PROJECTIVELY FULL IDEALS IN NOETHERIAN RINGS

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Abstract

Let R be a Noetherian commutative ring with unit $1 \neq 0$, and let I be a regular proper ideal of R. The set $\mathbf{P}(I)$ of integrally closed ideals projectively equivalent to I is linearly ordered by inclusion and discrete. There is naturally associated to $\mathbf{P}(I)$ a numerical semigroup $S(I)$; we have $S(I) = \mathbb{N}$ if and only if every element of $\mathbf{P}(I)$ is the integral closure of a power of the largest element J of $\mathbf{P}(I)$. If this holds, the ideal J and the set $\mathbf{P}(I)$ are said to be projectively full. If I is invertible and R is integrally closed, we prove that $\mathbf{P}(I)$ is projectively full. We investigate the behavior of projectively full ideals in various types of ring extensions. We prove that a normal ideal I of a local ring (R, M) is projectively full if $I \not\subseteq M^2$ and both the associated graded ring $G(M)$ and the fiber cone ring $F(I)$ are reduced. We present examples of normal local domains (R, M) of altitude two for which the maximal ideal M is not projectively full.

1 INTRODUCTION.

All rings in this paper are commutative with a unit $1 \neq 0$. Let I be a regular proper ideal of the Noetherian ring R (that is, I contains a regular element of R and $I \neq R$). The concept of projective equivalence of ideals and the study of ideals projectively equivalent to I was introduced by Samuel in [19] and further developed by Nagata in [12]. Making use of interesting work of Rees in [17], McAdam, Ratliff, and Sally in [11, Corollary 2.4] prove that the set $\mathbf{P}(I)$ of integrally closed ideals projectively equivalent to I is linearly ordered by inclusion (and discrete). They also prove that if I and J are projectively equivalent, then the set Rees I of Rees valuation rings of I is equal to the set Rees J of Rees valuation rings of J and the values of I and J with respect to these Rees valuation rings are proportional [11, Proposition 2.10]. We observe in [1] that the converse also holds and further develop the connections between projectively equivalent ideals and their Rees valuation rings. For this purpose, we define in [1] the ideal I to be **projectively full** if the set $\mathbf{P}(I)$ of integrally closed ideals projectively equivalent to I is precisely the set $\{(I^n)_a\}$ consisting of the integral closures of the powers of I. If there exists a projectively full ideal J that is projectively equivalent to I, we say that $\mathbf{P}(I)$ is **projectively full**. As described in [1], there is naturally associated to I and to the projective equivalence class of I a numerical semigroup $S(I)$. One has $S(I) = \mathbb{N}$, the semigroup of nonnegative integers under addition, if and only if $\mathbf{P}(I)$ is projectively full.

Our goal in the present paper is to build on the work in [11] and [1] by further developing the concept of projectively full ideals and examining the numerical semigroup $S(I)$. In Section 2 we present several results relating $\mathbf{P}(I)$ and $\mathbf{P}(IA)$, for certain R-algebras A. In Section 3, these results are applied to explore the relationship between the projective fullness of $\mathbf{P}(I)$ and $\mathbf{P}(IA)$, for certain R-algebras A. Several methods are given for obtaining projectively full ideals. We prove that an integrally closed complete intersection ideal of a local ring is projectively full. We also prove that if I is a proper invertible ideal of an integrally closed Noetherian domain, then $\mathbf{P}(I)$ is projectively full. In Section 4 we present classes of examples of projectively full ideals. For instance, we prove that if R_P is a regular local domain, then PI is projectively full for all regular ideals $I \nsubseteq P$. In Section 5 we present a family of examples of integrally closed local domains (R, M) of altitude two for which M, and therefore $\mathbf{P}(M)$, is not projectively full.

Our notation is as in [13] and [6]. A ring is said to be **integrally closed** if it is integrally closed in its total quotient ring. In particular, a ring that is equal to its total quotient ring is integrally closed. If we are given a ring homomorphism from a ring R to a ring A, then we say that A with respect to this homomorphism is an R-**algebra.**

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2 PROJECTIVELY EQUIVALENT IDEALS.

In this section we prove several elementary results about projectively equivalent ideals. For this, we need the following definitions. (Throughout, $\mathbb{N}$ denotes the set of nonnegative integers, and $\mathbb{N}_+$ (resp., $\mathbb{Q}_+$, $\mathbb{R}_+$) denotes the set of positive integers (resp., rational numbers, real numbers).)

Definition 2.1 Let I be an ideal in a Noetherian ring R.

(2.1.1) The **Rees ring $\mathbf{R}(R, I)$ of R with respect to I** is the graded subring $\mathbf{R}(R, I) = R[u, tI]$ of $R[u, t]$, where t is an indeterminate and $u = 1/t$.

(2.1.2) R' denotes the **integral closure** of R in its total quotient ring.

(2.1.3) If R is local with maximal ideal M, then $a(I)$ denotes the **analytic spread** of I (so $a(I) = \text{altitude}(\mathbf{R}(R, I)/((u, M)\mathbf{R}(R, I)))$ (see (2.1.1))).

(2.1.4) I_a denotes the **integral closure of I in R**, so $I_a = \{b \in R \mid b$ satisfies an equation of the form $b^n + i_1 b^{n-1} + \cdots + i_n = 0$, where $i_k \in I^k$ for $k = 1, \ldots, n\}$. The ideal I is said to be **integrally closed** in case $I = I_a$, and I is **normal** in case $(I^i)_a = I^i$ for all $i \in \mathbb{N}_+$.

(2.1.5) An ideal J in R is a **reduction** of I in case $J \subseteq I$ and $JI^n = I^{n+1}$ for some $n \in \mathbb{N}$.

(2.1.6) An ideal J in R is **projectively equivalent** to I in case $(I^i)_a = (J^j)_a$ for some $i, j \in \mathbb{N}_+$.

Concerning (2.1.6), Samuel introduced projectively equivalent ideals in 1952 in [19], and a number of properties of projective equivalence can be found in [4], [5], [8], [9], [10], [11], [15], [16].

Remark 2.2 Let R be a Noetherian ring. Then

(2.2.1) The relation "I is projectively equivalent to J" is an equivalence relation on $\mathbf{I} = \{I \mid I$ is an ideal of $R\}$.

(2.2.2) [11, (2.1)(b)]: If I and J are ideals in R and if $i, j, k, l \in \mathbb{N}_+$ with $\frac{i}{j} = \frac{k}{l}$, then $(I^i)_a = (J^j)_a$ if and only if $(I^k)_a = (J^l)_a$.

(2.2.3) Assume that I and J are projectively equivalent in R and let K be an ideal in R. Then $(I + K)/K$ and $(J + K)/K$ are projectively equivalent in R/K.

(2.2.4) Let A be an R-algebra. If I, J are projectively equivalent in R, then IA, JA are projectively equivalent in A.

Concerning (2.2.4), it is not true in general that integral closedness of ideals is preserved under a faithfully flat ring extension. This need not be true even if (R, M) is a regular local domain and $A = \widehat{R}$ is the M-adic completion of R [3]. Thus there exists a regular local domain R and an integrally closed ideal I of R such that $\mathbf{P}(I\widehat{R}) \neq \{J\widehat{R} \mid J \in \mathbf{P}(I)\}$ (see (2.4.3)).

With the preceding paragraph in mind, we note in (2.8) below that $\mathbf{P}(I)$, Rees I (see (2.3)), and $d(I)$ (see (2.4.4)) behave nicely when passing to: $R[X]$; $R(X)$; R_S (for certain multiplicatively closed subsets S of R); and, R/K (where $K \subseteq \mathrm{Rad}(R)$, the nilradical of R), and we also show that $d(I) \leq d(IA)$ for certain types of R-algebras A. For this, we first define, for a regular proper ideal I in a Noetherian ring R, the Rees valuation rings of I, the set $\mathbf{P}(I)$, and the positive integer $d(I)$.

Definition 2.3 Let I be a regular proper ideal in a Noetherian ring R, for each $x \in R$ let $v_I(x) = \max\{k \in \mathbb{N} \mid x \in I^k\}$ (as usual, $I^0 = R$ and $v_I(x) = \infty$ in case $x \in I^k$ for all $k \in \mathbb{N}$), and let $\overline{v}_I(x) = \lim_{k \to \infty}(\frac{v_I(x^k)}{k})$. Rees shows in [17] that: (a) $\overline{v}_I(x)$ is well defined; (b) for each $k \in \mathbb{N}$ and $x \in R$, $\overline{v}_I(x) \geq k$ if and only if $x \in (I^k)_a$ (as usual, $(I^0)_a = R$); and, (c) there exist valuations $v_1, \dots, v_g$ defined on R (with values in $\mathbb{N} \cup \{\infty\}$) and positive integers $e_1, \dots, e_g$ such that, for each $x \in R$, $\overline{v}_I(x) = \min\{\frac{v_i(x)}{e_i} \mid i = 1, \dots, g\}$. (These v_i and e_i are described as follows: let $z_1, \dots, z_d$ be the minimal prime ideals z in R such that $z + I \neq R$, for $i = 1, \dots, d$, let $R_i = R/z_i$, let F_i be the quotient field of R_i, let $\mathbf{R}_i = \mathbf{R}(R_i, (I + z_i)/z_i)$, let $p_{i,1}, \dots, p_{i,h_i}$ be the (height one) prime divisors of $u\mathbf{R}_i{}'$ (see (2.1.2)), let $w_{i,j}$ be the valuation of the discrete valuation ring $W_{i,j} = \mathbf{R}_i{}'_{p_{i,j}}$, let $e_{i,j} = w_{i,j}(u)$, let $V_{i,j} = W_{i,j} \cap F_i$, and define $v_{i,j}$ on R by $v_{i,j}(x) = w_{i,j}(x + z_i)$. Then $v_1, \dots, v_g$ are the valuations $v_{1,1}, \dots, v_{d,h_d}$ resubscripted and $e_1, \dots, e_g$ are the corresponding $e_{i,j}$ resubscripted, and **Rees I** $= \{(V_1, N_1), \dots, (V_g, N_g)\}$, where V_i is the valuation ring of the valuation v_i; note that $IV_i = N_i{}^{e_i} V_i$.

