R/X)\}$. (4.1) yields a simple proof of the following closely related result.

(4.5) Corollary. *If I is as in (4.1), then $\cup\{Ass(R/I^n); n \geq 1\} \subseteq Ass(R/X)$.*

Proof. It is well known that if Y and Z are ideals that are generated by R-sequences such that $Rad(Y) = Rad(Z)$, then $Ass(R/Y) = Ass(R/Z)$. It therefore follows that if $z_1, \ldots, z_m$ are the I-corner-elements, then $Ass(R/X) = Ass(R/P(z_j))$ for $j = 1, \ldots, m$ (since each $P(z_j)$ is generated by powers of $x_1, \ldots, x_d$). Therefore, since $I = \cap\{P(z_j); j = 1, \ldots, m\}$, it follows that $Ass(R/I) = Ass(R/(\cap\{P(z_j); j = 1, \ldots, m\})) \subseteq \cup\{Ass(R/P(z_j)); j = 1, \ldots, m\} = Ass(R/X)$. Finally, I^n is generated by monomials for all $n \geq 1$, so it follows from what was just shown that $Ass(R/I^n) \subseteq Ass(R/X)$, $\square$

For the proof of the next corollary we need the following definition.

(4.6) Definition. If P is a prime divisor of an ideal I in a Noetherian ring, then $\mathbf{D_P(I)}$ denotes the number of P-primary ideals in a decomposition of I as an irredundant intersection of irreducible ideals.

Concerning (4.6), a classical result of E. Noether [N, Satz VII] says that $D_P(I)$ is well defined (that is, $D_P(I)$ is independent of the particular irredundant irreducible decomposition of I).

(4.7) Corollary. *Let (R, M) be a Gorenstein local ring, let X be an ideal generated by a system of parameters $x_1, \ldots, x_d$, and let Q be an open monomial ideal. Then $v((Q : M)/Q) = v((Q : X)/Q)$.*

Proof. Let m be the number of Q-corner-elements. Then (3.7) shows that $v((Q : X)/Q) = m$, and (4.1) shows that Q is the irredundant intersection of m parameter ideals. Since R is Gorenstein, each of these parameter ideals is irreducible, so Q is the irredundant intersection of m irreducible ideals, so $D_M(Q) = m$ (see (4.6)). However, [HRS2, (2.4)] shows that $D_M(Q) = v((Q : M)/Q)$. Therefore $v((Q : X)/Q) = m = v((Q : M)/Q)$, $\square$

(4.1) shows that the I-corner-elements determine a decomposition of I as an irredundant intersection of parameter ideals. (4.8) shows that the converse also holds.

(4.8) Proposition. *For $j = 1, \dots, m$ let $\mathbf{a}_j = (a_{j,1}, \dots, a_{j,d})$ be a d-tuple of positive integers and let $I = \cap\{P(\mathbf{a}_j); j = 1, \dots, m\}$ be a decomposition of I as an irredundant intersection of parameter ideals. Then the I -corner-elements are the m elements $x_1^{a_{j,1}-1} \cdots x_d^{a_{j,d}-1}$.*

Proof. (Note: $Rad(I) = Rad(X)$, since $Rad(P(\mathbf{a}_j)) = Rad(X)$ for $j = 1, \dots, m$.)

It will first be shown that each of the m elements $z_j = x_1^{a_{j,1}-1} \cdots x_d^{a_{j,d}-1}$ is an I -corner-element.

For this, note first that $P(z_j) = P(\mathbf{a}_j)$ for $j = 1, \dots, m$. Therefore $z_i \notin z_jR$ for all $i \neq j \in \{1, \dots, m\}$ (for if $z_i \in z_jR$, then $P(\mathbf{a}_i) = P(z_i) \subseteq P(z_j) = P(\mathbf{a}_j)$, and this is a contradiction). Therefore (2.3) shows that $z_j \notin P(z_j) = P(\mathbf{a}_j)$ (so $z_j \notin I$) and that $z_j \in P(z_k) = P(\mathbf{a}_k)$ for $k \in \{1, \dots, j-1, j+1, \dots, m\}$. Also, $z_jx_i \in x_i^{(a_{j.1}-1)+1}R \subseteq P(\mathbf{a}_j)$ for $i = 1, \dots, d$, so $z_jx_i \in \cap\{P(\mathbf{a}_h); h = 1, \dots, m\} = I$. Therefore z_j is an I -corner-element, so it follows that $z_1, \dots, z_m$ are among the I -corner-elements.

