Power Series Over Noetherian Rings May 3, 2012

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Abstract. In this monograph the authors gather together results and examples from their work of the past two decades related to power series rings and to completions of Noetherian integral domains.

A major theme is the creation of examples that are appropriate intersections of a field with a homomorphic image of a power series ring over a Noetherian domain. This technique goes back to work of Akizuki in the 1930s and Nagata in the 1950s. In certain circumstances, such an intersection is computable as a directed union, and the Noetherian property for the associated directed union is equivalent to a flatness condition. This flatness criterion simplifies the analysis of several classical examples and yields new examples such as

- A catenary Noetherian local integral domain of any specified dimension bigger than one that has geometrically regular formal fibers and is not universally catenary.
- A three-dimensional non-Noetherian unique factorization domain B such that the unique maximal ideal of B has two generators; B has precisely n prime ideals of height two, where n is an arbitrary positive integer; and each prime ideal of B of height two is not finitely generated but all the other prime ideals of B are finitely generated.
- A two-dimensional Noetherian local domain that is a birational extension of a polynomial ring in three variables over a field yet fails to have Cohen-Macaulay formal fibers. This also demonstrates that Serre's condition S_1 need not lift to the completion. It is related to an example of Ogoma.

Another theme is an analysis of extensions of integral domains $R \hookrightarrow S$ having trivial generic fiber, that is, every nonzero prime ideal of S has a nonzero intersection with R. Motivated by a question of Hochster and Yao, we present in Chapters 29, 30 and 31 results about

- The height of prime ideals maximal in the generic fiber of certain extensions involving mixed power series/polynomial rings.
- The prime ideal spectrum of a power series ring in one variable over a one-dimensional Noetherian domain.
- The dimension of S if $R \hookrightarrow S$ is a local map of complete local domains having trivial generic fiber.

A third theme relates to the questions:

- What properties of a Noetherian domain extend to a completion?
- What properties of an ideal pass to its extension in a completion?
- What properties extend for a more general multi-adic completion?

We give an example of a three-dimensional regular local domain R having a prime ideal P of height two with the property that the extension of P to the completion of R is not integrally closed.

All of these themes are relevant to the study of prime spectra of Noetherian rings and of the induced spectral maps associated with various extensions of Noetherian rings. We describe the prime spectra of various extensions involving power series.

Contents

Preface

The authors have had as a long-term project the creation of examples using power series to analyze and distinguish several properties of commutative rings and their spectra. This monograph is our attempt to expose the results that have been obtained in this endeavor, to put these results in better perspective and to clarify their proofs. We hope in this way to assist current and future researchers in commutative algebra in utilizing the techniques described here.

William Heinzer, Christel Rotthaus, Sylvia Wiegand

CHAPTER 1

Introduction

Over the past eighty years, important examples of Noetherian integral domains have been constructed using power series, homomorphic images and intersections. The basic idea is that, starting with a typical Noetherian integral domain R such as a polynomial ring in several indeterminates over a field, we look for more unusual Noetherian and non-Noetherian extension rings inside a homomorphic image S of an ideal-adic completion of R. An ideal-adic completion of R is a homomorphic image of a power series ring over R; see Section 3.1 of Chapter 3. The constructed ring A has the form $A := L \cap S$, where L is a field between the field of fractions of R and the total quotient ring of S.

We have the following major goals:

(1) To construct new examples of Noetherian rings, continuing a tradition that goes back to Akizuki and Schmidt in the 1930s and Nagata in the 1950s.
(2) To construct new non-Noetherian integral domains that illustrate recent advances in ideal theory.
(3) To study birational extensions of Noetherian local domains; this is related to Zariski's constructions of valuation rings that are birational approximations of Noetherian local domains.
(4) To consider the fibers of an extension $R \hookrightarrow R^*$, where R is a Noetherian domain and R^* is the completion of R with respect to an ideal-adic topology, and to relate these fibers with birational extensions of R.

These objectives form a complete circle, since (4) is used to accomplish (1).

The development of this popular technique to create interesting new rings from well-known ones goes back to work of Akizuki in [**7**], Schmidt in [**98**] and Nagata in [**78**]. This work is continued by Ferrand-Raynaud [**29**], Lequain [**66**], Nishimura [**80**], Rotthaus [**91**], [**92**], Heitmann [**58**], Ogoma [**84**], [**85**], Brodmann-Rotthaus [**11**], [**12**], and Weston [**109**]. The present authors have been captivated by this technique and have been examining it for a number of years. Several chapters of this monograph (for example, see Chapters 4, 5, 6, 8, 22, 23, 17) contain a reorganized development of previous work of the present authors on this technique.

This construction technique is *universal* in the following sense: Every Noetherian local domain A having a coefficient field k and having field of fractions L such that L is finitely generated over k is such an intersection $L \cap S$, where $S = \widehat{R}/I$ and I is a suitable ideal of the $\mathbf{m}$-adic completion $\widehat{R}$ of a Noetherian local domain $(R, \mathbf{m})$. Furthermore we can choose R so that k is also a coefficient field for the ring R, L is the field of fractions of R and R is essentially finitely generated over k; see Section 5.2 of Chapter 5.

In his 1935 paper Akizuki gives an example in characteristic zero of a one-dimensional Noetherian local domain that is analytically ramified [**7**], while Schmidt gives such an example in positive characteristic [**98**, pp. 445-447].[1] An example due to Nagata is given in [**79**, Example 3, pp. 205-207]. (See also [**79**, (32.2), p. 114].)

In Chapter 4 (Example 4.8) we consider another example constructed by Nagata. This is the first occurence of a two-dimensional regular local domain containing a field of characteristic zero that fails to be a Nagata domain. Because the example is not Nagata, it is also not excellent. For the definition and information on Nagata rings and excellent rings see Definitions 2.3.1 and 3.27 in Sections 2.1 and 3.3 We describe in Example 4.10 a construction due to Rotthaus of a Nagata domain that is not excellent.

An interesting construction first introduced by Ray Heitmann in [**58**, page 126] shows how to adjoin a transcendental power series in an element a to the ring R in a way that is consistent with the (a)-adic completion. In a 1997 article [**39**, Theorem 2.8], the present authors adapted the construction of Heitmann to prove a semilocal inclusion version of the Noetherian Flatness Theorem 8.8 of Chapter 8; see Remark 8.9.2.

We consider this construction in the more general context of an arbitrary Noetherian integral domain in Chapter 5. We apply the construction, and its natural generalization to finitely many transcendental power series in the (a)-adic completion R^* of R in order to exhibit Noetherian extension domains inside R^*.

In the foundational work of Akizuki, Nagata and Rotthaus (and indeed in most of the papers cited above) the description of the constructed ring A as an intersection $A := L \cap S$, where L is a field between the field of fractions of R and the total quotient ring of S, is not explicitly stated. Instead A is defined as a direct limit or directed union of subrings. The fact that in certain circumstances the intersection domain A may be computable as a directed union is crucial to our development of this technique; the description of A as an intersection is often unfathomable. There is sometimes a natural direct limit domain B associated with A such that $B \subseteq A$. We examine conditions for A to equal B. This motivates our formulation of the limit-intersecting property that we define in Chapter 6 (Definition 6.5) and discuss in Chapter 12.

In Chapters 4 to 12, we develop the construction using an ideal-adic completion with respect to a principal ideal. We construct two integral domains:

1. An "intersection" integral domain of the form $A := L \cap S$ as above, the intersection of a field L with a power series ring S, and
2. An integral domain B that approximates A and is more easily computable, sometimes as a nested union of localized polynomial rings.[2]

To see a specific example, consider the ring $R := \mathbb{Q}[x, y]$, the polynomial ring in the variables x and y over the field $\mathbb{Q}$ of rational numbers. Let S be the formal power series ring $\mathbb{Q}[[x, y]]$ and let L be the field $\mathbb{Q}(x, y, e^x, e^y)$.[3] Since $x-y$ is a factor of $e^x - e^y$, the intersection domain $A = \mathbb{Q}(x, y, e^x, e^y) \cap \mathbb{Q}[[x, y]]$ is larger than might

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[1] For the definition of analytically ramified and most other terminology used here, see Chapters 2 and 3.

[2] The details of the construction of B are given in Chapter 6.

[3] This example with power series in two variables may be realized by taking first the (x)-adic completion R^* and then taking the (y)-adic completion of R^*.

be expected. In this example, the intersection domain A is Noetherian, whereas the approximation domain B is not Noetherian. More details about this example are given in Chapter 4 (Example 4.4, Theorem 4.12), Chapter 7 (Theorem 7.2, Example 7.3) and Chapter 8 (Example 8.11).

A primary task of our study is to determine for a given Noetherian domain R whether the ring $A := L \cap S$ is Noetherian. Here the ring S is a homomorphic image of an ideal-adic completion of R and L is a subfield of the total quotient ring of S that contains R. An important observation related to this task is that the Noetherian property for the associated direct limit ring B is equivalent to a flatness condition; see Chapter 8, Theorem 8.3. Whereas it took only about a page for Nagata [**79**, page 210] to establish the Noetherian property of his example, for the example of Rotthaus [**91**, pages 112-118], the proof of the Noetherian property took 7 pages. The results presented in Chapter 8 establish the Noetherian property rather quickly for this and other examples.

In the case when $S = R^*$, and L is an intermediate field between R and the total quotient ring of R^*, the integral domain $A = L \cap R^*$ sometimes inherits nice properties from R^* such as the Noetherian property. If the approximation domain B is Noetherian, then B is equal to the intersection domain A. However, as we demonstrate with examples, the converse fails; it is possible for B to be equal to A and not be Noetherian, If B is not Noetherian, we can sometimes identify the prime ideals of B that are not finitely generated. If a ring has exactly one prime ideal that is not finitely generated, that prime ideal contains all nonfinitely generated ideals of the ring.

In Chapters 7, 10, and 17, we adjust the construction from Chapters 4-9. An "insider" technique is introduced in Chapter 10 for building new examples inside more straightforward examples constructed as above. Using the insider process, the verification of the Noetherian property for the constructed rings is streamlined. Even if one of the constructed rings is not Noetherian, the proof is simplified. We analyze classical examples of Nagata and others from this viewpoint. Chapter 7 contains an investigation of more general rings that involve power series in two variables x and y over a field k as with the specific example mentioned above. In Chapters 17 and 18, we construct low-dimensional non-Noetherian integral domains that are strangely close to being Noetherian: One example is a three-dimensional local unique factorization domain B inside $k[[x, y]]$; the ring B has maximal ideal $(x, y)B$ and exactly one prime ideal that is not finitely generated.

Recently there has been considerable interest in non-Noetherian analogues of Noetherian notions such as the concept of a "regular" ring. Rotthaus and Sega in [**97**] show that the rings B constructed in Chapters 17 and 18, even though non-Noetherian, are *coherent regular* local rings in the sense that every finitely generated submodule of a free module has a finite free resolution; see [**97**] and Remark 17.11.[4]

One of our additional goals is to consider the two questions: "What properties of a ring extend to a completion?" and "What properties of the base ring R are preserved by the construction?" Chapter 15 contains an example of a three-dimensional regular local domain $(A, \mathbf{n})$ with a height-two prime ideal P such that the extension

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[4] Rotthaus and Sega show more generally that rings constructed with Insider Construction 13.1 are coherent regular if $R = k[x, y_1 \dots, y_r]_{(x,y_1 \dots, y_r])}$ is a localized polynomial ring over a field k, $m = 1$, $r, n \in \mathbb{N}$ and $\tau_1, \dots, \tau_n$ are algebraically independent elements of $xk[[x]]$. Such rings can have arbitrarily large Krull dimension, whereas the rings constructed in Chapters 17 and 18 have dimension 3 or 4.

$P\widehat{A}$ to the $\mathbf{n}$-adic completion of A is not integrally closed. In Chapter 16 we prove that the Henselization of a Noetherian local ring having geometrically normal formal fibers is universally catenary; we also present for each integer $n \geq 2$ a catenary Noetherian local integral domain having geometrically normal formal fibers that is not universally catenary. We present in Chapter 14 a brief exposition of excellent rings. In some cases we identify when the constructed ring is excellent; for example, see Chapter 9 (Polynomial Example Theorems 9.2, 9.5 and 9.7) and Chapter 17.

Assume the ring R is a unique factorization domain (UFD) and R^* is the (a)-adic completion of R with respect to a prime element a of R. We observe that the approximation domain B is then a UFD; see Proposition 6.21.

Since the Noetherian property for the approximation domain is equivalent to the flatness of a certain homomorphism, we devote considerable time and space to exploring flat extensions. We present results involving flatness in Chapters 9, 11 and 12.

The idealwise construction in Chapter 19 is of a nature different from the construction in Chapters 4 and 5. Let $(R, \mathbf{m})$ be a excellent normal local domain and let $\widehat{R}$ denote the $\mathbf{m}$-adic completion of R, we consider $D := L \cap \widehat{R}$, where L is a purely transcendental extension of the field of fractions K of R; say $L = K(\tau_1, \tau_2, \ldots, \tau_n)$, where $\tau_1, \tau_2, \ldots, \tau_n \in \widehat{R}$ are algebraically independent over K. The elements $\tau_1, \ldots, \tau_n$ are said to be *idealwise independent*, if $K(\tau_1, \ldots, \tau_n) \cap \widehat{R}$ equals the localized polynomial ring $R[\tau_1, \ldots, \tau_n]_{(\mathbf{m}, \tau_1, \ldots, \tau_n)}$. The idealwise construction demonstrates that the intersection domain can sometimes be small or large, depending on whether expressions in the power series have additional prime divisors. The consideration of idealwise independent elements leads to other related flatness conditions, namely two other concepts of independence over R for algebraically independent elements $\tau_1, \ldots, \tau_n$ of $\widehat{\mathbf{m}}$. A summary of the analysis and properties related to the idealwise independence definition is given in Chapter 19.

In later chapters of this monograph, we study prime ideals and their relations in mixed power series/polynomial extensions of low-dimensional rings. For example, we describe the prime ideal structure of the power series ring $R[[x]]$ over a one-dimensional Noetherian domain R, as well as the prime ideal structure of $k[[x]][y]$, where k is a field in Chapter 30. We analyze the generic formal fibers of mixed power series in Chapter 29. Motivated by a question of Hochster and Yao, we consider in Chapter 31 ring extensions $S \hookrightarrow T$ having *trivial generic fiber*; that is, every nonzero prime ideal of T intersects S in a nonzero prime ideal.

The topics of this book include the following:

1. Flatness properties of maps of rings, Chapters 8, 9, 11, 12, 19 and 22.
2. Preservation of properties of rings and ideals under passage to completion, Chapters 15 32 and 28.
3. The catenary and universally catenary property of Noetherian rings, Chapter 9 and 16.
4. Excellent rings and geometrically regular and geometrically normal formal fibers, Chapters 9, 14 and 16.
5. Examples of non-Noetherian local rings having Noetherian completions, Chapter 17 and 18.
6. The prime ideal structure of certain rings, Chapters 17, 29, 30, and 31.
7. The approximation of a DVR using higher dimensional regular local rings, Chapter 25.

(8) Trivial generic fiber extensions, Chapters 29, 30, and 31.
(9) Transfer of the excellence property, Chapter 23.
(10) Birational extensions of Noetherian integral domains, Chapter 8.
(11) Multi-ideal-adic completions, Chapter 32.

CHAPTER 2

Tools

In this chapter we review conventions and terminology, state several basic theorems and review the concept of flatness of modules and maps.

2.1. Conventions and terminology

We generally follow the notation of Matsumura [**73**]. We also reference Dummit and Foote [**24**] for more elementary concepts. Thus by a ring we mean a commutative ring with identity, and a ring homomorphism $R \to S$ maps the identity element of R to the identity element of S. The set of prime ideals of a ring R is denoted $\operatorname{Spec} R$. The set $\operatorname{Spec} R$ is naturally a partially ordered set with respect to inclusion. For an ideal I of a ring R, let

$$\mathcal{V}(I) \ = \ \{P \in \operatorname{Spec} R \mid I \subseteq P \ \}.$$

The *Zariski topology* on $\operatorname{Spec} R$ is obtained by defining the closed subsets to be the sets of the form $\mathcal{V}(I)$ as I varies over all the ideals of R.

We use $\mathbb{Z}$ to denote the ring of integers, $\mathbb{N}$ for the positive integers, $\mathbb{N}_0$ the nonnegative integers, $\mathbb{Q}$ the rational numbers, $\mathbb{R}$ the real numbers and $\mathbb{C}$ the complex numbers.

Regular elements. An element r of a ring R is said to be a *zerodivisor* if there exists a nonzero element $a \in R$ such that $ar = 0$, and r is a *regular element* if r is not a zerodivisor. The total ring of fractions of the ring R, denoted $\mathcal{Q}(R)$, is the localization of R at the multiplicatively closed set of regular elements, thus $\mathcal{Q}(R) := \{\ a/b \mid a, b \in R \text{ and } b \text{ is a regular element }\}$. There is a natural embedding $R \hookrightarrow \mathcal{Q}(R)$ of a ring R into its total ring of fractions $\mathcal{Q}(R)$, where $r \mapsto \frac{r}{1}$ for every $r \in R$.

An *integral domain*, sometimes called a *domain* or an *entire* ring, is a nonzero ring in which every nonzero element is a regular element. If R is a subring of an integral domain S and S is a subring of $\mathcal{Q}(R)$, we say S is *birational over* R, or a *birational extension* of R.

Krull dimension. The *Krull dimension*, or briefly *dimension*, of a ring R, denoted $\dim R$, is n if there exists a chain $P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n$ of prime ideals of R and there is no such chain of length greater than n. We say that $\dim R = \infty$ if there exists a chain of prime ideals of R of length greater than n for each $n \in \mathbb{N}$. For a prime ideal P of a ring R, the *height* of P, denoted $\operatorname{ht} P$, is $\dim R_P$, where R_P is the localization of R at the multiplicatively closed set $R \setminus P$.

Unique factorization domains. An integral domain R is a *unique factorization domain* (UFD) if every nonzero nonunit of R is a finite product of prime elements; an element $p \in R$ is *prime* if pR is a prime ideal.

In a UFD every height-one prime ideal is principal.

Local rings. If a ring R (not necessarily Noetherian) has a unique maximal ideal $\mathbf{m}$, we say R is *local* and write $(R, \mathbf{m})$ to denote that R is local with maximal ideal $\mathbf{m}$. If $(R, \mathbf{m})$ and $(S, \mathbf{n})$ are local rings, a ring homomorphism $f : R \to S$ is a *local homomorphism* if $f(\mathbf{m}) \subseteq \mathbf{n}$.

Let $(R, \mathbf{m})$ be a local ring. A subfield k of R is said to be a *coefficient field for* R if the composite map $k \hookrightarrow R \to R/\mathbf{m}$ defines an isomorphism of k onto $R/\mathbf{m}$.

If $(R, \mathbf{m})$ is a subring of a local ring $(S, \mathbf{n})$, then S is said to *dominate* R if $\mathbf{m} = \mathbf{n} \cap R$, or equivalently, if the inclusion map $R \hookrightarrow S$ is a local homomorphism. The local ring $(S, \mathbf{n})$ is said to *birationally dominate* $(R, \mathbf{m})$ if S is an integral domain that dominates R and S is contained in the field of fractions of R.

Jacobson radical. The *Jacobson radical* $\mathcal{J}(R)$ of a ring R is the intersection of all maximal ideals of R. An element z of R is in $\mathcal{J}(R)$ if and only if $1 + zr$ is a unit of R for all $r \in R$.

If I is a proper ideal of R, then $1 + I := \{ 1 + a \mid a \in I \}$ is a multiplicatively closed subset of R that does not contain 0. Let $(1 + I)^{-1}R$ denote the localization $R_{(1+I)}$ of R at the multiplicatively closed set $1 + I$, [**73**, Section 4]. If P is a prime ideal of R and $P \cap (1 + I) = \emptyset$, then $(P + I) \cap (1 + I) = \emptyset$. Therefore I is contained in every maximal ideal of $(1 + I)^{-1}R$, so $I \subseteq \mathcal{J}((1 + I)^{-1}R)$. In particular for the principal ideal $I = zR$, where z is a nonunit of R, we have $z \in \mathcal{J}((1 + zR)^{-1}R)$.

Finite type. An extension ring S of a ring R is of *finite type* over R if S is finitely generated as an R-algebra. An extension ring S of R is *essentially of finite type* over R if S is a localization at some multiplicatively closed subset of an extension ring of R of finite type. We also say that S is *essentially finitely generated* in this case.

Symbolic powers. If P is a prime ideal of a ring R and e is a positive integer, the e^{th} *symbolic power of* P, denoted $P^{(e)}$, is defined as

$$P^{(e)} := \{a \in R \mid ab \in P^e \text{ for some } b \in R \setminus P \}.$$

Valuation domains. An integral domain R is a *valuation domain* if for each element $a \in \mathcal{Q}(R) \setminus R$, we have $a^{-1} \in R$. A valuation domain R is called a *discrete valuation ring* (DVR) if R is Noetherian and not a field, or equivalently, if R is a local principal ideal domain (PID) and not a field.

Remark 2.1. If R is a valuation domain with field of fractions K and F is a subfield of K, then $R \cap F$ is again a valuation domain and has field of fractions F [**79**, (11.5)]. If R is a DVR and the field F is not contained in R, then $R \cap F$ is again a DVR [**79**, (33.7)].

Integral closure, normal domains. An integral domain R is said to be *integrally closed* provided it satisfies the following condition: for every monic polynomial $f(x)$ in the polynomial ring $R[x]$, if $a \in \mathcal{Q}(R)$ is a root of $f(x)$, then $a \in R$. The ring R is a *normal ring* if for each $P \in \operatorname{Spec} R$ the localization R_P is an integrally closed domain [**73**, page 64]. If R is a Noetherian normal ring and $\mathbf{p}_1, \dots, \mathbf{p}_r$ are the minimal primes of R, then R is isomorphic to the direct product $R/\mathbf{p}_1 \times \cdots \times R/\mathbf{p}_r$ and each $R/\mathbf{p}_i$ is an integrally closed domain. Since a nontrivial direct product is not local, a normal Noetherian local ring is a normal domain.

If R is a normal Noetherian integral domain and L is a finite separable algebraic field extension of $\mathcal{Q}(R)$, then the integral closure of R in L is a finite R-module by [**73**, Lemma 1, page 262] or [**79**, (10.16)]. Thus, if R is a normal Noetherian integral domain of characteristic zero, then the integral closure of R in a finite algebraic field extension is a finite R-module.

The order function associated to an ideal. Let I be a nonzero ideal of a ring R such that $\bigcap_{n=0}^{\infty} I^n = (0)$. Adopt the convention that $I^0 = R$, and for each nonzero element $r \in R$ define

$$\operatorname{ord}_{R,I}(r) \ := \ n \quad \text{if} \quad r \in I^n \ \setminus \ I^{n+1}.$$

Remark 2.2. With R, I and $\operatorname{ord}_{R,I}$ as above, consider the following two properties for nonzero elements a, b in R:

1. If $a + b \neq 0$, then $\operatorname{ord}_{R,I}(a + b) \geq \min\{\operatorname{ord}_{R,I}(a), \operatorname{ord}_{R,I}(b)\}$.
2. $\operatorname{ord}_{R,I}(ab) = \operatorname{ord}_{R,I}(a) + \operatorname{ord}_{R,I}(b)$.

Clearly the function $\operatorname{ord}_{R,I}$ always satisfies item 1. If $\operatorname{ord}_{R,I}$ satisfies item (2) for all nonzero a, b in R, then the function $\operatorname{ord}_{R,I}$ extends uniquely to a function on $\mathcal{Q}(R) \setminus (0)$ by defining $\operatorname{ord}_{R,I}(\frac{a}{b}) := \operatorname{ord}_{R,I}(a) - \operatorname{ord}_{R,I}(b)$ for nonzero elements $a, b \in R$, and the set

$$V \ := \ \{q \ \in \mathcal{Q}(R) \ \setminus \ (0) \mid \operatorname{ord}_{R,I}(q) \ \geq \ 0 \ \} \ \cup \ \{0\}$$

is a DVR. Therefore if item (2) holds for all nonzero a, b in R, then R is an integral domain and I is a prime ideal of R.

Let A be a commutative ring and let $R := A[[x]] = \{f = \sum_{i=0}^{\infty} f_i x^i \mid f_i \in R\}$, the *formal power series ring* over A in the variable x. With $I := xR$ and f a nonzero element in R, we write $\operatorname{ord} f$ for $\operatorname{ord}_{R,I}(f)$. Thus $\operatorname{ord} f$ is the least integer $i \geq 0$ such that $f_i \neq 0$. The element f_i is called *leading form* of f.

In the case where $(R, \mathbf{m})$ is a local ring, we abbreviate $\operatorname{ord}_{R,\mathbf{m}}$ by ord_R.

Regular local rings. A local ring $(R, \mathbf{m})$ is a *regular local ring* (RLR) if R is Noetherian and $\mathbf{m}$ can be generated by $\dim R$ elements. If $(R, \mathbf{m})$ is a regular local ring then R is an integral domain, and the function ord_R satisfies the conditions of Remark 2.2. The associated valuation domain

$$V := \{q \in \mathcal{Q}(R) \setminus \{0\} \mid \operatorname{ord}_R(q) \geq 0\} \cup \{0\}$$

is a DVR that birationally dominates R. If $x \in \mathbf{m} \setminus \mathbf{m}^2$, then $V = R[\mathbf{m}/x]_{xR[\mathbf{m}/x]}$, where $\mathbf{m}/x = \{y/x \mid y \in \mathbf{m}\}$.

We record the following definitions.

Definitions 2.3. Let R be a commutative ring.

1. R is called a *Nagata* ring if R is Noetherian and, for every $P \in \operatorname{Spec} R$ and every finite extension field L of $\mathcal{Q}(R/P)$, the integral closure of R/P in L is finitely generated as a module over R/P. In Nagata's book [**79**] a Nagata ring is called *pseudo-geometric*. It is clear from the definition that a homomorphic image of a Nagata ring is again a Nagata ring. An important result proved by Nagata is that a polynomial ring in finitely many variables over a Nagata ring is again a Nagata ring [**79**, Theorem 36.5, page 132].
2. An integral domain R is said to be a *Krull domain* if there exists a family $\mathcal{F} = \{V_\lambda\}_{\lambda \in \Lambda}$ of DVRs of its field of fractions $\mathcal{Q}(R)$ such that
    - $R = \bigcap_{\lambda \in \Lambda} V_\lambda$, and
    - Every nonzero element of $\mathcal{Q}(R)$ is a unit in all but finitely many of the V_λ.

A unique factorization domain (UFD) is a Krull domain, and a Noetherian integral domain is a Krull domain if and only if it is integrally closed. An integral domain R is a Krull domain if and only if it satisfies the following three properties:

- R_P is a DVR for each prime ideal P of R of height one.
- $R = \bigcap\{ R_P \mid P \text{ is a height-one prime } \}$.
- Every nonzero element of R is contained in only finitely many height-one primes of R.

If R is a Krull domain, then $\mathcal{F} = \{R_P \mid P \text{ is a height-one prime }\}$ is the unique minimal set of DVRs satisfying the properties in the definition of a Krull domain [**73**, Theorem 12.3]. Moreover, for each nonzero nonunit a of R the principal ideal aR has no embedded associated prime ideals and a unique irredundant primary decomposition $aR = q_1 \cap \cdots \cap q_t$. If $p_i = \text{rad}(q_i)$, then $R_{p_i} \in \mathcal{F}$ and q_i is a symbolic power of p_i; that is, $q_i = p_i^{(e_i)}$, where $e_i \in \mathbb{N}$, [**73**, Corollary, page 88].

(3) Let R be a Krull domain and let $R \hookrightarrow S$ be an inclusion map of R into a Krull domain S. The extension $R \hookrightarrow S$ satisfies the **PDE** condition ("pas d'éclatement", or in English "no blowing up") provided that for every height-one prime ideal Q in S, the height of $Q \cap R$ is at most one [**28**, page 30].

(4) A local ring $(R, \mathbf{m})$ is *Henselian* provided the following holds: for every monic polynomial $f(x) \in R[x]$ satisfying $f(x) \equiv g_0(x)h_0(x)$ modulo $\mathbf{m}[x]$, where g_0 and h_0 are monic polynomials in $R[x]$ such that

$$g_0R[x] + h_0R[x] + \mathbf{m}[x] = R[x],$$

there exist monic polynomials $g(x)$ and $h(x)$ in $R[x]$ such that $f(x) = g(x)h(x)$ and such that both

$$g(x) - g_0(x) \quad \text{and} \quad h(x) - h_0(x) \in \mathbf{m}[x].$$

In other words, if $f(x)$ factors modulo $\mathbf{m}[x]$ into two comaximal factors, then this factorization can be lifted back to $R[x]$. Alternatively, Henselian rings are rings for which the conclusion to Hensel's Lemma holds [**73**, Theorem 8.3].

DEFINITIONS 2.4. Let I be an ideal of a ring R.

(1) An element $r \in R$ is *integral over I* if there exists a monic polynomial $f(x) = x^n + \sum_{i=1}^n a_i x^{n-i}$ such that $f(r) = 0$ and such that $a_i \in I^i$ for each i with $1 \leq i \leq n$.
(2) The *integral closure* $\overline{I}$ of I is the set of elements of R integral over I; $\overline{I}$ is an ideal.
(3) The integral closure of $\overline{I}$ is equal to $\overline{I}$.
(4) If $I = \overline{I}$, then I is said to be *integrally closed*.
(5) The ideal I is said to be *normal* if I^n is integrally closed for every $n \geq 1$.
(6) If J is an ideal contained in I and $JI^{n-1} = I^n$, then J is said to be a *reduction* of I.

2.2. Basic theorems

Theorem 2.5 is a famous result proved by Krull that is now called the Krull intersection theorem.

Theorem 2.5 (Krull [**73**, Theorem 8.10]). *Let I be an ideal of a Noetherian ring R.*

1. *If I is contained in the Jacobson radical $\mathcal{J}(R)$ of R, then $\bigcap_{n=1}^{\infty} I^n = 0$, and, for each finite R-module M, we have $\bigcap_{n=1}^{\infty} I^n M = 0$.*
2. *If I is a proper ideal of a Noetherian integral domain, then $\bigcap_{n=1}^{\infty} I^n = 0$.*

Theorem 2.6 is another famous result of Krull that is now called the Krull Altitude theorem.

Theorem 2.6 (Krull [**73**, Theorem 13.5]). *Let R be a Noetherian ring and let $I = (a_1, \dots, a_r)R$ be an ideal generated by r elements. If P is a minimal prime divisor of I, then $\operatorname{ht} P \leq r$. Hence the height of a proper ideal of R is finite.*

To prove that a ring is Noetherian, it suffices by the following well-known result of Cohen to prove that every prime ideal of the ring is finitely generated.

Theorem 2.7 (Cohen [**19**]). *If each prime ideal of the ring R is finitely generated, then R is Noetherian.*

Theorem 2.8 is another important result proved by Cohen.

Theorem 2.8 (Cohen [**20**]). *Let R be a Noetherian integral domain and let S be an extension domain of R. For $P \in \operatorname{Spec} S$ and $\mathbf{p} = P \cap R$, we have*

$$\operatorname{ht} P \ + \ \textit{tr.deg.}_{k(\mathbf{p})} k(P) \ \leq \ \operatorname{ht} \mathbf{p} \ + \ \textit{tr.deg.}_R S,$$

where $k(\mathbf{p})$ is the field of fractions of $R/\mathbf{p}$ and $k(P)$ is the field of fractions of S/P.

Theorem 2.9 is a useful result due to Nagata about Krull domains and UFDs.

Theorem 2.9. [**99**, Theorem 6.3, p. 21] *Let R be a Krull domain. If S is a multiplicatively closed subset of R generated by prime elements and $S^{-1}R$ is a UFD, then R is a UFD.*

We use the following:

Fact 2.10. If D is an integral domain and c is a nonzero element of D such that cD is a prime ideal, then $D = D[1/c] \cap D_{cD}$.

Proof. Let $\beta \in D[1/c] \cap D_{cD}$. Then $\beta = \frac{b}{c^n} = \frac{b_1}{s}$ for some $b, b_1 \in D$, $s \in D \setminus cD$ and integer $n \geq 0$. If $n > 0$, we have $sb = c^n b_1 \implies b \in cD$. Thus we may reduce to the case where $n = 0$; it follows that $D = D[1/c] \cap D_{cD}$. □

Remarks 2.11. (1) If R is a Noetherian integral domain and S is a multiplicatively closed subset of R generated by prime elements, then $S^{-1}R$ a UFD implies that R is a UFD [**99**, Theorem 6.3] or [**73**, Theorem 20.2].

(2) If x is a nonzero prime element in an integral domain R such that R_{xR} is a DVR and $R[1/x]$ is a Krull, then by Fact 2.10, R is a Krull domain; and by Theorem 2.9, R is a UFD if $R[1/x]$ is a UFD.