Definition 2.4 Let I be a regular proper ideal in a Noetherian ring R.
(2.4.1) For $\alpha \in \mathbb{R}_+$ let $I_\alpha = \{x \in R \mid \overline{v}_I(x) \geq \alpha\}$ (see (2.3)).
(2.4.2) $\mathbf{W}(I) = \{\alpha \in \mathbb{R}_+ \mid \overline{v}_I(x) = \alpha$ for some $x \in R\}$.
(2.4.3) $\mathbf{U}(I) = \{\alpha \in \mathbf{W}(I) \mid I_\alpha$ is projectively equivalent to $I\}$ (see (2.4.1), (2.4.2), and (2.1.6)), and $\mathbf{P}(I) = \{I_\alpha \mid \alpha \in \mathbf{U}(I)\}$.
(2.4.4) $d(I)$ is the smallest positive integer d such that, for all $J \in \mathbf{P}(I)$, $(J^d)_a = (I^j)_a$ for some $j \in \mathbb{N}_+$. (See [11, Theorem 2.8].)

Remark 2.5 Let R be a Noetherian ring and let I be a regular proper ideal in R.

(2.5.1) Concerning (2.4.1), for each $\alpha \in \mathbb{R}_+$ the ideal I_α is an integrally closed ideal ($= (I_\alpha)_a$) in R, $I_n = (I^n)_a$ for all $n \in \mathbb{N}_+$, and for all $k \in \mathbb{N}_+$ and for all $I_\alpha \in \mathbf{P}(I)$ it holds that $({I_\alpha}^k)_a = I_{k\alpha}$, by [11, (2.1)(g), (2.1)(c) and (2.5)].

(2.5.2) The sets $\mathbf{W}(I)$ and $\mathbf{U}(I)$ of (2.4.2) and (2.4.3) are discrete subsets of $\mathbb{Q}_+$, by [11, (1.1) and (2.8)].

(2.5.3) For the set $\mathbf{P}(I)$ of (2.4.3), $\mathbf{P}(I) = \{J \mid J$ is an integrally closed ideal in R that is projectively equivalent to $I\}$, and $\mathbf{P}(I)$ is linearly ordered by inclusion, by [11, (2.4)].

(2.5.4) Concerning (2.4.4), it is shown in [11, (2.8) and (2.9)] that there exists a unique smallest positive integer $d(I)$ that is a common divisor (but not necessarily the greatest common divisor) of the integers $e_1, \dots, e_g$ of (2.3) such that, for each ideal J in R that is projectively equivalent to I, $(J^{d(I)})_a = (I^j)_a$ for some $j \in \mathbb{N}_+$. Also, $d(I)\alpha \in \mathbb{N}_+$ for all $\alpha \in \mathbf{U}(I)$. Further, if $H, J \in \mathbf{P}(I)$ and if $d(H), d(J)$ are the corresponding unique positive integers for H, J, then $(H^{d(J)})_a = (J^{d(H)})_a$, by [1, (4.8.3)]. Finally, there exist $n^*(I) \in \mathbb{N}_+$ such that $\{\alpha \in \mathbf{U}(I) \mid \alpha \geq n^*(I)\} = \{n^*(I) + \frac{k}{d(I)} \mid k \in \mathbb{N}\}$ (in fact, each large $n \in \mathbb{N}_+$ is a suitable choice for $n^*(I)$), by [11, (2.8)].

In Theorem 2.6 (together with its corollary) we show that $d(I) \leq d(IA)$ for certain standard types of R-algebras A.

Theorem 2.6 *Let I be a regular proper ideal in a Noetherian ring R and let A be a Noetherian ring that is an R-algebra having the property that: (a) IA is regular and proper; and, (b) if $H \subsetneq J$ in $\mathbf{P}(I)$, then $(HA)_a \subsetneq (JA)_a$ in $\mathbf{P}(IA)$. Then $d(IA) \geq d(I)$. If $d(IA) > d(I)$, then $d(IA) = kd(I)$ for some $k \in \mathbb{N}_+$.*

Proof. By the last sentence in (2.5.4) let n be a large enough positive integer so that, for all integers $m \geq n$, there are exactly $d = d(I)$ ideals in $\mathbf{P}(I)$ between $(I^{m+1})_a$ and $(I^m)_a$ (counting one endpoint) and there are exactly $d' = d(IA)$ ideals in $\mathbf{P}(IA)$ between $(I^{m+1}A)_a$ and $(I^m A)_a$ (counting one endpoint). Let $H_0, H_1, \dots, H_d$ in $\mathbf{P}(I)$ such that $(I^{n+1})_a = H_d \subsetneq \cdots \subsetneq H_0 = (I^n)_a$. Then each $(H_i A)_a \in \mathbf{P}(IA)$ (by (2.2.4)) and $(I^{n+1}A)_a = (H_d A)_a \subsetneq \cdots \subsetneq (H_0 A)_a = (I^n A)_a$, by (b). It follows that $d' \geq d$.

Now assume that $d' > d$. Then to show that $d' = kd$ for some $k \in \mathbb{N}_+$ it suffices to show that there are exactly $\frac{d'}{d} - 1$ ideals in $\mathbf{P}(IA)$ that are strictly between $(H_{i+1}A)_a$ and

$(H_iA)_a$ for $i \in \{0, 1, \ldots, d-1\}$.

For this, it may clearly be assumed that $d \geq 2$, so it suffices to show that if $H_{i+1} \subsetneq H_i \subsetneq H_{i-1}$ are three consecutive ideals in $\mathbf{P}(I)$ between $(I^{n+1})_a$ and $(I^n)_a$, then the number of ideals in $\mathbf{P}(IA)$ between $(H_{i+1}A)_a$ and $(H_iA)_a$ is the same as the number of ideals in $\mathbf{P}(IA)$ between $(H_iA)_a$ and $(H_{i-1}A)_a$.

For this, recall that $H_j = I_{n+\frac{j}{d}}$ (see (2.5.4)) and similarly (since $(I^{n+1}A)_a = (IA)_{n+1} \subseteq (H_{i+1}A)_a \subsetneq (H_iA)_a \subsetneq (H_{i-1}A)_a \subseteq (IA)_n = (I^nA)_a$) it follows that, for $j \in \{i-1, i, i+1\}$, $(H_jA)_a = (IA)_{n+\frac{h_j}{d'}}$ for some integer h_j between j and d'. Therefore there exist exactly $h_{i+1} - h_i$ ideals in $\mathbf{P}(IA)$ between $(H_{i+1}A)_a$ and $(H_iA)_a$ (counting one endpoint), and there exist exactly $h_i - h_{i-1}$ ideals in $\mathbf{P}(IA)$ between $(H_iA)_a$ and $(H_{i-1}A)_a$ (counting one endpoint), so it suffices to show that $h_{i+1} - h_i = h_i - h_{i-1}$.

For this, it follows from (2.5.1) that, for all $m \in d\mathbb{N}_+$, $({H_j}^m)_a = ({I_{n+\frac{j}{d}}}^m)_a = (I^{nm+\frac{j}{d}m})_a$ and, for all $m \in d'\mathbb{N}_+$, $((H_jA)^m)_a = (((IA)_{n+\frac{h_j}{d'}})^m)_a = ((IA)^{nm+\frac{h_j}{d'}m})_a = ((I^{nm+\frac{h_j}{d'}m})_aA)_a$. Therefore with $m = dd'$ we get $((I^{ndd'+jd'})_aA)_a = (({H_j}^{dd'})_aA)_a = ((H_jA)^{d'd})_a = ((I^{ndd'+h_jd})_aA)_a$. Therefore it follows from (b) that $(I^{ndd'+jd'})_a = (I^{nd'd+h_jd})_a$, so we get: (i) $(i+1)d' = dh_{i+1} \in \mathbb{N}_+$ (for $j = i+1$); (ii) $id' = dh_i \in \mathbb{N}_+$ (for $j = i$); and, (iii) $(i-1)d' = dh_{i-1} \in \mathbb{N}_+$ (for $j = i-1$). It follows by subtracting (ii) from (i) and (iii) from (ii) that $h_{i+1} - h_i = \frac{d'}{d} = h_i - h_{i-1} \in \mathbb{N}_+$, so the number of ideals in $\mathbf{P}(IA)$ between $(H_{i+1}A)_a$ and $(H_iA)_a$ is the number of ideals in $\mathbf{P}(IA)$ between $(H_iA)_a$ and $(H_{i-1}A)_a$. Therefore $d(IA) = d' = kd = kd(I)$, where $k = h_i - h_{i-1} \geq 2$. ■

Corollary 2.7 *Let I be a regular proper ideal in a Noetherian ring R. Then (a) and (b) of Theorem 2.6 hold for the following types of R-algebras A:*

1. *A is a Noetherian integral extension ring of R such that IA is regular.*

2. *A is a faithfully flat Noetherian extension ring of R.*

3. *$A = R_S$ for some multiplicatively closed subset S of R such that $IR_S \neq R_S$.*

Therefore $d(IA) = kd(I)$ for some positive integer k (possibly $k = 1$) for such rings A.

Proof. For (1) and (2) it is well known that $(JA)_a \cap R = J_a$ for all regular proper ideals J in R, so (b) holds for such rings A. Also, (a) holds for the rings A in (1) by hypothesis, and (a) holds for the rings A in (2) by faithful flatness (since I is regular in R).

For the rings R_S as in (3), by hypothesis IR_S is proper, so (a) holds by flatness. For (b), let $d = d(I)$ and let n and $H_0, \ldots, H_d$ be as in the first paragraph of the proof of (2.6). Suppose that $H_{i+1}R_S = H_iR_S$ for some $i \in \{0, 1, \ldots, d-1\}$. (Note that $JR_S = (JR_S)_a$ if J is an ideal in R such that $J = J_a$.) Let $H = H_{i+1}$ and $J = H_i$. Since $H, J \in \mathbf{P}(I)$, there exist $h, j \in \mathbb{N}_+$ such that $(H^d)_a = (I^h)_a$ and $(J^d)_a = (I^j)_a$ (by (2.5.4)). Therefore $(I^hR_S)_a$ $= H^dR_S = J^dR_S$ (since $H_{i+1}R_S = H_iR_S$) $= (I^jR_S)_a$, hence $h = j$ (since IR_S is a regular proper ideal). Therefore, since $(H^d)_a = (I^h)_a$ and $(J^d)_a = (I^j)_a$, it follows that $(H^d)_a =$ $(J^d)_a$, so $H_a = J_a$, hence $H_{i+1} = H_i$, and this is a contradiction. It follows that the ideals H_iR_S are distinct (so (b) holds) and are ideals in $\mathbf{P}(IR_S)$ between $(I^{n+1}R_S)_a$ and $(I^nR_S)_a$, hence $d(IR_S) \geq d(I)$.