Now let w be an I -corner-element. Then $w \notin I$, so $w \notin P(z_j) = P(\mathbf{a}_j)$ for some $j = 1, \dots, m$. Therefore $z_j \in wR$, by (2.3), so $z_j = wg$ for some monomial g by (2.2.2). If $g \neq 1$, then $wg \in I$, since w is an I -corner-element. But this implies that $z_j \in I$, and this contradicts the fact that z_j is an I -corner-element. Therefore $g = 1$, so $w = z_j$, so $z_1, \dots, z_m$ are all the I -corner-elements, $\square$

(4.9) Corollary. *Let $z_1, \dots, z_m$ be monomials such that $z_i \notin z_jR$ for $i \neq j \in \{1, \dots, m\}$, let $J = (z_1, \dots, z_m)R$, and let $I = \cap\{P(z_j); j = 1, \dots, m\}$. Then $z_1, \dots, z_m$ are the I -corner-elements and $I : X = I + J$.*

Proof. If $\cap\{P(z_j); j = 1, \dots, m\}$ is an irredundant intersection, then it follows from (4.8) that the I -corner-elements are the elements $z_1, \dots, z_m$, so $I : X = I + J$ by (3.2). Therefore it remains to show that this intersection is irredundant.

For this, suppose, on the contrary, that it is redundant. Then by resubscripting the z_j, if necessary, it may be assumed that $I = \cap\{P(z_j); j = 1, \dots, k\}$ for some $k < m$ (so $z_1, \dots, z_k$ are the I -corner-elements, by (4.8)). Then $z_m \notin I$, since $z_m \notin P(z_m) \supseteq I$, so (3.15) shows that there exists a monomial g such that gz_m is an I -corner-element. Therefore $gz_m = z_i$ for some $i = 1, \dots, k$, and this contradicts the hypothesis that $z_i \notin z_j R$ for $i \neq j \in \{1, \ldots, m\}$. Therefore the intersection is irredundant, $\square$

In (4.10) we show that a decomposition as in (4.1) of a monomial ideal is unique.

(4.10) Theorem (Unique Factorization). *Let $z_1, \ldots, z_m$ and $w_1, \ldots, w_n$ be monomials such that $\cap\{P(z_j); j = 1, \ldots, m\} = \cap\{P(w_i); i = 1, \ldots, n\}$ are irredundant intersections of parameter ideals. Then $n = m$ and $\{z_1, \ldots, z_m\} = \{w_1, \ldots, w_n\}$.*

Proof. Let $I = \cap\{P(z_j); j = 1, \ldots, m\}$. Then it follows from (4.8) that $z_1, \ldots, z_m$ are the I -corner-elements, so they are the monomials in $(I : X) - I$ by (3.2). However, $I = \cap\{P(w_i); i = 1, \ldots, n\}$, by hypothesis, so similar statements hold for $w_1, \ldots, w_n$ in place of $z_1, \ldots, z_m$, hence it follows that $n = m$ and that $\{w_1, \ldots, w_n\}$ $= \{z_1, \ldots, z_m\}$, $\square$

In (4.11) we note two additional results concerning I -corner-elements.

(4.11) Proposition. *Assume that $x_1, \ldots, x_d$ is a permutable R -sequence, let I be a monomial ideal such that $Rad(I) = Rad(X)$, let $z_1, \ldots, z_m$ be the I -corner -elements, and let f be a monomial. Then:*

(4.11.1) *$fI = (\cap\{P(fz_j); j = 1, \ldots, m\}) \cap fR$ and $fz_1, \ldots, fz_m$ are fI -corner-elements.*

(4.11.2) *$I : fR = \cap\{P(w_j); j = 1, \ldots, k\}$, where $z_j = w_j f$ for $j = 1, \ldots, k$ and $z_j \notin fR$ for $j = k+1, \ldots, m$ (for some $k \in \{0, 1, \ldots, m\}$) and $w_1, \ldots, w_k$ are the $I : fR$ -corner-elements.*