(3) If R is a valuation domain with value group $\mathbb{Z} \oplus \mathbb{Z}$ ordered lexicographically, then the maximal ideal $\mathbf{m}$ of R is principal, say $\mathbf{m} = xR$. It follows that $R[1/x]$ is a DVR, however, R is not a Krull domain.

The Eakin-Nagata Theorem is useful for proving descent of the Noetherian property.

Theorem 2.12 (Eakin-Nagata [**73**, Theorem 3.7(i)]). *If B is a Noetherian ring and A is a subring of B such that B is a finitely generated A-module, then A is Noetherian.*

Krull domains have an approximation property with respect to the family of DVRs obtained by localizing at height-one primes.

Theorem 2.13 (Approximation Theorem [**73**, Theorem 12.6]). *Let A be a Krull domain with field of fractions K, let $P_1, \dots, P_r$ be height-one primes of A, and let v_i denote the valuation with value group $\mathbb{Z}$ associated to the DVR A_{P_i}, for each i with $1 \le i \le r$. For arbitrary integers $e_1, \dots, e_r$, there exists $x \in K$ such that*

$$v_i(x) \;=\; e_i \quad \textit{for} \quad 1 \;\le\; i \;\le\; r \quad \textit{and} \quad v(x) \;\ge\; 0,$$

for every valuation v associated to a height-one prime of A not in the set $\{P_1, \dots, P_r\}$.

An interesting result proved by Nishimura is

Theorem 2.14 (Nishimura [**80**, Theorem, page 397], or [**73**, Theorem 12.7]). *Let R be a Krull domain. If R/P is Noetherian for each $P \in \operatorname{Spec} R$ with $\operatorname{ht} P = 1$, then R is Noetherian.*

Remark 2.15. It is observed in [**37**, Lemma 1.5] that the conclusion of Theorem 2.14 still holds if it is assumed that R/P is Noetherian for all but at most finitely many of the height one primes P of R.

Theorem 2.16 is useful for describing the maximal ideals of a power series ring $R[[x]]$. It is related to the fact that an element $f = a_0 + a_1x + a_2x^2 + \cdots \in R[[x]]$ with the $a_i \in R$ is a unit of $R[[x]]$ if and only if a_0 is a unit of R.

Theorem 2.16 ([**79**, Theorem 15.1]). *Let $R[[x]]$ be the formal power series ring in a variable x over a commutative ring R. There is a one-to-one correspondence between the maximal ideals $\mathbf{m}$ of R and the maximal ideals $\mathbf{m}^*$ of $R[[x]]$ where $\mathbf{m}^*$ corresponds to $\mathbf{m}$ if and only if $\mathbf{m}^*$ is generated by $\mathbf{m}$ and x.*

As an immediate corollary of Theorem 2.16, we have

Corollary 2.17. *The element x is in the Jacobson radical $\mathcal{J}(R[[x]])$ of the power series ring $R[[x]]$. In the formal power series ring $S := R[[x_1, \dots, x_n]]$, the ideal $(x_1, \dots, x_n)S$ is contained in the Jacobson radical $\mathcal{J}(S)$ of S .*

Theorem 2.18 is an important result first proved by Chevalley.

Theorem 2.18 (Chevalley, [**17**]). *Let $(R, \mathbf{m})$ be a Noetherian local domain. There exists a DVR that birationally dominates R.*

More generally, let P be a prime ideal of a Noetherian integral domain R. There exists a DVR V that birationally contains R and has center P on R, that is, the maximal ideal of V intersects R in P.

2.3. Flatness

The concept of flatness was introduced by Serre in the 1950's in an appendix to his paper [**100**]. Mumford writes in [**74**, page 424]: "The concept of flatness is a riddle that comes out of algebra, but which technically is the answer to many prayers."

Definitions 2.19. A module M over a ring R is *flat* over R if tensoring with M preserves exactness of every exact sequence of R-modules. The R-module M is said to be *faithfully flat* over R if, for every sequence of R-modules

$$\mathcal{S}: \quad 0 \longrightarrow M_1 \longrightarrow M_2,$$

$\mathcal{S}$ is exact if and only if $\mathcal{S} \otimes_R M$ is exact.

A ring homomorphism $\phi : R \to S$ is said to be a *flat homomorphism* if S is flat as an R-module.

Flatness is preserved by several standard ring constructions as we record in Remarks 2.20. There is an interesting elementwise criterion for flatness that is stated as item 2 of Remarks 2.20.

Remarks 2.20. The following facts are useful for understanding flatness. We use these facts to obtain the results in Chapters 8 and 17.

(1) Since localization at prime ideals commutes with tensor products, the module M is flat as an R-module $\iff$ M_Q is flat as an R_Q-module, for every prime ideal Q of R.

(2) An R-module M is flat over R if and only if for every $m_1, \dots, m_n \in M$ and $a_1, \dots, a_n \in R$ such that $\sum a_i m_i = 0$, there exist a positive integer k, a subset $\{b_{ij}\}_{i=1}^{n},{}_{j=1}^{k} \subseteq R$, and elements $m'_1, \dots, m'_k \in M$ such that $m_i = \sum_{j=1}^{k} b_{ij} m'_j$ for each i and $\sum_{i=1}^{n} a_i b_{ij} = 0$ for each j, [**73**, Theorem 7.6] or [**71**, Theorem 1]. Thus every free module is flat.

(3) A finitely generated module over a local ring is flat if and only if it is free [**71**, Proposition 3.G].

(4) If the ring S is a localization of R, then S is flat as an R-module [**71**, (3.D), page 19].

(5) Let S be a flat R-algebra. Then S is faithfully flat over R $\iff$ one has $JS \neq S$ for every proper ideal J of R, [**71**, Theorem 3, page 28] or [**73**, Theorem 7.2].

(6) If the ring S is a flat R-algebra, then every regular element of R is regular on S [**71**, (3.F)].

(7) Let S be a faithfully flat R-algebra and let I be an ideal of R. Then $IS \cap R = I$ [**73**, Theorem 7.5].

(8) Let R be a subring of a ring S. If S is Noetherian and faithfully flat over R, then R is Noetherian. This is an easy exercise; see Exercise 6 at the end of this chapter.

(9) Let R be an integral domain with field of fractions K and let S be a faithfully flat R-algebra. By item (6), every nonzero element of R is regular on S and so K naturally embeds in the total quotient ring $\mathcal{Q}(S)$ of S. By item (7), all ideals in R extend and contract to themselves with respect to S, and thus $R = K \cap S$. In particular, if $S \subseteq K$, then $R = S$ [**71**, page 31].

(10) If $\phi : R \to S$ is a flat homomorphism of rings, then ϕ satisfies the going-down theorem [**71**, (5.D), page 33]. This implies for each $P \in \operatorname{Spec} S$ that the height of P is greater than or equal to the height in R of $\phi^{-1}(P)$.

(11) Let $R \to S$ be a flat homomorphism of rings and let I and J be ideals of R. Then $(I \cap J)S = IS \cap JS$. If J is finitely generated, then $(I :_R J)S = IS :_S JS$ [**73**, Theorem 7.4] or [**71**, (3.H) page 23].

(12) Consider the following short exact sequence of R-modules:

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0\,.$$

If A and C are flat over R, then so is B.

(13) If S is a flat R-algebra and M is a flat S-module, then M is a flat R-module [**73**, page 46].

The following standard result about flatness follows from what Matsumura calls "change of coefficient ring". It is convenient to refer to both the module and homomorphism versions.

FACT 2.21. Let C be a commutative ring, let D, E and F be C-algebras.

1. If $\psi : D \to E$ is a flat C-algebra homomorphism, then $\psi \otimes_C 1_F : D \otimes_C F \to E \otimes_C F$ is a flat C-algebra homomorphism.
2. If E is a flat D-module via the C-algebra homomorphism ψ, then $E \otimes_C F$ is a flat $D \otimes_C F$-module via the C-algebra homomorphism $\psi \otimes_C 1_F$.

PROOF. By the definition of flat homomorphism, the two statements are equivalent. Since E is a flat D-module, $E \otimes_D (D \otimes_C F)$ is a flat $(D \otimes_C F)$-module by [**73**, p. 46, Change of coefficient ring]. Since $E \otimes_D (D \otimes_C F) = E \otimes_C F$, Fact 2.21 follows. $\square$

We use Remark 2.22.3 in Chapter 7.

REMARKS 2.22. Let R be an integral domain.

1. Every flat R-module M is torsionfree, i.e., if $r \in R, x \in M$ and $rx = 0$, then $r = 0$ or $x = 0$ [**71**, (3.F), page 21]
2. Every finitely generated torsionfree module over a PID is free [**24**, Theorem 5. page 462].
3. Every torsionfree module over a PID is flat. This follows from item 2 and Remark 2.20.3.
4. Every injective homomorphism of R into a field is flat. This follows from Remarks 2.20.13 and 2.20.4.

In Chapter 3 we discuss other tools we will be using involving ideal-adic completions and properties of excellent rings.

Exercises

1. Prove that every height-one prime ideal of a UFD is principal.
2. Let V be a local domain with nonzero principal maximal ideal yV. Prove that V is a DVR if $\bigcap_{n=1}^{\infty} y^n V = (0)$.

   Comment: It is not being assumed that V is Noetherian, so it needs to be established that V has dimension one.
3. Prove as stated in Remark 2.1 that if R is a valuation domain with field of fractions K and F is a subfield of K, then $R \cap F$ is again a valuation domain and has field of fractions F; also prove that if R is a DVR and the field F is not contained in R, then $R \cap F$ is again a DVR.
4. Prove that a unique factorization domain is a Krull domain.
5. Prove the assertions in Remark 2.2.

(6) Let $R[[x]]$ be the formal power series ring in a variable x over a commutative ring R.
  (i) Prove that $a_0 + a_1x + a_2x^2 + \cdots \in R[[x]]$, where the $a_i \in R$, is a unit of $R[[x]]$ if and only if a_0 is a unit of R.
  (ii) Prove that x is contained in every maximal ideal of $R[[x]]$.
  (iii) Prove Theorem 2.16 that the maximal ideals $\mathbf{m}$ of R are in one-to-one correspondence with the maximal ideals $\mathbf{m}^*$ of $R[[x]]$, where $\mathbf{m}^*$ corresponds to $\mathbf{m}$ if and only if $\mathbf{m}^*$ is generated by $\mathbf{m}$ and x.

(7) Prove items 4-8 of Remarks 2.20.

Suggestion: For the proof of item 8, use item 7.

(8) Let $f : A \to B$ be a ring homomorphism and let P be a prime ideal of A. Prove that there exists a prime ideal Q in B that contracts in A to P if and only if the extended ideal $f(P)B$ contracts to P in A, i.e., $P = f(P)B \cap A$. (Here we are using the symbol $\cap$ as in Matsumura [**73**, item (3), page xiii].)

(9) Let $f : A \hookrightarrow B$ be an injective ring homomorphism and let P be a minimal prime of A.
  (i) Prove that there exists a prime ideal Q of B that contracts in A to P.
  (ii) Deduce that there exists a minimal prime Q of B that contracts in A to P.

Suggestion: Consider the multiplicatively closed set $A \setminus P$ in B.

(10) Let P be a height-one prime of a Krull domain A and let v denote the valuation with value group $\mathbb{Z}$ associated to the DVR A_P. If A/P is Noetherian, prove that $A/P^{(e)}$ is Noetherian for every positive integer e.

Suggestion: Using Theorem 2.13, show there exists $x \in \mathcal{Q}(A)$ such that $v(x) = 1$ and $1/x \in A_Q$ for every height-one prime Q of A different from P. Let $B = A[x]$.
  (i) Show that $P = xB \cap A$ and $B = A + xB$.
  (ii) Show that $A/P \cong B/xB \cong x^iB/x^{i+1}B$ for every positive integer i.
  (iii) Deduce that B/x^eB is a Noetherian B-module and thus a Noetherian ring.
  (iv) Prove that $x^eB \cap A \subseteq x^eA_P \cap A = P^{(e)}$ and B/x^eB is a finite $A/(x^eB \cap A)$-module generated by the images of $1, x, \ldots, x^{e-1}$.
  (v) Apply Theorem 2.12 to conclude that $A/(x^eB \cap A)$ and hence $A/P^{(e)}$ is Noetherian.

(11) Let A be a Krull domain having the property that A/P is Noetherian for all but at most finitely many of the $P \in \operatorname{Spec} A$ with $\operatorname{ht} P = 1$. Prove that A is Noetherian.

Suggestion: By Nishimura's result Theorem 2.14 and Cohen's result Theorem 2.7, it suffices to prove each prime ideal of A of height greater than one is finitely generated. Let $P_1, \ldots, P_n$ be the height-one prime ideals of A for which A/P_i may fail to be Noetherian. For each nonunit $a \in A \setminus (P_1 \cup \cdots \cup P_n)$, observe that $aA = Q_1^{(e_1)} \cap \cdots \cap Q_s^{(e_s)}$, where $Q_1, \ldots, Q_s$ are height-one prime ideals of A not in the set $\{P_1, \ldots, P_n\}$. Consider the embedding $A/aA \hookrightarrow \prod(A/Q_i^{(e_i)}$. By the previous exercise, each $A/Q_i^{(e_i)}$ is Noetherian. Apply Theorem 2.12 to conclude that A/aA is Noetherian. Deduce that every prime ideal of A of height greater than one is finitely generated.

CHAPTER 3

More Tools

In this chapter we discuss ideal-adic completions. We describe several results concerning complete local rings. We review the definitions of catenary and excellence and record several important results about catenary and excellent rings.

3.1. Introduction to ideal-adic completions

Definitions 3.1. Let R be a commutative ring with identity. A *filtration* on R is a decreasing sequence $\{I_n\}_{n=0}^{\infty}$ of ideals of R. Associated to a filtration there is a well-defined completion R^* that may be defined to be the inverse limit [1]

$$R^* = \varprojlim_n \ R/I_n \tag{3.1.0}$$

and a canonical homomorphism $\psi : R \to R^*$ [**83**, Chapter 9]. If $\bigcap_{n=0}^{\infty} I_n = (0)$, then ψ is injective and R may be regarded as a subring of R^* [**83**, page 401]. In the terminology of Northcott, a filtration $\{I_n\}_{n=0}^{\infty}$ is said to be *multiplicative* if $I_0 = R$ and $I_n I_m \subseteq I_{n+m}$, for all $m \geq 0$, $n \geq 0$ [**83**, page 408].

A well-known example of a multiplicative filtration on R is the I-adic filtration $\{I^n\}_{n=0}^{\infty}$, where I is a fixed ideal of R such that $\bigcap_{n=0}^{\infty} I^n = (0)$. In this case we say $R^* := \varprojlim_n \ R/I^n$ is the *I-adic completion* of R. If the canonical map $R \to R^*$ is an isomorphism, we say that R is *I-adically complete.* An ideal L of R is *closed in the* I-adic topology on R if $\bigcap_{n=1}^{\infty}(L + I^n) = L$.

Remark 3.2. In the special case where $I = zR$ is a principal ideal with $z \in R$ having the property that $\bigcap_{n=1}^{\infty} z^n R = (0)$, then the *$z$-adic completion* of R is the inverse limit

$$R^* := \varprojlim_n \ R/z^n R. \tag{3.2.}$$

Let y be an indeterminate over R. If the ideal $(y - z)R[[y]]$ is closed in the J-adic topology on $R[[y]]$, where $J := (y, z)R[[y]]$, then the J-adic completion R^* also has the form

$$R^* \ = \ \frac{R[[y]]}{(y - z)R[[y]]}. \tag{3.2.2}$$

This follows from [**79**, (17.5)].

We observe that if R is Noetherian, then the ideal $(y - z)R[[y]]$ is closed in the J-adic topology on $R[[y]]$. To see this, let $^{-}$ denote image in $R[[y]]/(y - z)R[[y]]$. It suffices to show that $\cap_{n=1}^{\infty}\overline{(y, z)^n R[[y]]} = (\overline{0})$.

---

[1] We refer to Appendix A of [**73**] for the definition of direct and inverse limits.

We have $\overline{(y,z)^n R[[y]]} = \overline{y^n R[[y]]}$, for every $n \in \mathbb{N}$. By Corollary 2.17, the element y is in the Jacobson radical of $R[[y]]$. Hence $\overline{y}$ is in the Jacobson radical of $\overline{R[[y]]}$, a Noetherian ring. We have

$$\bigcap_{n=1}^{\infty} \overline{(y,z)^n R[[y]]} = \bigcap_{n=1}^{\infty} \overline{y^n R[[y]]} = (\overline{0}).$$

The second equality follows from Theorem 2.5.1. Therefore $(y-z)R[[y]]$ is closed in the J-adic topology. Thus if R is Noetherian then R^* has the form of Equation 3.2.2.

In general, if R^* has the form of Equation 3.2.2, then the elements of R^* are power series in z with coefficients in R, but without the uniqueness of expression as power series that occurs in the formal power series ring $R[[y]]$. If R is already complete in its (z)-adic topology, then $R = R^*$, but often it is the case for a Noetherian integral domain R that there exist elements of R^* that are transcendental over the field of fractions of R.

Fact 3.3. If R is a countable Noetherian integral domain and z is a nonzero nonunit of R, then the (z)-adic completion R^* of R contains an uncountable subset that is algebraically independent over R.

Proof. The (z)-adic completion R^* of R is uncountable. By Fact 3.4 there exists a minimal prime P_0 of R^* such that R^*/P_0 is uncountable. Since R is a Noetherian integral domain, R^* is flat over R by Remark 3.7.2. Thus, by Remark 2.20.9, $P_0 \cap R = 0$. In R^*/P_0 there exists an uncountable subset of algebraically independent elements over R. By taking preimages in R^*, we get an uncountable subset of the (z)-adic completion R^* of R. This set is is algebraically independent over R since the algebraic closure of the field of fractions of the countable ring R is countable. □

Fact 3.4. If D is an uncountable Noetherian commutative ring, then there exists a prime ideal P of D such that D/P is uncountable. Hence there exists a minimal prime P_0 of D such that D/P_0 is uncountable.

Proof. The ring D contains a finite chain of submodules

$$0 = M_0 \subset M_1 \subset \cdots \subset M_\ell = D$$

such that each quotient $M_i/M_{i-1} = D/P_i$, for some prime ideal P_i of D, [**73**, Theorem 6.4]. If each of the quotients were countable then D would be countable. Thus D/P is uncountable for some prime ideal P of D, and hence D/P_0 is uncountable, for each minimal prime P_0 contained in P. □

We reserve the notation $\widehat{R}$ for the situation where R is a local ring with maximal ideal $\mathbf{m}$ and $\widehat{R}$ is the $\mathbf{m}$-adic completion of R. For a local ring $(R, \mathbf{m})$, we say that $\widehat{R}$ is "the" *completion* of R. If $\mathbf{m}$ is generated by elements $a_1, \ldots, a_n$, then $\widehat{R}$ is realizable by taking the a_1-adic completion R_1^* of R, then the a_2-adic completion R_2^* of R_1^*, ..., and then the a_n-adic completion of R_{n-1}^*.

Fact 3.5. If I is an ideal of a Noetherian ring R contained in the Jacobson radical of R, then every ideal L of R is closed in the I-adic topology on R.

Proof. Let $\overline{}$ denote image in R/L. It suffices to show that $\cap_{n=1}^{\infty} \overline{(L+I^n)} = (\overline{0})$. Notice that $\overline{(L+I)}^{\,n} = \overline{(L+I^n)}$ and that $\overline{L+I}$ is in the Jacobson radical of $\overline{R}$. Thus $\cap_{n=1}^{\infty} \overline{(L+I^n)} = \cap_{n=1}^{\infty} \overline{(L+I)}^{\,n} = (\overline{0})$ by Theorem 2.5, as desired. □

We use the following definitions.

Definition 3.6. A Noetherian local ring R is said to be

(1) *analytically unramified* if the completion $\widehat{R}$ is reduced, i.e., has no nonzero nilpotent elements;
(2) *analytically irreducible* if the completion $\widehat{R}$ is an integral domain;
(3) *analytically normal* if the completion $\widehat{R}$ is an integrally closed (i.e., normal) domain.

If R has any one of the properties of Definition 3.6, then R is reduced. If R has either of the last two properties, then R is an integral domain. If $\widehat{R}$ is not reduced, then R is *analytically ramified*; if $\widehat{R}$ is not an integral domain, then R is *analytically reducible.*

3.2. Basic theorems about completions

Remark 3.7. Let I be an ideal of a ring R.

(1) If R is I-adically complete, then I is contained in the Jacobson radical $\mathfrak{J}(R)$, where $\mathfrak{J}(R)$ is the intersection of the maximal ideals of R [**73**, Theorem 8.2] or [**71**, 24.B, pages 73-74].
(2) If R is a Noetherian ring, then the I-adic completion R^* of R is flat over R [**71**, Corollary 1, page 170].
(3) If R is Noetherian then the I-adic completion R^* of R is faithfully flat over $R \iff$ for each proper ideal J of R we have $JR^* \neq R^*$.
(4) If R is a Noetherian ring and $I \subseteq \mathfrak{J}(R)$, then the I-adic completion R^* is faithfully flat over R [**71**, Theorem 56, page 172].
(5) If $I = (a_1, \dots, a_n)R$ is an ideal of a Noetherian ring R, then the I-adic completion R^* of R is isomorphic to a quotient of the formal power series ring $R[[x_1, \dots, x_n]]$; namely,
$$R^* \;=\; \frac{R[[x_1, \dots, x_n]]}{(x_1 - a_1, \dots, x_n - a_n)R[[x_1, \dots, x_n]]}$$
[**73**, Theorem 8.12].

In Theorem 8 of Cohen's famous paper [**18**] on the structure and ideal theory of complete local rings a result similar to Nakayama's lemma is obtained without the usual finiteness condition of Nakayama's lemma. As formulated in [**73**, Theorem 8.4], the result is:

Theorem 3.8. (A version of Cohen's Theorem 8) *Let I be an ideal of a ring R and let M be an R-module. Assume that R is complete in the I-adic topology and $\bigcap_{n=1}^{\infty} I^n M = (0)$. If M/I is generated over R/I by elements $\overline{w}_1, \dots, \overline{w}_s$ and w_i is a preimage in M of $\overline{w}_i$ for $1 \le i \le s$, then M is generated over R by $w_1, \dots, w_s$.*

Let K be a field and let $R = K[[x_1, \dots, x_n]]$ be a formal power series ring in n variables over K. It is well-known that there exists a K-algebra embedding of R into the formal power series ring $K[[y, z]]$ in two variables over K [**117**, page 219]. We observe in Corollary 3.9 restrictions on such an embedding.

Corollary 3.9. *Let $(R, \mathbf{m})$ be a complete local ring and assume that the map $\varphi : (R, \mathbf{m}) \to (S, \mathbf{n})$ is a local homomorphism.*

(1) *If* $\mathbf{m}S$ *is* $\mathbf{n}$*-primary and* $S/\mathbf{n}$ *is finite over* $R/\mathbf{m}$*, then* S *is a finitely generated* R*-module.*
(2) *If* $\mathbf{m}S = \mathbf{n}$ *and* $R/\mathbf{m} = S/\mathbf{n}$*, then* φ *is surjective.*
(3) *If* $R = K[[x_1, \dots, x_n]]$ *is a formal power series ring in* $n > 2$ *variables over the field* K *and* $S = K[[y, z]]$ *is a formal power series ring in two variables over* K*, then* $\varphi(\mathbf{m})S$ *is not* $\mathbf{n}$*-primary.*

We record in Remarks 3.11 several consequences of Cohen's structure theorems for complete local rings. We use the following definitions.

Definitions 3.10. Let $(R, \mathbf{m})$ be a local ring.

(1) $(R, \mathbf{m})$ is said to be *equicharacteristic* if R has the same characteristic as its residue field $R/\mathbf{m}$.
(2) A subfield k of R is a *coefficient field* of R if the canonical map of $R \to R/\mathbf{m}$ restricts to an isomorphism of k onto $R/\mathbf{m}$.

Remarks 3.11.

(1) Every equicharacteristic complete Noetherian local ring has a coefficient field. [**18**], [**73**, Theorem 28.3], [**79**, (31.1)].
(2) If k is a coefficient field of a complete Noetherian local ring $(R, \mathbf{m})$ and $x_1, \dots, x_n$ are generators of $\mathbf{m}$, then every element of R can be expanded as a power series in $x_1, \dots, x_n$ with coefficients in k. [**79**, (31.1)] Thus R is a homomorphic image of a formal power series ring in n variables over k.
(3) (i) Every complete Noetherian local ring, whether equicharacteristic or not, is Henselian (defined in Definition 2.3.4) by [**79**, (30.3)].
(ii) Every complete Noetherian local ring is a homomorphic image of a complete regular local ring.
(iii) Every complete regular local ring is a power series ring over either a field or a complete discrete valuation ring [**18**], [**79**, (31.1)].
(4) If $(R, \mathbf{m})$ is a complete Noetherian local domain, then R is a finite integral extension of a complete regular local domain [**79**, (31.6)] and the integral closure of R in a finite algebraic field extension is a finite R-module [**79**, (32.1)].
(5) If a Noetherian local ring R is analytically unramified, then the integral closure of R is a finite R-module [**79**, (32.2)].
(6) Let $(R, \mathbf{m})$ be a one-dimensional Noetherian local domain. The following two statements then hold [**79**, Ex. 1 on page 122] and [**63**].
    (i) The integral closure $\overline{R}$ of R is a finite R-module if and only if the completion $\widehat{R}$ of R is reduced, i.e., if and only if R is analytically unramified.
    (ii) The minimal primes of $\widehat{R}$ are in one-to-one correspondence with the maximal ideals of $\overline{R}$.

A classical result of Rees describes necessary and sufficient conditions in order that a Noetherian local ring $(R, \mathbf{m})$ be analytically unramified.

Theorem 3.12. (Rees) [**90**] *Let* $(R, \mathbf{m})$ *be a reduced Noetherian local ring with total ring of fractions* $\mathcal{Q}(R)$*. Then the following are equivalent.*

(1) *The ring* R *is analytically unramified.*
(2) *For every choice of finitely many elements* $\lambda_1, \dots, \lambda_n$ *in* $\mathcal{Q}(R)$*, the integral closure of* $R[\lambda_1, \dots, \lambda_n]$ *is a finite* $R[\lambda_1, \dots, \lambda_n]$*-module.*

The following is an immediate corollary of Theorem 3.12.

Corollary 3.13. (Rees) [**90**] *Let* $(R, \mathbf{m})$ *be an analytically unramified Noetherian local ring and let* $\lambda_1, \dots, \lambda_n$ *be elements of* $\mathcal{Q}(R)$. *For every prime ideal* P *of* $A = R[\lambda_1, \dots, \lambda_n]$, *the local ring* A_P *is also analytically unramified.*

3.3. Chains of prime ideals, fibers of maps and excellence

We begin by discussing chains of prime ideals.

Definitions 3.14. Let P and Q be prime ideals of a ring A.

1. If $P \subsetneq Q$, we say that the inclusion $P \subsetneq Q$ is *saturated* if there is no prime ideal of A strictly between P and Q.
2. A possibly infinite chain of prime ideals $\cdots \subsetneq P_i \subsetneq P_{i+1} \subsetneq \cdots$ is called *saturated* if every inclusion $P_i \subsetneq P_{i+1}$ is saturated.
3. A ring A is *catenary* provided for every pair of prime ideals $P \subsetneq Q$ of A, every chain of prime ideals from P to Q can be extended to a saturated chain and every two saturated chains from P to Q have the same number of inclusions.
4. A ring A is *universally catenary* provided every finitely generated A-algebra is catenary.
5. A ring A is said to be *equidimensional* if $\dim A = \dim A/P$ for every minimal prime P of A.

Theorem 3.15 is a well-known result of Ratliff [**73**, Theorem 31.7].

Theorem 3.15. *A Noetherian local domain* A *is universally catenary if and only if its completion* $\widehat{A}$ *is equidimensional.*

A sharper result also due to Ratliff relating the universally catenary property to properties of the completion is Theorem 3.16.

Theorem 3.16. [**88**, Theorem 2.6] *A Noetherian local ring* $(R, \mathbf{m})$ *is universally catenary if and only if the completion of* $R/\mathbf{p}$ *is equidimensional for every minimal prime ideal* $\mathbf{p}$ *of* R.

Remark 3.17. Every Noetherian local ring that is a homomorphic image of a regular local ring, or even a homomorphic image of a Cohen-Macaulay local ring, is universally catenary [**73**, Theorem 17.9, page 137].

Discussion 3.18. Let $f : A \to B$ be a ring homomorphism. The map f can always be factored as the composite of the surjective map $A \to f(A)$ followed by the inclusion map $f(A) \hookrightarrow B$. This is often helpful for understanding the relationship of A and B. If J is an ideal of B, then $f^{-1}(J)$ is an ideal of A called the *contraction* of J to A with respect to f. If Q is a prime ideal of B, then $P := f^{-1}(Q)$ is a prime ideal of A. Thus associated with the ring homomorphism $f : A \to B$, there is a well-defined map $f^* : \operatorname{Spec} B \to \operatorname{Spec} A$ of topological spaces, where for $Q \in \operatorname{Spec} B$ we define $f^*(Q) = f^{-1}(Q) = P \in \operatorname{Spec} A$.

Let A be a ring and let $P \in \operatorname{Spec}(A)$. The *residue field of* A at P, denoted $k(P)$, is the field of fractions $\mathcal{Q}(A/P)$ of A/P. By permutability of localization and residue class formation we have $k(P) = A_P/PA_P$.

Given a ring homomorphism $f : A \to B$ and an ideal I of A, the ideal $f(I)B$ is called the *extension* of I to B with respect to f. For $P \in \operatorname{Spec} A$, the extension ideal $f(P)B$ is, in general, not a prime ideal of B. The *fiber* over P in $\operatorname{Spec} B$ is the set of all $Q \in \operatorname{Spec} B$ such that $f^*(Q) = P$. Exercise 7 of Chapter 2 asserts that the fiber over P is nonempty if and only if P is the contraction of the extended ideal $f(P)B$. In general, the fiber over P in $\operatorname{Spec} B$ is the spectrum of the ring $B \otimes_A k(P) = S^{-1}(B/f(P)B)$, where S is the multiplicatively closed set $A \setminus P$. Notice that a prime ideal Q of B contracts to P in A if and only if $f(P) \subseteq Q$ and $Q \cap S = \emptyset$. This describes exactly the prime ideals of $S^{-1}(B/f(P)B)$.

Definition 3.19. Let $f : A \to B$ be a ring homomorphism and let $P \in \operatorname{Spec} A$. The fiber over P with respect to the map f is said to be *regular* if the ring $B \otimes_A k(P)$ is a Noetherian regular ring, i.e., $B \otimes_A k(P)$ is a Noetherian ring with the property that its localization at every prime ideal is a regular local ring.

Definition 3.20. Let $f : A \to B$ be a ring homomorphism and let $P \in \operatorname{Spec} A$. The fiber over P with respect to the map f is said to be *geometrically regular* if for every finite extension field F of $k(P)$ the ring $B \otimes_A F$ is a Noetherian regular ring. The map $f : A \to B$ is said to have *geometrically regular* fibers if for each $P \in \operatorname{Spec} A$ the fiber over P is geometrically regular.

Remark 3.21. Let $f : A \to B$ be a ring homomorphism with A and B Noetherian rings and let $P \in \operatorname{Spec} A$. To check that the fiber of f over P is geometrically regular as in Definition 3.20, it suffices to show that $B \otimes_A F$ is a Noetherian regular ring for every finite purely inseparable field extension F of $k(P)$ [**32**, Théorème (22.5.8)].

Definition 3.22. A homomorphism $f : A \to B$ of Noetherian rings is said to be *regular* if it is flat with geometrically regular fibers. See Definition 2.19 for the definition of flat.

Example 3.23. Let x be an indeterminate over a field k of characteristic zero, and let

$$A \;:=\; k[x(x-1), x^2(x-1)]_{(x(x-1),x^2(x-1))} \;\subset\; k[x]_{(x)} \;=:\; B.$$

Then $(A, \mathbf{m}_A)$ and $(B, \mathbf{m}_B)$ are one-dimensional local domains with the same field of fractions $k(x)$ and with $\mathbf{m}_A B = \mathbf{m}_B$. Hence the inclusion map $f : A \hookrightarrow B$ has geometrically regular fibers. Since $A \neq B$, the map f is not flat by Remark 2.20.8. Hence f is not a regular morphism.

We present in Chapter 11 examples of maps of Noetherian rings that are regular, and other examples of maps that are flat but fail to be regular.

The formal fibers of a Noetherian local ring as in Definition 3.24 play an important role in the concept of excellence of a Noetherian ring.