The last statement is clear from (2.6) and what has already been shown. ■

It would be interesting to know whether Corollary 2.7 holds in general for a Noetherian ring A that is a flat R-algebra such that $IA \neq A$.

We next consider classes of R-algebras A for which $d(IA) = d(I)$. For these algebras, $\mathbf{P}(I)$ and Rees I also extend nicely. The proofs of these results ((2.8.1) - (2.8.4)) are straightforward, so they are omitted.

Proposition 2.8 *Let I be a regular proper ideal in a Noetherian ring R.*

(2.8.1) *Let K be an ideal in R such that $K \subseteq \mathrm{Rad}(R)$. Then $\mathbf{P}((I+K)/K) = \{(J+K)/K \mid J \in \mathbf{P}(I)\}$, Rees $(I+K)/K =$ Rees I, and $d((I+K)/K) = d(I)$.*

(2.8.2) *Let $P_1, \ldots, P_k$ be the centers in R of the Rees valuations of I, and let S be a multiplicatively closed subset of the nonzero elements of R. Assume that $S \cap P_i = \emptyset$ if and only if $i = 1, \ldots, h$ (for some integer h with $1 \leq h \leq k$). Then Rees $IR_S = \{(V, N) \in$ Rees $I \mid N \cap R \in \{P_1, \ldots, P_h\}\}$, and $d(IR_S) \geq d(I)$. Moreover, if $h = k$, then $\mathbf{P}(IR_S) = \{JR_S \mid J \in \mathbf{P}(I)\}$ and $d(IR_S) = d(I)$.*

(2.8.3) *Let $X_1, \ldots, X_g$ be independent indeterminates, and let $A = R[X_1, \ldots, X_g]$. Then $\mathbf{P}(IA) = \{JA \mid J \in \mathbf{P}(I)\}$, Rees $IA = \{V[X_1, \ldots, X_g]_{N[X_1, \ldots, X_g]} \mid (V, N) \in$ Rees $I\}$, and $d(IA) = d(I)$.*

(2.8.4) *Let $X_1, \ldots, X_g$ be independent indeterminates. As in [13, pp. 17-18], let $A =$*

$R(X_1, \dots, X_g)$ denote the quotient ring of the polynomial ring $R[X_1, \dots, X_g]$ with respect to the multiplicatively closed set of polynomials whose coefficients generate the unit ideal of R. Then $\mathbf{P}(IA) = \{JA \mid J \in \mathbf{P}(I)\}$, Rees $IA = \{V[X_1, \dots, X_g]_{N[X_1,\dots,X_g]} \mid (V, N) \in$ Rees $I\}$, *and $d(IA) = d(I)$*

Remark 2.9 Let I be a regular proper ideal in a Noetherian ring R. Let A be either $R[X_1, \dots, X_g]$ or $R(X_1, \dots, X_g)$ $(g \geq 1)$, and let J be an ideal in A. If there exist $i, j \in \mathbb{N}_+$ such that either $J^j = I^i A$ or $(J^j)_a = (I^i A)_a$, then there exists an ideal G in R such that $(G^j)_a = (I^i)_a$ and $((GA)^j)_a = ((G^j)_a)A = (J^j)_a$, so $(J^j)_a$ is the integral closure of the extension of the j-th power of an ideal in R.

Proof. If either $J^j = I^i A$ or $(J^j)_a = (I^i A)_a$, then IA, J are projectively equivalent in A, so $J_a \in \mathbf{P}(IA)$ (by (2.5.3)). But by (2.8.3) (if $A = R[X_1, \dots, X_g]$) or by (2.8.4) (if $A = R(X_1, \dots, X_g)$) $\mathbf{P}(IA) = \{HA \mid H \in \mathbf{P}(I)\}$. It follows that $J_a = GA$, where $G = J_a \cap R$. It therefore follows that $((I^i)_a)A = (J^j)_a = ((J_a)^j)_a = ((GA)^j)_a = ((G^j)_a)A$, so $(J^j)_a = ((G^j)_a)A$ and $(G^j)_a = (I^i)_a$. ■

In Proposition 2.10, we consider projective equivalence for invertible ideals in an integrally closed Noetherian ring. Concerning the hypothesis of (2.10), we remark that an integrally closed local ring R that contains a regular proper principal ideal is an integral domain. Moreover, if R is an integrally closed Noetherian ring with $\mathrm{Rad}(R) = (0)$, then R is a finite product of Noetherian integrally closed domains.

Proposition 2.10 *Let R be an integrally closed Noetherian ring and let I be a regular proper ideal of R.*

1. *If I is invertible, then every ideal projectively equivalent to I is invertible.*
2. *If R is local and $I = bR$ is principal, then every ideal J in R that is projectively equivalent to I is principal and invertible.*

Proof. Since a regular ideal I of a Noetherian ring R is invertible if and only if IR_M is principal for each maximal ideal M of R and since projective equivalence behaves well with respect to localization, to prove item 1 it suffices to prove item 2. Thus we may assume

(R, M) is an integrally closed local ring and $I = bR$ is a regular proper principal ideal (so R is an integral domain). Let J be an ideal in R that is projectively equivalent to bR, so $(J^i)_a = (b^m R)_a$ for some $i, m \in \mathbb{N}_+$. Then $a(J) = 1$ (see (2.1.3)), since $a(J) = a((J^i)_a) = a((b^m R)_a) = a(bR) = 1$. Assume temporarily that R/M is infinite and let xR be a minimal reduction of J. Then $xR = (xR)_a$, since R is integrally closed, and $xR \subseteq J \subseteq (xR)_a$, so $J = xR$ is principal. Now, if R/M is finite, then $T = R[X]_{MR[X]}$ is an integrally closed local domain with infinite residue field and JT is projectively equivalent to bT, by (2.2.4), so JT is principal. Since a minimal generating set for J is a generating set for JT, it can be reduced to a minimal (one element) generating set for JT, so J is principal (since $J = JT \cap R$). Since J is principal, J is invertible. ■

Remark 2.11 Without the assumption that R is integrally closed, easy examples show that Proposition 2.10 fails. For example, if (R, M) is a local domain of altitude one that is not integrally closed and $I = xR$ is a reduction of M, then M is projectively equivalent to I and M is not invertible. For a specific example, let t be an indeterminate over a field k and consider the subring $R = k[[t^2, t^3]]$ of the formal power series ring $k[[t]]$. Then $I = t^2 R$ is a reduction of $M = (t^2, t^3)R$, so M is projectively equivalent to I.

3 PROJECTIVELY FULL IDEALS.

Projectively full ideals are introduced in [1, Section 4]. It is observed in [1, (4.12)] that $\mathbf{P}(I)$ is projectively full for every nonzero proper ideal I in a regular local domain of altitude two; see also [11, (3.6)]). In this section we develop basic properties of projectively full ideals. We then determine various classes of ideals I for which either I or $\mathbf{P}(I)$ is projectively full. Concerning the basic properties, our main results show that: for certain R-algebras A, if IA is projectively full, then I is projectively full (see (3.2)); the converse holds for A as in (2.8) (by (3.5)); and, the converse need not hold, even if A is a finite free integral extension domain of R (see (4.2.1)). Concerning ideals I where either I or $\mathbf{P}(I)$ is projectively full, we show that: the integrally closed complete intersection ideals of a local ring are projectively full (see (3.6) and (3.7)); and $\mathbf{P}(I)$ is projectively full for an invertible ideal I in an integrally closed Noetherian ring (see (3.8)).

We begin with the following remark.

Remark 3.1 Let I be a regular proper ideal in a Noetherian ring R.

(3.1.1) It is immediate from the relevant definitions that for each $J \in \mathbf{P}(I)$ (see (2.3.3)) we have $\{(J^i)_a \mid i \in \mathbb{N}_+\} \subseteq \mathbf{P}(I)$, and $\mathbf{P}(I)$ is projectively full if and only if there exists $J \in \mathbf{P}(I)$ such that $\{(J^i)_a \mid i \in \mathbb{N}_+\} = \mathbf{P}(I)$. It is clear that if such an ideal J exists, then J must be the largest element in the linearly ordered (discrete) set $\mathbf{P}(I)$. The numerical semigroup $S(I) = d(I)\mathbf{U}(I) \cup \{0\}$ (where $\mathbf{U}(I)$ is as in (2.3.3)) is the semigroup of nonnegative integers under addition, if and only if $\mathbf{P}(I)$ is projectively full.

(3.1.2) It follows from (2.5.4) and (3.1.1) that $\mathbf{P}(I)$ is projectively full if and only if $d(K) = 1$, where K is the largest element in $\mathbf{P}(I)$; cf [1, (4.11)]. Also, I is projectively full if and only if $d(I) = 1$.

Proposition 3.2 is an immediate corollary of (2.7) and (3.1.2).

Proposition 3.2 *Let I be a regular proper ideal in a Noetherian ring R and let A be one of the following types of R-algebras:*

1. *A is a Noetherian integral extension ring of R such that IA is regular.*
2. *A is a faithfully flat Noetherian extension ring of R.*
3. *$A = R_S$ for some multiplicatively closed subset S of R such that $IR_S \neq R_S$.*

If IA is projectively full, then I is projectively full.

Proof. If IA is projectively full, then $d(IA) = 1$, by (3.1.2), so $d(I) = 1$, by (2.7), so I is projectively full, by (3.1.2). ■

Remark 3.3 **(3.3.1)** The converses of (3.2)(1) and (3.2)(2) are not true; in (4.2.1) below we give specific examples where A is a finite free integral extension and I is a projectively full ideal such that IA is not projectively full.