Proof. For (4.11.1), since each permutation of $x_1, \ldots, x_d$ is an R -sequence, each monomial is regular. Therefore since $z_j \in (I : X) - I$, by (3.2), it is readily checked that $fz_j \in (fI : X) - fI$ for $j = 1, \ldots, m$, so each fz_j is an fI -corner-element by (3.2). Also, $P(fz_j) : fR = P(z_j)$, for if $f = x_1^{b_1} \cdots x_d^{b_d}$ and $z_j = x_1^{a_1} \cdots x_d^{a_d}$, then $P(fz_j) : fR = (x_1^{a_1+b_1+1}, \ldots, x_d^{a_d+b_d+1})R : x_1^{b_1} \cdots x_d^{b_d} R$ $= (x_1^{a_1+1}, x_2^{a_2+b_2+1}, \ldots, x_d^{a_d+b_d+1})R : x_2^{b_2} \cdots x_d^{b_d} R = \cdots = (x_1^{a_1+1}, \ldots, x_d^{a_d+1})R =$ $P(z_j)$. Therefore it follows that $(\cap\{P(fz_j); j = 1, \ldots, m\}) \cap fR = f[(\cap\{P(fz_j); j = 1, \ldots, m\}) : fR] = f[\cap\{P(fz_j) : fR; j = 1, \ldots, m\}] = f[\cap\{P(z_j); j = 1, \ldots, m\}] =$ fI, the last equality by (4.1).

For (4.11.2), $z_j \in fR$ if and only if $f \notin P(z_j)$, by (2.3). Therefore if $z_j = fw_j$ for $j = 1, \ldots, k$ and if $z_j \notin fR$ (so $f \in P(z_j)$) for $j = k+1, \ldots, m$, then it follows from (4.1) that $I : fR = (\cap\{P(z_j); j = 1, \ldots, m\}) : fR = \cap\{P(z_j) : fR; j = 1, \ldots, m\} = \cap\{P(w_j f) : fR; j = 1, \ldots, k\} = \cap\{P(w_j); j = 1, \ldots, k\}$. Finally, it follows from (4.8) that if $\cap\{P(w_j); j = 1, \ldots, k\}$ is an irredundant intersection, then $w_1, \ldots, w_k$ are the $I : fR$ -corner-elements, so it remains to show that this intersection is irredundant.

For this, suppose that the intersection is redundant, so (by resubscripting, if necessary) there exists $h < k$ such that $I : fR = \cap\{P(w_j); j = 1, \ldots, h\}$ is an irredundant intersection, so $w_1, \ldots, w_h$ are the $I : fR$ -corner-elements by (4.8). Therefore either (a) $w_k \in I : fR$; or, (b) $w_k \notin I : fR$. If (b) holds, then gw_k is an $I : fR$ -corner-element for some monomial g by (3.15), so $gw_k = w_j$ for some $j = 1, \ldots, h$ (so $g \neq 1$). Therefore $gz_k = fgw_k = fw_j = z_j$, so $z_j = gz_k \in I$ (by the definition of I -corner-element, since $g \neq 1$ is a monomial), and this contradicts the fact that z_j in an I -corner-element. Therefore (b) does not hold, so (a) holds, hence $z_k = fw_k \in I$, and this contradicts the fact that z_k is an I -corner-element. Therefore neither (a) nor (b) holds, so it follows that $\cap\{P(w_j); j = 1, \ldots, k\}$ is an irredundant intersection, hence $w_1, \ldots, w_k$ are the $I : fR$ -corner-elements, $\square$

(4.12) Corollary. *With the notation of (4.11), let $J = (f_1, \ldots, f_n)R$ be a monomial ideal and let $w_{j,i}$ be monomials such that $z_j = w_{j,i} f_i$ (if $z_j \in f_i R$) or $w_{j,i} = 1$ (if $z_j \notin f_i R$). Then $I : J = \cap\{P(w_{j,i}); j = 1, \ldots, m$ and $i = 1, \ldots, n\}$, so the $I : J$ -corner-elements are among the mn monomials $w_{j,i}$.*

Proof. If $w_{j,i} = 1$, then $P(w_{j,i}) = X$, and X contains all other parameter ideals. Therefore the conclusion follows from (4.11.2) and the fact that $I : J = \cap\{I : f_i R; i = 1, \ldots, n\}$, $\square$

To prove the next theorem, which gives an irredundant parametric decomposition of the ideal generated by the k-th powers of the monomial generators of a monomial ideal, we need the following definition and lemma.

(4.13) Definition. If $J = (f_1, \ldots, f_n)R$ is a monomial ideal and k is a positive integer, then $\mathbf{J}^{[k]}$ denotes the ideal $(f_1^k, f_2^k, \ldots, f_n^k)R$.