Definition 3.24. Let $(R, \mathbf{m})$ be a Noetherian local ring and let $\widehat{R}$ be the $\mathbf{m}$-adic completion of R. The *formal fibers* of R are the fibers of the canonical inclusion map $R \hookrightarrow \widehat{R}$.

Definition 3.25. A Noetherian ring A is called a *G-ring* if for each prime ideal P of A the map of A_P to its PA_P-adic completion is regular, or, equivalently, the formal fibers of A_P are geometrically regular for each prime ideal P of A.

Remark 3.26. In Definition 3.25 it suffices that for every maximal ideal $\mathbf{m}$ of A, the map from $A_{\mathbf{m}}$ to its $\mathbf{m}A_{\mathbf{m}}$-adic completion is regular, by [**73**, Theorem 32.4]

Definition 3.27. A Noetherian ring A is *excellent*[2] if

- (i) A is universally catenary,
- (ii) A is a G-ring, and
- (iii) for every finitely generated A-algebra B, the set $\operatorname{Reg}(B)$ of primes P of B for which B_P is a regular local ring is an open subset of $\operatorname{Spec} B$.

Remark 3.28. The class of excellent rings includes the ring of integers as well as all fields and all complete Noetherian local rings [**73**, page 260]. All Dedekind domains of characteristic zero are excellent [**71**, (34.B)]. The usefulness of the concept of excellent rings is enhanced by the fact that the class of excellent rings is stable under the ring-theoretic operations of localization and passage to a finitely generated algebra [**32**, Chap. IV], [**71**, (33.G) and (34.A)]. An excellent ring is a Nagata ring [**71**, Theorem 78, page 257].

We give some motivation and explanations for the definition of excellence in Chapter 14.

Remark 3.29. In Corollary 9.14 of Chapter 9, we prove that the 2-dimensional Noetherian local ring B constructed in Example 9.11 has the property that the map $f : B \to \widehat{B}$ has geometrically regular fibers. This ring B of Example 9.11 is also an example of a catenary ring that is not universally catenary. Thus the property of having geometrically regular formal fibers does not imply that a Noetherian local ring is excellent.

Exercises

(1) ([**25**]) Let R be a commutative ring and let P be a prime ideal of the power series ring $R[[x]]$. Let $P(0)$ denote the ideal in R of constant terms of elements of P.

- (i) If $x \notin P$ and $P(0)$ is generated by n elements of R, prove that P is generated by n elements of $R[[x]]$.
- (ii) If $x \in P$ and $P(0)$ is generated by n elements of R, prove that P is generated by $n+1$ elements of $R[[x]]$.
- (iii) If R is a PID, prove that every prime ideal of $R[[x]]$ of height one is principal.

(2) Let R be a DVR with maximal ideal yR and let $S = R[[x]]$ be the formal power series ring over R in the variable x. Let $f \in S$. Recall that f is a unit in S if and only if the constant term of f is a unit in R by Exercise 4 of Chapter 2.

- (a) Show that S is a 2-dimensional RLR with maximal ideal $(x, y)S$.
- (b) If g is a factor of f and S/fS is a finite R-module, then S/gS is a finite R-module.
- (c) If n is a positive integer and $f := x^n + y$, then S/fS is a DVR that is a finite R-module if and only if $R = \widehat{R}$, i.e., R is complete.
- (d) If f is irreducible and $fS \neq xS$, then S/fS is a finite R-module implies that R is complete.
- (e) If R is complete, then S/fS is a finite R-module for each nonzero f in S.

Suggestion: For item (d) use that if R is not complete, then by Nakayama's lemma, the completion of R is not a finite R-module. For item (e) use Theorem 3.8.

Let f be a monic polynomial in x with coefficients in R.

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[2] For motivation and more information about excellent rings see Chapter 14.

What are necessary and sufficient conditions in order that the residue class ring S/fS is a finite R-module?

(3) ([**23**]) Let R be an integral domain and let $f \in R[[x]]$ be a nonzero nonunit of the formal power series ring $R[[x]]$. Prove that the principal ideal $fR[[x]]$ is closed in the (x)-adic topology, that is, $fR[[x]] = \bigcap_{m\geq 0}(f, x^m)R[[x]]$.

Suggestion: Reduce to the case where $c = f(0)$ is nonzero. Then f is a unit in the formal power series ring $R[\frac{1}{c}][[x]]$. If $g \in \bigcap_{m\geq 0}(f, x^m)R[[x]]$, then $g = fh$ for some $h \in R[\frac{1}{c}][[x]]$, say $h = \sum_{n\geq 0} h_n x^n$, with $h_n \in R[\frac{1}{c}]$. Let $m \geq 1$. As $g \in (f, x^m)R[[x]]$, $g = fq + x^m r$, for some $q, r \in R[[x]]$. Thus $g = fh = fq + x^m r$, hence $f(h - q) = x^m r$. As $f(0) \neq 0$, $h - q = x^m s$, for some $s \in R[\frac{1}{c}[[x]]$. Hence $h_0, h_1, \dots, h_{m-1} \in R$.

(4) Let R be a commutative ring and let $f = \sum_{n\geq 0} f_n x^n \in R[[x]]$ be a power series having the property that its leading form f_r is a regular element of R, that is, $\operatorname{ord} f = r$, so $f_0 = f_1 = \cdots = f_{r-1} = 0$, and f_r is a regular element of R. As in the previous exercise, prove that the principal ideal $fR[[x]]$ is closed in the (x)-adic topology.

(5) Let $f : A \hookrightarrow B$ be as in Example 3.23.
  (i) Prove as asserted in the text that f has geometrically regular fibers but is not flat.
  (ii) Prove that the inclusion map of $C := k[x(x-1)]_{(x(x-1))} \hookrightarrow k[x]_{(x)} = B$ is flat and has geometrically regular fibers. Deduce that the map $C \hookrightarrow B$ is a regular map.

(6) Let $\phi : (R, \mathbf{m}) \hookrightarrow (S, \mathbf{n})$ be an injective local map of the Noetherian local ring $(R, \mathbf{m})$ into the Noetherian local ring $(S, \mathbf{n})$. Let $\widehat{R} = \varprojlim_n R/\mathbf{m}^n$ denote the $\mathbf{m}$-adic completion of R and let $\widehat{S} = \varprojlim_n S/\mathbf{n}^n$ denote the $\mathbf{n}$-adic completion of S.
  (i) Prove that there exists a map $\widehat{\phi} : \widehat{R} \to \widehat{S}$ that extends the map $\phi : R \hookrightarrow S$.
  (ii) Prove that $\widehat{\phi}$ is injective if and only if for each positive integer n there exists a positive integer s_n such that $\mathbf{n}^{s_n} \cap R \subseteq \mathbf{m}^n$.
  (iii) Prove that $\widehat{\phi}$ is injective if and only if for each positive integer n the ideal $\mathbf{m}^n$ is closed in the topology on R defined by the ideals $\{\mathbf{n}^n \cap R\}_{n\in\mathbb{N}}$, i.e., the topology on R that defines R as a subspace of S.

Suggestion: For each $n \in \mathbb{N}$, we have $\mathbf{m}^n \subseteq \mathbf{n}^n \cap R$. Hence there exists a map $\phi_n : R/\mathbf{m}^n \to R/(\mathbf{n}^n \cap R) \hookrightarrow S/\mathbf{n}^n$, for each $n \in N$. The family of maps $\{\phi_n\}_{n\in\mathbb{N}}$ determine a map $\widehat{\phi} : \widehat{R} \to \widehat{S}$. Since $R/\mathbf{n}^n$ is Artinian, the descending chain of ideals $\{\mathbf{m}^n + (\mathbf{n}^s \cap R)\}_{s\in\mathbb{N}}$ stabilizes, and $\mathbf{m}^n$ is closed in the subspace topology if and only if there exists a positive integer s_n such that $\mathbf{n}^{s_n} \cap R \subseteq \mathbf{m}^n$. This holds for each $n \in \mathbb{N}$ if and only if the $\mathbf{m}$-adic topology on R is the subspace topology from S.

(7) Let $(R, \mathbf{m})$, $(S, \mathbf{n})$ and $(T, \mathbf{q})$ be Noetherian local rings. Assume there exist injective local maps $f : R \hookrightarrow S$ and $g : S \hookrightarrow T$, and let $h := gf : R \hookrightarrow T$ be the composite map. For $\widehat{f} : \widehat{R} \to \widehat{S}$ and $\widehat{g} : \widehat{S} \to \widehat{T}$ and $\widehat{h} : \widehat{R} \to \widehat{T}$ as in the previous exercise, prove that $\widehat{h} = \widehat{g}\widehat{f}$.

(8) Let $(R, \mathbf{m})$ and $(S, \mathbf{n})$ be Noetherian local rings such that S dominates R and the $\mathbf{m}$-adic completion $\widehat{R}$ of R dominates S.
  (i) Prove that R is a subspace of S.
  (ii) Prove that $\widehat{R}$ is an algebraic retract of $\widehat{S}$, i.e., $\widehat{R} \hookrightarrow \widehat{S}$ and there exists a surjective map $\pi : \widehat{S} \to \widehat{R}$ such that π restricts to the identity map on the subring $\widehat{R}$ of $\widehat{S}$.

(9) Let k be a field and let R be the localized polynomial ring $k[x]_{xk[x]}$, and thus $\widehat{R} = k[[x]]$. Let $n \geq 2$ be a positive integer. If $\operatorname{char} k = p > 0$, assume that n is not a multiple of p.
  (i) Prove that there exists $y \in k[[x]]$ such that $y^n = 1 + x$.
  (ii) For y as in (i), let $S := R[yx] \hookrightarrow k[[x]]$. Prove that S is a local ring integral over R with maximal ideal $(x, yx)S$. By the previous exercise, $\widehat{R} = k[[x]]$ is an algebraic retract of $\widehat{S}$.
  (iii) Prove that the integral closure $\overline{S}$ of S is not local. Indeed, if the field k contains a primitive n-th root of unity, then $\overline{S}$ has n distinct maximal ideals. Deduce that $\widehat{R} \neq \widehat{S}$, so $\widehat{R}$ is a nontrivial algebraic retract of $\widehat{S}$.

Suggestion: Use Remark 3.11 parts (3) and (6ii).

CHAPTER 4

First Examples of the Construction

The basic idea of the Inclusion Construction 5.3 defined in the next chapter is: Start with a well understood Noetherian domain R, then take an ideal-adic completion R^* of R and intersect R^* with an appropriate field L between R and the total quotient ring of R^*. Define $A := L \cap R^*$. This is made more explicit in Section 5.1 of Chapter 5.

In this chapter we illustrate the construction with several examples.

4.1. Elementary examples

We first consider examples where R is a polynomial ring over the field $\mathbb{Q}$ of rational numbers. In the case of one variable the situation is well understood:

Example 4.1. Let y be a variable over $\mathbb{Q}$, let $R := \mathbb{Q}[y]$, and let L be a subfield of the field of fractions of $\mathbb{Q}[[y]]$ such that $\mathbb{Q}(y) \subseteq L$. Then the intersection domain $A := L \cap \mathbb{Q}[[y]]$ is a rank-one discrete valuation domain (DVR) with field of fractions L (see Remark 2.1), maximal ideal yA and y-adic completion $A^* = \mathbb{Q}[[y]]$. For example, if we work with our favorite transcendental function e^y and put $L = \mathbb{Q}(y, e^y)$, then A is a DVR having residue field $\mathbb{Q}$ and field of fractions L of transcendence degree 2 over $\mathbb{Q}$.

The integral domain A of Example 4.1 is perhaps the simplest example of a Noetherian local domain on an algebraic function field $L/\mathbb{Q}$ of two variables that is not essentially finitely generated over its ground field $\mathbb{Q}$, i.e., A is not the localization of a finitely generated $\mathbb{Q}$-algebra. However A does have a nice description as an infinite nested union of localized polynomial rings in 2 variables over $\mathbb{Q}$; see Example 6.6. Thus in a certain sense there is a good description of the elements of the intersection domain A in this case.

The two-dimensional (two variable) case is more interesting. The following theorem of Valabrega [**107**] is useful in considering this case.

Theorem 4.2. (Valabrega) *Let C be a DVR, let y be an indeterminate over C, and let L be a subfield of $\mathcal{Q}(C[[y]])$ such that $C[y] \subset L$. Then the integral domain $D = L \cap C[[y]]$ is a two-dimensional regular local domain having completion $\widehat{D} = \widehat{C}[[y]]$, where $\widehat{C}$ is the completion of C.*

Exercise 4 of this chapter outlines a proof for Theorem 4.2. Applying Valabrega's Theorem 4.2, we see that the intersection domain is a two-dimensional regular local domain with the "right" completion in the following two examples:

Example 4.3. Let x and y be indeterminates over $\mathbb{Q}$ and let C be the DVR $\mathbb{Q}(x, e^x) \cap \mathbb{Q}[[x]]$. Then $A_1 := \mathbb{Q}(x, e^x, y) \cap C[[y]] = C[y]_{(x,y)}$ is a two-dimensional regular local domain with maximal ideal $(x, y)A_1$ and completion $\mathbb{Q}[[x, y]]$.

Example 4.4 is mentioned in Chapter 1 and is generalized in Theorem 9.5; see Remark 9.8.2.

EXAMPLE 4.4. Let x and y be indeterminates over $\mathbb{Q}$ and let C be the DVR $\mathbb{Q}(x, e^x) \cap \mathbb{Q}[[x]]$ as in Example 4.3. Then $A_2 := \mathbb{Q}(x, y, e^x, e^y) \cap C[[y]]$ is a two-dimensional regular local domain with maximal ideal $(x, y)A_2$ and completion $\mathbb{Q}[[x, y]]$. See Theorem 4.12.

REMARKS 4.5. (1) There is a significant difference between the integral domains A_1 of Example 4.3 and A_2 of Example 4.4. As is shown in Theorem 9.5, the two-dimensional regular local domain A_1 of Example 4.3 is, in a natural way, a nested union of three-dimensional regular local domains. It is possible therefore to describe A_1 rather explicitly. On the other hand, the two-dimensional regular local domain A_2 of Example 4.4 contains, for example, the element $\frac{e^x - e^y}{x-y}$. As discussed in Example 7.3, the associated nested union domain B naturally associated with A_2 is a nested union of four-dimensional RLRs, is three-dimensional and is not Noetherian. Notice that the two-dimensional regular local ring A_1 is a subring of an algebraic function field in three variables over $\mathbb{Q}$, while A_2 is a subring of an algebraic function field in four variables over $\mathbb{Q}$. Since the field $\mathbb{Q}(x, e^x, y)$ is contained in the field $\mathbb{Q}(x, e^x, y, e^y)$, the local ring A_1 is dominated by the local ring A_2.

(2) It is shown in Chapter 19 Theorem 19.30 and Corollary 19.33 that if we go outside the range of Valabrega's theorem, that is, if we take more general subfields L of the field of fractions of $\mathbb{Q}[[x, y]]$ such that $\mathbb{Q}(x, y) \subseteq L$, then the intersection domain $A = L \cap \mathbb{Q}[[x, y]]$ can be, depending on L, a localized polynomial ring in $n \geq 3$ variables over $\mathbb{Q}$ or even a localized polynomial ring in infinitely many variables over $\mathbb{Q}$. In particular, $A = L \cap \mathbb{Q}[[x, y]]$ need not be Noetherian. Theorem 4.12 describes possibilities for the intersection domain A in this setting.

4.2. Historical examples

There are classical examples, related to singularities of algebraic curves, of one-dimensional Noetherian local domains $(R, \mathbf{m})$ such that the $\mathbf{m}$-adic completion $\widehat{R}$ is not an integral domain, that is, R is analytically reducible; see Section 2.2, Remarks 3.11, and Theorem 3.12. We demonstrate this in Example 4.6.

EXAMPLE 4.6. Let X and Y be variables over $\mathbb{Q}$ and consider the localized polynomial ring

$$S := \mathbb{Q}[X, Y]_{(X,Y)\mathbb{Q}[X,Y]} \quad \text{and the quotient ring} \quad R := \frac{S}{(X^2 - Y^2 - Y^3)S}.$$

Since the polynomial $X^2 - Y^2 - Y^3$ is irreducible in the polynomial ring $\mathbb{Q}[X, Y]$, the ring R is a one-dimensional Noetherian local domain. Let x and y denote the images in R of X and Y, respectively. The principal ideal yR is primary for the maximal ideal $\mathbf{m} = (x, y)R$, and so the $\mathbf{m}$-adic completion $\widehat{R}$ is also the y-adic completion of R. Thus

$$\widehat{R} = \frac{\mathbb{Q}[X][[Y]]}{(X^2 - Y^2((1 + Y))}.$$

Since $1+Y$ has a square root $(1+Y)^{1/2} \in \mathbb{Q}[[Y]]$, we see that $X^2 - Y^2(1+Y)$ factors in $\mathbb{Q}[X][[Y]]$ as

$$X^2 - Y^2(1+Y) \;=\; (X - Y(1+Y)^{1/2}) \cdot (X + Y(1+Y)^{1/2}).$$

Thus $\widehat{R}$ is not an integral domain. Since the polynomial $Z^2 - (1+y) \in R[Z]$ has x/y as a root and $x/y \notin R$, the integral domain R is not normal; see Section 2.1. The birational integral extension $\overline{R} := R[\frac{x}{y}]$ has two maximal ideals,

$$\mathbf{m}_1 \;:=\; (\mathbf{m}, \frac{x}{y} - 1)\overline{R} \;=\; (\frac{x-y}{y})\overline{R} \quad \text{and} \quad \mathbf{m}_2 \;:=\; (\mathbf{m}, \frac{x}{y} + 1)\overline{R} \;=\; (\frac{x+y}{y})\overline{R}.$$

To see, for example, that $\mathbf{m}_1 = (\frac{x-y}{y})\overline{R}$, it suffices to show that $\mathbf{m} \subset (\frac{x-y}{y})\overline{R}$. It is obvious that $x - y \in (\frac{x-y}{y})\overline{R}$. We also clearly have $\frac{x^2-y^2}{y^2} \in (\frac{x-y}{y})\overline{R}$, and $x^2 - y^2 = y^3$. Hence $\frac{y^3}{y^2} = y \in (\frac{x-y}{y})\overline{R}$, and so $\mathbf{m}_1$ is principal and generated by $\frac{x-y}{y}$. Similarly, the maximal ideal $\mathbf{m}_2$ is principal and is generated by $\frac{x+y}{y}$. Thus $\overline{R} = R[\frac{x}{y}]$ is a PID, and hence is integrally closed. To better understand the structure of R and $\overline{R}$, it is instructive to extend the homomorphism

$$\varphi:\; S \;\longrightarrow\; \frac{S}{(X^2 - Y^2 - Y^3)S} \;=\; R.$$

Let $X_1 := X/Y$ and $S' := S[X_1]$. Then S' is a regular integral domain and the map φ can be extended to a map $\psi : S' \to R[\frac{x}{y}]$ such that $\psi(X_1) = \frac{x}{y}$. The kernel of ψ is a prime ideal of S' that contains $X^2 - Y^2 - Y^3$. Since $X = YX_1$, and Y^2 is not in ker ψ, we see that ker $\psi = (X_1^2 - 1 - Y)S'$. Thus

$$\psi:\; S' \;\longrightarrow\; \frac{S'}{(X_1^2 - 1 - Y)S'} \;=\; R[\frac{x}{y}] \;=\; \overline{R}.$$

Notice that $X_1^2 - 1 - Y$ is contained in exactly two maximal ideals of S', namely

$$\mathbf{n}_1 :=\; (X_1 - 1,\; Y)S' \quad \text{and} \quad \mathbf{n}_2 :=\; (X_1 + 1,\; Y)S'.$$

The rings $S_1 := S'_{\mathbf{n}_1}$ and $S_2 := S'_{\mathbf{n}_2}$ are two-dimensional RLRs that are local quadratic transformations of S, and the map ψ localizes to define maps

$$\psi_{\mathbf{n}_1} :\; S_1 \;\to\; \frac{S_1}{(X_1^2 - 1 - Y)S_1} \;=\; \overline{R}_{\mathbf{m}_1} \text{ and } \psi_{\mathbf{n}_2} :\; S_2 \;\to\; \frac{S_2}{(X_1^2 - 1 - Y)S_2} \;=\; \overline{R}_{\mathbf{m}_2}.$$

REMARK 4.7. Examples given by Akizuki [**7**] and Schmidt [**98**], provide one-dimensional Noetherian local domains R such that the integral closure $\overline{R}$ is not finitely generated as an R-module; equivalently, the completion $\widehat{R}$ of R has nonzero nilpotents [**79**, (32.2) and Ex. 1, page 122] and [**63**, Corollary 5].

If R is a normal one-dimensional Noetherian local domain, then R is a rank-one discrete valuation domain (DVR) and it is well-known that the completion of R is again a DVR. Thus R is analytically irreducible. Zariski showed that the normal Noetherian local domains that occur in algebraic geometry are analytically normal; see [**117**, pages 313-320] and Section 3.3. In particular, the normal local domains occuring in algebraic geometry are analytically irreducible.

This motivated the question of whether there exists a normal Noetherian local domain for which the completion is not a domain. Nagata produced such examples in [**78**]. He also pinpointed sufficient conditions for a normal Noetherian local domain to be analytically irreducible [**79**, (37.8)].

In Example 4.8, we present a special case of a construction of Nagata [**78**], [**79**, Example 7, pages 209-211] of a two-dimensional regular local domain A that is not excellent and a two-dimensional normal Noetherian local domain D that is analytically reducible. These concepts are defined in Section 3.3 and Section 3.1. Nagata constructs A as a nested union of subrings. He proves that A is Noetherian with completion $\widehat{A} = \mathbb{Q}[[x,y]]$. Since the completion of a Noetherian local ring is a faithfully flat extension, it follows by Remark 2.20.9 that A is also an intersection as defined in Example 4.8.

Example 4.8. (Nagata) [**79**, Example 7, pages 209-211] Let x and y be algebraically independent over $\mathbb{Q}$ and let R be the localized polynomial ring $R = \mathbb{Q}[x,y]_{(x,y)}$. Then the completion of R is $\widehat{R} = \mathbb{Q}[[x,y]]$. Let $\tau \in x\mathbb{Q}[[x]]$ be an element that is transcendental over $\mathbb{Q}(x,y)$, e.g., $\tau = e^x - 1$. Let $\rho := y + \tau$ and $f := \rho^2 = (y+\tau)^2$. Now define

$$A := \mathbb{Q}(x,y,f) \cap \mathbb{Q}[[x,y]] \quad \text{and} \quad D := \frac{A[z]}{(z^2 - f)A[Z]},$$

where z is an indeterminate. It is clear that A is a quasilocal integral domain. Nagata proves that f is a prime element of A and that A is a two-dimensional regular local domain with completion $\widehat{A} = \mathbb{Q}[[x,y]]$; see Proposition 10.2. He also shows that D is a normal Noetherian local domain. We return to this example in Section 10.1.

Remarks 4.9. (1) The integral domain D in Example 4.8 is analytically reducible. This is because the element f factors as a square in the completion $\widehat{A}$ of A. Thus

$$\widehat{D} = \frac{\mathbb{Q}[[x,y,z]]}{(z-(y+\tau))(z+(y+\tau))},$$

which is not an integral domain.

(2) The two-dimensional regular local domain A of Example 4.8 is not a Nagata ring and therefore is not excellent. (For the definition of a Nagata ring, see Definition 2.3.1, and for the definition of excellence, see Section 3.3, Definition 3.27.) To see that A is not a Nagata ring, notice that A has a principal prime ideal generated by f that factors as a square in $\widehat{A} = \mathbb{Q}[[x,y]]$; namely f is the square of the prime element ρ of $\widehat{A}$. Therefore the one-dimensional local domain A/fA has the property that its completion $\widehat{A}/f\widehat{A}$ has a nonzero nilpotent element. This implies that the integral closure of the one-dimensional Noetherian domain A/fA is not finitely generated over A/fA by Remark 3.11.6(i). Hence A is not a Nagata ring. Moreover, the map $A \to \widehat{A} = \mathbb{Q}[[x,y]]$ is not a regular morphism; see Section 3.3.

The existence of examples such as the normal Noetherian local domain D of Example 4.8 naturally motivated the question: Is a Nagata domain necessarily excellent? Rotthaus shows in [**91**] that the answer is “no” as described below.

In Example 4.10, we present a special case of the construction of Rotthaus. In [**91**] the ring A is constructed as a direct limit. The fact that A is Noetherian implies that A is also an intersection as defined in Example 4.10. To see this, we use that A is Noetherian implies that its completion $\widehat{A}$ is a faithfully flat extension, and then we apply Remark 2.20.8.

Example 4.10. (Rotthaus) Let x, y, z be algebraically independent over $\mathbb{Q}$ and let R be the localized polynomial ring $R = \mathbb{Q}[x,y,z]_{(x,y,z)}$. Let $\tau_1 = \sum_{i=1}^{\infty} a_i y^i \in$

$\mathbb{Q}[[y]]$ and $\tau_2 = \sum_{i=1}^{\infty} b_i y^i \in \mathbb{Q}[[y]]$ be power series such that y, τ_1, τ_2 are algebraically independent over $\mathbb{Q}$, for example, $\tau_1 = e^y - 1$ and $\tau_2 = e^{y^2} - 1$. Let $u := x + \tau_1$ and $v := z + \tau_2$. Define

$$A \ := \ \mathbb{Q}(x, y, z, uv) \ \cap \ (\mathbb{Q}[x, z]_{(x,z))}[[y]]).$$

Remark 4.11. The integral domain A of Example 4.10 is a Nagata domain that is not excellent. Rotthaus shows in [**91**] that A is Noetherian and that the completion $\widehat{A}$ of A is $\mathbb{Q}[[x, y, z]]$, so A is a 3-dimensional regular local domain. Moreover she shows the formal fibers of A are geometrically reduced, but are not geometrically regular. Since u, v are part of a regular system of parameters of $\widehat{A}$, it is clear that $(u, v)\widehat{A}$ is a prime ideal of height two. It is shown in [**91**], that $(u, v)\widehat{A} \cap A = uvA$. Thus uvA is a prime ideal and $\widehat{A}_{(u,v)\widehat{A}}/uv\widehat{A}_{(u,v)\widehat{A}}$ is a non-regular formal fiber of A. Therefore A is not excellent.

Since A contains a field of characteristic zero, to see that A is a Nagata domain it suffices to show for each prime ideal P of A that the integral closure of A/P is a finite A/P-module. Since the formal fibers of A are reduced, the integral closure of A/P is a finite A/P-module; see Remark 3.11.5.

4.3. Iterative examples

We present a family of examples contained in $k[[x, y]]$, where k is a field and x and y are indeterminates. We show that for certain values of the parameters that occur in the examples, one obtains an example of a 3-dimensional local Krull domain $(B, \mathbf{n})$ such that B is not Noetherian, $\mathbf{n} = (x, y)B$ is 2-generated and the $\mathbf{n}$-adic completion $\widehat{B}$ of B is a two-dimensional regular local domain.

Let R be the localized polynomial ring $R := k[x, y]_{(x,y)}$. If $\sigma, \tau \in \widehat{R} = k[[x, y]]$ are algebraically independent over R, then the polynomial ring $R[t_1, t_2]$ in the two variables t_1, t_2 over R, can be identified with a subring of $\widehat{R}$ by means of an R-algebra isomorphism mapping $t_1 \to \sigma$ and $t_2 \to \tau$. The structure of the local domain $A = k(x, y, \sigma, \tau) \cap \widehat{R}$ depends on the residual behavior of σ and τ with respect to prime ideals of $\widehat{R}$. Theorem 4.12 illustrates this in the special case where $\sigma \in k[[x]]$ and $\tau \in k[[y]]$. A special case of this is given in Example 4.4. Theorem 4.12 introduces a technique that is further developed in Chapters 5, 6, 7 and 8.

Theorem 4.12. *Let k be a field, let x and y be indeterminates over k, and let*

$$\sigma := \sum_{i=0}^{\infty} a_i x^i \in k[[x]] \qquad \text{and} \qquad \tau := \sum_{i=0}^{\infty} b_i y^i \in k[[y]]$$

be formal power series that are algebraically independent over the fields $k(x)$ and $k(y)$, respectively. Then $A := k(x, y, \sigma, \tau) \cap k[[x, y]]$ is a two-dimensional regular local domain with maximal ideal $(x, y)A$ and completion $\widehat{A} = k[[x, y]]$. The intersection ring A has a subring B with the following properties:

- *The local ring A birationally dominates the local ring B.*
- *The ring B is the directed union of a chain of four-dimensional regular local domains that are localized polynomial rings in four variables over k and are birationally dominated by both A and B.*
- *The local ring B is a Krull domain with maximal ideal $\mathbf{n} = (x, y)B$.*
- *The dimension of B is either 2 or 3.*

- *The local ring B is Hausdorff in the topology defined by the powers of $\mathbf{n}$.*
- *The $\mathbf{n}$-adic completion $\widehat{B}$ of B is canonically isomorphic to $k[[x,y]]$.*

Moreover the following statements are equivalent:

1. *The ring B is Noetherian.*
2. *The dimension of B is 2.*
3. *The rings B and A are equal.*
4. *Every finitely generated ideal of B is closed in the $\mathbf{n}$-adic topology on B.*
5. *Every principal ideal of B is closed in the $\mathbf{n}$-adic topology on B.*

Depending on the choice of σ and τ, B may or may not be Noetherian. In particular there exist certain values for σ and τ such that $B \neq A$ and other values such that $B = A$.

To establish the asserted properties of the ring A of Theorem 4.12, we use the following consequence of the useful result of Valabrega stated as Theorem 4.2 above. Since the construction of A can be realized by a succession of two principal ideal-adic completions, first using power series in x, then using power series in y, we consider it an "iterative" example.

PROPOSITION 4.13. *With the notation of Theorem 4.12, let $C = k(x,\sigma) \cap k[[x]]$ and let L be the field of fractions of $C[y,\tau]$. Then the ring $A = L \cap C[[y]]$ and A is a two-dimensional regular local domain with maximal ideal $(x,y)A$ and completion $\widehat{A} = k[[x,y]]$.*

PROOF. The ring C is a rank-one discrete valuation domain with completion $k[[x]]$, and the field $k(x,y,\sigma,\tau) = L$ is an intermediate field between the fields of fractions of the rings $C[y]$ and $C[[y]]$. Hence, by Theorem 4.2, $A = L \cap C[[y]]$ is a regular local domain with completion $k[[x,y]]$. $\square$

REMARK 4.14. In examining properties of subrings of the formal power series ring $k[[x,y]]$ over the field k, we use that the subfields $k((x))$ and $k((y))$ of the field $\mathcal{Q}(k[[x,y]])$ are linearly disjoint over k as defined for example in [**116**, page 109]. It follows that if $\alpha_1,\dots,\alpha_n \in k[[x]]$ are algebraically independent over $k(x)$ and $\beta_1,\dots,\beta_m \in k[[y]]$ are algebraically independent over $k(y)$, then the elements $\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_m$ are algebraically independent over $k(x,y)$.

We return to the proof of Theorem 4.12 in Chapter 7.

Exercises

(1) Prove that the intersection domain A of Example 4.1 is a DVR with field of fractions L and (y)-adic completion $A^* = \mathbb{Q}[[y]]$.

Comment. Exercise 2 of Chapter 2 implies that A is a DVR. With the additional hypothesis of Example 4.1, it is true that the (y)-adic completion of A is $\mathbb{Q}[[y]]$.

(2) Let R be an integral domain with field of fractions K. Let F be a subfield of K and let $S := F \cap R$. For each principal ideal aS of S, prove that $aS = aR \cap S$.

(3) Let R be a local domain with maximal ideal $\mathbf{m}$ and field of fractions K. Let F be a subfield of K and let $S := F \cap R$. Prove that S is local with maximal ideal $\mathbf{m} \cap S$, and thus conclude that R dominates S. Give an example where R is not Noetherian, but S is Noetherian.

Remark. It can happen that R is Noetherian while S is not Noetherian; see Chapter 17.

(4) Assume the notation of Theorem 4.2. Thus y is an indeterminate over the DVR C and $D = C[[y]] \cap L$, where L is a subfield of the field of fractions of $C[[y]]$ with $C[y] \subset L$. Let x be a generator of the maximal ideal of C and let $R := C[y]_{(x,y)C[y]}$. Observe that R is a two-dimensional RLR with maximal ideal $(x, y)R$ and that $C[[y]]$ is a two-dimensional RLR with maximal ideal $(x, y)C[[y]]$ that dominates R. Let $\mathbf{m} := (x, y)C[[y]] \cap D$.

  (i) Using Exercise 2, prove that
$$C \;\cong\; \frac{R}{yR} \;\hookrightarrow\; \frac{D}{yD} \;\hookrightarrow\; \frac{C[[y]]}{yC[[y]]} \;\cong\; C.$$

  (ii) Deduce that $C \cong \frac{D}{yD}$, and that $\mathbf{m} = (x, y)D$.