(3.3.2) The converse of (3.2)(3) is not true; in fact, it may happen that:

1. I is projectively full, but IR_S is not projectively full.

2. $\mathbf{P}(I)$ is projectively full, but $\mathbf{P}(IR_S)$ is not projectively full.

Proof. (of 3.3)(2) For (1), let $(R, M = (x, y, z)R)$ be a regular local domain of altitude three, let $p = zR$, let $P = (x, y)R$, and let $I = pP^2$. Then I is projectively full, by (4.1.3) below, p and P are the centers in R of two of the Rees valuation rings of I, and $IR_P = P^2R_P$ is not projectively full. (However, in this example $\mathbf{P}(IR_P) = \{P^iR_P \mid i \in \mathbb{N}_+\}$ is projectively full.)

For (2), let A be a Noetherian ring having a regular proper ideal J such that $\mathbf{P}(J)$ is not projectively full. For example, let A be the normal local domain of [1, Example 4.14]. Let X be an indeterminate over A and let $R = A[X]$. Then the ideal $I = XJR$ is projectively full by (4.1.4), so $\mathbf{P}(I)$ is projectively full. Let S be the multiplicative system generated by X. Then $IR_S = JR_S$, and $\mathbf{P}(JR_S)$ is not projectively full. ■

In the case where A is a faithfully flat Noetherian extension of R, or the case where $A = R_S$ and $IA \neq A$, it would be interesting to know if $\mathbf{P}(IA)$ is projectively full implies that $\mathbf{P}(I)$ is projectively full.

Remark 3.4 There are often many ideals that localize to the same ideal in a localization. With this in mind, if $JR_S = IR_S$ and IR_S is projectively full, then so are both I and J, by (3.2)(3). So assume, for example, that (R, M) is a regular local domain, let $P \neq M$ be a nonzero prime ideal in R, and let $b \in R - P$. Then for all positive integers n it holds that *every* ideal between b^nP and P is projectively full (since $b^nPR_P = PR_P$ is projectively full).

In Proposition 3.5 we consider extension rings A of R for which the converse of (3.2) holds. This result is an immediate corollary of (2.8) and (3.1.2).

Proposition 3.5 *Let I be a regular proper ideal in a Noetherian ring R.*

(3.5.1) *Let K be an ideal in R such that $K \subseteq \mathrm{Rad}(R)$. Then I (respectively $\mathbf{P}(I)$) is projectively full in R if and only if $(I + K)/K$ (respectively $\mathbf{P}((I + K)/K)$) is projectively full in R/K.*

(3.5.2) *Let $P_1, \dots, P_k$ be the centers in R of the Rees valuations of R and let S be a multiplicatively closed subset of R such that $S \cap (P_1 \cup \cdots \cup P_k) = \emptyset$. Then I (respectively*

$\mathbf{P}(I)$) is projectively full in R if and only if IR_S (respectively $\mathbf{P}(IR_S)$) is projectively full in R_S.

(3.5.3) *Let $X_1, \ldots, X_g$ be indeterminates and let $A = R[X_1, \ldots, X_g]$. Then I (respectively $\mathbf{P}(I)$) is projectively full in R if and only if IA (respectively $\mathbf{P}(IA)$) is projectively full in A.*

(3.5.4) *Let $X_1, \ldots, X_g$ be indeterminates, and let $A = R(X_1, \ldots, X_g)$. Then I (respectively $\mathbf{P}(I)$) is projectively full in R if and only if IA (respectively $\mathbf{P}(IA)$) is projectively full in A.*

Proof. For (3.5.1), I is projectively full if and only if $d(I) = 1$ (by (3.1.2)) if and only if $d((I+K)/K) = 1$ (by (2.8.1)) if and only if $(I+K)/K$ is projectively full (by (3.1.2)). Also, $\mathbf{P}(I)$ is projectively full if and only if there exists $J \in \mathbf{P}(I)$ such that $d(J) = 1$ (by (3.1.2)) if and only if $d((J+K)/K) = 1$ (by (2.8.1)) if and only if $\mathbf{P}((I+K)/K)$ is projectively full (by (3.1.2), since $(J+K)/K \in \mathbf{P}((I+K)/K)$).

The proof of (3.5.2) (resp., (3.5.3), (3.5.4)) is similar using (2.8.2) (resp., (2.8.3), (2.8.4)) in place of (2.8.1). ■

In Proposition 3.6 (and its corollary) we show that the integrally closed complete intersection ideals of a local ring as classified by Goto are projectively full.

Proposition 3.6 *Let (R, M) be a local ring and I a normal ideal in R with $I \not\subseteq M^2$. If both the associated graded ring $G(M) = \bigoplus_{n \geq 0} M^n/M^{n+1}$ and the fiber cone ring $F(I) = \bigoplus_{n \geq 0} I^n/MI^n$ are reduced, then I is projectively full.*

Proof. Since $G(M)$ is reduced, the maximal ideal M is also normal. For it is shown in [18, Theorem 2.1] that $G(M) = \mathbf{R}(R,M)/u\mathbf{R}(R,M)$, so $G(M)$ is reduced if and only if $u\mathbf{R}(R,M)$ is the intersection of its minimal prime divisors if and only if $p\mathbf{R}(R,M)_p = u\mathbf{R}(R,M)_p$ for all prime divisors p of $u\mathbf{R}(R,M)$; it follows that each such $\mathbf{R}(R,M)_p$ is a discrete valuation ring, so $u\mathbf{R}(R,M) = (u\mathbf{R}(R,M))_a$, by [14, Theorem 2.10], so $M^n = u^n\mathbf{R}(R,M) \cap R = (u^n\mathbf{R}(R,M))_a \cap R = {M^n}_a$ for all $n \in \mathbb{N}_+$, hence M is normal. Let J be an integrally closed ideal such that $(J^t)_a = I^s$ for some positive integers t, s. Then we must have $s \geq t$. If not, then $I^s \subseteq (J^{s+1})_a \subseteq M^{s+1}$. Choose $a \in I \setminus M^2$. If a' denotes the image of a in $M/M^2 \subseteq G(M)$, then $(a')^s = 0$, hence $a' = 0$, contradicting the choice of a.

In particular, since $(J^t)_a \subseteq I^t$, we have $J \subseteq I$. Let k be the positive integer such that $J \subseteq I^k$ and $J \nsubseteq I^{k+1}$. We will prove that $J = I^k$.

We first show that $J \nsubseteq MI^k$. Assume that $J \subseteq MI^k$. Then $I^s = (J^t)_a \subseteq (M^t I^{kt})_a$. This implies that $kt \leq s - t$, otherwise $I^s \subseteq (M^t I^{s-t+1})_a \subseteq M^{s+1}$, which, as shown above, is not true. But then $(J^t)_a = I^s \subseteq I^{(k+1)t}$ and hence $J \subseteq I^{k+1}$, contradicting the choice of k.

We now consider the fiber cone ring $F(I) = \bigoplus_{n \geq 0} I^n/MI^n$. Let $x \in J \setminus MI^k$. The image of x in $\left(J + MI^k/MI^k\right) \subseteq \left(I^k/MI^k\right)$ is nonzero and, since $F(I)$ is reduced, we have $x^t \in J^t \setminus MI^{kt}$. This shows that $J^t \nsubseteq MI^{kt}$, and since $J^t \subseteq I^s$, we get $s \leq kt$. On the other hand, since $J \subseteq I^k$, we have $I^s = (J^t)_a \subseteq I^{kt}$, and therefore $s \geq kt$. In conclusion, $s = kt$ and from $(J^t)_a = I^{kt}$ we obtain $I^k \subseteq J_a = J$. ■

In [2], Goto described the M-primary integrally closed complete intersection ideals in a local ring (R, M). He proves that such ideals I exist only when the ring R is regular, in which case there exist regular parameters $x_1, \dots, x_d$ and a positive integer n such that $I = (x_1^n, x_2, \dots, x_d)R$. Moreover, all the powers of such an ideal are integrally closed, i.e., the ideal is normal. An immediate corollary of Proposition 3.6 shows that the ideals of this type are also projectively full.

Corollary 3.7 *Let (R, M) be a regular local domain of altitude $d \geq 2$ and let $x_1, \dots, x_d$ be a regular system of parameters. Then every ideal of the form $I = (x_1^n, x_2, \dots, x_d)R$ is projectively full.*

Proof. Under these assumptions, both $G(M) = \mathbf{R}(R, M)/u\mathbf{R}(R, M)$ and $F(I) = \mathbf{R}(R, M)/(u, M)\mathbf{R}(R, M)$ are polynomial rings in d variables with coefficients in R/M, and therefore reduced. Also, I is normal, so the conclusion follows from (3.6). ■

In Proposition 3.8, we show that the projective equivalence class of each proper invertible ideal in an integrally closed Noetherian ring is projectively full.

Proposition 3.8 *Let R be an integrally closed Noetherian ring and let I be a nonzero proper ideal of R.*

1. *If I is invertible, then $\mathbf{P}(I)$ is projectively full.*

2. If R is local and I is principal (or equivalently invertible), say $I = bR$, then there exists $x \in R$ such that $\mathbf{P}(bR) = \{x^i R \mid i \in \mathbb{N}_+\}$.

Proof. Assume that I is invertible. By Proposition 2.10 every ideal projectively equivalent to I is invertible. Let K be the largest element of $\mathbf{P}(I)$ and let $H \in \mathbf{P}(I)$. Since H is an arbitrary element of $\mathbf{P}(I)$, to prove that $\mathbf{P}(I)$ is projectively full it suffices to prove that H is a power of K. By the linear order of $\mathbf{P}(I)$, there exists $n \in \mathbb{N}_+$ such that $K^{n+1} \subseteq H \subseteq K^n$. If $H = K^n$ we are done. If $H \subsetneq K^n$, then $H = JK^n$ for some invertible ideal J such that $K \subseteq J \subsetneq R$. Since H and K are projectively equivalent and since invertible ideals of an integrally closed ring are integrally closed, there exist positive integers h and k such that $H^h = K^k$. Therefore $K^k = H^h = J^h K^{nh}$. If $k \leq nh$, then multiplying this equation by K^{-k} gives a contradiction to the fact that J is a proper ideal. Therefore $k > nh$. Multiplying by K^{-nh}, we obtain $K^{k-nh} = J^h$. Thus K and J are projectively equivalent. Since K is the largest ideal projectively equivalent to H, we have $K = J$ and $H = K^{n+1}$. This proves item 1. Since an invertible ideal of a local domain is principal, item 2 is an immediate consequence of item 1. ■

In connection with Proposition 3.8, a question we have considered, but not resolved, is whether $\mathbf{P}(I)$ is always projectively full if I is a divisorial ideal of an integrally closed Noetherian domain. A complicating factor here is that the integral closure of a power of the divisorial ideal I may fail to be divisorial. We discuss this in Remark 3.9.