(4.14) Lemma. *Let J be a monomial ideal, let g be a monomial, and let k be a positive integer. Then $g \in J$ if and only if $g^k \in J^{[k]}$.*

Proof. It is clear that $g \in J$ implies that $g^k \in J^{[k]}$.

For the converse assume that $g^k \in J^{[k]}$. Let $J = (f_1, \ldots, f_n)R$, where each f_i is a monomial. Then the hypothesis and (2.2.1) imply that $g^k \in f_i^k R$ for some $i = 1, \ldots, n$. Now g^k and f_i^k are monomials in the R -sequence $x_1^k, \ldots, x_d^k$, so by (2.2.2) there exists a monomial s in $x_1^k, \ldots, x_d^k$ such that $g^k = s f_i^k$. Then it is clear that there exists a monomial t in $x_1, \ldots, x_d$ such that $t^k = s$, so $g^k = t^k f_i^k$, hence $g = t f_i \in J$, as desired, $\square$

(4.15) Theorem. *Let I be a monomial ideal such that $Rad(I) = Rad(X)$, let $z_1, \ldots, z_m$ be the I -corner-elements, and let k be a positive integer. Then $I^{[k]} = \cap\{(P(z_j))^{[k]}; j = 1, \ldots, m\}$ is a decomposition of $I^{[k]}$ as an irredundant intersection of parameter ideals.*

Proof. Let $I = (f_1, \ldots, f_n)R$ and note that $x_1^k, \ldots, x_d^k$ is an R -sequence. Therefore since each f_i is a monomial (in $x_1, \ldots, x_d$) and since $I^{[k]} = (f_1^k, \ldots, f_n^k)R$, it follows that $I^{[k]}$ is generated by monomials in $x_1^k, \ldots, x_d^k$, and $Rad(I^{[k]}) = Rad(X^{[k]})$ (since $Rad(I) = Rad(X)$). Also, for each $j = 1, \ldots, m$ it holds that z_j is a monomial in $x_1, \ldots, x_d$ such that $z_j \notin I$ and $z_j x_i \in I$ for $i = 1, \ldots, d$, so it follows that z_j^k is a monomial in $x_1^k, \ldots, x_d^k$ such that $z_j^k \notin I^{[k]}$ (by (4.14)) and $z_j^k x_i^k \in I^{[k]}$ for $i = 1, \ldots, d$. Therefore the m elements $z_1^k, \ldots, z_m^k$ are among the $I^{[k]}$ -corner-elements (for the R -sequence $x_1^k, \ldots, x_d^k$).

Now let z^* be an $I^{[k]}$ -corner-element (for the R -sequence $x_1^k, \ldots, x_d^k$), so z^* is a monomial in $x_1^k, \ldots, x_d^k$ such that $z^* \notin I^{[k]}$ and $z^* x_i^k \in I^{[k]}$ for $i = 1, \ldots, d$. Then it is clear that there exists a monomial z in $x_1, \ldots, x_d$ such that $z^k = z^*$, so $z \notin I$ (since $z^k \notin I^{[k]}$) and $z x_i \in I$ (by (4.14), since $z^k x_i^k \in I^{[k]}$). Therefore z is an I -corner-element, so $z = z_p$ for some $p = 1, \ldots, m$, so $z^* = z^k = z_p^k$. Therefore it follows that $z_1^k, \ldots, z_m^k$ are all the $I^{[k]}$ -corner-elements (for $x_1^k, \ldots, x_d^k$) so it follows from (4.1) that $I^{[k]} = \cap\{P(z_j^k); j = 1, \ldots, m\}$.

Finally, fix $j \in \{1, \ldots, m\}$ and let $z_j = x_1^{a_1} \cdots x_d^{a_d}$. Then $z_j^k = x_1^{k a_1} \cdots x_d^{k a_d} = (x_1^k)^{a_1} \cdots (x_d^k)^{a_d}$, so it follows from (2.1.2) that $P(z_j^k) = ((x_1^k)^{a_1+1}, \ldots, (x_d^k)^{a_d+1})R$ and that $P(z_j) = (x_1^{a_1+1}, \ldots, x_d^{a_d+1})R$, so it follows that $P(z_j^k) = (P(z_j))^{[k]}$.

Therefore it follows from the preceding paragraph that $I^{[k]} = \cap\{(P(z_j))^{[k]}); j = 1, \ldots, m\}$, $\square$

The final result in this section shows that the I -corner-elements determine the $I^{[k]}$ -corner-elements.