  (iii) Let $k := \frac{C}{xC}$ denote the residue field of C. Prove that $\frac{D}{xD}$ is a DVR and that
$$k[y] \;\hookrightarrow\; \frac{R}{xR} \;\hookrightarrow\; \frac{D}{xD} \;\hookrightarrow\; \frac{C[[y]]}{xC[[y]]} \;\cong\; k[[y]].$$

  (iv) For each positive integer n, prove that
$$\frac{R}{(x,y)^nR} \;\cong\; \frac{D}{(x,y)^nD} \;\cong\; \frac{C[[y]]}{(x,y)^nC[[y]]}.$$
Deduce that $\widehat{R} = \widehat{D} = \widehat{C}[[y]]$, where $\widehat{C}$ is the completion of C.

  (v) Let P be a prime ideal of D such that $x \notin P$. Prove that there exists $b \in P$ such that $b(D/xD) = y^r(D/xD)$ for some positive integer r, and deduce that $P \subset (b, x)D$.

  (vi) For $a \in P$, observe that $a = c_1b + a_1x$, where c_1 and a_1 are in D. Since $x \notin P$, deduce that $a_1 \in P$ and hence $a_1 = c_2b + a_2x$, where c_2 and a_2 are in D. Conclude that $P \subset (b, x^2)D$. Continuing this process, deduce that
$$bD \;\subseteq P \;\subseteq\; \bigcap_{n=1}^{\infty}(b, x^n)D.$$

  (vii) Extending the ideals to $C[[y]]$, observe that
$$bC[[y]] \;\subseteq\; PC[[y]] \;\subseteq\; \bigcap_{n=1}^{\infty}(b, x^n)C[[y]] \;=\; bC[[y]],$$
where the last equality is because the ideal $bC[[y]]$ is closed in the topology defined by the ideals generated by the powers of x on the Noetherian local ring $C[[y]]$. Deduce that $P = bD$.

  (viii) Conclude by Theorem 2.7 that D is Noetherian and hence a two-dimensional regular local domain with completion $\widehat{D} = \widehat{C}[[y]]$.

(5) Let k be a field and let $f \in k[[x, y]]$ be a formal power series of order $r \geq 2$. Let $f = \sum_{n=r}^{\infty} f_n$, where $f_n \in k[x, y]$ is a homogeneous form of degree n. If the leading form f_r factors in $k[x, y]$ as $f_r = \alpha \cdot \beta$, where α and β are coprime homogeneous polynomials in $k[x, y]$ of positive degree, prove that f factors in $k[[x, y]]$ as $f = g \cdot h$, where g has leading form α and h has leading form β.

Suggestion. Let $G = \bigoplus_{n \geq 0} G_n$ represent the polynomial ring $k[x, y]$ as a graded ring obtained by defining $\deg x = \deg y = 1$. Notice that G_n has dimension $n + 1$ as a vector space over k. Let $\deg \alpha = a$ and $\deg \beta = b$. Then $a + b = r$ and for each integer $n \geq r + 1$, we have $\dim(\alpha \cdot G_{n-a}) = n - a + 1$ and $\dim(\beta \cdot G_{n-b}) = n - b + 1$. Since α and β are coprime, we have

$$(\alpha \cdot G_{n-a}) \cap (\beta \cdot G_{n-b}) = f_r \cdot G_{n-r}.$$

Conclude that $\alpha \cdot G_{n-a} + \beta \cdot G_{n-b}$ is a subspace of G_n of dimension $n+1$ and hence that $G_n = \alpha \cdot G_{n-a} + \beta \cdot G_{n-b}$. Let $g_a := \alpha$ and $h_b := \beta$. Since $f_{r+1} \in G_{r+1} = \alpha \cdot G_{r+1-a} + \beta \cdot G_{r+1-b} = g_a \cdot G_{b+1} + h_b \cdot G_{a+1}$, there exist forms $h_{b+1} \in G_{b+1}$ and $g_{a+1} \in G_{a+1}$ such that $f_{r+1} = g_a \cdot h_{b+1} + h_b \cdot g_{a+1}$. Since $G_{r+2} = g_a \cdot G_{b+2} + h_b \cdot G_{a+2}$, there exist forms $h_{b+2} \in G_{b+2}$ and $g_{a+2} \in G_{a+2}$ such that $f_{r+2} - g_{a+1} \cdot h_{b+1} = g_a \cdot h_{b+2} + h_b \cdot g_{a+2}$. Proceeding by induction, assume for a positive integer s that there exist forms $g_a, g_{a+1}, \ldots, g_{a+s}$ and $h_b, h_{b+1}, \ldots, h_{b+s}$ such that the power series $f - (g_a + \cdots + g_{a+s})(h_b + \cdots + h_{b+s})$ has order greater than or equal to $r + s + 1$. Using that

$$G_{r+s+1} = g_a \cdot G_{b+s+1} + h_b \cdot G_{a+s+1},$$

deduce the existence of forms $g_{a+s+1} \in G_{a+s+1}$ and $h_{b+s+1} \in G_{b+s+1}$ such that the power series $f - (g_a + \cdots + g_{a+s+1})(h_b + \cdots + h_{b+s+1})$ has order greater than or equal to $r + s + 2$.

(6) Let k be a field of characteristic zero. Prove that both

$$xy + z^3 \quad \text{and} \quad xyz + x^4 + y^4 + z^4$$

are irreducible in the formal power series ring $k[[x, y, z]]$. Thus there does not appear to be any natural generalization to the case of three variables of the result in the previous exercise.

CHAPTER 5

Describing the Construction

We discuss a universal technique that yields the examples of the previous section and also leads to more examples. There are two versions of the construction that we use. At first they appear to be quite different:

- The *Inclusion Construction* 5.3 defines an intersection $A = A_{\text{inc}} := R^* \cap L$ included in R^*, where R^* is an ideal-adic completion of a Noetherian integral domain R and L is the subfield of the total quotient ring of R^* that is generated over $\mathcal{Q}(R)$ by elements $\tau_1, \dots, \tau_s$ of R^* that are algebraically independent over R.
- The *Homomorphic Image Construction* 5.4 is an intersection $A = A_{\text{hom}}$ of a homomorphic image of an ideal-adic completion R^* of a Noetherian integral domain R with the field of fractions of R.

The two versions are explained in more detail below. Construction 5.4 includes, up to isomorphism, Construction 5.3 as a special case.

We begin with the setting for both constructions. This setting will also be used in later chapters.

Setting 5.1. Let R be a Noetherian integral domain with field of fractions $K := \mathcal{Q}(R)$, let $z \in R$ be a nonzero nonunit, and let R^* denote the (z)-adic completion of R.

Remark 5.2. Section 3.1 contains material on the (z)-adic completion R^* of a Noetherian integral domain R with respect to a nonzero nonunit z of R. Remark 3.2 of Chapter 2 implies that R^* has the form

$$R^* \;=\; \frac{R[[y]]}{(y-z)R[[y]]},$$

where y is an indeterminate over R. It is natural to ask for conditions in order that R^* is an integral domain, or equivalently, that $(y-z)R[[y]]$ is a prime ideal. The element $y-z$ obviously generates a prime ideal of the polynomial ring $R[y]$. Our assumption that z is a nonunit of R implies that $(y-z)R[[y]]$ is a proper ideal. We consider in Exercise 3 of this chapter examples where $(y-z)R[[y]]$ is prime and examples where it is not prime.

5.1. Two construction methods and a picture

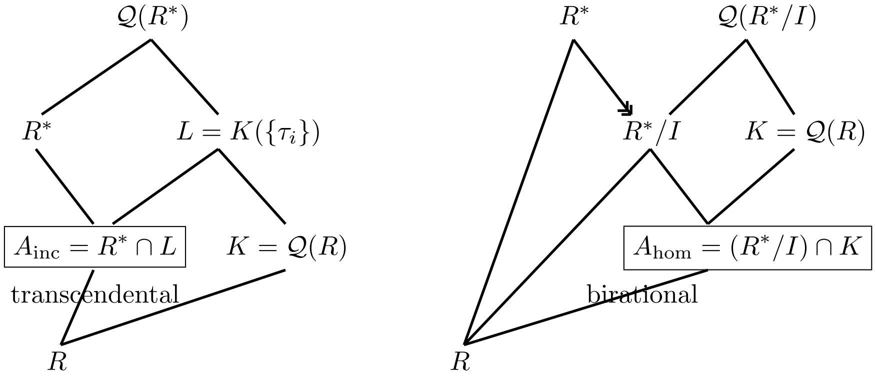
With R, z and R^* as in Setting 5.1 we describe two methods for the construction associated with R^*. The Inclusion Construction A_{inc} results in a domain transcendental over R and contained in a power series extension of R, whereas the Homomorphic Image Construction A_{hom} results in a domain that is birational over R and is contained in a homomorphic image of a power series extension of R.

Construction 5.3. *Inclusion Construction*: Let $\tau_1, \dots, \tau_s \in zR^*$ be algebraically independent elements over R such that $K(\tau_1, \dots, \tau_s) \subseteq \mathcal{Q}(R^*)$. We define A to be the intersection domain $A = A_{\text{inc}} := K(\tau_1, \dots, \tau_s) \cap R^*$, so that A_{inc} is contained in R^* and is a transcendental extension of R.

Construction 5.4. *Homomorphic Image Construction*: Let I be an ideal of R^* having the property that $P \cap R = (0)$ for each $P \in \operatorname{Spec} R^*$ that is associated to I. Define $A = A_{\text{hom}} := K \cap (R^*/I)$. Thus A_{hom} is contained in a homomorphic image of R^* and is a birational extension of R.

Note 5.5. The condition in (5.4), that $P \cap R = (0)$ for every prime ideal P of R^* that is associated to I, implies that the field of fractions K of R embeds in the total quotient ring $\mathcal{Q}(R^*/I)$ of R^*/I. To see this, observe that the canonical map $R \to R^*/I$ is an injection and that regular elements of R remain regular as elements of R^*/I. In this connection see Exercise 1.

Picture 5.6. The diagram below shows the relationships among these rings.



(5.3) $A := L \cap R^*$ (5.4) $A := K \cap (R^*/I)$

Remarks 5.7. (1) Chapters 4, 17, 22 and 23 feature the Inclusion Construction 5.3 that realizes the intersection domain $A = A_{\text{inc}} := K(\tau_1, \dots, \tau_s) \cap R^*$, while Chapter 8 features the Homomorphic Image Construction 5.4. Construction 5.3 yields the examples of Sections 4.1, 4.2 and 4.3; e.g., Nagata's Example 4.8 fits the format of Construction 5.3, since

$$A := \mathbb{Q}(x, y, \rho^2) \cap \mathbb{Q}[[x, y]] = \mathbb{Q}(x, y, \rho^2) \cap \mathbb{Q}[x][[y]].$$

(2) Construction 5.4 includes, up to isomorphism, Construction 5.3 as a special case. To see this, let R, z, R^* and $\tau_1, \dots, \tau_s \in zR^*$ be as in Construction 5.3. Let $t_1, \dots, t_s$ be indeterminates over R, define $S := R[t_1, \dots, t_s]$, let S^* be the z-adic completion of S and let I denote the ideal $(t_1 - \tau_1, \dots, t_s - \tau_s)S^*$. Consider the following diagram, where D_{hom} is the intersection domain result of Construction 5.4, when R and R^* there are replaced by S and S^*, and where λ is the R-algebra isomorphism of S into R^* that maps $t_i \to \tau_i$ for $i = 1, \dots, s$. The map λ naturally extends to a homomorphism of S^* onto R^* with the kernel of this extension being the ideal I. Thus there is an induced isomorphism of S^*/I onto R^* that we also label as λ.

(5.7.2.1)

$$\begin{tikzcd}
 & S := R[t_1, \dots, t_s] \arrow[r] \arrow[d, "\lambda"'] & D_{\text{hom}} := K(t_1, \dots, t_s) \cap (S^*/I) \arrow[r] \arrow[d, "\lambda"'] & S^*/I \arrow[d, "\lambda"'] \\
R \arrow[r] & R[\tau_1, \dots, \tau_s] \arrow[r] & A_{\text{inc}} := K(\tau_1, \dots, \tau_s) \cap R^* \arrow[r] & R^*.
\end{tikzcd}$$

Since λ maps D_{hom} isomorphically onto A_{int}, we see that Construction 5.4 includes as a special case Construction 5.3.

(3) Many of the classical examples of interesting Noetherian local domains can be obtained with Construction 5.3 where R is a regular local domain. Construction 5.3, however, is not sufficient to obtain certain other types of rings such as Ogoma's celebrated example [**84**] of a normal non-catenary Noetherian local domain. Ogoma's example can be realized, however, as an intersection with the homomorphic image of a completion as described in Construction 5.4. In Example 9.11 of Chapter 9, we construct a Noetherian local domain with geometrically regular formal fibers that is not universally catenary; this requires the homomorphic image construction.

5.2. Universality

In this section we describe how the methods of Section 5.1 can be regarded as universal for the construction of certain Noetherian local domains. Consider the following general question.

QUESTION 5.8. Let k be a field and let L/k be a finitely generated field extension. What are the Noetherian local domains $(A, \mathbf{n})$ such that

1. L is the field of fractions A, and
2. k is a coefficient field for A?

Recall from Section 2.1, that k is a coefficient field of $(A, \mathbf{n})$ if the composite map $k \hookrightarrow A \to A/\mathbf{n}$ defines an isomorphism of k onto $A/\mathbf{n}$.

In relation to Question 5.8, we observe in Theorem 5.9 the following general facts.

THEOREM 5.9. *Let $(A, \mathbf{n})$ be a Noetherian local domain having a coefficient field k. Then there exists a Noetherian local subring $(R, \mathbf{m})$ of A such that:*

1. *The local ring R is essentially finitely generated over k.*
2. *If $\mathcal{Q}(A) = L$ is finitely generated over k, then R has field of fractions L.*
3. *The field k is a coefficient field for R.*
4. *The local ring A dominates R and $\mathbf{m}A = \mathbf{n}$.*
5. *The inclusion map $\varphi : R \hookrightarrow A$ extends to a surjective homomorphism $\widehat{\varphi} : \widehat{R} \to \widehat{A}$ of the $\mathbf{m}$-adic completion $\widehat{R}$ of R onto the $\mathbf{n}$-adic completion $\widehat{A}$ of A.*
6. *For the ideal $I := \ker(\widehat{\varphi})$, of the completion $\widehat{R}$ of R from item 5, we have:*
    - (a) *$\widehat{R}/I \cong \widehat{A}$, so $\widehat{R}/I$ dominates A, and*
    - (b) *$P \cap A = (0)$ for every $P \in \operatorname{Ass}(\widehat{R}/I)$, and so the field of fractions $\mathcal{Q}(A)$ of A embeds in the total ring of quotients $\mathcal{Q}(\widehat{R}/I)$ of $\widehat{R}/I$, and*

(c) $A = \mathcal{Q}(A) \cap (\widehat{R}/I)$.

Proof. Since A is Noetherian, there exist elements $t_1, \dots, t_n \in \mathbf{n}$ such that $(t_1, \dots, t_n)A = \mathbf{n}$. For item 2, we may assume that $L = k(t_1, \dots, t_n)$, since every element of $\mathcal{Q}(A)$ has the form a/b, where $a, b \in \mathbf{n}$. To see the existence of the integral domain $(R, \mathbf{m})$ and to establish item 1, we set $T := k[t_1, \dots, t_n]$ and $\mathbf{p} := \mathbf{n} \cap T$. Define $R := T_{\mathbf{p}}$ and $\mathbf{m} := \mathbf{n} \cap R$. Then $k \subseteq R \subseteq A$, $\mathbf{m}A = \mathbf{n}$, R is essentially finitely generated over k and k is a coefficient field for R. Thus we have established items 1- 4. Even without the assumption that $\mathcal{Q}(A)$ is finitely generated over k, there is a relationship between R and A that is realized by passing to completions. Let φ be the inclusion map $R \hookrightarrow A$. The map φ extends to a map $\widehat{\varphi} : \widehat{R} \to \widehat{A}$, and by Corollary 3.9.2, the map $\widehat{\varphi}$ is surjective; thus item 5 holds. Let $I := \ker \widehat{\varphi}$. Then $\widehat{R}/I = \widehat{A}$, for the first part of item 6. The remaining assertions in item 6 follow from the fact that A is a Noetherian local domain and $\widehat{A} \cong \widehat{R}/I$. Applying Remarks 3.7, we have $\widehat{R}/I$ is faithfully flat over A, and by Remark 2.20.6 the nonzero elements of A are regular on $\widehat{R}/I$.

The following commutative diagram, where the vertical maps are injections, displays the relationships among these rings:

$$\begin{tikzcd}
 & \widehat{R} \arrow[r, "\widehat{\varphi}"] & \widehat{A} \cong \widehat{R}/I \arrow[r] & \mathcal{Q}(\widehat{R}/I) \\
k \arrow[r, "\subseteq"] & R \arrow[u] \arrow[r, "\varphi"] & A := \mathcal{Q}(A) \cap (\widehat{R}/I) \arrow[u] \arrow[r] & \mathcal{Q}(A) \arrow[u]
\end{tikzcd} \quad . \tag{5.1}$$

This completes the proof of Theorem 5.9. ☐

In relation to Question 5.8, we summarize as follows.

Summary 5.10. Every Noetherian local domain $(A, \mathbf{n})$ having a coefficient field k, and having the property that the field of fractions L of A is finitely generated over k is realizable as an intersection $L \cap \widehat{R}/I$, where I is an ideal in the completion $\widehat{R}$ of a Noetherian local domain R essentially finitely generated over k and $\mathcal{Q}(R) = L$.

Remark 5.11. Summary 5.10 is a first start towards a classification of the Noetherian local domains A with the stated properties. The drawback with (5.10), however, is that it is not true for every triple R, L, I as in (5.10) that $L \cap (\widehat{R}/I)$ is Noetherian (see Chapter 26, Section 26.4 below). In order to have a more satisfying classification an important goal is to identify the ideals I of $\widehat{R}$ and fields L such that $L \cap (\widehat{R}/I)$ is Noetherian.

If a Noetherian local domain R is essentially finitely generated over a field k, then there often exist ideals I in the completion $\widehat{R}$ of R such that the intersection domain $\mathcal{Q}(R) \cap (\widehat{R}/I)$ is an interesting Noetherian local domain that birationally dominates R. This technique may be used to describe the example given by Nagata [**78**] or [**79**, Example 7, page 209], and other examples given in [**11**], [**12**], [**58**], [**84**], [**85**], [**91**], [**92**], and [**109**].

In order to give a more precise criterion for the intersection domain A to be Noetherian, we restrict Constructions 5.3 and 5.4 to a completion R^* of R with respect to a nonzero nonunit z of R. This sometimes permits an explicit description of the intersection via the approximation methods of Chapter 6. In Chapter 8 we give necessary and sufficient conditions for A to be computable as a nested union of subrings of a specific form. We also prove that the Noetherian property for the associated nested union is equivalent to flatness of a certain map.

Exercises

(1) Let A be an integral domain and let $A \hookrightarrow B$ be an injective map to an extension ring B. For an ideal I of B, prove that the following are equivalent:

  (i) The induced map $A \to B/I$ is injective, and each nonzero element of A is regular on B/I.

  (ii) The field of fractions $\mathcal{Q}(A)$ of A naturally embeds in the total quotient ring $\mathcal{Q}(B/I)$ of B/I.

  If A and B are Noetherian, prove that conditions (i) and (ii) are also equivalent to the following condition:

  (iii) For each prime ideal P of B that is associated to I we have $P \cap A = (0)$.

(2) Let $A := k[x] \hookrightarrow k[x,y] =: B$ be a map of the polynomial ring $k[x]$ into the polynomial ring $k[x,y]$ and let $I = xyB$. Prove that the induced map $A \to B/I$ is injective, but the field of fractions $\mathcal{Q}(A)$ of A does not embed into the total quotient ring $\mathcal{Q}(B/I)$ of B/I.

(3) Let z be a nonzero nonunit of a Noetherian integral domain R, let y be an indeterminate, and let $R^* = \frac{R[[y]]}{(z-y)R[[y]]}$ be the (z)-adic completion of R.

  (i) If $z = ab$, where $a, b \in R$ are nonunits such that $aR + bR = R$, prove that there exists a factorization

$$z - y \;=\; (a + a_1 y + \cdots) \cdot (b + b_1 y + \cdots) \;=\; (\sum_{i=0}^{\infty} a_i y^i) \cdot (\sum_{i=0}^{\infty} b_i y^i),$$

  where the $a_i, b_i \in R$, $a_0 = a$ and $b_0 = b$.

  (ii) If R is a principal ideal domain, prove that R^* is an integral domain if and only if zR has prime radical.

(4) Let A be an integral domain and let $A \hookrightarrow B$ be an injective map to an extension ring B. Let I be an ideal of B having the property that $I \cap A = (0)$ and every nonzero element of A is a regular element on B/I. Let $C := \mathcal{Q}(A) \cap (B/I)$.

  (i) Prove that $C = \{a/b \mid a, b \in A,\ b \neq 0 \text{ and } a \in I + bB\}$.

  (ii) Assume that $J \subseteq I$ is an ideal of B having the property that every nonzero element of A is a regular element on B/J. Let $D := \mathcal{Q}(A) \cap (B/J)$. Prove that $D \subseteq C$.

  Suggestion: Item ii is immediate from item i. To see item i, observe that $bC = b(B/I) \cap \mathcal{Q}(A)$, and $a \in bC \iff a \in b(B/I) \iff a \in I + bB$.

CHAPTER 6

Two Approximations and their Connection

Our goal in this chapter is to describe in detail Inclusion Construction 5.3 and Homomorphic Image Construction 5.4. The first difficulty we face is identifying precisely what we have constructed—because, while the form of the example as an intersection as given in Constructions 5.3 and 5.4 of Chapter 5 is wonderfully concise, sometimes it is difficult to fathom.

We use the notation of Setting 5.1. The approximation methods in this chapter describe a subring B inside the constructed intersection domain A of Constructions 5.3 and 5.4. This subring is useful for describing A. We discuss two approximation methods corresponding to the two construction methods of Chapter 5. In Chapter 7 we use the approximation from Section 6.1 to continue the proof of Theorem 4.12 concerning the iterative examples of Section 4.3.

Associated to each of the Constructions 5.3 and 5.4, the computable subrings B contained in the intersection domains A approximate A in each of the constructions. In the case of Construction 5.3 we obtain an “approximation” domain B that is a nested union of localizations of polynomial rings in s variables over R. In the case of Construction 5.4, the “approximation” domain B is a nested union of birational extensions of R that are essentially finitely generated R-algebras. In both constructions B is a subring of the corresponding intersection domain A.

6.1. Approximations for Inclusion Construction

In this section we give an explicit description of the approximation domain B of the previous paragraph for Construction 5.3. We use the last parts, the *endpieces*, of the power series $\tau_1, \dots, \tau_s$. First we describe the endpieces for a general element γ of R^*.

Endpiece Notation 6.1. Let R be a Noetherian integral domain and let z be a nonzero nonunit element of R. Let R^* be the z-adic completion of R; see Remark 3.2. Each $\gamma \in zR^*$ has an expansion as a power series in z over R,

$$\gamma \;:=\; \sum_{i=1}^{\infty} c_i z^i, \quad \text{where} \quad c_i \in R.$$[1]

For each nonnegative integer n we define the n^{th} *endpiece* γ_n of γ with respect to this expansion:

$$\gamma_n \;:=\; \sum_{i=n+1}^{\infty} c_i z^{i-n}. \tag{6.1.1}$$

---

[1] We are interested in the ring $R[\gamma]$, where γ is algebraically independent over R and nonzero elements of $R[\gamma]$ are regular in R^*. By modifying γ by an element in R, we may assume $\gamma \in zR^*$.

It follows that, for each $n \in \mathbb{N}$, we have the relation

$$\gamma_n \;=\; c_{n+1}z \;+\; z\gamma_{n+1}. \tag{6.1.2}$$

In Construction 5.3, the elements $\tau_1, \dots \tau_s \in zR^*$ are algebraically independent over the field of fractions $\mathcal{Q}(R)$ of R and have the property that every nonzero element of the polynomial ring $R[\tau_1, \dots, \tau_s]$ is a regular element of R^*. Thus $\mathcal{Q}(R[\tau_1, \dots, \tau_s])$ is contained in the total quotient ring $\mathcal{Q}(R^*)$. Hence it makes sense to define the *intersection domain* $A := R^* \cap \mathcal{Q}(R[\tau_1, \dots, \tau_s])$ inside $\mathcal{Q}(R^*)$. We set

$$U_0 \;:=\; R[\tau_1, \dots, \tau_s] \;\subseteq\; R^* \cap \mathcal{Q}(R[\tau_1, \dots, \tau_s]) \;=:\; A,$$

and U_0 is a polynomial ring in s variables over R. Each $\tau_i \in zR^*$ has a representation $\tau_i := \sum_{j=1}^{\infty} r_{ij}z^j$, where the $r_{ij} \in R$. For each positive integer n, we associate with this representation of τ_i the n^{th} endpiece,

$$\tau_{in} \;:=\; \sum_{j=n+1}^{\infty} r_{ij}z^{j-n}. \tag{6.1.3}$$

We define

$$U_n \;:=\; R[\tau_{1n}, \dots, \tau_{sn}] \quad \text{and} \quad B_n \;:=\; (1 + zU_n)^{-1}U_n \tag{6.1.4}$$

Then U_n is a polynomial ring in s variables over R, and z is in every maximal ideal of B_n, so $z \in \mathcal{J}(B_n)$, the Jacobson radical of B_n; see Section 2.1 Using (6.1.2), we have for each positive integer n a birational inclusion of polynomial rings $U_n \subseteq U_{n+1}$. We also have $U_{n+1} \subseteq U_n[1/z]$. By Remark 3.7.1, the element z is in $\mathcal{J}(R^*)$. Hence the localization B_n of U_n is also a subring of A and $B_n \subseteq B_{n+1}$. We define

$$U \;:=\; \bigcup_{n=1}^{\infty} U_n \;=\; \bigcup_{n=1}^{\infty} R[\tau_{1n}, \dots, \tau_{sn}] \quad \text{and} \quad B \;:=\; \bigcup_{n=1}^{\infty} B_n. \tag{6.1.5}$$

It follows that

$$B = (1 + zU)^{-1}U \quad \text{and} \quad B \;\subseteq\; A \;:=\; R^* \cap \mathcal{Q}(R[\tau_1, \dots, \tau_s]). \tag{6.1.6}$$

REMARK 6.2. With the notation and setting of (6.1), the representation

$$\tau_i \;=\; \sum_{j=1}^{\infty} r_{ij}z^j$$

of τ_i as a power series in z with coefficients in R is not unique. Indeed, since $z \in R$, it is always possible to modify finitely many of the coefficients r_{ij} in this representation. It follows that the endpiece τ_{in} is also not unique. However, as we observe in Proposition 6.3 the rings U and U_n are uniquely determined by the τ_i.

PROPOSITION 6.3. *Assume the notation and setting of (6.1). Then the ring U and the rings U_n are independent of the representation of the τ_i as power series in z with coefficients in R. Hence also the ring B and the rings B_n are independent of the representation of the τ_i as power series in z with coefficients in R.*

PROOF. For $1 \le i \le s$, assume that τ_i and $\omega_i = \tau_i$ have representations

$$\tau_i := \sum_{j=1}^{\infty} a_{ij} z^j \qquad \text{and} \qquad \omega_i := \sum_{j=1}^{\infty} b_{ij} z^j,$$

where each $a_{ij}, b_{ij} \in R$. We define the n^{th}-endpieces τ_{in} and ω_{in} as in (6.1):

$$\tau_{in} = \sum_{j=n+1}^{\infty} a_{ij} z^{j-n} \qquad \text{and} \qquad \omega_{in} = \sum_{j=n+1}^{\infty} b_{ij} z^{j-n}.$$

Then we have

$$\tau_i = \sum_{j=1}^{\infty} a_{ij} z^j = \sum_{j=1}^{n} a_{ij} z^j + z^n \tau_{in} = \sum_{j=1}^{\infty} b_{ij} z^j = \sum_{j=1}^{n} b_{ij} z^j + z^n \omega_{in} = \omega_i.$$

Therefore, for $1 \le i \le s$ and each positive integer n,

$$z^n \tau_{in} - z^n \omega_{in} = \sum_{j=1}^{n} b_{ij} z^j - \sum_{j=1}^{n} a_{ij} z^j, \qquad \text{and so} \qquad \tau_{in} - \omega_{in} = \frac{\sum_{j=1}^{n} (b_{ij} - a_{ij}) z^j}{z^n}.$$

Thus $\sum_{j=1}^{n} (b_{ij} - a_{ij}) z^j \in R$ is divisible by z^n in R^* . Since $z^n R$ is closed in the (z)-adic topology on R, we have $z^n R = R \cap z^n R^*$. It follows that z^n divides the sum $\sum_{j=1}^{n} (b_{ij} - a_{ij}) z^j$ in R. Therefore $\tau_{in} - \omega_{in} \in R$. Thus the rings U_n and $U = \bigcup_{n=1}^{\infty} U_n$ are independent of the representation of the τ_i. Since $B_n = (1 + zU_n)^{-1} U_n$ and $B = \bigcup_{n=1}^{\infty} B_n$, the rings B_n and the ring B are also independent of the representation of the τ_i. □

REMARK 6.4. Let R be a Noetherian integral domain, let $z \in R$ be a nonzero nonunit, let R^* denote the (z)-adic completion of R, and let $\tau_1, \dots, \tau_s \in zR^*$ be algebraically independent elements over R, as in the setting of Construction 5.3.

(1) Assume in addition that R is local with maximal ideal $\mathbf{m}$. We observe in this case that the ring B defined in Equation 6.1.5 is the directed union of the localized polynomial rings $C_n := (U_n)_{P_n}$, where $P_n := (\mathbf{m}, \tau_{1n}, \dots, \tau_{sn}) U_n$ and U_n is as defined in Equation 6.1.4. It is clear that $C_n \subseteq C_{n+1}$, and $P_n \cap (1 + zU_n) = \emptyset$ implies that $B_n \subseteq C_n$. We show that $C_n \subseteq B_{n+1}$: Let $\frac{a}{d} \in C_n$, where $a \in U_n$ and $d \in U_n \setminus P_n$. Then $d = d_0 + \sum_{i=1}^{s} \tau_{in} b_i$, where $d_0 \in R$ and each $b_i \in U_n$. Notice that $d_0 \notin \mathbf{m}$ since $d \notin P_n$, and so $d_0^{-1} \in R$. Thus

$$d d_0^{-1} = 1 + \sum_{i=1}^{s} \tau_{in} b_i d_0^{-1} \in (1 + zU_{n+1}),$$

since each $\tau_{in} \in zU_{n+1}$ by (6.1.2). Hence $\frac{a}{d} = \frac{ad_0}{dd_0} \in B_{n+1}$, and so

$$B = \bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} C_n.$$

(2) In the special case where $R = k[x]$, then $R^* = k[[x]]$ is the (x)-adic completion of R and $U_n = k[x, \tau_{1n}, \dots, \tau_{sn}]$, as defined in Equation 6.1.4, using Endpiece Notation 6.1. We have that $B = \bigcup k[x, \tau_{1n}, \dots, \tau_{sn}]_{P_n}$ where $P_n := (x, \tau_{1n}, \dots, \tau_{sn}) k[x, \tau_{1n}, \dots, \tau_{sn}]$ by a proof similar to that for item 1.

(3) Let R be the localized polynomial ring $k[x, y_1, \dots, y_r]_{(x, y_1, \dots, y_r)}$ over a field k with variables $x, y_1, \dots, y_r$ and let $z = x$. Then $R^* = k[y_1, \dots, y_r]_{(y_1, \dots, y_r)}[[x]]$ is the (x)-adic completion of R. Define $U'_n = k[x, y_1, \dots, y_r, \tau_{1n}, \dots, \tau_{sn}]$ and $U' = \bigcup U'_n$,

using Endpiece Notation 6.1, and use U'_n in place of U_n from Equation 6.1.4. Then the ring B of Equation 6.1.5 satisfies $B = \bigcup C_n$, where each $C_n = (U'_n)_{P_n}$ and each $P_n := (x, y_1, \dots, y_r, \tau_{1n}, \dots, \tau_{sn})U'_n$.

It is important to identify when the approximation domain B equals the intersection domain A of Construction 5.3. In Definition 6.5, we introduce the term "limit-intersecting" for this situation. We use the same term for Homomorphic Images Construction 5.4 in Definition 6.12.

Definition 6.5. Using the notation in (6.1.3) and (6.1.5), we say Construction 5.3 is *limit-intersecting* over R with respect to the τ_i if $B = A$. In this case, we refer to the sequence of elements $\tau_1, \dots \tau_s \in zR^*$ as *limit-intersecting* over R, or briefly, as *limit-intersecting* for A.

Example 6.6 illustrates the limit-intersecting property. Notice that Example 4.1 is a special case of Example 6.6 .