Remark 3.9 Let R be an integrally closed Noetherian domain. Invertible ideals of R are divisorial, and a divisorial ideal I of R is uniquely representable as the intersection of symbolic powers of the height-one prime ideals that contain it, say

$$I = P_1^{(e_1)} \cap P_2^{(e_2)} \cap \cdots \cap P_g^{(e_g)}.$$

Let v_i denote the normalized valuation associated to the valuation domain R_{P_i}. Then $v_i(I) = e_i$. The valuations v_i and positive integers e_i are a subset of the valuations and associated positive integers mentioned in Definition 2.3. If I is invertible, then this subset is all the valuations of Definition 2.3, and Rees $I = \{R_{P_1}, \ldots, R_{P_g}\}$ since I is invertible if and only if IR_P has analytic spread one for every prime ideal P of R that contains I.

Notice that $(I^n)_a$ is divisorial for all $n \in \mathbb{N}_+$ if and only if Rees $I = \{R_{P_1}, \dots, R_{P_g}\}$. A prime ideal P is the center of a Rees valuation ring of I if the analytic spread of IR_P is equal to the height of P, and the converse holds if R_P is quasi-unmixed [7, Prop. 4.1]. If $I \subset P$ and P is of height 2, then R_P is Cohen-Macaulay and thus unmixed, so P is the center of a Rees valuation ring of I if and only if IR_P has analytic spread 2. Thus if R has altitude two, then the Rees valuation rings of a divisorial ideal I of R are all centered on height-one primes of R if and only if I is invertible. However, there exists a normal local domain (R, M) of altitude 3 that has a divisorial height-one prime ideal P such that P has analytic spread 2 (so P is not invertible) and yet R_P is the unique Rees valuation ring of P (so $(P^n)_a = P^{(n)}$ is divisorial for all $n \in \mathbb{N}_+$). For a specific example, let k be a field and let $R = k[[x, y, z, w]]$, where $xy = zw$. Then $P = (x, z)R$ is a height-one prime of R and P has analytic spread 2. Since the localization of R at any nonmaximal prime ideal is a regular local domain, if $P \subset Q$ with Q a prime of R of height 2, then PR_Q is principal. Therefore R_P is the unique Rees valuation ring of P.

In connection with Remark 3.9, it seems natural to ask:

Question 3.10 If I is a divisorial ideal of an integrally closed Noetherian domain and if the integral closure of I^n is divisorial for every $n \in \mathbb{N}_+$, does it follow that $\mathbf{P}(I)$ is projectively full?

Remark 3.11 Without the assumption that R is integrally closed, easy examples show that Proposition 3.8 fails. Indeed, if (R, M) is a local domain of altitude one such that the integral closure of R is a valuation domain, then all the M-primary ideals of R are projectively equivalent, and given an arbitrary numerical semigroup S, it is possible to construct a local domain (R, M) of altitude one such that the integral closure of R is a valuation domain and such that $S(M) = S$. Let $b_1 < b_2 < \cdots < b_r$ be positive integers having greatest common divisor 1, and let $S = \langle b_1, b_2, \dots, b_r \rangle$ denote the numerical semigroup determined by $b_1, \dots, b_r$. Let t be an indeterminate over the field k and let R be the subring $k[[t^{b_1}, t^{b_2}, \dots, t^{b_r}]]$ of the formal power series ring $k[[t]]$. Then R is local with maximal ideal $M = (t^{b_1}, \dots, t^{b_r})R$ and the integral closure of R is the valuation domain

$k[[t]]$. The integrally closed M-primary ideals of R are precisely the M-primary ideals that are contracted from $k[[t]]$. If I is M-primary, then $I_a = Ik[[t]] \cap R$ and $Ik[[t]] = t^b k[[t]]$, where $b \in S$. Thus the integrally closed M-primary ideals are in one-to-one correspondence with the elements of $S = \langle b_1, \dots b_r \rangle$. Therefore $S(M) = S$. We conclude that every numerical semigroup S is realizable as $S(M)$ for a local domain (R, M) of altitude one. In particular, M is projectively full if and only if $S = \mathbb{N}$, or, equivalently, if and only if $R = k[[t]]$.

In Remark 3.12 we obtain a partial extension of Proposition 3.8 to integrally closed regular principal ideals in Noetherian rings that are not integrally closed.

Remark 3.12 Let R be a Noetherian ring and let $I = bR$ be an integrally closed regular proper principal ideal. Then the following hold:

(3.12.1) For each ideal H in R that is projectively equivalent to I, H^n is principal for infinitely many $n \in \mathbb{N}_+$, and $H^n = (H^n)_a$ for all large $n \in \mathbb{N}_+$.

(3.12.2) If the largest ideal in $\mathbf{P}(I)$ is principal, say xR, then $\mathbf{P}(I)$ is projectively full and each ideal projectively equivalent to I is a power of xR.

(3.12.3) If c is a regular element in R such that $\mathrm{Rad}(cR) = \mathrm{Rad}(bR)$, then $bR = (bR)_a$ if and only if $cR = (cR)_a$. Therefore if b is a regular element in R such that $bR = (bR)_a$, then the conclusion of (3.12.1) holds for all principal ideals cR such that $\mathrm{Rad}(cR) = \mathrm{Rad}(bR)$.

Proof. For (3.12.1), let H be an ideal in R that is projectively equivalent to I. Then $(H^h)_a = b^d R$ for some $d, h \in \mathbb{N}_+$. (It follows from [14, Theorem 2.10] that $bR = (bR)_a$ if and only if R_p is integrally closed for all prime divisors p of bR; it follows from this (and the fact that bR and $b^i R$ have the same prime divisors for all $i \in \mathbb{N}_+$) that if $bR = (bR)_a$, then $b^i R = (b^i R)_a$ for all $i \in \mathbb{N}_+$.) Therefore $H^h \subseteq (H^h)_a = b^d R$, so there exists an ideal J in R such that $H^h = b^d J$. Therefore $(b^d R)_a = b^d R = (H^h)_a = (b^d J)_a$, hence $R = b^d R : b^d R = (b^d J)_a : b^d R = J_a$, so $J = R$ and $H^h = b^d R$. Therefore $H^{hn} = b^{dn} R$ is integrally closed and principal for all $n \in \mathbb{N}_+$. It therefore follows from [7, (11.15)] that $H^i = (H^i)_a$ for all large $i \in \mathbb{N}_+$.

For (3.12.2), assume that xR is the largest ideal in $\mathbf{P}(I)$ and let H be an ideal in R that is projectively equivalent to I. To complete the proof it must be shown that $H = x^i R$ for some $i \in \mathbb{N}_+$.

For this, by the linear order of $\mathbf{P}(I)$ there exists $n \in \mathbb{N}_+$ such that $x^{n+1}R \subseteq H \subseteq x^nR$. If $H = x^nR$, we are done. If $H \subsetneq x^nR$, then $H = x^nJ$ for some ideal J in R such that $xR \subseteq J \subsetneq R$. Since H are xR are projectively equivalent, $(H^h)_a = x^iR$ for some $h, i \in \mathbb{N}_+$. Therefore $(x^{hn}J^h)_a = x^iR$. If $i \leq hn$, then $(x^{hn-i}J^h)_a = (x^{hn}J^h)_a : x^iR = x^iR : x^iR = R$, and this contradicts $J \neq R$. Therefore $i > hn$, so $(J^n)_a = (x^{hn}J^n)_a : x^{hn}R = x^iR : x^{hn}R = x^{i-hn}R$, so J is projectively equivalent to xR. Since $J \supseteq xR$ and xR is the largest ideal in $\mathbf{P}(I)$, it follows that $J = xR$, hence $H = x^nJ = x^{n+1}R$.

(3.12.3) follows as in the parenthetical part of the proof of (3.12.1). ∎

4 EXAMPLES OF PROJECTIVELY FULL IDEALS.

We use Remark 3.1.2 to obtain several classes of examples of projectively full ideals.

Example 4.1 **(4.1.1)** If the integer $e_i = 1$ for some Rees valuation ring V_i of I (see (2.3)), then I is projectively full. Moreover, if J is a regular proper ideal in R that is not contained in the center P_i in R of V_i, then IJ is projectively full.

(4.1.2) Let P be a prime ideal in a Noetherian ring R such that R_P is a regular local domain. Then P is projectively full. Moreover, PI is projectively full for all regular ideals I in R such that $I \not\subseteq P$. In particular, if (R, M) is a regular local domain, then M^nP is projectively full for all $n \in \mathbb{N}_+$ and for all nonzero prime ideals $P \neq M$.

(4.1.3) Let x be a regular parameter in a regular local domain R and let I be an ideal in R such that $\mathrm{ht}(I) \geq 2$. Then xI is projectively full.

(4.1.4) Let R be a Noetherian domain and let X be an indeterminate. Then $XIR[X]$ is projectively full in $R[X]$ for every nonzero ideal I in R.

(4.1.5) Let R be a Noetherian domain, let X be an indeterminate, and let I be a regular proper ideal in R that is not projectively full. Therefore $d := d(I) > 1$. For each $n > 1$ in $\mathbb{N}_+$ it holds that $X^nIR[X]$ is projectively full if and only if n and d are relatively prime. In particular, $X^nIR[X]$ is projectively full for all $n \in \mathbb{N}_+$ if and only if I is projectively full in R.