(4.16) Corollary. *With the notation of (4.15), $c(I) = c(I^{[k]})$ (see (3.9)). More specifically, if $z_1, \ldots, z_m$ are the I -corner-elements, and if $z_j = x_1^{a_{j,1}} \cdots x_d^{a_{j,d}}$, then the $I^{[k]}$ -corner-elements are the m monomials $z_j^{(k)} = x_1^{ka_{j,1}+k-1} \cdots x_d^{ka_{j,d}+k-1}$.*

Proof. This follows immediately from (4.15) and (4.8), $\square$

5. A Related Irredundant Irreducible Decomposition. Let I be a monomial ideal in a local ring (R, M) such that $Rad(I) = Rad(X)$. Then the main result in this section, (5.1), gives a decomposition of $I + M(I : X)$ that is closely related to (4.1).

(5.1) Theorem. *Assume that R is local with maximal ideal M, let I be a monomial ideal such that $Rad(I) = Rad(X)$, let $z_1, \ldots, z_m$ be the I -corner-elements, and for $j = 1, \ldots, m$ let Q_j be maximal in $S_j = \{Q; Q$ is an ideal in $R, P(z_j) \subseteq Q$, and $z_j \notin Q\}$. Then each Q_j is irreducible, $\bigcap_{j=1}^{m} Q_j$ is an irredundant intersection, and $(\bigcap_{j=1}^{m} Q_j) \cap (I : X) = I + M(I : X)$.*

Proof. Fix $j \in \{1, \ldots, m\}$. Then $P(z_j) \in S_j$, by (2.3), so S_j is not empty, so there exists an ideal Q_j that is maximal with respect to being in S_j. Then each ideal that properly contains Q_j must contain z_j, so Q_j is irreducible.

Also, $z_j \in P(z_i) \subseteq Q_i$ for $i \in \{1, \ldots, j-1, j+1, \ldots, m\}$ (by (3.3)) and $z_j \notin Q_j$, so $\bigcap_{j=1}^{m} Q_j$ is an irredundant intersection.

Further, since $z_j M \subset z_j R$, it follows that $z_j M \subseteq Q_j$, and $I + (z_1, \ldots, z_{j-1}, z_{j+1}, \ldots z_m)R \subseteq P(z_j) \subseteq Q_j$, by (3.3) and (4.1), so it follows from (3.7) that $I + M(I : X) \subseteq I + (z_1, \ldots, z_{j-1}, z_{j+1}, \ldots z_m)R + z_j M \subseteq Q_j$. Therefore it follows that $I + M(I : X) \subseteq (\bigcap_{j=1}^{m} Q_j) \cap (I : M)$.

Finally, if $y \in (\bigcap_{j=1}^{m} Q_j) \cap (I : M)$, then $y = \sum_{i=1}^{n} s_i f_i + \sum_{j=1}^{m} r_j z_j$ (by (3.7), where $I = (f_1, \ldots, f_n)R)$ and y is in each Q_j. However, since $I + (z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_m)R + z_j M \subseteq Q_j$ and $z_j \notin Q_j$, it follows that $r_j z_j \in Q_j$, hence $r_j \in Q_j : z_j R = M$. Since this holds for each $j = 1, \ldots, m$ it follows that $y \in I + M(I : X)$, $\square$

(5.2) Remark. It is readily checked that the following are equivalent for two ideals J and Y in a local ring (R, M): (a) $M(J : Y) \subseteq J$; (b) $J : Y = J : M$; (c) $J : (J : Y) = M$. If any of (a) - (c) hold with I and X in place of J and Y, then $X = M$ and R is a regular local ring.

Proof. It follows from [T, Theorem 6] that $I : (I : X)$ is generated by monomials, so if any of (a) - (c) hold, then in particular (c) holds, so M is generated by monomials. But every ideal generated by monomials is contained in X, so $M = X$ is generated by an R -sequence, hence R is a regular local ring, □

(5.3) Corollary. *With the notation of (5.1) assume that R is a regular local ring with maximal ideal $M = X$. Then $I = \cap\{Q_j; j = 1, \ldots, m\}$ is an irredundant irreducible decomposition of I and $Q_j = P(z_j)$ for $j = 1, \ldots, m$.*