Example 6.6. Let $R = k[x]$ where k is a field and x is an indeterminate. Let

$$\tau_1 \ := \ \sum_{j=1}^{\infty} r_{1j}x^j, \qquad \dots \qquad , \tau_s \ := \ \sum_{j=1}^{\infty} r_{sj}x^j \ \in \ xk[[x]]$$

be algebraically independent over $k(x)$. By Remark 2.1, $A := k(x, \tau_1, \dots, \tau_s) \cap k[[x]]$ is a DVR with (x)-adic completion $\widehat{A} = A^* = k[[x]]$. The elements $\tau_1, \dots, \tau_s$ are limit-intersecting over R, that is, $A = B$, where B is as defined in (6.1.5). This follows from Theorem 9.2. We give the following direct proof.

Proof. Let $C_n = k[x, \tau_{1n}, \dots, \tau_{sn}]_{(x, \tau_{1n}, \dots, \tau_{sn})}$. By Remark 6.4, it suffices to prove that $A = \bigcup_{n=1}^{\infty} C_n$. Notice that C_n is a RLR of dimension $s+1$, and $C_n \subseteq C_{n+1} \subseteq A$ with A birationally dominating C_{n+1} and C_{n+1} birationally dominating C_n. Since C_n is a regular local ring, the function that associates to each nonzero $f \in C_n$ the nonnegative integer $e := \operatorname{ord}_{C_n}(f)$, where f is in the e-th power of the maximal ideal of C_n, but not in the $(e+1)$-th power, defines a valuation; see Section 2.1.

Let $a \in A$. Using that A is birational over C_n and that C_n is a UFD, we can write $a = f_n/g_n$, where $f_n, g_n \in C_n$ are relatively prime. Notice that $a \in C_n$ if and only if $\operatorname{ord}_{C_n}(g_n) = 0$. Let ord_A denote the order valuation associated to the one-dimensional regular local ring A. Since $a \in A$ and since ord_A is a valuation, we have $\operatorname{ord}_A(a) = \operatorname{ord}_A(f_n) - \operatorname{ord}_A(g_n) \geq 0$; see Remark 2.2. Since A dominates C_n, we have $\operatorname{ord}_A(g_n) \geq 0$ and $\operatorname{ord}_A(g_n) = 0$ if and only if $\operatorname{ord}_{C_n}(g_n) = 0$. These statements hold for each of the rings C_n.

To show that $a \in \bigcup_{n=1}^{\infty} C_n$ and hence that $\bigcup_{n=1}^{\infty} C_n = A$, it suffices to show that $\operatorname{ord}_{C_m}(g_m) = 0$ for some positive integer m. If $\operatorname{ord}_A(g_n) > 0$, it suffices to show that with the representation $a = \frac{f_{n+1}}{g_{n+1}}$, where $f_{n+1}, g_{n+1} \in C_{n+1}$ are relatively prime, then $\operatorname{ord}_A(g_{n+1}) < \operatorname{ord}_A(g_n)$. By (6.1.1) we see that the maximal ideal $(x, \tau_{1n}, \dots, \tau_{sn})C_n$ of C_n is contained in xC_{n+1}. Thus there exist elements f'_{n+1}, g'_{n+1} in C_{n+1} such that

$$f_n = xf'_{n+1} \text{ and } g_n = xg'_{n+1}. \quad \text{Hence} \quad a \ = \ \frac{f_{n+1}}{g_{n+1}} \ = \ \frac{f_n}{g_n} \ = \ \frac{xf'_{n+1}}{xg'_{n+1}} \ = \ \frac{f'_{n+1}}{g'_{n+1}}.$$

Since C_{n+1} is a UFD and f_{n+1} and g_{n+1} are relatively prime in C_{n+1}, we see that $g'_{n+1} = bg_{n+1}$ for some $b \in C_{n+1}$. Since A dominates C_{n+1} we have

$$\operatorname{ord}_A g_n \;=\; \operatorname{ord}_A x \;+\; \operatorname{ord}_A g'_{n+1} \;=\; 1 \;+\; \operatorname{ord}_A g'_{n+1} \;\geq\; 1 \;+\; \operatorname{ord}_A g_{n+1}.$$

This completes the proof that $\tau_1, \dots, \tau_s$ are limit-intersecting over R. $\square$

Example 6.7 extends and generalizes Example 6.6 to a situation where R is a polynomial ring in several variables over $k[x]$.

Example 6.7. Let m be a positive integer and let $R := k[x, y_1, \dots, y_m]$ be a polynomial ring in independent variables $x, y_1, \dots, y_m$ over a field k. Let

$$\tau_1 \;:=\; \sum_{j=1}^{\infty} r_{1j}x^j, \qquad \dots \qquad, \tau_s \;:=\; \sum_{j=1}^{\infty} r_{sj}x^j \;\in\; xk[[x]]$$

be algebraically independent over $k(x)$. Then $C := k(x, \tau_1, \dots, \tau_s) \cap k[[x]]$ is a DVR with (x)-adic completion $\widehat{C} = C^* = k[[x]]$ by Remark 2.1. Let R^* denote the (x)-adic completion of R, and let

$$A \;:=\; k(x, \tau_1, \dots, \tau_s, y_1, \dots, y_m) \;\cap\; R^*.$$

Notice that $C[y_1, \dots, y_n] \subsetneq A$. The inclusion is proper since elements such as $1 - xy_1$ are units of A, but are not units of the polynomial ring $C[y_1, \dots, y_m]$. Let $U_n := k[x, y_1, \dots, y_m, \tau_{1n}, \dots, \tau_{sn}]$, where the τ_{in} are the n^{th} endpieces of the τ_i. Theorem 9.2 implies that

$$A \;=\; B \;:=\; \bigcup_{n=1}^{\infty} B_n, \quad \text{where} \quad B_n \;:=\; (1 + xU_n)^{-1}U_n.$$

Thus the elements $\tau_1, \dots, \tau_s$ are limit-intersecting over R.

Remark 6.8. The limit-intersecting property depends on the choice of the elements $\tau_1, \dots, \tau_s \in zR^*$. For example, if R is the polynomial ring $\mathbb{Q}[z, y]$, then $R^* = \mathbb{Q}[y][[z]]$. Let $s = 1$, and let $\tau_1 = \tau := e^z - 1 \in zR^*$. Then τ is algebraically independent over $\mathbb{Q}(z, y)$. Let $U_0 = R[\tau]$. Example 6.7 shows that τ is limit-intersecting. On the other hand, the element $y\tau$ is not limit-intersecting. For if $U'_0 := R[y\tau,]$, then $\mathcal{Q}(U_0) = \mathcal{Q}(U'_0)$ and the intersection domain

$$A \;=\; \mathcal{Q}(U_0) \;\cap\; R^* \;=\; \mathcal{Q}(U'_0) \;\cap\; R^*$$

is the same for τ and $y\tau$. However the approximation domain B' associated to U'_0 does not contain τ. Indeed, $\tau \notin R[y\tau][1/z]$. Hence B' is properly contained in the approximation domain B associated to U_0. We have $B' \subsetneq B = A$ and the limit-intersecting property fails for the element $y\tau$.

6.2. Approximations for Homomorphic Image Construction

Applying the notation from Setting 5.1 to Homomorphic Image Construction 5.4, we again approach A using a sequence of "computable rings" over R, but we use the *fronts* of the power series involved, rather than the ends. The computable rings that are so obtained are not localizations of polynomial rings over R; instead they are localizations of finitely generated birational extensions within the fraction field of R. Nevertheless, as we explain below, there is a sequence of computable rings associated to Homomorphic Image Construction 5.4.

A goal of the computations here is to develop machinery for a proof that flatness of a certain extension implies that the integral domain A of Homomorphic Image Construction 5.4 is Noetherian and is a localization of a subring of $R[1/z]$. This goal is realized in Theorem 8.1 of Chapter 8.

For Construction 5.4 one no longer has an approximation of A by a nested union of polynomial rings over R. Indeed, in Construction 5.4 the extension $R \subseteq A$ is birational. However, there is an analogous approximation.

Frontpiece Notation 6.9. Let R be a Noetherian integral domain with field of fractions $K := \mathcal{Q}(R)$, let $z \in R$ be a nonzero nonunit, and let R^* denote the (z)-adic completion of R. Let I be an ideal of R^* having the property that $P \cap R = (0)$ for each $P \in \operatorname{Spec} R^*$ that is associated to I. As in Construction 5.4, define $A = A_{\text{hom}} := K \cap (R^*/I)$.

Since $I \subset R^*$, each $\gamma \in I$ has an expansion as a power series in z over R,

$$\gamma \ := \ \sum_{i=0}^{\infty} a_i z^i, \quad \text{where} \quad a_i \in R.$$

For each positive integer n we define the n^{th} *frontpiece* γ_n of γ with respect to this expansion:

$$\gamma_n \ := \ \sum_{j=0}^{n} \frac{a_j z^j}{z^n}.$$

Thus with $I := (\sigma_1, \dots, \sigma_t)R^*$, then for each σ_i we have

$$\sigma_i \ := \ \sum_{j=0}^{\infty} a_{ij} z^j, \qquad \text{where the} \qquad a_{ij} \ \in \ R,$$

and the n^{th} frontpiece σ_{in} of σ_i is

$$\sigma_{in} \ := \ \sum_{j=0}^{n} \frac{a_{ij} z^j}{z^n}. \tag{6.9.1}$$

For Homomorphic Image Construction 5.4, we obtain approximating rings as follows: We define

$$U_n := R[\sigma_{1n}, \dots, \sigma_{tn}], \qquad \text{and } B_n := (1 + zU_n)^{-1} U_n. \tag{6.9.2}$$

The rings U_n and B_n are subrings of K. We observe in Proposition 6.11 that they may also be considered to be subrings of R^*/I. First we show in Proposition 6.10 that the approximating rings U_n and B_n are nested.

Proposition 6.10. *With the setting of Frontpiece Notation 6.9, for each integer $n \geq 0$ and for each integer i with $1 \leq i \leq t$, we have*

1. $\sigma_{in} = -z a_{i,n+1} + z\sigma_{i,n+1}$.
2. $(z, \sigma_i)R^* = (z, a_{i0})R^*$ *and hence* $(z, I)R^* = (z, a_{10}, \dots, a_{t0})R^*$.
3. $(z, \sigma_i)R^* = (z, z^n \sigma_{in})R^*$ *and hence* $(z, I)R^* = (z, z^n\sigma_{1n}, \dots, z^n\sigma_{tn})R^*$.

Thus $R \subseteq U_0$ and we have $U_n \subseteq U_{n+1}$ and $B_n \subseteq B_{n+1}$ for each positive integer n.

Proof. For item 1, by Definition 6.9.1, we have $\sigma_{i,n+1} := \sum_{j=0}^{n+1} \frac{a_{ij} z^j}{z^{n+1}}$. Thus

$$z\sigma_{i,n+1} = \sum_{j=0}^{n+1} \frac{a_{ij} z^{j+1}}{z^{n+1}} = \sum_{j=0}^{n} \frac{a_{ij} z^j}{z^n} + z a_{i,n+1} = \sigma_{in} + z a_{i,n+1}.$$

For item 2, by definition

$$\sigma_i := \sum_{j=0}^{\infty} a_{ij} z^j = a_{i0} + z\big(\sum_{j=1}^{\infty} a_{ij} z^{j-1}\big).$$

For item 3, observe that

$$\sigma_i = \sum_{j=0}^{\infty} a_{ij} z^j = z^n \sigma_{in} + z^{n+1}\big(\sum_{j=n+1}^{\infty} a_{ij} z^{j-n-1}\big).$$

The asserted inclusions follow from this. □

Proposition 6.11. *Assume the setting of Frontpiece Notation 6.9 and let* n *be a positive integer. As an element of the total quotient ring of* R^*/I *the* n^{th} *frontpiece* σ_{in} *is the negative of the* n^{th} *endpiece of* σ_i *defined in Endpiece Notation 6.1, that is, for* $\sigma_i := \sum_{j=0}^{\infty} a_{ij} z^j$, *where each* $a_{ij} \in R$,

$$\sigma_{in} = -\sum_{j=n+1}^{\infty} \frac{a_{ij} z^j}{z^n} = -\sum_{j=n+1}^{\infty} a_{ij} z^{j-n} \pmod{I}.$$

It follows that $\sigma_{in} \in K \cap (R^*/I)$, *and so* U_n *and* B_n *are subrings of* R^*/I.

Proof. Let π denote the natural homomorphism from R^* onto R^*/I. Using that the restriction of π to R is the identity map on R, we have

$$\begin{aligned}\sigma_i = z^n\sigma_{in} + \sum_{j=n+1}^{\infty} a_{ij} z^j \implies & z^n\sigma_{in} = \sigma_i - \sum_{j=n+1}^{\infty} a_{ij} z^j \\ \implies & \pi(z^n\sigma_{in}) = \pi(\sigma_i) - \pi\big(\sum_{j=n+1}^{\infty} a_{ij} z^j\big) \\ \implies & z^n\sigma_{in} = -z^n\pi\big(\sum_{j=n+1}^{\infty} a_{ij} z^{j-n}\big).\end{aligned}$$

Therefore $z^n\sigma_{in} \in z^n\pi(R^*) = z^n(R^*/I)$. Since z is a regular element of R^*/I, we have $\sigma_{in} = -\pi(\sum_{j=n+1}^{\infty} a_{ij} z^{j-n})$, is an element of R^*/I. □

Definition 6.12. Assume the setting of Frontpiece Notation 6.9. We define

$$U := \bigcup_{n=1}^{\infty} U_n, \qquad B := \bigcup_{n=1}^{\infty} B_n = (1+zU_n)^{-1}U_n, \qquad A := K \cap (R^*/I). \tag{6.12.1}$$

By Remark 3.7.1, the element z is in the Jacobson radical of R^*. Hence $B \subseteq A$. Construction 5.4 is said to be *limit-intersecting* if $B = A$.

6.3. The Inclusion Construct is a Homomorphic Image Construct

Even though they appear different, Proposition 6.11 shows for each of the power series in the ideal I that the approximation in Frontpiece Notation 6.9 is, modulo the ideal I, the negative of the approximation in Endpiece Notation 6.1.

In view of Remark 5.7.2 of Chapter 5, Construction 5.4 includes Construction 5.3 as a special case. When Inclusion Construction 5.3 is translated to Homomorphic Image Construction 5.4 under the correspondence described in (5.7.2), Proposition 6.14 shows that the approximating rings using frontpieces as in (6.9) correspond to the approximating rings using endpieces as in (6.1).

We revise the notation slightly so that we can view Inclusion Construction as a Homomorphic Image Construction as in Remark 5.7.2:

Revised Notation 6.13. Let R, z, R^* and $\tau_1, \dots, \tau_s$ be as in Construction 5.3. Thus

$$\tau_i \ := \ \sum_{j=1}^{\infty} r_{ij} z^j \quad \text{where} \quad r_{ij} \ \in \ R.$$

Let $t_1, \dots, t_s$ be indeterminates over R, define $S := R[t_1, \dots, t_s]$, let S^* be the (z)-adic completion of S and let I denote the ideal $(t_1 - \tau_1, \dots, t_s - \tau_s)S^*$. Notice that $S^*/I \cong R^*$ implies that $P \cap S = (0)$ for each prime ideal $P \in \operatorname{Ass}(S^*/I)$. Thus we are in the setting of Homomorphic Image Construction and can define $D := \mathcal{Q}(R)(t_1, \dots, t_s) \cap (S^*/I)$. Let $\sigma_i := t_i - \tau_i$, for each i with $1 \le i \le s$.

As in (6.9), define σ_{in} to be the n^{th} frontpiece for σ_i over S. We consider (5.7.2.1), the same diagram as before, where λ is the R-algebra isomomorphism that maps $t_i \to \tau_i$ for $i = 1, \dots, s$:

$$\begin{tikzcd}
 & S := R[t_1, \dots, t_s] \arrow[r] \arrow[d, "\lambda"'] & D := \mathcal{Q}(R)(t_1, \dots, t_s) \cap (S^*/I) \arrow[r] \arrow[d, "\lambda"'] & S^*/I \arrow[d, "\lambda"'] \\
R \arrow[r] & S' := R[\tau_1, \dots, \tau_s] \arrow[r] & A := \mathcal{Q}(R)(\tau_1, \dots, \tau_s) \cap R^* \arrow[r] & R^*.
\end{tikzcd}$$

As before, λ maps D isomorphically onto A via $\lambda(t_i) = \tau_i$ for every i with $1 \le i \le s$. Define V_n, C_n, V, C using Frontspiece Notation 6.9 and Equation 6.12.1 over S, that is, with S and S^* in place of R and R^*. Define U_n, B_n, U, B using Endpiece Notation 6.1 and Equations 6.1.4 and Equation 6.1.5 over R. Then

$$\begin{aligned}
V_n &: = S[\sigma_{1n}, \dots, \sigma_{sn}] = R[t_1, \dots, t_s][\sigma_{1n}, \dots, \sigma_{sn}], \\
C_n &: = (1 + zV_n)^{-1} V_n, \\
V &: = \bigcup_{n=1}^{\infty} V_n, \qquad C := \bigcup_{n=1}^{\infty} C_n = (1 + zV)^{-1} V \\
U &: = \bigcup_{n=1}^{\infty} U_n = \bigcup_{n=1}^{\infty} R[\tau_{1n}, \dots, \tau_{sn}] \quad \text{and} \quad B := \bigcup_{n=1}^{\infty} B_n = (1 + zU)^{-1} U.
\end{aligned} \tag{6.1}$$

Proposition 6.14. *With the setting of Revised Notation 6.13, the R-algebra isomorphism λ has the following properties:*

$$\lambda(D) = A, \quad \lambda(\sigma_{in}) = \tau_{in}, \quad \lambda(V_n) = U_n, \quad \lambda(C_n) = B_n, \quad \lambda(V) = U, \quad \lambda(C) = B,$$

for all i with $1 \le i \le s$ and all $n \in \mathbb{N}$.

PROOF. We have elements $r_{ij} \in R$ so that

$$\begin{aligned}
\tau_i &:= \sum_{j=1}^{\infty} r_{ij} z^j, \qquad \tau_{in} := \sum_{j=n+1}^{\infty} r_{ij} z^{j-n} \\
\sigma_i &:= t_i - \tau_i = t_i - \sum_{j=1}^{\infty} r_{ij} z^j, \qquad \sigma_{in} := \frac{t_i - \sum_{j=1}^{n} r_{ij} z^j}{z^n} \\
\Longrightarrow \lambda(\sigma_{in}) &= \frac{\tau_i - \sum_{j=1}^{n} r_{ij} z^j}{z^n} \\
&= \tau_{in}.
\end{aligned}$$

The remaining statements of Proposition 6.14 now follow. ☐

REMARK 6.15. With the setting of Revised Notation 6.13, since U_n is a polynomial ring over R in the variables $\tau_{1n}, \dots \tau_{sn}$, Proposition 6.14 implies that V_n is a polynomial ring over R in the variables $\sigma_{1n}, \dots, \sigma_{sn}$. Thus

$$V_n := S[\sigma_{1n}, \dots, \sigma_{sn}] = R[t_1, \dots, t_s][\sigma_{1n}, \dots, \sigma_{sn}] = R[\sigma_{1n}, \dots, \sigma_{sn}].$$

6.4. Basic properties of the approximation domains

The following general lemma is used in Theorem 6.17 to establish more detailed descriptions for our setting.

LEMMA 6.16. *Let S be a subring of a ring T and let $x \in S$ be a regular element of T such that $\bigcap_{n=1}^{\infty} x^n T = (0)$. The following conditions are equivalent.*

1. (i) *$xS = xT \cap S$, and* (ii) *$S/xS = T/xT$.*
2. *For each positive integer n we have:*
   $x^n S = x^n T \cap S, \quad S/x^n S = T/x^n T \quad$ *and* $\quad T = S + x^n T.$
3. *The rings S and T have the same (x)-adic completion.*
4. (i) *$S = S[1/x] \cap T$, and* (ii) *$T[1/x] = S[1/x] + T$.*

PROOF. To see that item 1 implies item 2, observe that

$$x^n T \cap S = x^n T \cap xS = x(x^{n-1} T \cap S),$$

so the equality $x^{n-1}S = x^{n-1}T \cap S$ implies the equality $x^n S = x^n T \cap S$. Moreover $S/xS = T/xT$ implies $T = S + xT = S + x(S + xT) = \cdots = S + x^n T$, so $S/x^n S = T/x^n T$ for every $n \in \mathbb{N}$. Therefore (1) implies (2).

It is clear that item 2 is equivalent to item 3.

To see that item 2 implies (4i), let $s/x^n \in S[1/x] \cap T$ with $s \in S$ and $n \geq 0$. Item 2 implies that $s \in x^n T \cap S = x^n S$ and therefore $s/x^n \in S$. To see (4ii), let $\frac{t}{x^n} \in T[1/x]$ with $t \in T$ and $n \geq 0$. Item 2 implies that $t = s + x^n t_1$ for some $s \in S$ and $t_1 \in T$. Therefore $\frac{t}{x^n} = \frac{s}{x^n} + t_1$. Thus (2) implies (4).

It remains to show that item 4 implies item 1. To see that (4) implies (1i), let $t \in T$ and $s \in S$ be such that $xt = s$. Then $t = s/x \in S[1/x] \cap T = S$, by (4i). Thus $xt \in xS$. To see that (4) implies (1ii), let $t \in T$. Then $\frac{t}{x} = \frac{s}{x^n} + t'$, for some $n \in \mathbb{N}$, $s \in S$ and $t' \in T$ by (4ii). Thus $t = \frac{s}{x^{n-1}} + t'x$. Hence $t - t'x = \frac{s}{x^{n-1}} \in S[1/x] \cap T = S$, by (4ii). ☐

Theorem 6.17 relates to the analysis of the homomorphic image construction. This analysis is useful for the development of examples in later chapters.

Construction Properties Theorem 6.17. (Homomorphic Image Version) *Let R be a Noetherian integral domain with field of fractions $K := \mathcal{Q}(R)$, let $z \in R$ be a nonzero nonunit, and let R^* denote the (z)-adic completion of R. Let I be an ideal of R^* having the property that $P \cap R = (0)$ for each $P \in \operatorname{Spec} R^*$ that is associated to I. With the notation of Frontpiece Notation 6.9 and Definition 6.12, we have for each positive integer n:*

1. *The ideals of R containing z^n are in one-to-one inclusion preserving correspondence with the ideals of R^* containing z^n. In particular, we have $(I,z)R^* = (a_{10}, \dots, a_{t0}, z)R^*$ and*
$$(I,z)R^* \ \cap \ R \ = \ (a_{10}, \dots, a_{t0}, z)R^* \ \cap \ R \ = \ (a_{10}, \dots, a_{t0}, z)R.$$
2. *The ideal $(a_{10}, \dots, a_{t0}, z)R$ equals $z(R^*/I) \cap R$ under the identification of R as a subring of R^*/I, and the element z is in the Jacobson radical of both R^*/I and B.*
3. $z^n(R^*/I) \cap A = z^n A, \qquad z^n(R^*/I) \cap U = z^n U, \qquad z^n(R^*/I) \cap B = z^n B.$
4. $U/z^nU = B/z^nB = A/z^nA = R^*/(z^nR^* + I)$. *The rings A, U and B all have (z)-adic completion R^*/I, that is, $A^* = U^* = B^* = R^*/I$.*
5. $R[1/z] = U[1/z]$, $U = R[1/z] \cap B = R[1/z] \cap A = R[1/z] \cap (R^*/I)$ *and the integral domains R, U, B and A all have the same field of fractions K.*

Proof. The first assertion of item 1 follows because R/z^nR is canonically isomorphic to R^*/z^nR^*. The next assertion of (1) follows from part 2 of Proposition 6.10. If $\gamma = \sum_{i=1}^t \sigma_i\beta_i + z\tau \in (I,z)R^* \cap R$, where $\tau, \beta_i \in R^*$, then write each $\beta_i = b_i + z\beta_i'$, where $b_i \in R$, $\beta_i' \in R^*$. Thus $\gamma - \sum_{i=1}^t a_{i0}b_i \in zR^* \cap R = zR$, and so $\gamma \in (a_{10}, \dots, a_{t0}, z)R$.

Since $z(R^*/I) = (z,I)R^*/I$, we have $(a_{10}, \dots, a_{t0}, z)R \subseteq z(R^*/I) \cap R$. The reverse inclusion in item 2 follows from $(I,z)R^* = (a_{10}, \dots, a_{t0}, z)R^*$. For the last part of item 2, we have that $z \in \mathcal{J}(R^*)$ and so $1 + az$ is outside every maximal ideal of R^* for every $a \in R^*$. Thus $z \in \mathcal{J}(R^*/I)$. By the definition of B in Equation 6.12.1, $z \in \mathcal{J}(B)$.

The first assertion of item 3 follows from the definition of A as $(R^*/I) \cap K$. To see that $z(R^*/I) \cap U \subseteq zU$, let $g \in z(R^*/I) \cap U$. Then $g \in U_n$, for some n, implies $g = r_0 + g_0$, where $r_0 \in R$, $g_0 \in (\sigma_{1n}, \dots, \sigma_{tn})U_n$. Also $\sigma_{in} = -za_{i,n+1} + z\sigma_{i,n+1}$, and so $g_0 \in zU_{n+1} \subseteq z(R^*/I)$. Now $r_0 \in (z, \sigma_1, \dots, \sigma_t)R^* = (I,z)R^*$. Thus by item 1, $r_0 \in (a_{10}, \dots, a_{t0}, z)R$. Also each $a_{i0} = \sigma_i - z\sum_{j=1}^\infty a_{ij}z^{j-1} \in zU$ because $a_{i0} = z\sigma_{i1} - za_{i1}$.

Thus $r_0 \in zU$, as desired. This proves that $z^n(R^*/I) \cap U = z^nU$. Since $B = (1+zU)^{-1}U$, we also have $z^n(R^*/I) \cap B = z^nB$.

Item 4 follows from item 3 and Lemma 6.16.

For (5): if $g \in U$, then $g \in U_n$, for some n. Clearly each $\sigma_{in} \in R[1/z]$, and so $g \in R[1/z]$. To see that $U = R[1/z] \cap B$, apply Lemma 6.16 with $S = U$, $B = T$. Similarly we see that $U = R[1/z] \cap A$, since
$$R[1/z] \ \cap \ A \ = \ R[1/z] \ \cap \ (\mathcal{Q}(R) \ \cap \ (R^*/I)) \ = \ R[1/z] \ \cap \ (R^*/I).$$
It is clear that the integral domains R, U, B and A all have the same field of fractions K. □

Remark 6.18. We note the following implications from Theorem 6.17 .

1. Item 5 of Theorem 6.17 implies that the definitions in (6.12.1) of B and U are independent of

   (a) the choice of generators for I, and
   (b) the representation of the generators of I as power series in z.
(2) Item 5 of Theorem 6.17 implies that the rings $U = R[1/z] \cap (R^*/I)$ and $B = (1 + zU)^{-1}U$ are uniquely determined by z and the ideal I of R^*.
(3) Since z is in the Jacobson radical of B, item 4 of Theorem 6.17 implies that if $b \in B$ is a unit of A, then b is already a unit of B.
(4) We have the following diagram displaying the relationships among the rings. Recall that $B = (1 + zU)^{-1}U$.

$$\begin{tikzcd}
\mathcal{Q}(R) \arrow[r, equal] & \mathcal{Q}(U) \arrow[r, equal] & \mathcal{Q}(B) \arrow[r, equal] & \mathcal{Q}(A) \arrow[r, "\subseteq"] & \mathcal{Q}(R^*/I) \\
R[1/z] \arrow[u] \arrow[r, equal] & U[1/z] \arrow[u] \arrow[r, "\subseteq"] & B[1/z] \arrow[u] \arrow[r, "\subseteq"] & A[1/z] \arrow[u] \arrow[r, "\subseteq"] & (R^*/I)[1/z] \arrow[u] \\
R \arrow[u] \arrow[r, "\subseteq"] & U = \cup\, U_n \arrow[u] \arrow[r, "\subseteq"] & B \arrow[u] \arrow[r, "\subseteq"] & A \arrow[u] \arrow[r, "\subseteq"] & R^*/I. \arrow[u]
\end{tikzcd}$$

We record in Construction Properties Theorem 6.19 (Inclusion Version) some basic properties of the integral domains associated with Inclusion Construction 5.3 that follow from Construction Properties Theorem 6.17 (Homomorphic Image Version), Remark 6.18, Proposition 6.3 and Proposition 6.14.

Construction Properties Theorem 6.19. (Inclusion Version) *Let R be a Noetherian integral domain with field of fractions $K := \mathcal{Q}(R)$, let $z \in R$ be a nonzero nonunit, and let R^* denote the (z)-adic completion of R. Let $\underline{\tau} = \{\tau_1, \dots, \tau_s\}$ be a set of elements of R^* that are algebraically independent over K, so that $R[\underline{\tau}]$ is a polynomial ring in s variables over R. Define $A = A_{inc} := K(\underline{\tau}) \cap R^*$, as in Inclusion Construction 5.3. Let U_n, B_n, B and U be defined as in Equations 6.1.4 and 6.1.5. Then:*

(1) $z^nR^* \cap A = z^nA$, $\quad z^nR^* \cap B = z^nB \quad$ *and* $\quad z^nR^* \cap U = z^nU$, *for each* $n \in N$.
(2) $R[\underline{\tau}][1/z] = U[1/z]$, $\quad U = R[\underline{\tau}][1/z] \cap B = R[\underline{\tau}]1/z] \cap A$, *and* $B[1/z]$ *is a localization of* $R[\underline{\tau}]$. *The integral domains* $R[\underline{\tau}], U, B$ *and* A *all have the same field of fractions, namely* $K(\underline{\tau})$.
(3) $R/z^nR = U/z^nU = B/z^nB = A/z^nA = R^*/z^nR^*$, *for each* $n \in N$.
(4) *The* (z)*-adic completions of* U, B *and* A *are all equal to* R^*, *that is,* $R^* = U^* = B^* = A^*$.
(5) *The definitions in Equation 6.1.5 of* B *and* U *are independent of the representations given in Notation 6.1 for the* τ_i *as power series in* R^*.
(6) *If* $(R, \mathbf{m})$ *is local, then* B *is local and, in view of Remark 6.4.1, we have*

$$B \;=\; (1 + zU)^{-1}U \;=\; \bigcup_{n=1}^{\infty} (U_n)_{(\mathbf{m}, \tau_{1n}, \dots, \tau_{sn})U_n}.$$

Proposition 6.20 concerns the extension to R^* of a prime ideal of either A or B that does not contain z, and provides information about the maps from $\operatorname{Spec} R^*$ to $\operatorname{Spec} A$ and to $\operatorname{Spec} B$. We use Proposition 6.20 in Chapters 7 and 17.

PROPOSITION 6.20. *With the notation of Construction Properties Theorem 6.19 we have:*

(1) *z is in the Jacobson radical of B, in the Jacobson radical of A, and in the Jacobson radical of R^*.*
(2) *If q is a prime ideal of R, then qU is a prime ideal of U, and either $qB = B$ or qB is a prime ideal of B.*
(3) *Let I be an ideal of B or of A and let $t \in \mathbb{N}$. If $z^t \in IR^*$, then $z^t \in I$.*
(4) *Let $P \in \operatorname{Spec} B$ or $P \in \operatorname{Spec} A$ with $z \notin P$. Then z is a nonzerodivisor on R^*/PR^*. Thus $z \notin Q$ for each associated prime of the ideal PR^*. Since z is in the Jacobson radical of R^*, it follows that PR^* is contained in a nonmaximal prime ideal of R^*.*
(5) *If R is local, then R^*, A and B are local. Let $\mathbf{m}_R$, $\mathbf{m}_{R^*}, \mathbf{m}_A$ and $\mathbf{m}_B$ denote the maximal ideals of R, R^*, A and B, respectively. In this case*
    - *$\mathbf{m}_B = \mathbf{m}_R B$, $\mathbf{m}_A = \mathbf{m}_R A$ and each prime ideal P of B such that $\operatorname{ht}(\mathbf{m}_B/P) = 1$ is contracted from R^*.*
    - *If an ideal I of B is such that IR^* is primary for $\mathbf{m}_{R^*}$, then I is primary for $\mathbf{m}_B$.*

PROOF. For item 1, since $B_n = (1 + zU_N)^{-1}U_n$, it follows that $1 + zb$ is a unit of B_n for each $b \in B_n$. Therefore z is in the Jacobson radical of B_n for each n and thus z is in the Jacobson radical of B. The statement for R^* follows from Remark 3.7.1.

For item 2, since each U_n is a polynomial ring over R, qU_n is a prime ideal of U_n and thus $qU = \bigcup_{n=0}^{\infty} qU_n$ is a prime ideal of U. Since B is a localization of U, qB is either B or a prime ideal of B.