Proof. It is noted in (2.4.4) that the integer $d = d(I)$ is a common divisor of the integers $e_1, \ldots, e_g$ of (2.3). Therefore, since I is projectively full if and only if $d(I) = 1$ (by (3.1.2)), the first statement in (4.1.1) is clear.

For the second statement in (4.1.1), it is shown in [1, (3.6)] that for all regular proper ideals I and J in R one has Rees $I \cup$ Rees $J \subseteq$ Rees IJ. Therefore V_i is a Rees valuation ring of IJ, and if $J \not\subseteq P_i$, then the integer e_i of V_i for IJ is also one (since $(IJ)V_i = IV_i = N^{e_i} = N$, where N is the maximal ideal of V_i), so IJ is projectively full by the first statement of this remark.

For (4.1.2), the integer e of the order valuation ring V of P is one, so the first two statements follow immediately from the second statement in (4.1.1), and the third statement is a special case of the second statement.

For (4.1.3), if x is a regular parameter in a regular local domain R, then R_{xR} is a regular local domain, so it follows from (4.1.2) that if $\mathrm{ht}(I) > 1$, then xI is projectively full.

For (4.1.4), $P = XR[X]$ is a prime ideal such that $R[X]_P$ is a regular local domain. Therefore, since $IR[X] \not\subseteq XR[X]$ for all nonzero ideals I of R, it follows from (4.1.2) that $XIR[X]$ is projectively full.

For (4.1.5), assume first that $e > 1$ is a divisor of $d = d(I)$, say $d = eq$. Let $n^*(I) \in \mathbb{N}_+$ such that $\{n^*(I) + (i/d) \mid i \in \mathbb{N}\} \subseteq \mathbf{U}(I)$ (see (2.5.4)). Then $n^*(I) + (q/d)) \in \mathbf{U}(I)$, so the ideal $H := I_{n^*(I)+(q/d)} \in \mathbf{P}(I)$, by (2.4.3), hence $(H^d)_a = (I^{(n^*(I))d+q})_a$, by [11, (2.3)(b)]. However, $((n^*(I))d+q)/d) = (n^*(I)e+1)/e$, so $(H^e)_a = (I^{n^*(I)e+1})_a$, by (2.5.1). Therefore, for all $h \in \mathbb{N}_+$ it holds that

$$((X^{(n^*(I)e+1)h}H)^e)_a = ((X^{eh(n^*(I)e+1)}H^e))_a = ((X^{eh})^{(n^*(I)e+1)}H^e))_a = ((X^{eh}I)^{n^*(I)e+1})_a,$$

so $X^{(n^*(I)e+1)h}H$ is projectively equivalent to $X^{eh}I$, and it is clear that $(X^{(n^*(I)e+1)h}H)_a$ is not the integral closure of any power of $X^{eh}I$, so $X^{eh}I$ is not projectively full.

Conversely, assume that $d \in \mathbb{N}_+$ is such that X^dI is not projectively full, so it must be shown that $d = d_ie$ for some divisor $d_i > 1$ of $d = d(I)$ and for some $e \in \mathbb{N}_+$. By hypothesis there exists an ideal J in $R[X]$ that is projectively equivalent to X^dI such that $J_a \neq ((X^dI)^n)_a$ for all $n \in \mathbb{N}_+$. Now $(J^j)_a = ((X^dI)^n)_a = X^{nd}(I^n)_a$, for some $j, n \in \mathbb{N}_+$, and by [11, (2.1)(b)] it may be assumed that j, n are relatively prime (and $j > 1$, by the preceding sentence). Now $J^j \subseteq (J^j)_a = X^{nd}(I^n)_a \subseteq X^{nd}R[X]$, hence $J^j = X^{nd}K$ for some ideal K in $R[X]$ such that $K \not\subseteq XR[X]$ (since $IR[X] \not\subseteq XR[X]$). Let $m \in \mathbb{N}$ such that J

$\subseteq X^m R[X]$ and $J \not\subseteq X^{m+1}R[X]$, so $J = X^m H$ for some ideal H in $R[X]$. Then $X^{mj}H^j = J^j = X^{nd}K$. Therefore $mj = nd$ $(= nd(I))$, since $K \not\subseteq XR[X]$ and $H^j \not\subseteq XR[X]$ (since $H \not\subseteq XR[X]$ and $XR[X]$ is a prime ideal). Therefore $K = H^j$, so $J^j = X^{mj}H^j$, so $X^{mj}(H^j)_a = (J^j)_a = X^{nd}(I^n)_a$, so $(H^j)_a = (I^n)_a$. Therefore by (2.9), if we let $G = H_a \cap R$, then $H_a = GR[X]$, $(G^j)_a R[X] = (H^j)_a$, and $(G^j)_a = (I^n)_a$ (so G is projectively equivalent to I (in R)). Therefore $(G^{d(I)})_a = (I^i)_a$ for some $i \in \mathbb{N}_+$, by (2.5.4).

Since $(G^j)_a = (I^n)_a$ and $(G^{d(I)})_a = (I^i)_a$, it follows from [11, (2.1)(b)] that $ji = nd(I)$. Therefore $nd(I) = mj$ (resp., $ji = nd(I)$) and n, j are relatively prime, so it follows that $m = en$ (resp., $d(I) = jf$) for some e (resp., f) $\in \mathbb{N}_+$. Therefore j is a divisor of $d(I)$, and since $nd = mj = enj$ it follows that $d = je$ for some divisor j of $d(I)$ and for some $e \in \mathbb{N}_+$, and it was noted in the preceding paragraph that $j > 1$.

The final statement is clear from what has already been shown. ■

It follows from (4.1.2) that every nonzero prime ideal of a regular local domain is projectively full. If (R, M) is a regular local domain and $\dim R = 2$, then $\mathbf{P}(I)$ is projectively full for every nonzero proper ideal I of R [1, (4.12)]. It would be interesting to know whether this is also true when $\dim R \geq 3$.

Remark 4.2 **(4.2.1)** A projectively full ideal may fail to extend to a projectively full ideal in a finite free integral extension domain.

(4.2.2) Concerning (4.1.1), there exist regular ideals bA and J in a Noetherian domain A and positive integers $d > 1$ such that: bA has a Rees valuation ring with integer $e = 1$; $b^{d-1}J$, and $b^{d+1}J$ are projectively full; and, $b^d J$ is not projectively full. Therefore: (a) the product of an ideal H that has a Rees valuation with integer $e = 1$ and an ideal which is projectively full and is contained in H need not be projectively full; and, (b) the product of an ideal H that has a Rees valuation with integer $e = 1$ and an ideal which is not projectively full and is contained in H may be projectively full.

(4.2.3) Concerning (4.1.2), it is often the case for a height-one prime ideal P of an integrally closed Noetherian domain R that the rank-one discrete valuation domain $V = R_P$ is not the only Rees valuation ring of P.

Proof. For (4.2.1), let I be a nonzero ideal in a Noetherian domain R that is not projectively full (so $d = d(I) > 1$), so $(X^d I)R[X^d]$ is projectively full (by (4.1.4)). However, (4.1.5) shows that $(X^d I)R[X]$ is not projectively full in the free (of degree d) integral extension domain $R[X]$ of $R[X^d]$.

For (4.2.2), let R and I be as in (4.1.5), let $d > 1$ be a divisor of $d(I)$ such that $d-1$ and $d+1$ are relatively prime to $d(I)$, let $b = X$, let $A = R[X]$, and let $J = IA$. Then $V = A_{bA}$ is a regular local domain, so bA has V as a Rees valuation ring with $e = 1$. Also, it follows from (4.1.5) that $b^{d-1}J$ is projectively full in A, $b^d J$ is not projectively full in A, and $b^{d+1}J$ is projectively full in A. The last statement (concerning (a) and (b)) clearly follows from this.

For (4.2.3), let P be a height-one prime ideal of an integrally closed local domain (R, M) of altitude 2. Then $V = R_P$ is the unique Rees valuation ring of P if and only if P is invertible if and only if the analytic spread $a(P)$ of P is one. With this in mind, let x, y be independent indeterminates over a field k and let R be the subring $k[[x^2, xy, y^2]]$ of the formal power series ring $k[[x, y]]$. Then R is a normal local domain and $P = (x^2, xy)R$ is a height-one prime ideal of R such that $a(P) = 2$. Therefore P is projectively full and has more than one Rees valuation ring. ■

One problem we have not been able to solve is: given a nonzero ideal in a Noetherian domain R, does there always exist a finite integral extension domain A of R such that $\mathbf{P}(IA)$ is projectively full? In Proposition 4.3 we give a "logical" candidate for A and show that, at least, for each $J \in \mathbf{P}(I)$ it holds that $(JA)_a$ is the power of some fixed ideal in $\mathbf{P}(IA)$.

In Proposition 4.3, for elements $b_1, \dots, b_g$ of a Noetherian ring R, we let $R[{b_1}^{1/k}, \dots, {b_g}^{1/k}]$ denote an integral extension ring of R generated by elements $b_1^{1/k}, \dots, b_g^{1/k}$ that are k-th roots of $b_1, \dots, b_g$, respectively. This integral extension ring of R can be obtained in several ways. If R is an integral domain, the extension $A = R[{b_1}^{1/k}, \dots, {b_g}^{1/k}]$ can be constructed to also be an integral domain. On the other hand, one can also construct $R[{b_1}^{1/k}, \dots, {b_g}^{1/k}]$ so that it is a finite free R-module of rank gk. In any case, we note that if R is local with maximal ideal M, then A is local with maximal ideal $(M, {b_1}^{1/k}, \dots, {b_g}^{1/k})A$.