Proof. Since $X = M$, $M(I : X) \subseteq I$, so (5.1) shows that $(\overset{m}{\underset{j=1}{\cap}} Q_j) \cap (I : M) = I$, hence $(\overset{m}{\underset{j=1}{\cap}} (Q_j/I)) \cap ((I : M)/I) = I/I$. Also, $(I : X)/I = (I : M)/I$ is the socle of R/I, and it is shown in [HRS3, (3.3.2)] that $(\overset{m}{\underset{j=1}{\cap}} (Q_j/I)) \cap ((I : M)/I) = (0)$ if and only if $\overset{m}{\underset{j=1}{\cap}} (Q_j/I) = (0)$. It therefore follows that $I = \cap\{Q_j; j = 1, \ldots, m\}$, and (5.1) shows that this is an irredundant intersection of irreducible ideals.

To see that $Q_j = P(z_j)$ for $j = 1, \ldots, m$, fix $j \in \{1, \ldots, m\}$ and note that it is shown in (4.1) that $I = \cap\{P(z_j); j = 1, \ldots, m\}$. Since R is regular, it follows that each parameter ideal $P(z_j)$ is irreducible. Also, $P(z_j) \subseteq Q_j$, by construction (see (5.1)). Further, it is shown in [HRS3, (3.6)] that there are no containment relations among the ideals in $IC(I) = \{q$; there exists an irredundant irreducible decomposition of I with q as a factor$\}$. Therefore it follows that $Q_j = P(z_j)$ for $j = 1, \ldots, m$, □

(5.4) Remark. If either $M \neq X$ or if R is a Gorenstein local ring, but not regular, in (5.3), then the parameter ideal $P(z_j)$ is irreducible and every monomial ideal that contains $P(z_j)$ must contain z_j (by (3.16), since z_j is the unique $P(z_j)$ -corner-element, by (4.3)). However, there are ideals that contain $P(z_j)$ that are not monomial ideals, so the unique cover of $P(z_j)$ is properly contained in $(P(z_j), z_j)R$, and hence $P(z_j) \subset Q_j$ in (5.1).

6. Parametric and Irreducible ideals. In this section we prove a few additional results concerning parameter ideals and their relation to irreducible ideals.

(6.1) Proposition. *Consider the following statements about a monomial ideal I such that $Rad(I) = Rad(X)$:*

(6.1.1) *I has exactly one corner-element.*

(6.1.2) *I is a parameter ideal.*

(6.1.3) *I is irreducible.*

Then (6.1.3) $\Rightarrow$ (6.1.1) $\Leftrightarrow$ (6.1.2), and all three statements are equivalent when R is Cohen-Macaulay and $Rad(X) = P$ is a prime ideal such that R_P is a Gorenstein local ring of altitude d.

Proof. Since $Rad(I) = Rad(X)$, (4.1) shows that $I = \cap\{P(z_j); j = 1, \ldots, m\}$, where $z_1, \ldots, z_m$ are the I -corner-elements. Therefore if I is irreducible, then $m = 1$, so (6.1.3) $\Rightarrow$(6.1.1).

(4.1) shows that (6.1.1) $\Rightarrow$ (6.1.2).

Assume that (6.1.2) holds and let $I = (x_1^{a_1}, \ldots, x_d^{a_d})R$. Then $x_1^{a_1-1} \cdots x_d^{a_d-1}$ is the unique I -corner-element (since $I : X = x_1^{a_1-1} \cdots x_d^{a_d-1}R$), so (6.1.2) $\Rightarrow$ (6.1.1).

Finally, if R is Cohen-Macaulay and $Rad(X) = P$ is a prime ideal such that R_P is a Gorenstein local ring, then it readily follows from (4.3) that (6.1.1) $\Rightarrow$ (6.1.3), □

(6.2) Remark. (6.1) provides an alternate proof that open monomial ideals are finite intersections of parameter ideals in Gorenstein local rings. Specifically, let Q be such an ideal. If Q is irreducible, then Q is a parameter ideal by (6.1). If Q is not irreducible, then Q is the intersection of two monomial ideals that properly contain it. (For if $Q = (f_1, \ldots, f_n)R$ and $f_i = x_1^{e_{i,1}} \cdots x_d^{e_{i,d}}$ is such that $e_{i,j} \geq 1$ and $e_{i,k} \geq 1$, then $Q = Q_1 \cap Q_2$, where $Q_1 = (f_1, \ldots, f_{i-1}, x_1^{e_{i,1}} \cdots x_j^{e_{i,j}-1} \cdots x_d^{e_{i,d}}, f_{i+1}, \ldots, f_n)R$ and $Q_2 = (f_1, \ldots, f_{i-1}, x_1^{e_{i,1}} \cdots x_k^{e_{i,k}-1} \cdots x_d^{e_{i,d}}, f_{i+1}, \ldots, f_n)R$.) Therefore by induction on the (finite) number of monomial ideals between X and Q it follows that the open monomial ideals Q_1 and Q_2 are finite intersections of parameter ideals, so Q is.