To see item 3, let I be an ideal of B. The proof for A is identical. We observe that there exist elements $b_1, \dots, b_s \in I$ such that $IR^* = (b_1, \dots, b_s)R^*$. If $z^t \in IR^*$, there exist $\alpha_i \in R^*$ such that

$$z^t \;=\; \alpha_1 b_1 + \cdots + \alpha_s b_s.$$

We have $\alpha_i = a_i + z^{t+1}\lambda_i$ for each i, where $a_i \in B$ and $\lambda_i \in R^*$. Thus

$$z^t[1 - z(b_1\lambda_1 + \cdots + b_s\lambda_s)] \;=\; a_1b_1 + \cdots + a_sb_s \in B \cap z^tB^* = z^tB.$$

Therefore $\gamma := 1 - z(b_1\lambda_1 + \cdots + b_s\lambda_s) \in B$. Thus $z(b_1\lambda_1 + \cdots + b_s\lambda_s) \in B \cap zR^* = zB$, and so $b_1\lambda_1 + \cdots + b_s\lambda_s \in B$. By item 1, the element z is in the Jacobson radical of B. Hence γ is invertible in B. Since $\gamma z^t \in (b_1, \cdots, b_s)B$, it follows that $z^t \in I$.

For item 4, assume that $P \in \operatorname{Spec} B$. The proof for $P \in \operatorname{Spec} A$ is identical. We have that

$$P \cap zB = zP \quad \text{and so} \quad \frac{P}{zP} = \frac{P}{P \cap zB} \cong \frac{P + zB}{zB}$$

By Construction Properties Theorem 6.19.3, B/zB is Noetherian. Hence the B-module P/zP is finitely generated. Let $g_1, \dots, g_t \in P$ be such that $P = (g_1, \dots, g_t)B + zP$. Then also $PR^* = (g_1, \dots, g_t)R^* + zR^* = (g_1, \dots, g_t)R^*$, the last equality by Nakayama's Lemma.

Let $\widehat{f} \in R^*$ be such that $z\widehat{f} \in PR^*$. we show that $\widehat{f} \in PR^*$.

Since $\widehat{f} \in R^*$, we have $\widehat{f} := \sum_{i=0}^{\infty} c_i z^i$, where each $c_i \in R$. For each $m > 1$, let $f_m := \sum_{i=0}^{m} c_i z^i$, the first $m + 1$ terms of this expansion of $\widehat{f}$. Then $f_m \in R \subseteq B$

and there exists an element $\widehat{h}_1 \in R^*$ so that.

$$\widehat{f} = f_m + z^{m+1}\widehat{h_1}.$$

Since $z\widehat{f} \in PR^*$, we have

$$z\widehat{f} = \widehat{a_1}g_1 + \cdots + \widehat{a_t}g_t,$$

where $\widehat{a_i} \in R^*$. The $\widehat{a_i}$ have power series expansions in z over R, and thus there exist elements $a_{im} \in R$ such that $\widehat{a_i} - a_{im} \in z^{m+1}R^*$. Thus

$$z\widehat{f} = a_{1m}g_1 + \cdots + a_{tm}g_t + z^{m+1}\widehat{h_2},$$

where $\widehat{h}_2 \in R^*$, and

$$zf_m = a_{1m}g_1 + \cdots + a_{tm}g_t + z^{m+1}\widehat{h_3},$$

where $\widehat{h_3} = \widehat{h_2} - z\widehat{h_1} \in R^*$. Since the g_i are in B, we have $z^{m+1}\widehat{h_3} \in z^{m+1}R^* \cap B = z^{m+1}B$, the last equality by Construction Properties Theorem 6.19.1. Therefore $\widehat{h_3} \in B$. Rearranging the last set-off equation above, we obtain

$$z(f_m - z^m\widehat{h_3}) = a_{1m}g_1 + \cdots + a_{tm}g_t \in P.$$

Since $z \notin P$, we have $f_m - z^m\widehat{h_3} \in P$. It follows that $\widehat{f} \in P + z^mR^* \subseteq PR^* + z^mR^*$, for each $m > 1$. Hence we have that $\widehat{f} \in PR^*$, as desired.

For item 5, if R is local, then B is local, A is local, $\mathbf{m}_B = \mathbf{m}_R B$ and $\mathbf{m}_A = \mathbf{m}_R A$ since $R/zR = B/zB = A/zA = R^*/zR^*$ and z is in the Jacobson radical of B and of A. If $z \notin P$, then item 4 implies that no power of z is in PR^*. Hence PR^* is contained in a prime ideal Q of R^* that does not meet the multiplicatively closed set $\{z^n\}_{n=1}^{\infty}$. Thus $P \subseteq Q \ \cap B \subsetneq \mathbf{m}_B$. Since $\operatorname{ht}(\mathbf{m}_B/P) = 1$, we have $P = Q \cap B$, so P is contracted from R^*. If $z \in P$, then $B/zB = R^*/zR^*$ implies that PR^* is a prime ideal of R^* and $P = PR^* \cap B$.

For the second part of item 5, if IR^* is $\mathbf{m}$-primary then $z^t \in IR^*$. Thus $z^t \in I$ by item 3. By Theorem 6.19.3, $B/z^tB = R^*/z^tR^*$ and so I/z^tB is primary for the maximal ideal of B/a^tB. Therefore I is primary for the maximal ideal of B. □

In many of the examples constructed in this book, the ring R is a polynomial ring (or a localized polynomial ring) in finitely many variables over a field; such rings are UFDs. We observe in Proposition 6.21 that the constructed ring B is a UFD if R is a UFD and z is a prime element.

PROPOSITION 6.21. *With the notation of Construction Properties Theorem 6.19:*

(1) *If R is a UFD and z is a prime element of R, then zB is a prime ideal, $B[1/z]$ is a Noetherian UFD and B is a UFD.*
(2) *If in addition R is regular, then $B[1/z]$ is a regular Noetherian UFD.*

PROOF. By Proposition 6.20.2, zB is a prime ideal. Since R is a Noetherian UFD and $R[\underline{\tau}]$ is a polynomial ring in finitely many variables over R, it follows that $R[\underline{\tau}]$ is a Noetherian UFD. By Theorem 6.19.2, the ring $B[1/z]$ is a localization of $R[\underline{\tau}]$ and thus is a Noetherian UFD; moreover $B[1/z]$ is regular if R is. By Theorem 6.19.4, The (z)-adic completion of B is R^* . By Proposition 6.20.1, z is in the Jacobson radical of R^*. Since R^* is Noetherian, it follows that $\bigcap_{n=1}^{\infty} z^nR^* = (0)$. Thus $\bigcap_{n=1}^{\infty} z^nB = (0)$. It follows that B_{zB} is a DVR [**79**, (31.5)].

By Fact 2.10, we have $B = B[1/z] \cap B_{zB}$. Therefore B is a Krull domain. Since $B[1/z]$ is a UFD and B is a Krull domain, Theorem 2.9 implies that B is a UFD. □

Exercises

(1) Prove item 2 of Remark 6.4, that is, with R a polynomial ring $k[x]$ over a field k and $R^* = k[[x]]$ the (x)-adic completion of R, show that with the notation of Construction 5.3 we have $B = \bigcup C_n$, where $C_n = (U_n)_{P_n}$ and $P_n := (x, \tau_{1n}, \ldots, \tau_{sn})U_n$. Thus B is a DVR that is the directed union of a birational family of localized polynomial rings in $n+1$ indeterminates.

(2) Assume the setting of Frontpiece Notation 6.9 and Definition 6.12. If J is a proper ideal of B, prove that JB^* is a proper ideal of B^*, where B^* is the (z)-adic completion of B.

(3) Assume the setting of Frontpiece Notation 6.9, and let W denote the set of elements of R^* that are regular on R^*/I. Prove that the natural homomorphism $\pi : R^* \to R^*/I$ extends to a homomorphism $\overline{\pi} : W^{-1}R^* \to W^{-1}(R^*/I)$.

CHAPTER 7

Revisiting the Iterative Examples

In this chapter we continue the proof of Theorem 4.12 stated in Chapter 4. We give specific examples that illustrate possible outcomes of the construction in Chapter 5.

7.1. Properties of the iterative examples

We begin by fixing notation. We include several remarks concerning the integral domains that are used in the proof of Theorem 4.12.

Notation and Remarks 7.1. Let k be a field, let x and y be indeterminates over k, and let

$$\sigma := \sum_{i=0}^{\infty} a_i x^i \in k[[x]] \qquad \text{and} \qquad \tau := \sum_{i=0}^{\infty} b_i y^i \in k[[y]]$$

be formal power series that are algebraically independent over the fields $k(x)$ and $k(y)$, respectively. Let $R := k[x,y]_{(x,y)}$, and let σ_n, τ_n be the n^{th} endpieces of σ, τ defined as in (6.1). Define

$$\begin{aligned}
C_n &:= k[x,\sigma_n]_{(x,\sigma_n)}, \quad C := k(x,\sigma) \cap k[[x]] = \varinjlim(C_n) = \bigcup_{n=1}^{\infty} C_n \ ;\\
D_n &:= k[y,\tau_n]_{(y,\tau_n)}, \quad D := k(y,\tau) \cap k[[y]] = \varinjlim(D_n) = \bigcup_{n=1}^{\infty} D_n;\\
U_n &:= k[x,y,\sigma_n,\tau_n], \quad U := \varinjlim U_n = \bigcup_{n=1}^{\infty} U_n;\\
B_n &:= k[x,y,\sigma_n,\tau_n]_{(x,y,\sigma_n,\tau_n)} \quad B := \varinjlim(B_n) = \bigcup_{n=1}^{\infty} B_n;\\
A &:= k(x,y,\sigma,\tau) \cap k[[x,y]].
\end{aligned} \tag{7.1.1}$$

(i) Since $k[[x,y]]$ is the (x,y)-adic completion of the Noetherian ring R, Remark 3.7.4 implies that $k[[x,y]]$ is faithfully flat over R. By Remark 2.20.7

$$(x,y)^n k[[x,y]] \ \cap \ R \ = \ (x,y)^n R$$

for each $n \in \mathbb{N}$. Equation 6.1.2 implies for each positive integer n the inclusions

$$C_n \ \subset \ C_{n+1}, \quad D_n \ \subset \ D_{n+1}, \quad \text{and} \quad B_n \ \subset \ B_{n+1}.$$

Moreover, for each of these inclusions we have birational domination of the larger local ring over the smaller, and the local rings C_n, D_n, B_n are all dominated by $k[[x,y]]$.

(ii) We observe that $(x, y)B \cap B_n = (x, y, \sigma_n, \tau_n)B_n$, for each $n \in \mathbb{N}$. For if $h \in (x, y, \sigma_n, \tau_n)U_n$, then Equation 6.1.2 implies that

$$\sigma_n \;=\; -xa_{n+1} + x\sigma_{n+1} \quad \text{and} \quad \tau_n = -yb_{n+1} + y\tau_{n+1}.$$

Hence $h \in (x, y)U_{n+1} \cap U_n \subseteq (x, y)U \cap U_n$. Since $(x, y, \sigma_n, \tau_n)U_n$ is a maximal ideal of U_n contained in $(x, y)U$, a proper ideal of U, it follows that $(x, y)U \cap U_n = (x, y, \sigma_n, \tau_n)U_n$. Thus also $(x, y)B \cap B_n = (x, y, \sigma_n, \tau_n)B$ for each $n \in \mathbb{N}$.

(iii) We also observe that $U_n[\frac{1}{xy}] = U[\frac{1}{xy}]$ for each $n \in \mathbb{N}$. For Equation 6.1.2 implies that $\sigma_{n+1} \in U_n[\frac{1}{x}] \subseteq U_n[\frac{1}{xy}]$ and $\tau_{n+1} \in U_n[\frac{1}{y}] \subseteq U_n[\frac{1}{xy}]$. Hence $U_{n+1} \subseteq U_n[\frac{1}{xy}]$ for each $n \in \mathbb{N}$. Hence $U \subseteq U_n[\frac{1}{xy}]$, and $U_n[\frac{1}{xy}] = U[\frac{1}{xy}]$.

(iv) By Remark 2.1, the rings C and D are rank-one discrete valuation domains; as in Example 6.6, they are the asserted directed unions. The rings B_n are four-dimensional regular local domains that are localized polynomial rings over the field k. Thus B is the directed union of a chain of four-dimensional regular local rings, with each ring birational over the previous ring.

We prove in Theorem 7.2 other parts of Theorem 4.12.

Theorem 7.2. *Assume the setting of Notation 7.1. Then the ring A is a two-dimensional regular local domain that birationally dominates the ring B; A has maximal ideal $(x, y)A$ and completion $\widehat{A} = k[[x, y]]$. Moreover we have:*

1. *The rings U and B are UFDs,*
2. *B is a local Krull domain with maximal ideal $\mathbf{n} = (x, y)B$,*
3. *The dimension of B is either 2 or 3,*
4. *B is Hausdorff in the topology defined by the powers of $\mathbf{n}$,*
5. *The $\mathbf{n}$-adic completion $\widehat{B}$ of B is canonically isomorphic to $k[[x, y]]$, and*
6. *The following statements are equivalent:*
    - (a) *$B = A$.*
    - (b) *B is a two-dimensional regular local domain.*
    - (c) *B is Noetherian.*
    - (d) *Every finitely generated ideal of B is closed in the $\mathbf{n}$-adic topology on B.*
    - (e) *Every principal ideal of B is closed in the $\mathbf{n}$-adic topology on B.*

Proof. The assertions about A follow from Proposition 4.13. Since U_0 has field of fractions $k(x, y, \sigma, \tau) = \mathcal{Q}(A)$ and $U_0 \subseteq B \subseteq A$, the extension $B \hookrightarrow A$ is birational. Since B is the directed union of the four-dimensional regular local domains B_n and $(x, y)B \cap B_n = (x, y, \sigma_n, \tau_n)B$ for each $n \in \mathbb{N}$ by (4.7.n.iv) and (4.7.n.ii), we see that B is local with maximal ideal $\mathbf{n} = (x, y)B$. Since B and A are both dominated by $k[[x, y]]$, it follows that A dominates B.

We now prove that U and B are UFDs. Since U_n is a polynomial ring over a field and $U_n[\frac{1}{xy}] = U[\frac{1}{xy}]$ by (4.7.n.iii), we have $U[\frac{1}{xy}]$ is a UFD. For each $n \in \mathbb{N}$, the principal ideals xU_n and yU_n are prime ideals in the polynomial ring U_n. Therefore xU and yU are principal prime ideals of U. Moreover, $U_{xU} = B_{xB}$ and $U_{yU} = B_{yB}$ are DVRs since each is the contraction to the field $k(x, y, \sigma, \tau)$ of the (x)-adic or the (y)-adic valuations of $k[[x, y]]$; see Remark 2.2.

Applying Fact 2.10 with $S = U$ and $c = x$, and then with $S = U[\frac{1}{x}]$ and $c = y$, we obtain $U = U_{xU} \cap U_{yU} \cap U[\frac{1}{xy}]$. It follows that U is a Krull domain; see Definition 2.3.2 and the comments there. Hence, by Theorem 2.9, U is a UFD.

Since B is a localization of U, the ring B is a UFD. This completes the proof of items 1 and 2.

Since B is dominated by $k[[x,y]]$, B is Hausdorff in the topology defined by the powers of $\mathbf{n}$. Since $\mathbf{n}$ is finitely generated, $\widehat{B}$ is Noetherian [**79**, 31.7]. Therefore $\widehat{B}$ is a two-dimensional regular local domain with residue field k that canonically maps onto $k[[x,y]]$. This canonical surjection must have kernel (0), and so we have $\widehat{B} = k[[x,y]]$. Notice that B is a birational extension of the three-dimensional Noetherian domain $C[y,\tau]$. Theorem 2.8 implies that the dimension of B is at most 3. This completes the proof of items 3, 4 and 5.

For item 6, we have A is a two-dimensional RLR by Proposition 4.13. Thus $(a) \implies (b)$. Clearly $(b) \implies (c)$. Since B is local by item 2, and since the completion of a Noetherian local ring is a faithfully flat extension by Remark 3.7.4, we have $(c) \implies (d)$ by Remark 2.207. It is clear that $(d) \implies (e)$. To complete the proof of Theorem 7.2, it suffices to show that $(e) \implies (a)$. Since A birationally dominates B, we have $B = A$ if and only if $bA \cap B = bB$ for every element $b \in \mathbf{n}$. The principal ideal bB is closed in the $\mathbf{n}$-adic topology on B if and only if $bB = b\widehat{B} \cap B$. Also $\widehat{B} = \widehat{A}$ and $bA = b\widehat{A} \cap A$, for every $b \in B$. Thus (e) implies, for every $b \in B$,

$$bB = b\widehat{B} \cap B = b\widehat{A} \cap B = b\widehat{A} \cap A \cap B = bA \cap B,$$

and so $B = A$. This completes the proof. $\square$

Depending on the choice of σ and τ, the ring B may fail to be Noetherian. Example 7.3 shows that in the setting of Theorem 7.2 the ring B can be strictly smaller than $A := k(x,y,\sigma,\tau) \cap k[[x,y]]$.

EXAMPLE 7.3. Using the setting of Notation 7.1, let $\tau \in k[[y]]$ be defined to be $\sigma(y)$, that is, set $b_i := a_i$ for every $i \in \mathbb{N}$. We then have that $\theta := \frac{\sigma-\tau}{x-y} \in A$. Indeed,

$$\sigma - \tau = a_1(x-y) + a_2(x^2-y^2) + \cdots + a_n(x^n - y^n) + \cdots ,$$

and so $\theta = \frac{\sigma-\tau}{x-y} \in k[[x,y]] \cap k(x,y,\sigma,\tau) = A$. As a specific example, one may take $k := \mathbb{Q}$ and set $\sigma := e^x - 1$ and $\tau := e^y - 1$. The ring B is a localization of the ring $U := \bigcup_{n \in \mathbb{N}} k[x,y,\sigma_n,\tau_n]$.

CLAIM 7.4. *The element θ is not in B.*

PROOF. If θ is an element of B, then

$$\sigma - \tau \ \in \ (x-y)B \ \cap \ U \ = \ (x-y)U.$$

Let $S := k[x,y,\sigma,\tau]$ and let $U_n := k[x,y,\sigma_n,\tau_n]$ for each positive integer n. We have

$$U \ = \ \bigcup_{n \in \mathbb{N}} U_n \ \subseteq S[\frac{1}{xy}] \ \subseteq S_{(x-y)S},$$

where the last inclusion is because $xy \notin (x-y)S$. Thus $\theta \in B$ implies that

$$\sigma - \tau \in (x-y)S_{(x-y)S} \ \cap \ S \ = \ (x-y)S,$$

but this contradicts the fact that x, y, σ, τ are algebraically independent over k. $\square$

Therefore $\frac{\sigma-\tau}{x-y} \notin B$, and so $B \subsetneq A$ and $(x-y)B \subsetneq (x-y)A \cap B$. Since an ideal of B is closed in the $\mathbf{n}$-adic topology if and only if the ideal is contracted from $\widehat{B}$ and since $\widehat{B} = \widehat{A}$, the principal ideal $(x-y)B$ is not closed in the $\mathbf{n}$-adic topology on B. Using Theorem 7.2, we obtain a non-Noetherian three-dimensional quasilocal

Krull domain B having a two-generated maximal ideal such that B birationally dominates a four-dimensional regular local domain. We discuss this example in more detail in Example 8.11 of Chapter 8.

On the other hand, the ring B may be Noetherian. To complete the proof of Theorem 4.12, we establish in Example 7.13 below with $k = \mathbb{Q}$ that the elements $\sigma := e^x - 1$ and $\tau := e^{(e^y-1)} - 1$ give an example where $B = A$. The property of σ and τ we use in Example 7.13 to prove $A = B$ is that for each height-one prime ideal Q of $\widehat{R} = \mathbb{Q}[[x, y]]$ with $Q \cap R \neq (0)$ the images of σ and τ in $\widehat{R}/Q$ are algebraically independent over $R/(Q \cap R)$.

Remark 7.5. In the setting of Notation 7.1, It seems natural to consider the ring compositum of $\widehat{C} = k[[x]]$ and $\widehat{D} = k[[y]]$. We outline in Exercise 2 of this chapter a proof due to Kunz that the subring $\widehat{C}[\widehat{D}]$ of $k[[x, y]]$ is not Noetherian.

7.2. Residual algebraic independence

Recall that an extension of Krull domains $R \hookrightarrow S$ satisfies the condition **PDE** ("pas d'éclatement", or in English "no blowing up") provided that $\text{ht}(Q \cap R) \leq 1$ for each prime ideal of height one Q in S; see Definition 2.3. The iterative example leads us to consider in this section a related property as in the following definition.

Definition 7.6. Let $R \hookrightarrow S$ denote an extension of Krull domains. An element $\nu \in S$ is *residually algebraically independent with respect to S over R* if ν is algebraically independent over R and for every height-one prime ideal Q of S such that $Q \cap R \neq 0$, the image of ν in S/Q is algebraically independent over $R/(Q \cap R)$.

Remark 7.7. If $(R, \mathbf{m})$ is a two-dimensional regular local domain, or more generally an analytically normal Noetherian local domain of dimension at least two, it is naturally to consider the extension of Krull domains $R \hookrightarrow \widehat{R}$ and ask about the existence of an element $\nu \in \widehat{R}$ that is residually algebraically independent with respect to $\widehat{R}$ over R. If R has countable cardinality, for example, if $R = \mathbb{Q}[x, y]_{(x,y)}$, then a cardinality argument implies the existence of an element $\nu \in \widehat{R}$ that is residually algebraically independent with respect to $\widehat{R}$ over R.

We show in Proposition 19.25 and Theorem 19.37 of Chapter 19, that if $\nu \in \widehat{\mathbf{m}}$ is residually algebraically independent with respect to $\widehat{R}$ over R, then the intersection ring $\widehat{R} \cap \mathcal{Q}(R[\nu])$ is the localized polynomial ring $R[\nu]_{(\mathbf{m},\nu)}$. Therefore this intersection ring is Noetherian, but has dimension one more than the dimension of R and hence is not a subspace of $\widehat{R}$.

The existence of a residually algebraically independent element is important for completing the proof of the iterative example as we demonstrate in Theorem 7.9.

Setting 7.8. Let $R := \mathbb{Q}[x, y]_{(x,y)}$ be the localized polynomial ring in two variables x and y over the field $\mathbb{Q}$ of rational numbers. Then the completion of R with respect to the maximal ideal $\mathbf{m} := (x, y)R$ is $\widehat{R} := \mathbb{Q}[[x, y]]$, the formal power series ring in the variables x and y.

Theorem 7.9. *Let $\sigma \in x\mathbb{Q}[[x]]$ and $\tau \in y\mathbb{Q}[[y]]$ be such that the following two conditions are satisfied:*

(i) *σ is algebraically independent over $\mathbb{Q}(x)$ and τ is algebraically independent over $\mathbb{Q}(y)$.*

(ii) $\operatorname{trdeg}_{\mathbb{Q}} \mathbb{Q}(y, \tau, \{\frac{\partial^n \tau}{\partial y^n}\}_{n\in\mathbb{N}}) > r := \operatorname{trdeg}_{\mathbb{Q}} \mathbb{Q}(x, \sigma, \{\frac{\partial^n \sigma}{\partial x^n}\}_{n\in\mathbb{N}})$.

Following the notation of (7.1), let $C = k(x,\sigma) \cap k[[x]]$ *and let* $T = C[y]_{(x,y)}$. *Then* $\nu := \sigma + \tau$ *is residually algebraically independent over* T. *Hence* ν *is also residually algebraically independent over* $\mathbb{Q}[x,y]_{(x,y)}$.

Before proving Theorem 7.9, we establish the existence of elements σ and τ satisfying properties (i) and (ii) of Theorem 7.9. Let $\sigma = e^x - 1 \in \mathbb{Q}[[x]]$ and choose for τ a hypertranscendental element in $\mathbb{Q}[[y]]$. Recall that a power series $\tau = \sum_{i=0}^{\infty} b_i y^i \in \mathbb{Q}[[y]]$ is called *hypertranscendental* over $\mathbb{Q}[y]$ if the set of all partial derivatives $\{\frac{\partial^n \tau}{\partial y^n}\}_{n\in\mathbb{N}}$ is infinite and algebraically independent over $\mathbb{Q}(y)$. (Two examples of hypertranscendental elements are the Gamma function and the Riemann Zeta function.[1]) Thus there exist elements σ, τ that satisfy the conditions of Theorem 7.9. Another way to obtain such elements is to set $\sigma = e^x - 1$ and $\tau = e^{(e^y - 1)} - 1$. In this case, the conditions of Theorem 7.9 follow from [**9**].

In either case, Theorem 7.9 implies that $\sigma + \tau$ is residually algebraically independent, and, by Remark 7.7, we have the following corollary.

COROLLARY 7.10. *There exist elements* $\sigma \in \mathbb{Q}[[x]]$ *and* $\tau \in \mathbb{Q}[[y]]$ *such that the element* $\sigma + \tau \in (x,y)\mathbb{Q}[[x,y]]$ *is residually algebraically independent over* $\mathbb{Q}[x,y]_{(x,y)}$. *Therefore by Remark 7.7, the three-dimensional localized polynomial ring* $\mathbb{Q}[x,y,\sigma+\tau]_{(x,y,\sigma+\tau)}$ *is the intersection* $\mathbb{Q}(x,y,\sigma+\tau) \cap \widehat{R}$.

PROOF. We begin the proof of Theorem 7.9. The ring $T := C[y]_{(x,y)}$ is between R and its completion $\widehat{R}$ and has completion $\widehat{T} = \widehat{R}$:

$$R = \mathbb{Q}[x,y]_{(x,y)} \longrightarrow T = C[y]_{(x,y)} \longrightarrow \widehat{R} = \widehat{T} = \mathbb{Q}[[x,y]].$$

We display the relationships among these rings.

$$\begin{tikzcd}
& \widehat{R} = \mathbb{Q}[[x,y]] \arrow[d, dash] & \\
& T := C[y]_{(x,y)} \arrow[dl, dash] \arrow[dr, dash] & \\
\begin{array}{c} C := \mathbb{Q}(x,\sigma) \cap \mathbb{Q}[[x]] \\ = \cup\, \mathbb{Q}[x,\sigma_n]_{(x,\sigma_n)} \end{array} & & R := \mathbb{Q}[x,y]_{(x,y)}
\end{tikzcd}$$

The rings of the example

Let $\widehat{P}$ be a height-one prime ideal of $\widehat{R}$, let bars (for example, $\bar{x}$), denote images in $\overline{\widehat{R}} = \widehat{R}/\widehat{P}$ and set $P := \widehat{P} \cap R$ and $P_1 := \widehat{P} \cap T$. Assume that $P_1 \neq 0$. It is easy to see for $P_1 = xT$ or $P_1 = yT$ that the image $\bar{\sigma} + \bar{\tau}$ of $\sigma + \tau$ in $\overline{\widehat{R}} := \mathbb{Q}[[x,y]]/\widehat{P}$ is algebraically independent over $\overline{T} := T/P_1$. Thus we assume that $xy \notin \widehat{P}$.

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[1] The exponential function is, of course, far from being hypertranscendental.

In the following commutative diagram, we identify $\mathbb{Q}[[x]]$ with $\mathbb{Q}[[\bar{x}]]$ and $\mathbb{Q}[[y]]$ with $\mathbb{Q}[[\bar{y}]]$ etc.

$$\begin{tikzcd}
\mathbb{Q}[[y]] \arrow[r, "\psi_y"] & \overline{\widehat{R}} = \widehat{R}/\widehat{P} & \mathbb{Q}[[x]] \arrow[l, "\psi_x"'] \\
& \overline{T} = T/P_1 \arrow[u] & \\
\mathbb{Q}[y]_{(y)} \arrow[r, "\phi_y"] \arrow[uu] & \overline{R} = R/P \arrow[u] & \mathbb{Q}[x]_{(x)} \arrow[l, "\phi_x"'] \arrow[uu]
\end{tikzcd}$$

All morphisms in the diagram are injective and $\overline{\widehat{R}}$ is finite over both of the rings $\mathbb{Q}[[x]]$ and $\mathbb{Q}[[y]]$. Also $\overline{R}$ is algebraic over $\mathbb{Q}[\bar{x}]_{(\bar{x})}$, since $\operatorname{trdeg}_{\mathbb{Q}} \mathcal{Q}(\overline{R}) = 1$.

Let L denote the field of fractions of $\overline{\widehat{R}}$. We may consider $\mathbb{Q}(y, \tau, \{\frac{\partial^n \tau}{\partial y^n}\}_{n \in \mathbb{N}})$ and $\mathbb{Q}(x, \sigma, \{\frac{\partial^n \sigma}{\partial x^n}\}_{n \in \mathbb{N}})$ as subfields of L, where

$$\operatorname{trdeg}_{\mathbb{Q}}(\mathbb{Q}(y, \tau, \{\frac{\partial^n \tau}{\partial y^n}\}_{n \in \mathbb{N}}) > \operatorname{trdeg}_{\mathbb{Q}}(\mathbb{Q}(x, \sigma, \{\frac{\partial^n \sigma}{\partial x^n}\}_{n \in \mathbb{N}}).$$

Let d denote the partial derivative map $\frac{\partial}{\partial x}$ on $\mathbb{Q}((x))$. Since the extension L of $\mathbb{Q}((x))$ is finite and separable, d extends uniquely to a derivation $\widehat{d} : L \to L$. Let H denote the algebraic closure in L of the field $\mathbb{Q}(x, \sigma, \{\frac{\partial^n \sigma}{\partial x^n}\}_{n \in \mathbb{N}})$. Let $\widehat{p}(x, y) \in \mathbb{Q}[[x, y]]$ be a prime element generating $\widehat{P}$. Claim 7.11 asserts that the images of H and $\overline{\widehat{R}}$ under $\widehat{d}$ are inside H and $(1/p')\overline{\widehat{R}}$, respectively, as shown in Picture 7.11.1.

$$\begin{tikzcd}
& L := \mathcal{Q}(\overline{\widehat{R}}) \arrow[r, "\widehat{d}"'] & L & \\
& \overline{\widehat{R}} := \overline{\mathbb{Q}[[x, y]]} \arrow[r, "\widehat{d}"] \arrow[u] & \boxed{\frac{1}{p'(\bar{y})} \overline{\widehat{R}}} \arrow[u] & \\
& \mathbb{Q}[[\bar{x}]] \cong \mathbb{Q}[[x]] \arrow[r, "d := \frac{\partial}{\partial x}"] \arrow[u] & \mathbb{Q}[[\bar{x}]] \arrow[u] & \\
H \arrow[rrr, "\widehat{d}"] \arrow[uuur] & & & H := \left(\mathbb{Q}(x, \sigma, \{\frac{\partial^n \sigma}{\partial x^n}\}_{n=1}^{\infty})\right)^a \cap L \arrow[uuul] \\
& \mathbb{Q}[x, \sigma, \{\frac{\partial^n \sigma}{\partial x^n}\}_{n=1}^{\infty}] \arrow[r, "d"] \arrow[uu] \arrow[ul] & \mathbb{Q}[x, \sigma, \{\frac{\partial^n \sigma}{\partial x^n}\}_{n=1}^{\infty}] \arrow[uu] \arrow[ur] & \\
& \mathbb{Q} \arrow[r, "1_{\mathbb{Q}}"] \arrow[u] & \mathbb{Q} \arrow[u] &
\end{tikzcd}$$

Picture 7.11.1 The image of subrings of L via the extension $\widehat{d}$ of $d := \frac{\partial}{\partial x}$.

Claim 7.11. *With the notation above:* *(i)* $\widehat{d}(H) \subseteq H$.
(ii) There exists a polynomial $p(x, y) \in \mathbb{Q}[[x]][y]$ *with* $p\mathbb{Q}[[x, y]] = \widehat{P}$ *and* $p(\bar{y}) = 0$.
(iii) $\widehat{d}(\bar{y}) \neq 0$ *and* $p'(\bar{y})\widehat{d}(\bar{y}) \in \overline{\widehat{R}}$, *where* $p'(y) := \frac{\partial p(x,y)}{\partial y}$.

(iv) For every element $\lambda \in \overline{\widehat{R}}$*, we have* $p'(\bar{y})\widehat{d}(\lambda) \in \overline{\widehat{R}}$*, and so* $\widehat{d}(\overline{\widehat{R}}) \subseteq (1/p')\overline{\widehat{R}}$*.*

Proof. For item i, since d maps $\mathbb{Q}(x, \sigma, \{\frac{\partial^n \sigma}{\partial x^n}\}_{n\in\mathbb{N}})$ into itself, $\widehat{d}(H) \subseteq H$.