Proposition 4.3 *Let I be a regular proper ideal in a Noetherian ring R and assume that*

$\mathbf{P}(I)$ *is not projectively full. Let* K *be the largest ideal in* $\mathbf{P}(I)$*, let* $b_1, \ldots, b_g$ *be regular elements in* A *that generate* K*, and let* $d(K) = k$ *(so* $k > 1$*). Let* $A = R[{b_1}^{1/k}, \ldots, {b_g}^{1/k}]$ *and let* $H = ({b_1}^{1/k}, \ldots, {b_g}^{1/k})A$*. Then for each ideal* $J \in \mathbf{P}(K) = \mathbf{P}(I)$ *it holds that* $(JA)_a = (H^n)_a$ *for some* $n \in \mathbb{N}_+$*.*

Proof. Note first that $H^{[k]} = (({b_1}^{1/k})^k, ({b_2}^{1/k})^k, \ldots, ({b_g}^{1/k})^k)A = KA$, and it is clear that $H^{[k]}$ is a reduction of H^k, so $(KA)_a = (H^k)_a$. Now let $J \in \mathbf{P}(K)$, so $(J^k)_a = (K^n)_a$ for some $n \in \mathbb{N}_+$, by (2.5.4). Therefore $((JA)^k)_a = ((J^k)_a A)_a = ((K^n)_a A)_a = (((KA)_a)^n)_a = (((H^k)_a)^n)_a = (H^{kn})_a$. Thus $((JA)^k)_a = (H^{kn})_a$, so $(JA)_a = (H^n)_a$. ∎

Concerning (4.3), the only relation between $d(K)$ and $d(H)$ we have been able to determine is $d(K) \leq kd(H)$. This follows from: $d(K) \leq d(KA)$ (by (2.7)(1)) $= d(H^k)$ (since $KA = H^h$) $= kd(H)$ (by [1, (4.8.3)]).

For a regular proper ideal $J = (b_1, \ldots, b_g)R$ in a Noetherian ring R, we present in Example 4.4 a construction for obtaining a finite integral extension ring A of $R[X]$ such that $\mathbf{P}(KA)$ is projectively full, where $K = (b_1X^k, \ldots, b_gX^k)R[X]$ and $k = d(J)$.

Example 4.4 Let J be a regular proper ideal in a Noetherian ring R, assume that J is not projectively full and is the largest ideal in $\mathbf{P}(J)$, let $d(J) = k$ (so $k > 1$, by (3.1.2)), let $b_1, \ldots, b_g$ be regular elements in R that generate J, and let $B = R[{b_1}^{1/k}, \ldots, {b_g}^{1/k}]$. Let X be an indeterminate, let $K = (b_1X^k, b_2X^k, \ldots, b_gX^k)R[X]$, let $A = R[X, {b_1}^{1/k}X, \ldots, {b_g}^{1/k}X]$, and let $H = ({b_1}^{1/k}X, \ldots, {b_g}^{1/k}X)A$. Then K is not projectively full in $R[X]$, $d(K) = k$, A is obtained from $R[X]$ by adjoining the k-th root ${b_i}^{1/k}X = (b_iX^k)^{1/k}$ of each generator b_iX^k of K to $R[X]$, and H is projectively full in A and is projectively equivalent to KA.

Proof. By (4.1.5), K is not projectively full and $d(K) = k$. Let $C = B[X]$, so C is a finite integral extension ring of A and $HC = ({b_1}^{1/k}X, \ldots, {b_g}^{1/k}X)C$ is projectively full (by (4.1.4), since $({b_1}^{1/k}, \ldots, {b_g}^{1/k})B$ is an ideal in B). Therefore H is projectively full in A, by (3.2)(1), and H is projectively equivalent to KA, by (4.3). ∎

5 NON-PROJECTIVELY FULL MAXIMAL IDEALS.

In Sections 3 and 4 a number of examples of regular ideals I of a Noetherian ring are constructed for which I or $\mathbf{P}(I)$ is projectively full. The main result in this section gives a family of integrally closed local domains (R, M) of altitude two for which the maximal ideal M is not projectively full.

Example 5.1 Let x, y, Z, W be independent indeterminates, let F be a field whose characteristic is not 2, and let (R_0, M_0) denote the regular local domain $F[x, y]_{(x,y)}$ of altitude two. Let $k < i \leq j$ be positive integers that are units in F and set $R = R_0[z, w] = R_0[Z, W]/(Z^k - x^i - y^j, W^k - x^i + y^j)R_0[Z, W]$. Then the following hold:

(i) R is an integrally closed Cohen-Macaulay local domain (with maximal ideal $M = (x, y, z, w)R$) of altitude two.

(ii) If $j = i$ and if i, k are relatively prime, then M has a unique Rees valuation ring V.

(iii) If $j = i$ and if i, k are relatively prime, then $M = (x, y, z, w)R$ is not projectively full, $d(M) = k$, and $((z, w)R)_a = M_{i/k}$.

Proof. We first show that R is a Cohen-Macaulay local ring of altitude two. For this, let $K = (Z^k - x^i - y^j, W^k - x^i + y^j)R_0[Z, W]$, let $z = Z + K$ and $w = W + K$ in $R_0[Z, W]/K$, and let $R = R_0[z, w]$, so $z^k = x^i + y^j$ and $w^k = x^i - y^j$ are in R_0, but possibly R is not an integral domain (since we do not yet know that K is a prime ideal). Also, it is clear that R is an integral extension ring of R_0 and that $M = (x, y, z, w)R$ is a maximal ideal in R, and since $z^k = x^i + y^j \in (x, y)R_0 = M_0$ and $w^k = x^i - y^j \in (x, y)R_0 = M_0$, by integral dependence it follows that every maximal ideal in R contains $(x, y, z, w)R$, hence R is local. Further, K is a height two (not necessarily prime) ideal in the locally regular UFD $R_0[Z, W]$, so R is a free (of degree k^2) integral extension ring of the altitude two regular local domain R_0 (with $\{z^m w^n \mid m = 0, \ldots, k-1 \text{ and } n = 0, \ldots, k-1\}$ as a free basis), so altitude$(R) = 2$ and R is Cohen-Macaulay, by [13, (25.16)], so R is a Cohen-Macaulay local ring of altitude two. In particular, R satisfies (S_i) for all $i \in \mathbb{N}$ (that is, the maximum length of a prime sequence in R_p is $\text{ht}(p)$ for all $p \in \text{Spec}(R)$.

Since R is (S_2), to show that R is integrally closed it suffices (by [6, (23.8)]) to show that R satisfies (R_1) (that is, R_p is a regular local domain for all height one prime ideals p in R). We do this in the next four paragraphs.

Let p be a height one prime ideal in R, let P be the preimage of p in $R_0[Z, W]$, and let $T = R_0[Z, W]_P$. Then T is a regular local domain of altitude three and $f = Z^k - x^i - y^j$,

$g = W^k - x^i + y^j$ are in $Q = PT$. Also, $f_Z = kZ^{k-1}$, $f_W = 0$, $g_Z = 0$, and $g_W = kW^{k-1}$, and the determinant ($= k^2 Z^{k-1} W^{k-1}$) of the two by two matrix consisting of these four polynomials is in Q if and only if either: (a) $Z \in Q$; or, (b) $W \in Q$ (since k is a unit in R_0). That is, if and only if either: (a$'$) $z \in p$; or, (b$'$) $w \in p$. Therefore by [6, (30.4)], R_p is a regular local domain except, possibly, in cases (a$'$) or (b$'$).

To handle cases (a$'$) and (b$'$), note first that if $z \in p$, then $z^k = x^i + y^j \in p$, so $xy \notin p$ (since, otherwise, the height two M-primary ideal $(x, y)R$ is contained in the height one prime ideal p). Similarly, if $w \in p$, then $w^k = x^i - y^j \in p$, so $xy \notin p$. Therefore if q is a height one prime ideal in R that contains either x or y, then R_q is a regular local domain (by the preceding paragraph).

Next (as just above) let T be the altitude three regular local domain $R_0[Z, W]_P$ and let $Q = PT$ be the maximal ideal of T (so $(f = Z^k - x^i - y^j,\ g = W^k - x^i + y^j)T \subseteq Q$). Also, $f_x = -ix^{i-1}$, $f_y = -jy^{j-1}$, $g_x = -ix^{i-1}$, $g_y = jy^{j-1}$, and the two by two determinant ($= -2ijx^{i-1}y^{j-1}$) consisting of these four elements is in Q if and only if either: (c) $x \in Q$; or, (d) $y \in Q$ (since $2ij$ is a unit in R_0). Modulo K, it follows that either: (c$'$) $x \in p$; or, (d$'$) $y \in p$. Therefore by [6, (30.4)], R_p is a regular local domain except, possibly, in cases (c$'$) or (d$'$). However, the preceding paragraph shows that R_p is a regular local domain in both cases (c$'$) and (d$'$).

It follows that R_p is a regular local domain for all height one prime ideals p in R, so R is normal (by (R_1), (S_2) (see [6, Theorem 23.8])). Therefore R is an integrally closed reduced local ring, so R is a local domain (by [6, (9.11)]), hence K is a prime ideal and R is an integrally closed Cohen-Macaulay local domain. This completes the proof of (i).

For (ii), assume that $j = i$ and that i, k are relatively prime, and notice that $z^k \in (x^i, y^i)R \subseteq (x, y)^k R$ and $w^k \in (x^i, y^i)R \subseteq (x, y)^k R$, so $(x, y)R$ is a reduction of $M = (x, y, z, w)R$. It follows that each Rees valuation ring of M is an extension of the order valuation ring $V_0 = R_0[y/x]_{xR_0[y/x]} = R_0[x/y]_{yR_0[x/y]}$ of R_0. (Notice that $xV_0 = yV_0$ is the maximal ideal N_0 of V_0 and that if we let $t = y/x$ and $\bar{t} = t + N_0$ in the field V_0/N_0, then $V_0/N_0 = (R_0/M_0)(\bar{t})$ and $\bar{t}$ is transcendental over R_0/M_0.)

With this in mind, it follows from [20, Theorem 19, p. 55] that $k^2 = [R : R_0] \geq \sum_{i=1}{}^{g} e_i f_i$, where $V_1, \ldots, V_g$ are the extensions (to the quotient field of R) of V_0, e_i is the ramification index of N_0 in V_i (so $N_0V_i = N_i{}^{e_i}$, where N_i is the maximal ideal of V_i), and f_i is the relative degree $[(V_i/N_i) : (V_0/N_0)]$. It will now be shown that $g = 1$ and that $e_1 = k = f_1$

For this, if v is the valuation of any of the valuation rings $V \in \{V_1, \ldots, V_g\}$, then $v(x^i) = iv(x)$, $v(y^i) = iv(y)$, and $kv(z) = v(z^k) = v(x^i + y^i) = iv(x)$ (since v is an extension of the order valuation v_0). Therefore $kv(z) = iv(x)$ and, similarly, $kv(w) = iv(x)$. Since k, i are relatively prime (by hypothesis), it follows that $v(z) = v(w) \geq i$ and $v(x) \geq k$. Also, $v(x) = v(N_0)$ (since $xV_0 = N_0$, as noted above), so the ramification index e of N_0 in V is at least k.