(6.3) characterizes the parameter ideals that are minimal with respect to containing a given monomial ideal I.

(6.3) Proposition. *Let I be a monomial ideal such that $Rad(I) = Rad(X)$ and let Q be an ideal that is minimal in $\{q; I \subseteq q$ and q is a parameter ideal$\}$. Then $Q = P(z)$ for some I -corner-element z.*

Proof. By (4.1) and (6.1), $Q = P(w)$ for the unique Q -corner-element w. Then $w \notin Q$, so $w \notin I$, hence fw is an I -corner-element for some monomial f by (3.15). Then $I \subseteq P(fw) \subseteq P(w) = Q$, and $P(fw)$ is a parameter ideal. Therefore the definition of Q shows that $P(fw) = P(w)$, so $w = fw$ is an I -corner-element, □

(6.4) Corollary. *Let $I \subseteq J$ be monomial ideals such that $Rad(I) = Rad(X)$, and let $z_1, \dots z_m$ (resp., $w_1, \dots, w_n$) be the I (resp. J) -corner-elements. Then each $P(w_i)$ contains some $P(z_j)$ and then $z_j \in w_i R$.*

Proof. Let $\mathbf{P}(I) = \{q; I \subseteq q$ and q is a parameter ideal$\}$. Then $I \subseteq J \subseteq P(w_i)$ for $i = 1, \dots, n$, by (4.1), so each $P(w_i) \in \mathbf{P}(I)$. Fix $i \in \{1, \dots, n\}$. Then $P(w_i) \in \mathbf{P}(I)$, so $P(w_i)$ contains an ideal q that is minimal in $\mathbf{P}(I)$. Then $q = P(z_j)$ for some I -corner-element z_j, by (6.3), so $P(z_j) \subseteq P(w_i)$, and it is readily checked that this implies that $z_j \in w_i R$, □

In our final result, by "Q is an irreducible component of I" we mean that there exists a decomposition $\cap\{Q_j; j = 1, \dots, m\}$ of I as an irredundant finite intersection of irreducible ideals Q_j such that $Q = Q_j$ for some $j = 1, \dots, m$.

(6.5) Corollary. *Let I be a monomial ideal such that $Rad(I) = Rad(X)$ and let Q be minimal in $\{q;\ I \subseteq q$ and q is an irreducible monomial ideal in $R\}$. If R is a Gorenstein local ring, then Q is an irreducible component of I.*

Proof. If R is a Gorenstein local ring, and if Q is an irreducible monomial ideal, then Q is a parameter ideal, by (6.1), so this follows immediately from (6.3) and (4.1), □

References

[EH]. J. A. Eagon and M. Hochster, *R -sequences and indeterminates*, Quart. J. Math **25** (1974), 61–71.

[HMRS]. W. Heinzer, Ahmad Mirbagheri, L. J. Ratliff, Jr., and K. Shah, *Parametric decompositions of monomial ideals (II)*, (in preparation).

[HRS1]. W. Heinzer, L. J. Ratliff, Jr., and K. Shah, *On the embedded primary components of ideals (I)*, J. Algebra **167** (1994), 724–744.

[HRS2]. ———, *On the embedded primary components of ideals (II)*, J. Pure Appl. Algebra (to appear).

[HRS3]. ———, *On the embedded primary components of ideals (III)*, J. Algebra (to appear).

[HRS4]. ———, *On the embedded primary components of ideals (IV)*, Trans. Amer. Math. Soc. (to appear).

[HS]. C. Huneke and J. Sally, *Birational extensions in dimension two and integrally closed ideals*, J. Algebra **115** (1988), 481–500.

[N]. E. Noether, *Idealtheorie in Ringbereichen*, Math. Ann **83** (1921), 24–66.

[T]. D. Taylor, *Ideals Generated By Monomials In An R-Sequence*, Ph.D. Dissertation, U. Chicago, 1966.

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