For item ii, since x and y are not contained in P, the element $\widehat{p}(x,y)$ is regular in y as a power series in $\mathbb{Q}[[x,y]]$ (in the sense of Zariski-Samuel [**117**, p.145]). Thus by [**117**, Corollary 1, p.145] the element $\widehat{p}(x,y)$ can be written as:

$$\widehat{p}(x,y) = \epsilon(x,y)(y^n + b_{n-1}(x)y^{n-1} + \ldots + b_0(x)),$$

where $\epsilon(x,y)$ is a unit of $\mathbb{Q}[[x,y]]$ and each $b_i(x) \in \mathbb{Q}[[x]]$. Hence $\widehat{P}$ is also generated by

$$p(x,y) = p(y) := \epsilon^{-1}\widehat{p} \; = \; y^n + c_{n-1}y^{n-1} + \cdots + c_0,$$

where the $c_i \in \mathbb{Q}[[x]]$. Since $p(y)$ is the minimal polynomial of $\bar{y}$ over the field $\mathbb{Q}((x))$, we have $0 = p(\bar{y}) := \bar{y}^n + c_{n-1}\bar{y}^{n-1} + \cdots + c_1\bar{y} + c_0$.

For item iii, we observe that

$$p'(y) = ny^{n-1} + c_{n-1}(n-1)y^{n-2} + \cdots + c_1,$$

and $p'(\bar{y}) \neq 0$ by minimality. Now

$$\begin{aligned}
0 = \widehat{d}(p(\bar{y})) &= \widehat{d}(\bar{y}^n + c_{n-1}\bar{y}^{n-1} + \cdots + c_1\bar{y} + c_0) \\
&= n\bar{y}^{n-1}\widehat{d}(\bar{y}) + c_{n-1}(n-1)\bar{y}^{n-2}\widehat{d}(\bar{y}) + d(c_{n-1})\bar{y}^{n-1} + \cdots \\
&\qquad\qquad + c_1\widehat{d}(\bar{y}) + d(c_1)\bar{y} + d(c_0) \\
&= \widehat{d}(\bar{y})(n\bar{y}^{n-1} + c_{n-1}(n-1)\bar{y}^{n-2} + \cdots + c_1) + \\
&\qquad\qquad + d(c_{n-1})\bar{y}^{n-1} + \cdots + d(c_1)\bar{y} + d(c_0) \\
&= \widehat{d}(\bar{y})(p'(\bar{y})) + \sum_{i=0}^{n-1} d(c_i)\bar{y}^i
\end{aligned}$$

$$\Longrightarrow \; \widehat{d}(\bar{y})(p'(\bar{y})) = -\big(\sum_{i=0}^{n-1} d(c_i)\bar{y}^i\big) \quad \text{and} \quad \widehat{d}(\bar{y}) = \frac{-1}{p'(\bar{y})}\sum_{i=0}^{n-1} d(c_i)\bar{y}^i.$$

In particular, $p'(\bar{y})\widehat{d}(\bar{y}) \in \overline{\widehat{R}}$ and $\widehat{d}(\bar{y}) \neq 0$, as desired for item iii.

For item iv, we observe that every element $\lambda \in \overline{\widehat{R}}$ has the form:

$$\lambda = e_{n-1}(x)\bar{y}^{n-1} + \cdots + e_1(x)\bar{y} + e_0(x), \text{ where } e_i \in \mathbb{Q}[[x]].$$

Therefore:

$$\widehat{d}(\lambda) = \widehat{d}(\bar{y})\big[(n-1)e_{n-1}(x)\bar{y}^{n-2} + \cdots + e_1(x)\big] + \sum_{i=0}^{n-1} \widehat{d}(e_i(x))\bar{y}^i.$$

The sum expression on the right is in $\overline{\widehat{R}}$ and, as established above, $p'(\bar{y})\widehat{d}(\bar{y}) \in \overline{\widehat{R}}$, and so $p'(\bar{y})\widehat{d}(\lambda) \in \overline{\widehat{R}}$. $\square$

The next claim asserts an expression for $\widehat{d}(\bar{\tau})$ in terms of the partial derivative $\frac{\partial \bar{\tau}}{\partial \bar{y}}$ of $\bar{\tau}$ with respect to $\bar{y}$.

Claim 7.12. $\widehat{d}(\tau) = \widehat{d}(\bar{y})\frac{\partial \bar{\tau}}{\partial \bar{y}}$.

Proof. For every $m \in \mathbb{N}$, we have $\tau = \sum_{i=0}^{m} b_i y^i + y^{m+1}\tau_m$, where the $b_i \in \mathbb{Q}$ and $\tau_m \in \mathbb{Q}[[y]]$ is defined as in (6.1.1). Therefore

$$\widehat{d}(\bar{\tau}) = \widehat{d}(\bar{y}) \cdot \big(\sum_{i=1}^{m} ib_i\bar{y}^{i-1}\big) + \widehat{d}(\bar{y})(m+1)\bar{y}^m\bar{\tau}_m + \bar{y}^{m+1}\widehat{d}(\bar{\tau}_m).$$

Thus

$$p'(\bar{y})\widehat{d}(\bar{\tau}) = p'(\bar{y})\widehat{d}(\bar{y}) \cdot \sum_{i=1}^{m} ib_i\bar{y}^{i-1} + \bar{y}^m(p'(\bar{y})\widehat{d}(\bar{y})(m+1)\bar{\tau}_m + \bar{y}p'(\bar{y})\widehat{d}(\bar{\tau}_m)).$$

Since $\bar{\tau} = \sum_{i=0}^{\infty} b_i\bar{y}^i$ with $b_i \in \mathbb{Q}$, we have

$$\frac{\partial\bar{\tau}}{\partial\bar{y}} = \sum_{i=1}^{m} ib_i\bar{y}^{i-1} + \bar{y}^m \sum_{i=m+1}^{\infty} ib_i\bar{y}^{i-m-1}.$$

Thus $\frac{\partial\bar{\tau}}{\partial\bar{y}} - \sum_{i=1}^{m} ib_i\bar{y}^{i-1} \in \bar{y}^m\overline{\widehat{R}}$ and hence

$$p'(\bar{y})\widehat{d}(\bar{\tau}) - p'(\bar{y})\widehat{d}(\bar{y})\frac{\partial\bar{\tau}}{\partial\bar{y}} \in \bar{y}^m(\overline{\widehat{R}})$$

for every $m \in \mathbb{N}$. Since $p'(\bar{y}) \neq 0$ and $\overline{\widehat{R}}$ is an integral domain, $\widehat{d}(\tau) = \widehat{d}(\bar{y})\frac{\partial\bar{\tau}}{\partial\bar{y}}$. ☐

Completion of proof of Theorem 7.9. Recall that $\overline{R}$ is algebraic over $\mathbb{Q}[x]_{(x)}$, and so $\bar{y}$ is algebraic over $\mathbb{Q}[x]_{(x)}$. Also $T := C[y]_{(x,y)}$, where $C := \mathbb{Q}(x,\sigma) \cap Q[[x]]$. Thus $\overline{T} \subseteq H$, and so H is the algebraic closure of the field $\mathcal{Q}(\overline{T}, \{\frac{\partial^n\bar{\sigma}}{\partial\bar{x}^n}\}_{n\in\mathbb{N}})$ in L. Therefore $\bar{\tau} \notin H$ if and only if $\bar{\tau}$, or equivalently $\bar{\nu} = \bar{\sigma} + \bar{\tau}$, is transcendental over H. By hypothesis, the transcendence degree of $H/\mathbb{Q}$ is r. Since $\widehat{d}(H) \subseteq H$, if $\bar{\tau}$ were in H, then $\frac{\partial^n\bar{\tau}}{\partial\bar{y}^n} \in H$ for all $n \in \mathbb{N}$. This implies that the field $\mathbb{Q}(y,\tau,\{\frac{\partial^n\tau}{\partial y^n}\}_{n\in\mathbb{N}})$ is contained in H. This contradicts our hypothesis that $\operatorname{trdeg}_{\mathbb{Q}} \mathbb{Q}(y,\tau,\{\frac{\partial^n\tau}{\partial y^n}\}_{n\in\mathbb{N}}) > r$. Therefore ν is residually algebraically independent over T. This completes the proof of Theorem 7.9. ☐

Example 7.13. Let $\sigma \in \mathbb{Q}[[x]]$ and $\tau \in \mathbb{Q}[[y]]$ satisfy the conditions of Theorem 7.9. Then with the setting of Notation 7.1 we have $B = A$. In particular, if $\sigma = e^x - 1$ and $\tau = e^{(e^y-1)} - 1$, then $B = A$.

Proof. By Theorem 7.2, it suffices to prove that B is Noetherian. Recall that $T := C[y]_{(x,y)}$, where $C := \mathbb{Q}(x,\sigma) \cap Q[[x]]$, a DVR by Notation and Remarks 7.1. In order to show B is Noetherian we show that the morphism $\phi_y : T[\tau] \longrightarrow C[[y]][1/y]$ is flat; see Definition 2.19. Since flatness is a local property by Remark 2.20.1, it suffices to show for each prime ideal P of $C[[y]]$ with $y \notin P$ that the induced map $\phi_P : T[\tau]_{P\cap T[\tau]} \longrightarrow C[[y]]_P$ is flat. If $\operatorname{ht}(P \cap T[\tau]) \leq 1$, then $T[\tau]_{P\cap T[\tau]}$ is a field or a DVR and ϕ_P is flat by (2.22.3) since $C[[y]]_P$ is torsionfree over $T[\tau]_{P\cap T[\tau]}$. If $\operatorname{ht}(P \cap T[\tau])$ were greater than one, then $P \cap T \neq (0)$. But then Theorem 7.9 implies that the image of $\nu = \sigma + \tau$ in $\widehat{R}/P\widehat{R}$ is transcendental over $T/(P \cap T)$, and this implies that $\operatorname{ht}(P \cap T[\tau]) \leq 1$. Thus we must have $\operatorname{ht}(P \cap T[\tau]) \leq 1$. Therefore $\phi_y : T[\tau] \longrightarrow C[[y]][1/y]$ is flat. We prove in the Noetherian Flatness Theorem 8.8 of Chapter 8 that this flatness implies B is Noetherian and hence $A = B$. ☐

Exercises

(1) Let x and y be indeterminates over a field k and let R be the two-dimensional RLR obtained by localizing the mixed power series-polynomial ring $k[[x]][y]$ at the maximal ideal $(x,y)k[[x]][y]$.

  (i) For each height-one prime ideal P of R different from xR, prove that R/P is a one-dimensional complete local domain.

  (ii) For each nonzero prime ideal Q of $\widehat{R} = k[[x,y]]$ prove that $Q \cap R \neq (0)$. Conclude that the generic formal fiber of R is zero-dimensional.

Suggestion. For part (ii), use Theorem 3.8. For more information about the dimension of the formal fibers, see [**72**] and [**94**].

(2) (Kunz) Let L/k be a field extension with L having infinite transcendence degree over k. Prove that the ring $L \otimes_k L$ is not Noetherian. Deduce that the ring $k[[x]] \otimes_k k[[x]]$, which has $k((x)) \otimes_k k((x))$ as a localization is not Noetherian.

Suggestion. Let $\{x_\lambda\}_{\lambda \in \Lambda}$ be a transcendence basis for L/k and consider the subfield $F = k(\{x_\lambda\})$ of L. The ring $L \otimes_k L$ is faithfully flat over its subring $F \otimes_k F$, and if $F \otimes_k F$ is not Noetherian, then $L \otimes_k L$ is not Noetherian. Hence it suffices to show that $F \otimes_k F$ is not Noetherian. The module of differentials $\Omega^1_{F/k}$ is known to be infinite dimensional as a vector space over F [**65**, 5.4], and $\Omega^1_{F/k} \cong I/I^2$, where I is the kernel of the map $F \otimes_k F \to F$, defined by sending $a \otimes b \mapsto ab$. Thus the ideal I of $F \otimes_k F$ is not finitely generated.

CHAPTER 8

Building Noetherian domains

In this chapter we establish that flatness of a certain map implies that the integral domain obtained via Constructions 5.3 or 5.4 is Noetherian. We use this result as formulated in Noetherian Flatness Theorem 8.8 (Inclusion Version) to complete the proofs of Theorems 4.12 and 7.2 concerning the iterative examples. More generally, we prove the following theorem.

THEOREM 8.1. *Let R be a Noetherian integral domain with field of fractions K. Let z be a nonzero nonunit of R and let R^* denote the (z)-adic completion of R. Let I be an ideal of R^* having the property that $\mathbf{p} \cap R = (0)$ for each $\mathbf{p} \in \operatorname{Ass}(R^*/I)$. The inclusion map $R \hookrightarrow (R^*/I)[1/z]$ is flat if and only if $A := K \cap (R^*/I)$ is both Noetherian and realizable as a localization of a subring of $R[1/z]$.*

In the statement of Theorem 8.1, the ring $(R^*/I)[1/z]$ is the localization of R^*/I at the multiplicative system generated by z, and the intersection $K \cap (R^*/I)$ is taken inside the total quotient ring $\mathcal{Q}(R^*/I)$ of R^*/I. As discussed in Note 5.5, the hypotheses of Theorem 8.1 imply that the field K embeds in $\mathcal{Q}(R^*/I)$.

Theorem 8.1 is is stated in terms of the Homomorphic Image Construction 5.4. There are two Noetherian Flatness Theorems given in this chapter, one for each of the Constructions 5.3 and 5.4. [1] In Section 8.2, we state in Noetherian FlatnessTheorem 8.8 the corresponding version of this result for the Inclusion Construction 5.3. Thus the intersection domain A constructed using the Inclusion Construction 5.3 is both Noetherian and computable as a localization of an infinite directed union $U = \cup_{n=0}^{\infty} U_n$ of polynomial extension rings of R if and only if the map $U_0 \hookrightarrow R^*[1/z]$ is flat. The polynomial rings U_n are defined in Section 6.1 of Chapter 6.

Theorem 8.1 is implied by Noetherian Flatness Theorem 8.3 (Homomorphic Image Version) that is proved in Section 8.1. We make use of the approximation ring B of A defined in Section 6.2. The ring B is an infinite nested union of "computable rings" — while not localized polynomial rings over R, they are localizations of finitely generated birational extensions of R. In Section 8.1 we also prove a crucial lemma relating flatness and the Noetherian property.

8.1. Flatness and the Noetherian property

We use Lemma 8.2 in order to prove Noetherian Flatness Theorem 8.3 (Homomorphic Image Version). This lemma is crucial for our proof of Theorem 8.1. We thank Roger Wiegand for observing Lemma 8.2 and its proof. For an introduction to flatness see Section 2.3 of Chapter 2.

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[1] When this duplication seems confusing we distinguish between the versions by labeling the one for Construction 5.3 as the "Inclusion Version" or just inserting the word "inclusion", and by labeling the theorem for Construction 5.4 as the "Homomorphic Image Version" or just inserting the word "image".

LEMMA 8.2. *Let S be a subring of a ring T and let $z \in S$ be a regular element of T. Assume that the equivalent conditions of Lemma 6.16 are satisfied, that is $zS = zT \cap S$ and $S/zS = T/zT$. Then*

1. *$T[1/z]$ is flat over $S \iff T$ is flat over S.*
2. *If T is flat over S, then $D := (1+zS)^{-1}T$ is faithfully flat over $C := (1+zS)^{-1}S$.*
3. *If T is Noetherian and T is flat over S, then $C = (1+zS)^{-1}S$ is Noetherian.*
4. *If T and $S[1/z]$ are both Noetherian and T is flat over S, then S is Noetherian.*

PROOF. For item 1, if T flat over S, then by transitivity of flatness, Remark 2.20.13, the ring $T[1/z]$ is flat over S. For the converse, consider the exact sequence (using (6.16.3))

$$0 \to S = S[1/z] \cap T \xrightarrow{\alpha} S[1/z] \oplus T \xrightarrow{\beta} T[1/z] = S[1/z] + T \to 0,$$

where $\alpha(b) = (b, -b)$ for all $b \in S$ and $\beta(c,d) = c + d$ for all $c \in S[1/z]$, $d \in T$. Since the two end terms are flat S-modules, the middle term $S[1/z] \oplus T$ is also S-flat. Therefore the direct summand T is S-flat.

For item 2, since $S \to T$ is flat, the embedding $C = (1+zS)^{-1}S \hookrightarrow (1+zS)^{-1}T = D$ is flat. Since zC is in the Jacobson radical of C and $C/zC = S/zS = T/zT = D/zD$, each maximal ideal of C is contained in a maximal ideal of D, and so D is faithfully flat over C. This establishes item 2. If also T is Noetherian, then D is Noetherian, and, since D is faithfully flat over C, it follows that C is Noetherian by Remark 2.20.8, and thus item 3 holds.

For item 4, let J be an ideal of S. Since C is Noetherian by item 3 and $S[1/z]$ is Noetherian by hypothesis, there exists a finitely generated ideal $J_0 \subseteq J$ such that $J_0S[1/z] = JS[1/z]$ and $J_0C = JC$. To show $J_0 = J$, it suffices to show for each maximal ideal $\mathbf{m}$ of S that $J_0S_{\mathbf{m}} = JS_{\mathbf{m}}$. If $z \notin \mathbf{m}$, then $S_{\mathbf{m}}$ is a localization of $S[1/z]$, and so $J_0S_{\mathbf{m}} = JS_{\mathbf{m}}$, while if $z \in \mathbf{m}$, then $S_{\mathbf{m}}$ is a localization of C, and so $JCS_{\mathbf{m}} = J_0S_{\mathbf{m}}$. Therefore $J = J_0$ is finitely generated. It follows that S is Noetherian. ☐

Noetherian Flatness Theorem 8.3 (Homomorphic Image Version) gives precise conditions for the approximation ring B of the Homomorphic Image Construction 5.4 to be Noetherian.

NOETHERIAN FLATNESS THEOREM 8.3. (Homomorphic Image Version) *Let R be a Noetherian integral domain with field of fractions K. Let z be a nonzero nonunit of R and let R^* denote the (z)-adic completion of R. Let I be an ideal of R^* having the property that $\mathbf{p} \cap R = (0)$ for each $\mathbf{p} \in \operatorname{Ass}(R^*/I)$. As in (6.9.2), let*

$$U := \cup_{n=1}^{\infty} U_n, \quad B := \cup_{n=1}^{\infty} B_n = (1+zU)^{-1}U, \quad \text{and} \quad A := K \cap (R^*/I).$$

The following statements are equivalent:

1. *The extension $R \hookrightarrow (R^*/I)[1/z]$ is flat.*
2. *The ring B is Noetherian.*
3. *The extension $B \hookrightarrow R^*/I$ is faithfully flat.*
4. *The ring $A := K \cap (R^*/I)$ is Noetherian and $A = B$.*
5. *The ring U is Noetherian*
6. *The ring A is both Noetherian and a localization of a subring of $R[1/z]$.*

Proof. For (1) $\implies$ (2), if $R \hookrightarrow (R^*/I)[1/z]$ is flat, by factoring through $U[1/z] = R[1/z] \hookrightarrow (R^*/I)[1/z]$, we see that $U \hookrightarrow (R^*/I)[1/z]$ and $B \hookrightarrow (R^*/I)[1/z]$ are flat. By Lemma 8.2.2, where we let $S = U$ and $T = R^*/I$, the ring B is Noetherian.

For (2) $\implies$ (3), $B^* = R^*/I$ is flat over B, by Theorem 6.17.4 and Remark 3.7.3. By Proposition 6.20.1, $z \in \mathcal{J}(B)$, and so, using Remark 3.7.4, we have $B^* = R^*/I$ is faithfully flat over B.

For (3) $\implies$ (4), again Theorem 6.17.4 yields $B^* = R^*/I$, and so B^* is faithfully flat over B. Then

$$B = \mathcal{Q}(B) \cap R^* = \mathcal{Q}(A) \cap R^* = K(\underline{\tau}) \cap R^* = A$$

by Remark 2.20.9 and Theorem 6.19.2. By Remark 2.20.8, A is Noetherian.

For (4) $\implies$ (5), the composite embedding

$$U \hookrightarrow B = A \hookrightarrow B^* = A^* = R^*/I$$

is flat because B is a localization of U and A is Noetherian; see Remark 3.7.3. By Remark 3.7.4 again, A^* is faithfully flat over A. Thus by Lemma 8.2, parts 1 and 3, where again we let $S = U$ and $T = R^*/I$, we have $S[1/z] = U[1/z] = R[1/z]$ is Noetherian, and hence U is Noetherian by Lemma 8.2.4.

If U is Noetherian, then the localization B of U is Noetherian, and as above $B = A$. Hence A is a localization of U, a subring of $R[1/z]$. Thus (5) $\implies$ (6).

For (6) $\implies$ (1): since A is a localization of a subring D of $R[1/z]$, we have $A := \Gamma^{-1}D$, where Γ is a multiplicatively closed subset of D. Now

$$R \subseteq A = \Gamma^{-1}D \subseteq \Gamma^{-1}R[1/z] = \Gamma^{-1}A[1/z] = A[1/z],$$

and so $A[1/z]$ is a localization of R. Since A is Noetherian, $A \hookrightarrow A^* = R^*/I$ is flat by Remark 3.7.2. Thus $A[1/z] \hookrightarrow (R^*/I)[1/z]$ is flat. It follows that $R \hookrightarrow (R^*/I)[1/z]$ is flat. ☐

Corollary 8.4. *Let R, I and z be as in Noetherian Flatness Theorem 8.3 (Homomorphic Image version). If $\dim(R^*/I) = 1$, then $\varphi : R \hookrightarrow W := (R^*/I)[1/z]$ is flat and therefore the equivalent conditions of Theorem 8.3 all hold.*

Proof. We have z is in the Jacobson radical of R^*/I by Construction Properties Theorem 6.17.2. Thus $\dim(R^*/I) = 1$ implies that $\dim W = 0$. The hypothesis on the ideal I implies that every prime ideal P of W contracts to (0) in R. Hence

$$\varphi_P : R_{P\cap R} = R_{(0)} = K \hookrightarrow W_P.$$

Thus W_P is a K-module and so a vector space over K. By Remark 2.20.2, φ_P is flat. Since flatness is a local property by Remark 2.201, the map φ is flat. ☐

Remark 8.5. With R, I and z as in Noetherian Flatness Theorem 8.3:

(1) There are examples where A is Noetherian and yet $B \neq A$. Thus B in such examples is non-Noetherian (see Theorem 7.2 and Example 8.11 below).
(2) There are examples where $A = B$ is non-Noetherian (see Theorem 24.4 and Remark 8.10 below).
(3) A necessary and sufficient condition that $A = B$ is that A is a localization of $R[1/z] \cap A$. Indeed, Theorem 6.17.5 implies that $R[1/z] \cap A = U$ and, by Definition 6.12.1, $B = (1 + zU)^{-1}U$. Therefore the condition is sufficient. On the other hand, if $A = \Gamma^{-1}U$, where Γ is a multiplicatively closed

subset of U, then by Remark 6.18.3, each $y \in \Gamma$ is a unit of B, and so $\Gamma^{-1}U \subseteq B$ and $A = B$.

(4) We discuss in Chapter 9 a family of examples where $R \hookrightarrow (R^*/I)[1/z]$ is flat.

Theorem 8.6 extends the range of applications of Noetherian Flatness Theorem 8.3.

THEOREM 8.6. *Let R be a Noetherian integral domain with field of fractions K. Let z be a nonzero nonunit of R and let R^* denote the (z)-adic completion of R. Let I be an ideal of R^* having the property that $\mathbf{p} \cap R = (0)$ for each $\mathbf{p} \in \operatorname{Ass}(R^*/I)$. Assume that I is generated by a regular sequence of R^*. If $R \hookrightarrow (R^*/I)[1/z]$ is flat, then for each $n \in \mathbb{N}$ we have*

(1) $\operatorname{Ass}(R^*/I^n) = \operatorname{Ass}(R^*/I)$,
(2) *R canonically embeds in R^*/I^n, and*
(3) *$R \hookrightarrow (R^*/I^n)[1/z]$ is flat.*

PROOF. Let $I = (\sigma_1, \dots, \sigma_r)R^*$ where $\sigma_1, \dots, \sigma_r$ is a regular sequence in R^*. Then the sequence $\sigma_1, \dots, \sigma_r$ is quasi-regular in the sense of [**73**, Theorem 16.2, page 125]. It follows that I^n/I^{n+1} is a free R^*/I-module for each positive integer n. Consider the exact sequence

$$0 \to I^n/I^{n+1} \hookrightarrow R^*/I^{n+1} \to R^*/I^n \to 0. \tag{8.6.0}$$

Proceeding by induction, assume $\operatorname{Ass}(R^*/I^n) = \operatorname{Ass}(R^*/I)$. Then

$$\operatorname{Ass}(R^*/I^{n+1}) \subseteq \operatorname{Ass}(I^n/I^{n+1}) \cup \operatorname{Ass}(R^*/I^n).$$

Since I^n/I^{n+1} is a free (R^*/I)-module, it follows that $\operatorname{Ass}(R^*/I^{n+1}) = \operatorname{Ass}(R^*/I)$. Thus R canonically embeds in R^*/I^n for each $n \in \mathbb{N}$.

That $R \hookrightarrow (R^*/I^n)[1/z]$ is flat for every $n \in \mathbb{N}$ now follows by induction on n and considering the exact sequence obtained by tensoring over R the short exact sequence (8.6.0) with $R[1/z]$. $\square$

EXAMPLE 8.7. Let $R = k[x, y]$ be the polynomial ring in the variables x, y over a field k and let $R^* = k[y][[x]]$ be the (x)-adic completion of R. Fix an element $\tau \in xk[[x]]$ such that x and τ are algebraically independent over k, and define the $k[[x]]$-algebra homomorphism $\phi : k[y][[x]] \to k[[x]]$, by setting $\phi(y) = \tau$. Then $\ker(\phi) = (y - \tau)R^* =: I$. Notice that $\phi(R) = k[x, \tau] \cong R$ because x and τ are algebraically independent over k. Hence $I \cap R = (0)$. Also I is a prime ideal generated by a regular element of R^*, and $(I, x)R^* = (y, x)R^*$ is a maximal ideal of R^*. Corollary 8.4 and Theorem 8.6 imply that for each positive integer n, the intersection ring $A_n := (R^*/I^n) \cap k(x, y)$ is a one-dimensional Noetherian local domain having (x)-adic completion R^*/I^n. Since x generates an ideal primary for the unique maximal ideal of R^*/I^n, the ring R^*/I^n is also the completion of A_n with respect to the powers of the unique maximal ideal $\mathbf{n}_n$ of A_n. The ring A_1 is a DVR since R^*/I is a DVR by Remark 2.1. For $n > 1$, the completion of A_n has nonzero nilpotent elements and hence the integral closure of A_n is not a finitely generated A_n-module, Remarks 3.11. The inclusion $I^{n+} \subsetneq I^n$ and the fact that A_n has completion R^*/I^n imply that $A_{n+1} \subsetneq A_n$ for each $n \in \mathbb{N}$. Hence the rings A_n form a strictly descending chain

$$A_1 \supset A_2 \supset \cdots \supset A_n \supset \cdots$$

of one-dimensional local birational extensions of $R = k[x, y]$.

8.2. The Inclusion Version

As remarked in (5.7.2), the Inclusion Construction 5.4 is a special case of the Homomorphic Images Construction. Thus we obtain the following result analogous to Noetherian Flatness Theorem 8.3 (Homomorphic Image Version) for Inclusion Constructions.

Noetherian Flatness Theorem 8.8. (Inclusion Version) *Let R be a Noetherian integral domain with field of fractions K. Let z be a nonzero nonunit of R and let R^* denote the (z)-adic completion of R. Let $\tau_1, \dots, \tau_s \in zR^*$ be algebraically independent elements over K such that the field $K(\tau_1, \dots, \tau_s)$ is a subring of the total quotient ring of R^*. As in Equations 6.1.4, 6.1.5 and 6.1.6 of Notation 6.1, we define* $U_n := R[\tau_{1n}, \dots, \tau_{sn}]$, $B_n = (1 + zU_n)^{-1}U_n$,

$A := K(\tau_1, \dots, \tau_s) \cap R^*$, $U := \cup_{n=1}^{\infty} U_n$, *and* $B := \cup_{n=1}^{\infty} B_n = (1+zU)^{-1}U$.

The following statements are equivalent:

1. *The extension $U_0 := R[\tau_1, \dots \tau_s] \hookrightarrow R^*[1/z]$ is flat.*
2. *The ring B is Noetherian.*
3. *The extension $B \hookrightarrow R^*$ is faithfully flat.*
4. *The ring A is Noetherian and $A = B$.*
5. *The ring U is Noetherian*
6. *The ring A is Noetherian and is a localization of a subring of $U_0[1/z]$.*
7. *The ring A is Noetherian and is a localization of a subring of $U[1/z]$.*

Proof. For the proof of Theorem 8.8, use the identifications in (6.13) and (6.14); then apply Theorem 8.3. □

Remarks 8.9. (1) Theorem 8.8 completes the proofs of the iterative examples Theorems 4.12 and 7.2; see Example 7.13.

(2) The original proof given for Theorem 8.8 in [**43**] is an adaptation of a proof given by Heitmann in [**58**, page 126]. Heitmann considers the case where there is one transcendental element τ and defines the corresponding extension U to be a *simple PS-extension of R for z.* Heitmann proves in this case that a certain monomorphism condition on a sequence of maps is equivalent to U being Noetherian [**58**, Theorem 1.4].

In Chapters 9, 10, 11, 13, and 23, for an element z in the Jacobson radical of R, we study the condition in Noetherian Flatness Theorem 8.8 that the embedding $U_0 \to R^*[1/z]$ is flat. In some of these chapters we show that flatness of a polynomial extension implies flatness of $U_0 \to R^*[1/z]$ and that sometimes non-flatness of this polynomial extension implies non-flatness of $U_0 \to R^*[1/z]$.

Remark 8.10. Examples where $A = B$ is not Noetherian show that it is possible for A to be a localization of U and yet for A, and therefore also U, to fail to be Noetherian. Thus the equivalent conditions of Noetherian Flatness Theorem 8.8 are not implied by the property that A is a localization of U.

Example 8.11. Recall that in Example 7.3 we may take $k = \mathbb{Q}$ and $\sigma := e^x - 1$ and $\tau := e^y - 1$. We have $\theta := \frac{\sigma - \tau}{x - y}$ is in $A := \mathbb{Q}[[x, y]] \cap \mathbb{Q}(x, y, \sigma, \tau)$ and not in $B := \bigcup_{n \in \mathbb{N}} \mathbb{Q}[x, y, \sigma_n, \tau_n]_{(x,y,\sigma_n,\tau_n)}$, where σ_n and τ_n are the endpieces defined in (6.1.1). Then A is Noetherian and $B \subsetneq A$. Set $C := \bigcup_{n \in \mathbb{N}} \mathbb{Q}[x, y, \sigma_n]_{(x,y,\sigma_n)}$. Then

$C = \mathbb{Q}[y]_{(y)}[[x]] \cap \mathbb{Q}(x, y, \sigma)$ is an excellent two-dimensional regular local domain; see Theorem 9.2 and Corollary 9.7 of Chapter 9. Inside the (y)-adic completion of C, we define $U := \bigcup_{n \in \mathbb{N}} C[\tau_n]$ as in (6.1.5). The ring B is the localization of U at the multiplicative system $1 + yU$, and the rings B and U are not Noetherian. It follows that A is not a localization of U by Theorem 8.8.

Proof. By Claim 7.4, $\theta \notin B$. If U were Noetherian, then B would be Noetherian. But the maximal ideal of B is $(x, y)B$, so if B were Noetherian, then it would be a regular local domain with completion $\mathbb{Q}[[x, y]]$. Since the completion of a local Noetherian ring is a faithfully flat extension by Remark 3.7.3, and since the field of fractions of B is $\mathbb{Q}(x, y, \sigma, \tau)$, then B would equal A by Remark 2.20.9.

That A is Noetherian follows from Valabrega's Theorem 4.2. If A were a localization of U, then A would be a localization of B. But each of A and B has a unique maximal ideal and the maximal ideal of A contains the maximal ideal of B. Therefore $B \subsetneq A$ implies that A is not a localization of B. □

The following diagram displays the situation concerning possible implications among certain statements for the Inclusion Construction 5.3 and the approximations in Section 6.1:

$$\begin{tikzcd}
\boxed{R^*[1/a] \text{ is flat over } U_0 = R[\underline{\tau}]} \arrow[r, Leftrightarrow] \arrow[d, shift left=2, Rightarrow] & \boxed{U \text{ Noetherian}} \arrow[d, shift left=2, Rightarrow] \\
\boxed{A \text{ is a localization of } U} \arrow[u, shift left=2, Rightarrow, "/" marking] \arrow[r, shift left, Rightarrow, "/" marking] & \boxed{A \text{ Noetherian}} \arrow[u, shift left=2, Rightarrow, "/" marking] \arrow[l, shift left, Rightarrow, "/" marking]
\end{tikzcd}$$

Remark 8.12. In connection with the flatness property, if $U_0 := R[\underline{\tau}] \hookrightarrow R^*[1/z]$ is flat, then for each $P \in \operatorname{Spec} R^*[1/z]$ one has that $\operatorname{ht} P \geq \operatorname{ht}(P \cap U_0)$. We show in Chapter 11 that conversely this height inequality in certain contexts implies flatness.