Also, as noted in the preceding paragraph, $v(z) = v(w)$, so $\frac{w}{z}$ is a unit in V, and $(\frac{w}{z})^k = \frac{w^k}{z^k} = \frac{x^i - y^i}{x^i + y^i}$. As above, let $t = \frac{y}{x}$, so t is a unit in V_0 whose residue class $\overline{t}$ in V_0/N_0 is transcendental over R_0/M_0 and $\frac{x^i - y^i}{x^i + y^i} = \frac{1 - t^i}{1 + t^i} \in V_0$, It follows that the residue class of $\frac{w}{z}$ in V/N is algebraic of degree k over V_0/N_0, so $[V/N : V_0/N_0] \geq k$.

Therefore, by the preceding two paragraphs, for each $(V_i, N_i) \in \{(V_1, N_1), \ldots, (V_g, N_g)\}$ it holds that $N_0V_i = N_i{}^{e_i} \subseteq N_i{}^k$ (so $e_i \geq k$) and $f_i = [V_i/N_i : V_0/N_0] \geq k$. But $k^2 = [R : R_0] \geq \sum_{i=1}{}^g e_i f_i$, so it follows that $g = 1$ and that $e_1 = k = f_1$. In what follows, we denote (V_1, N_1) by (V, N), e_1 by e, and f_1 by f. After normalizing v, we have $v(x) = v(y) = k$ (so $v(M) = k$) and $v(z) = v(w) = i$.)

For (iii), assume that $j = i$ and that i, k are relatively prime. To see that M is not projectively full, note that $M^k \supseteq (z^k, w^k)R = (x^i + y^i, x^i - y^i)R$ (since $j = i$), and $(x^i + y^i, x^i - y^i)R = (2x^i, 2y^i)R = (x^i, y^i)R$ (since 2 is a unit in R), and $(x^i, y^i)R$ is a reduction of M^i. It follows that $((z^k, w^k)R)_a = (M^i)_a$, so $((z, w)^k R)_a = (M^i)_a$. Therefore $(z, w)R$ and M are projectively equivalent and $((z, w)R)_a = M_{i/k}$, by [11, (2.3)], so $((z, w)R)_a$ is not the integral closure of any power of M (since k and i are relatively prime), hence M is not projectively full.

By the preceding paragraph $i/k \in \mathbf{U}(M)$ (see (2.4.3)). Therefore, since i, k are relatively prime and since $d(M)\mathbf{U}(M) \subseteq \mathbb{N}_+$ (by (2.5.4)), it follows that $d(M)$ is a multiple of k. On the other hand, $d(M)$ is a divisor of the integer e associated to the Rees valuation ring (V, N) of M, by (2.5.4). By (2.3), this integer e is given by $MV = N^e$, so $v(M) = e$. However, $v(M) = k$, by the second preceding paragraph, so it follows that $d(M)$ is a divisor of k. Therefore $d(M) = k$. ■

In the next remark we note (with brief indications of proofs) several properties of two rings related to the rings $R[z, w]$ of (5.1).

Remark 5.2 With notation as in (5.1), let $A = R[M/x] = R[y/x, z/x, w/x]$ and let $\mathbf{R} = R[u, tM] = R[u, tx, ty, tz, tw]$ (where t is an indeterminate and $u = 1/t$). Also, with the assumptions as in (5.1.2), let (V, N) be the unique Rees valuation ring of M. Then the following hold:

(5.2.1) xA' is $N \cap A'$-primary, A is Cohen-Macaulay, xA is p'-primary, where $p' = N \cap A$, $p' = (M, z/x, w/x)A$, and A is not integrally closed. Moreover, if $i - k$ is a unit in R_0, then $A_{p'}$ is not integrally closed, but A_p is a regular local domain for all height one prime ideals $p \neq p'$.

(5.2.2) $u\mathbf{R}'$ is primary, $\mathbf{R}$ is Cohen-Macaulay, $u\mathbf{R}$ is p^*-primary, where $p^* = (u, M, tz, tw)\mathbf{R}$, and $\mathbf{R}$ is not integrally closed. Moreover, if $i - k$ is a unit in R_0, then $\mathbf{R}_{p^*}$ is not integrally closed, but $\mathbf{R}_p$ is a regular local domain for all height one prime ideals $p \neq p^*$.

Proof. (We only sketch the proofs.)

xA' is $N \cap A'$-primary (by [11, (2.9)]) and $u\mathbf{R}'$ is primary (since M has a unique Rees valuation ring).

A (resp., $\mathbf{R}$) is a free (of degree k^2) integral extension domain of the locally regular UFD $A_0 = R_0[M_0/x] = R_0[y/x]$ (resp., $\mathbf{R}_0 = R_0[u, tM_0] = R_0[u, tx, ty]$) (since $(z/x)^k = x^{i-k} + y^{i-k}(y/x)^k$ and $(w/x)^k = x^{i-k} - y^{i-k}(y/x)^k$ imply $(f_1, g_1)A_0[Z, W] = (Z^k - x^{i-k} - y^{i-k}(y/x)^k, W^k - x^{i-k} + y^{i-k}(y/x)^k)A_0[Z, W]$ is a height two prime ideal) (resp., $(tz)^k = u^{i-k}(tx)^i + u^{i-k}(ty)^i$ and $(tw)^k = u^{i-k}(tx)^i - u^{i-k}(ty)^i$) imply $(f_2, g_2)\mathbf{R}_0[Z, W] = (Z^k - u^{i-k}(tx)^i - u^{i-k}(ty)^i, W^k - u^{i-k}(tx)^i + y^{i-k}(ty)^i)\mathbf{R}_0[Z, W]$ is a height two prime ideal)), so it follows from [13, (25.16)] that A (resp., $\mathbf{R}$) is Cohen-Macaulay.

Since A (resp., $\mathbf{R}$) is Cohen-Macaulay and xA' (resp., $u\mathbf{R}'$) has a unique (height one) prime divisor, it follows from the structure of A (resp., $\mathbf{R}$) that xA (resp., $u\mathbf{R}$) has a unique prime divisor, say p' (resp., p^*). And it then follows that $p' = N \cap A = (M, z/x, w/x)A$ (resp., $p^* = (u, M, tz, tw)\mathbf{R}$).

Using the determinant of $f_{1Z} = kZ^{k-1}$, $f_{1W} = 0$, $g_{1Z} = 0$, and $g_{1W} = kW^{k-1}$ and then of $f_{1x} = -(i-k)x^{i-k-1}$, $f_{1\tau} = -ky^{i-k}\tau^{k-1}$, $g_{1x} = -(i-k)x^{i-k-1}$, and $g_{1\tau} = ky^{i-k}\tau^{k-1}$ (with $\tau = y/x$), it follows from [6, (30.4)] that A_p is a regular local domain for all $p \neq p'$. However, $A_{p'}$ is not a regular local domain, since, otherwise, it would follow that $V/N = V_0/N_0$ (where (V_0, N_0) is the order valuation ring of R_0), and this contradicts the fact (shown in the proof of (5.1)(ii)) that w/z is algebraic of degree k over V_0/N_0. It therefore follows that A is not integrally closed.

It is clear that analogous statements hold for $B = R[M/y] = R[x/y, z/y, w/y]$ and y in place of A and x. With this in mind, since $\mathbf{R}[1/(tx)] = A[tx, 1/(tx)]$ (resp., $\mathbf{R}[1/(ty)] = B[ty, 1/(ty)]$), and since $uA[tx, 1/(tx)] = xA[tx, 1/(tx)]$ (resp., $uA[ty, 1/(ty)] = yA[ty, 1/(ty)]$), it follows that $\mathbf{R}$ is not integrally closed, that $\mathbf{R}_{p^*}$ is not integrally closed, and that $\mathbf{R}_p$ is a regular local domain for all height one prime ideals $p \neq p^*$. ■

In Example 5.1 M has only one Rees valuation ring. In the final remark in this paper we consider the ideals in $\mathbf{P}(I)$ in the case where I has only one Rees valuation ring.

Remark 5.3 Let I be a proper ideal in a Noetherian domain R and assume that I has only one Rees valuation ring, say (V, N). Then $\mathbf{P}(I) \subseteq \{N^i \cap R \mid i \in \mathbb{N}\}$, so if $P = N \cap R$ is the center of V on R, then each ideal in $\mathbf{P}(I)$ is a P-primary valuation ideal (that is, it is contracted from a valuation overring of R). Assume that I is maximal in $\mathbf{P}(I)$. If $IV = N^e$, then $I = N^e \cap R$ and I is projectively full if and only if each $J \in \mathbf{P}(I)$ has the property that $JV = N^{ne}$ for some $n \in \mathbb{N}$. In general, the inclusion $\mathbf{P}(I) \subseteq \{N^i \cap R \mid i \in \mathbb{N}\}$ need not be an equality. For example, if (R, M) is a regular local domain of altitude two and $M = (x, y)R$, then $I = (x, y^2)R$ has a unique Rees valuation ring. To see this one can apply [1, (2.9)] or [11, (3.1)]. Notice that $MR[x/y^2] = yR[x/y^2]$ is a height-one prime ideal and $V = R[x/y^2]_{yR[x/y^2]}$ is a valuation domain. Also, $MR[y^2/x]$ is a height-one prime ideal and $R[y^2/x]_{MR[y^2/x]} = V$. Thus V is the unique Rees valuation ring of $I = (x, y^2)R$. The ideal I is projectively full (by (3.7)) and $I \subsetneq M = N \cap R$. Also, $IV = N^2$, so $\mathbf{P}(I) = \{N^{2n} \cap R\}_{n=1}^{\infty}$.

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