Exercise

(1) For the strictly descending chain of one-dimensional local domains

$$A_1 \supset A_2 \supset \cdots \supset A_n \supset \cdots$$

that are birational extensions of $R = k[x, y]$ given in Example 8.7, describe the integral domain $D := \bigcap_{n=1}^{\infty} A_n$.

Suggestion: Since $\mathbf{n}_n \cap R = (x, y)R$, we have $R_{(x,y)R} \subset A_n$ for each $n \in \mathbb{N}$. By Exercise 4 of Chapter 5, the ring A_n may be described as

$$A_n = \{\ a/b \mid a, b \in R,\ b \neq 0 \ \text{ and } a \in I^n + bR^*\ \}.$$

Show that $a \in I^n + bR^*$ for all $n \in \mathbb{N}$ if and only if $a/b \in R_{(x,y)R}$.

CHAPTER 9

Examples where flatness holds

We continue the notation of the preceding chapters: R is a Noetherian integral domain with field of fractions K, z is a nonzero nonunit of R, and R^* is the (z)-adic completion of R. We consider R^* as a power series ring in z over R in the sense of Remark 3.2. Assume that I is an ideal of R^* such that $\mathbf{p} \cap R = (0)$ for each $\mathbf{p} \in \operatorname{Ass}(R^*/I)$.

In Sections 9.1 and 9.2 we present several examples of Homomorphic Image Construction 5.4 where the flatness condition of Noetherian Flatness Theorem 8.3 holds; that is, the map $R \hookrightarrow (R^*/I)[1/z]$ is flat. We also describe several of these examples using Inclusion Construction 5.3. Inclusion Construction 5.3 is in certain ways more transparent.

In Section 9.3 we investigate special properties of the intersection domain $A := K \cap (R^*/I)$ from Homomorphic Image Construction 5.4, such as having geometrically regular formal fibers. We present in Example 9.11 an example of a Noetherian local domain that is not universally catenary, but has geometrically regular formal fibers. Homomorphic Image Construction 5.4 permits the construction of such examples, whereas Inclusion Construction 5.3 does not.

9.1. Polynomial rings over special DVRs

In view of Noetherian Flatness Theorem 8.3, it is natural to ask about the existence of ideals I of the (z)-adic completion R^* such that $R \hookrightarrow (R^*/I)[1/z]$ is flat. In the case where R is a polynomial ring over a field and R^* is the completion with respect to one of the variables, Polynomial Example Theorem 9.5 presents explicit ideals I of R^* such that $R \hookrightarrow (R^*/I)[1/z]$ is flat. The intersection domains A obtained with Homomorphic Image Construction 5.4 using these ideals are equal to their approximation domains B. Moreover the rings A are polynomial rings over "special" DVRs—that is, of the form given in Example 6.6 and called A there.

First we prove Polynomial Example Theorem 9.2 for Inclusion Construction 5.3; this version is useful for many of our examples and it is vital to Insider Construction 13.1 discussed in Chapters 10 and13. The Insider Construction simplifies the verification of examples. The isomorphisms obtained from Notation 6.13 and Proposition 6.14 yield the corresponding result for Homomorphic Image Construction 5.4. We give localized versions of both Polynomial Example Theorems 9.5 and 9.2 in Localized Polynomial Example Theorem 9.7.

We present the setting used for the versions of the first two Polynomial Example Theorems together. For convenience we also include the definitions of the intersection and approximation domains corresponding to the two constructions from Sections 5.1, 6.1 and 6.2.

Setting and Notation 9.1. Let x be an indeterminate over a field k. Let r be a nonnegative integer and s a positive integer. Assume that $\tau_1, \dots, \tau_s \in xk[[x]]$ are algebraically independent over $k(x)$ and let $y_1, \dots, y_r$ and $t_1, \dots, t_s$ be additional indeterminates. We define the following rings:

$$(9.1.a) \quad R := k[x, y_1, \dots, y_r], \;\; R^* = k[y_1, \dots, y_r][[x]], \;\; V = k(x, \tau_1, \dots, \tau_s) \cap k[[x]].$$

Notice that R^* is the (x)-adic completion of R and V is a DVR by Remark 2.1.

We use the base ring R to define as in Construction 5.3 and Section 6.1

$$(9.1_{\text{incl}}) \qquad A_{\text{incl}} := k(x, y_1, \dots, y_r, \tau_1, \dots \tau_s) \cap R^*, \;\; B_{\text{incl}} := (1 + xU_{\text{incl}})^{-1} U_{\text{incl}},$$

where $U_{\text{incl}} := \bigcup_{n \in \mathbb{N}} R[\tau_{1n}, \dots \tau_{sn}]$, each τ_{in} is the n^{th} endpiece of τ_i and each $\tau_{in} \in R^*$, for $1 \le i \le s$. By Construction Properties Theorem 6.19.4, the ring R^* is the (x)-adic completion of each of the rings $A_{\text{incl}}, B_{\text{incl}}$ and U_{incl}.

Set $S := k[x, y_1, \dots, y_r, t_1, \dots, t_s]$, let S^* be the (x)-adic completion of S and let $\sigma_i := t_i - \tau_i$, for each i. We define $I := (t_1 - \tau_1, \dots, t_s - \tau_s)S^* = (\sigma_1, \dots, \sigma_s)S^*$.

With S as the base ring, we define as in Construction 5.4 and Section 6.2

$$(9.1_{\text{hom}})$$
$$A_{\text{hom}} := k(x, y_1, \dots, y_r, t_1, \dots, t_s) \cap (S^*/I), \;\; B_{\text{hom}} := (1 + xU_{\text{hom}})^{-1} U_{\text{hom}},$$

where $U_{\text{hom}} := \bigcup_{n \in \mathbb{N}} S[\sigma_{1n}, \dots, \sigma_{sn}]$, each σ_{in} is the n^{th} frontpiece of σ_i and each $\sigma_{in} \in \mathcal{Q}(S) \cap (S^*/I)$, for $1 \le i \le s$, by Proposition 6.11. By Construction Properties Theorem 6.17.4, the ring S^*/I is the (x)-adic completion of each of the rings $A_{\text{hom}}, B_{\text{hom}}$ and U_{hom}.

Proposition 6.14 and Notation 6.13 imply the following isomorphisms

$$(9.1.b) \quad U_{\text{incl}} \cong U_{\text{hom}}, \qquad A_{\text{incl}} \cong A_{\text{hom}}, \qquad B_{\text{incl}} \cong B_{\text{hom}} \qquad R^* \cong S^*/I.$$

In Polynomial Example Theorems 9.2, 9.5 and 9.7, the rings A and B are often excellent; see Definition 3.27. This is not always true by Remark 9.4 below.

Polynomial Example Theorem 9.2. (Inclusion Version) *Assume Setting and Notation 9.1 with* R, R^*, V, A_{inc} *and* B_{inc} *as defined in Equations 9.5.a and* 9.1_{incl}. *Then:*

1. *The canonical map* $\alpha : R[\tau_1, \dots \tau_s] \hookrightarrow R^*[1/x]$ *is flat.*
2. $B_{incl} = A_{incl}$ *is Noetherian of dimension* $r + 1$ *and is the localization* $(1 + xV[y_1, \dots, y_r])^{-1} V[y_1, \dots, y_r]$ *of the polynomial ring* $V[y_1, \dots, y_r]$ *over the DVR* V. *Thus* A_{incl} *is a regular integral domain.*
3. B_{incl} *is a directed union of localizations of polynomial rings in* $r + s + 1$ *variables over* k.
4. *If* k *has characteristic zero, then the ring* $B_{incl} = A_{incl}$ *is excellent.*

Proof. The map

$$k[x, \tau_1, \dots \tau_s] \hookrightarrow k[[x]][1/x]$$

is flat by Remark 2.20.4 since $k[[x]][1/x]$ is a field. By Fact 2.21

$$k[x, \tau_1, \dots \tau_s] \otimes_k k[y_1, \dots, y_r] \hookrightarrow k[[x]][1/x] \otimes_k k[y_1, \dots, y_r]$$

is flat. We also have $k[[x]][1/x] \otimes_k k[y_1, \dots, y_r] \cong k[[x]][y_1, \dots, y_r][1/x]$ and

$$k[x, \tau_1, \dots \tau_s] \otimes_k k[y_1, \dots, y_r] \cong k[x, y_1, \dots, y_r][\tau_1, \dots \tau_s].$$

Hence the natural inclusion map

$$k[x, y_1, \dots, y_r][\tau_1, \dots \tau_s] \overset{\beta}{\hookrightarrow} k[[x]][y_1, \dots, y_r][1/x]$$

is flat. Also $k[[x]][y_1,\dots,y_r] \hookrightarrow k[y_1,\dots,y_r][[x]]$ is flat since it is the map taking a Noetherian ring to an ideal-adic completion; see Remark 3.7.2. Therefore

$$k[[x]][y_1,\dots,y_r][1/x] \overset{\delta}{\hookrightarrow} k[x,y_1,\dots,y_r][[x]][1/x]$$

is flat. It follows that the map

$$k[x,y_1,\dots,y_r][\tau_1,\dots\tau_s] \overset{\delta\circ\beta}{\hookrightarrow} R^*[1/x] = k[y_1,\dots,y_r][[x]][1/x]$$

is flat. Thus Noetherian Flatness Theorem 8.8 implies items 1 to 3.

If k has characteristic zero, then V is excellent by Remark 3.28; hence item 4 follows from item 2 since excellence is preserved under localization of a finitely generated algebra by Remark 3.28. For more details see [**71**, (34.B),(33.G) and (34.a)], [**32**, Chap. IV]. □

We observe in Proposition 9.3 that over a perfect field k of characteristic $p>0$ (so that $k=k^{1/p}$) a one-dimensional form of the construction in Polynomial Example Theorem 9.2 yields a DVR that is not a Nagata ring, defined in Definition 2.3.1, and thus is not excellent; see Remark 3.28, [**73**, p. 264], [**71**, Theorem 78, Definition 34.8].

Proposition 9.3. *Let k be a perfect field of characteristic $p>0$, let the element τ of $xk[[x]]$ be such that x and τ are algebraically independent over k and set $V := k(x,\tau)\cap k[[x]]$. Then V is a DVR for which the integral closure $\overline{V}$ of V in the purely inseparable field extension $k(x^{1/p},\tau^{1/p})$ is not a finitely generated V-module. Hence V is not a Nagata ring and so is not excellent.*

Proof. It is clear that V is a DVR with maximal ideal xV. Since x and τ are algebraically independent over k, $[k(x^{1/p},\tau^{1/p}) : k(x,\tau)] = p^2$. Let W denote the integral closure of V in the field extension $k(x^{1/p},\tau)$ of degree p over $k(x,\tau)$. Notice that

$$W \;=\; k(x^{1/p},\tau) \;\cap\; k[[x^{1/p}]] \quad \text{and} \quad \overline{V} \;=\; k(x^{1/p},\tau^{1/p}) \;\cap\; k[[x^{1/p}]]$$

are both DVRs having residue field k and maximal ideal generated by $x^{1/p}$. Thus $\overline{V} = W + x^{1/p}\overline{V}$. If $\overline{V}$ were a finitely generated W-module, then by Nakayama's Lemma it would follow that $W=\overline{V}$. This is impossible because $\overline{V}$ is not birational over W. It follows that $\overline{V}$ is not a finitely generated V-module, and hence V is not a Nagata ring. □

Remark 9.4. Assume the field k is perfect with characteristic $p>0$. By Proposition 9.3, the ring V is not excellent. Since $A = V[y]_{(x,y)}$, the ring V is a homomorphic image of A. Since excellence is preserved under homomorphic image, the ring A is not excellent.

We now state and prove the homomorphic image version of the theorem.

Polynomial Example Theorem 9.5. (Homomorphic Image Version) *Assume Setting and Notation 9.1 with A_{hom} and B_{hom} as in Equation 9.1$_{hom}$. Then*

1. *The canonical map $\alpha : S \hookrightarrow (S^*/I)[1/x]$ is flat.*
2. *$B_{hom} = A_{hom}$ is Noetherian of dimension $r+1$ and is a localization of the polynomial ring $V[y_1,\dots,y_r]$ over the DVR V. Thus A_{hom} is a regular integral domain.*
3. *B_{hom} is a directed union of localizations of polynomial rings in $r+s+1$ variables over k.*

(4) *If k has characteristic zero, then $B_{hom} = A_{hom}$ is excellent.*

PROOF. The ideal $I := (t_1 - \tau_1, \dots, t_s - \tau_s)S^*$ is a prime ideal of S^* and $S^*/I \cong k[y_1, \dots, y_r][[x]]$. The fact that $\tau_1, \dots, \tau_s$ are algebraically independent over $k(x)$ implies that $I \cap S = (0)$.

We identify Homomorphic Image Construction 5.4 for the ring S with Inclusion Construction 5.3 for the ring R as in the following diagram slightly modified from Chapter 5, where λ is the R-algebra isomomorphism that maps $t_i \to \tau_i$ for $i = 1, \dots, s$, and $K := \mathcal{Q}(R)$.

(9.5.1)
$$\begin{tikzcd}
& S := R[t_1, \dots, t_s] \arrow[r] \arrow[d, "\lambda"'] & A_{\mathrm{hom}} := K(t_1, \dots, t_s) \cap (S^*/I) \arrow[r] \arrow[d, "\lambda"'] & S^*/I \arrow[d, "\lambda"'] \\
R \arrow[r] & R[\tau_1, \dots, \tau_s] \arrow[r] & A_{\mathrm{incl}} := K(\tau_1, \dots, \tau_s) \cap R^* \arrow[r] & R^*.
\end{tikzcd}$$

In view of the identifications displayed in this diagram, the items of the homomorphic image version of Polynomial Example Theorem 9.5 follow from the corresponding items of the inclusion version, Polynomial Example Theorem 9.2. □

REMARK 9.6. The identifications indicated in the proof of Theorem 9.5 yield that a one-dimensional example over a perfect field k of characteristic $p > 0$ can be constructed using Homomorphic Image Construction 5.4 that corresponds to the example described in Remark 9.4.

We give below a localized form of Polynomial Example Theorems 9.5 and 9.2, where the rings R, A, and B are local. The ring B is a localization of $U = U_n$, where each $U_n = R[\tau_{in}]$, and B is also a localization of $U' = U'_n$, where each $U_n = k[y_j, \tau_{in}]$. This simpler second form U' of U is used in Chapters 17 and 18.

LOCALIZED POLYNOMIAL EXAMPLE THEOREM 9.7. *If we adjust Setting and Notation 9.1 so that the base rings are the regular local rings*

$$R := k[x, y_1, \dots, y_r]_{(x, y_1, \dots, y_r)} \quad \text{and} \quad S := k[x, y_1, \dots, y_r, t_1, \dots, t_s]_{(x, y_1, \dots, y_r, t_1, \dots, t_s)},$$

then the conclusions of Polynomial Example Theorems 9.2 and 9.5 are still valid. In particular:

(1) *For Inclusion Construction 5.3, with the notation of Equation 9.1_{incl},*

$$A_{incl} = B_{incl} = V[y_1, \dots, y_r]_{(x, y_1, \dots, y_r])},$$

is a Noetherian regular local ring, and the extension $R[t_1, \dots, t_s] \to R^*[1/x]$ is flat. In addition,

$$\begin{aligned}
B_{incl} =& \bigcup (U_n)_{\mathbf{m}_n} = U_{\mathbf{m}_U} = \bigcup (U'_n)_{\mathbf{m}'_n} = U'_{\mathbf{m}_{U'}}, \ \text{where } U = \bigcup U_n, \ U' = \bigcup U'_n, \\
U_n =& k[x, y_1, \dots, y_r]_{(x, y_1, \dots, y_r)}[\tau_{1n}, \dots, \tau_{sn}], \quad \mathbf{m}_n = (x, y_1, \dots, y_r, \tau_{1n}, \dots, \tau_{sn})U_n, \\
U'_n =& k[x, y_1, \dots, y_r, \tau_{1n}, \dots, \tau_{sn}], \quad \mathbf{m}'_n = (x, y_1, \dots, y_r, \tau_{1n}, \dots, \tau_{sn})U'_n, \\
\mathbf{m}_U =& (x, y_1, \dots, y_r)U \quad \text{and} \quad \mathbf{m}'_U = (x, y_1, \dots, y_r)U'.
\end{aligned}$$

(2) *For Homomorphic Image Construction 5.6, with the notation of Equation 9.1_{hom}, $A_{hom} = V[y_1, \dots, y_r]_{(x, y_1, \dots, y_r])} \cong B_{hom}$. The flatness statement in Theorem 9.5.1 holds for the revised S and S^*.*

(3) $A_{incl} \cong A_{hom}$.

PROOF. The proofs of Theorems 9.2 and 9.5 apply to the localized polynomial rings. The statements about the rings U follow from Remark 6.4. Item 3 follows from Diagram 9.5.1. $\square$

EXAMPLE 9.8. Let S be as in Localized Polynomial Example Theorem 9.7. Then $t_1-\tau_1,\dots,t_s-\tau_s$ is a regular sequence in S^*. Let $I=(t_1-\tau_1,\dots,t_s-\tau_s)S^*$ as in Localized Polynomial Example Theorem 9.7. Then Theorem 8.6 implies that $S \hookrightarrow (S^*/I^n)[1/x]$ is flat for each positive integer n. Using I^n in place of I, Theorem 8.3.2 implies the existence for every r and n in $\mathbb{N}$ of a Noetherian local domain $A=\mathcal{Q}(R)\cap R^*/I^n$ such that the dimension of A is $r+1$ and the (x)-adic completion of A has nilradical $\mathbf{n}$ with $\mathbf{n}^{n-1}\neq(0)$. The statements about A follow since $A^*=R^*/I^n$ and since Remark 3.7.3 implies $A\hookrightarrow R^*/I^n$ is faithfully flat.

Here are some more specific examples to which Polynomial Example Theorems 9.5, 9.2 and 9.7 apply. Example 9.9 shows that the dimension of U can be greater than the dimension of B_{hom}.

EXAMPLES 9.9. Assume the setting and notation of (9.1).

(1) Let $S:=k[x,t_1,\dots,t_s]$, that is, there are no y variables, and let S^* denote the (x)-adic completion of S. Then $I=(t_1-\tau_i,\dots,t_s-\tau_s)$ and, by Polynomial Example Theorem 9.5,

$$(S^*/I)\cap\mathcal{Q}(S)=A_{\text{hom}}=B_{\text{hom}}$$

is the DVR obtained by localizing U at the prime ideal xU. In this example $S[1/x]=U[1/x]$ has dimension $s+1$ and so $\dim U=s+1$, while $\dim(S^*/I)=\dim A_{\text{hom}}=\dim B_{\text{hom}}=1$.

(2) Essentially the same example as in item 1 can be obtained by using Theorem 9.2 as follows. Let $R=k[x]$, then $R^*=k[[x]]$ and

$$A_{\text{incl}}=k(x,\tau_1,\dots,\tau_s)\cap k[[x]] \quad\text{and}\quad A_{\text{incl}}=B_{\text{incl}},$$

by Theorem 9.2. In this case U_{incl} is a directed union of polynomial rings over k,

$$U_{\text{incl}} \;=\; \bigcup_{n=1}^{\infty} k[x][\tau_{1n},\dots,\tau_{sn}],$$

where the τ_{in} are the $\mathbf{n}^{\text{th}}$ endpieces of the τ_i as in Section 6.1. By Proposition 6.11, the endpieces are related to the frontpieces of the homomorphic image construction.

(3) Applying Localized Polynomial Example Theorem 9.7, one can modify Example 9.9.1 by taking S to be the $(s+1)$-dimensional regular local domain $k[t_1,\dots,t_s,x]_{(t_1,\dots,t_s,x)}$. In this case $S[1/x]=U[1/x]$ has dimension s, while we still have $S^*/I\cong k[[x]]$. Thus $\dim U=s+1$ and $\dim(S^*/I)=1=\dim A_{\text{hom}}=\dim B_{\text{hom}}$.

One can also obtain a local version using the inclusion construction with $R=k[x]_{(x)}$ and applying Theorem 9.7. We again have $R^*=k[[x]]$.

With S as in either (9.9.1) or (9.9.3), the domains B_n constructed from S as in Section 6.2 of Chapter 6 are $(s+1)$-dimensional regular local domains dominated by $k[[x]]$ and having k as a coefficient field. By Localized Polynomial Example Theorem 9.7 the family $\{B_n\}_{n\in\mathbb{N}}$ is a directed union of $(s+1)$-dimensional regular local domains whose union $B=A$ is Noetherian. This can be seen directly using Noetherian Flatness Theorem 8.3: In either case, since $(S^*/I)[1/x]$ is a field, the extension $S \hookrightarrow (S^*/I)[1/x]$ is flat. Thus by Theorem 8.3 the family $\{B_n\}_{n\in\mathbb{N}}$ is a directed union of $(s+1)$-dimensional regular local domains whose union B is Noetherian, and is, in fact a DVR.

(4) In the notation of (9.1) with the adjustment of Localized Polynomial Example Theorem 9.7, let $r = 1$ and $y_1 = y$. Thus $S = k[x, y, t_1, \dots, t_s]_{(x,y,t_1,\dots,t_s)}$. Then $S^*/I \cong k[y]_{(y)}[[x]]$. By Theorem 9.7.2, the extension $S \hookrightarrow (S^*/I)[1/x]$ is flat. Let $V = k[[x]] \cap k(x, \tau_1, \dots, \tau_s)$. Then V is a DVR and $(S^*/I) \cap \mathcal{Q}(S) \cong V[y]_{(x,y)}$ is a 2-dimensional regular local domain that is the directed union of $(s+2)$-dimensional regular local domains.

9.2. Transfer to the intersection of two ideals

We use the homomorphic image construction and assume the notation of Noetherian Flatness Theorem 8.3. In Theorem 9.10 we show that in certain circumstances the flatness, Noetherian and computability properties associated with ideals I_1 and I_2 of R^* as described in Theorem 8.3 transfer to their intersection $I_1 \cap I_2$.[1] We use Theorem 9.10 in Section 9.3 to show in Theorem 9.13 that the property of regularity of the generic formal fibers also transfers in certain cases to the intersection domain associated with the ideal $I_1 \cap I_2$.

Theorem 9.10. *Let R be a Noetherian integral domain with field of fractions K, let z be a nonzero nonunit of R, and let R^* denote the (z)-adic completion of R. Let I_1 and I_2 be ideals of R^* such that*

- (i) *$z^n \in I_1 + I_2$ for some positive integer n,*
- (ii) *$\mathbf{p} \cap R = (0)$, for each $\mathbf{p} \in \operatorname{Ass}(R^*/I_1)$, and each $\mathbf{p} \in \operatorname{Ass}(R^*/I_1)$, and*
- (iii) *$R \hookrightarrow (R^*/I_i)[1/z]$ is flat for each i.*

We set $A_i := K \cap (R^*/I_i)$ and set $A := K \cap (R^*/(I_1 \cap I_2))$. Then:

1. *The ideal $I := I_1 \cap I_2$ satisfies the conditions of Noetherian Flatness Theorem 8.3 and the map $R \hookrightarrow (R^*/I))[1/z]$ is flat. The (z)-adic completion A^* of A is R^*/I, and the (z)-adic completion A_i^* of A_i is R^*/I_i, for $i = 1, 2$.*
2. *The ring $A^*[1/z] \cong (A_1^*)[1/z] \oplus (A_2^*)[1/z]$. If $Q \in \operatorname{Spec}(A^*)$ and $z \notin Q$, then $(A^*)_Q$ is a localization of either A_1^* or A_2^*.*
3. *We have $A \subseteq A_1 \cap A_2$, and $(A_1)[1/z] \cap (A_2)[1/z] \subseteq A_P$ for each $P \in \operatorname{Spec} A$ with $z \notin P$. Thus we have $A[1/z] = (A_1)[1/z] \cap (A_2)[1/z]$.*

Proof. By Theorem 8.3, the (z)-adic completion A_i^* of A_i is R^*/I_i. Since $\operatorname{Ass}(R^*/(I_1 \cap I_2)) \subseteq \operatorname{Ass}(R^*/I_1) \cup \operatorname{Ass}(R^*/I_2)$, the condition on assassinators of Theorem 8.3 holds for the ideal $I_1 \cap I_2$. The natural R-algebra homomorphism $\pi : R^* \to (R^*/I_1) \oplus (R^*/I_2)$ has kernel $I_1 \cap I_2$. Further, the localization of π at z is onto because $(I_1 + I_2)R^*[1/z] = R^*[1/z]$. Thus $(R^*/(I_1 \cap I_2))[1/z] \cong (R^*/I_1)[1/z] \oplus (R^*/I_2)[1/z] = (A_1^*)[1/z] \oplus (A_2^*)[1/z]$ is flat over R. Therefore A is Noetherian and $A^* = R^*/I$ is the (z)-adic completion of A.

If $Q \in \operatorname{Spec}(A^*)$ and $z \notin Q$, then A_Q^* is a localization of

$$A^*[1/z] \cong (A_1^*)[1/z] \oplus (A_2^*)[1/z].$$

Every prime ideal of $(A_1^*)[1/z] \oplus (A_2^*)[1/z]$ has the form either $(Q_1)A_1^*[1/z] \oplus (A_2^*)[1/z]$ or $(A_1^*)[1/z] \oplus Q_2A_2^*[1/z]$, where $Q_i \in \operatorname{Spec}(A_i^*)$. It follows that A_Q^* is a localization of either A_1^* or A_2^*.

---

[1] A generalization of Theorem 9.10 is given in Theorem 16.9.

Since R^*/I_i is a homomorphic image of $R^*/(I_1 \cap I_2)$, the intersection ring $A \subseteq A_i$ for $i = 1, 2$. Let $P \in \operatorname{Spec} A$ with $z \notin P$. Since $A^* = R^*/I$ is faithfully flat over A, there exists $P^* \in \operatorname{Spec}(A^*)$ with $P^* \cap A = P$. Then $z \notin P^*$ implies $A^*_{P^*}$ is either $(A^*_1)_{P^*_1}$ or $(A^*_2)_{P^*_2}$, where $P^*_i \in \operatorname{Spec}(A^*_i)$. By symmetry, we may assume $A^*_{P^*} = (A^*_1)_{P^*_1}$. Let $P_1 = P^*_1 \cap A_1$. Since $A_P \hookrightarrow A^*_{P^*}$ and $(A_1)_{P_1} \hookrightarrow (A^*_1)_{P^*_1}$ are faithfully flat, we have

$$A_P = A^*_{P^*} \cap K = (A^*_1)_{P^*_1} \cap K = (A_1)_{P_1} \supseteq (A_1)[1/z].$$

It follows that $(A_1)[1/z] \cap (A_2)[1/z] \subseteq A_P$. Thus we have

$$(A_1)[1/z] \ \cap \ (A_2)[1/z] \ \subseteq \ \bigcap \{A_P \mid P \in \operatorname{Spec} A \text{ and } z \notin P\} \ = \ A[1/z].$$

Since $A[1/z] \subseteq (A_i)[1/z]$, for $i = 1, 2$, we have $A[1/z] = (A_1)[1/z] \cap (A_2)[1/z]$. □

The ring B of Example 9.11 is a two-dimensional Noetherian local domain that birationally dominates a three-dimensional regular local domain and is such that B is not universally catenary. The completion of B has two minimal primes one of dimension one and one of dimension two and this implies B is not universally catenary by Theorem 3.15. It follows that B is not a homomorphic image of a regular local ring because every homomorphic image of a regular local ring, or even of a Cohen-Macaulay local ring, is universally catenary by Remark 3.17. We present in Chapter 16 other examples of Noetherian local domains of various dimensions that are not universally catenary and that have properties similar to those of the Noetherian local domain B of Example 9.11.

EXAMPLE 9.11. Let k be a field of characteristic zero[2] and let x, y, z be indeterminates over k. Let $R = k[x, y, z]_{(x,y,z)}$, let K denote the field of fractions of R, and let $\tau_1, \tau_2, \tau_3 \in xk[[x]]$ be algebraically independent over $k(x, y, z)$. Let R^* denote the (x)-adic completion of R. We take the ideal I to be the intersection of the two prime ideals $Q := (z - \tau_1, y - \tau_2)R^*$, which has height 2, and $P := (z - \tau_3)R^*$, which has height 1. Then R^*/P and R^*/Q are examples of the form considered in Examples 9.9. Thus $(R^*/P)[1/x]$ and $(R^*/Q)[1/x]$ are both flat over R. Here $R^*/P \cong k[y]_{(y)}[[x]]$; the ring $V := k[[x]] \cap k(x, \tau_3)$ is a DVR, and $A_1 := (R^*/P) \cap K \cong V[y]_{(x,y)}$ is a two-dimensional regular local domain that is a directed union of three-dimensional RLRs, while $A_2 := (R^*/Q) \cap K$ is a DVR.

Since $\tau_1, \tau_3 \in xk[[x]]$, the ideal $(z - \tau_1, z - \tau_3)R^*$ has radical $(x, z)R^*$. Hence the ideal $P + Q$ is primary for the maximal ideal $(x, y, z)R^*$, so, in particular, P is not contained in Q. Therefore the representation $I = P \cap Q$ is irredundant and $\operatorname{Ass}(R^*/I) = \{P, Q\}$. Since $P \cap R = Q \cap R = (0)$, the ring R injects into R^*/I. Let $A := K \cap (R^*/I)$.

By (9.10.1), the inclusion $R \hookrightarrow (R^*/I)[1/x]$ is flat. By Noetherian Flatness Theorem 8.3, the ring A is Noetherian and is a localization of a subring of $R[1/x]$. The map $A \hookrightarrow \widehat{A}$ of A into its completion factors through the map $A \hookrightarrow A^* = R^*/I$. Since R^*/I has minimal primes P/I and Q/I with $\dim R^*/P = 2$ and $\dim R^*/Q = 1$, and since $\widehat{A}$ is faithfully flat over $A^* = R^*/I$, the ring $\widehat{A}$ is not equidimensional. It follows that A is not universally catenary by Theorem 3.15.

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[2] The characteristic zero assumption implies that the intersection rings A_1 and A_2 as constructed below are excellent; see (9.5.4).

9.3. Regular maps and geometrically regular formal fibers

We show in Corollary 9.14 that the ring $A = B$ of Example 9.11 has geometrically regular formal fibers.[3] Another example of a Noetherian local domain that is not universally catenary but has geometrically regular formal fibers is given in [**32**, (18.7.7), page 144] using a gluing construction; also see [**31**, (1.1)].

Proposition 9.12. *Let R, z, R^*, A, B and I be as in Theorem 8.3. Assume that for each $P \in \operatorname{Spec}(R^*/I)$ with $z \notin P$, the map $\psi_P : R_{P\cap R} \longrightarrow (R^*/I)_P$ is regular. Then $A = B$ and moreover:*

1. *A is Noetherian and the map $A \longrightarrow A^* = R^*/I$ is regular.*
2. *If R is semilocal with geometrically regular formal fibers and z is in the Jacobson radical of R, then A has geometrically regular formal fibers.*

Proof. Since flatness is a local property by (2.20.1), and regularity of a map includes flatness, the map $\psi_z : R \longrightarrow (R^*/I)[1/z]$ is flat. By Theorem 8.3, the intersection ring A is Noetherian with (z)-adic completion $A^* = R^*/I$. Hence $A \longrightarrow A^*$ is flat.

Let $Q \in \operatorname{Spec}(A)$, let $k(Q)$ denote the field of fractions of A/Q and let $Q_0 = Q \cap R$.

Case 1: $z \in Q$. Then $R/Q_0 = A/Q = A^*/QA^*$ and the ring $A^*_{QA^*}/QA^*_{QA^*} = A^* \otimes_A k(Q) = A_Q/QA_Q$ is trivially geometrically regular over $k(Q)$.

Case 2: $z \notin Q$. Let $k(Q) \subseteq L$ be a finite algebraic field extension. We show the ring $A^* \otimes_A L$ is regular. Let $W \in \operatorname{Spec}(A^* \otimes_A L)$ and let $W' = P \cap (A^* \otimes_A k(Q))$. The prime W' corresponds to a prime ideal $P \in \operatorname{Spec}(A^*)$ with $P \cap A = Q$. By assumption the map

$$R_{Q_0} \longrightarrow (R^*/I)_P = A^*_P$$

is regular. Since $z \notin Q$ it follows that $R_{Q_0} = U_{Q\cap U} = A_Q$ and that $k(Q_0) = k(Q)$. Thus the ring $A^*_P \otimes_{A_Q} L$ is regular. Therefore $(A^* \otimes_A L)_W$, which is a localization of this ring, is regular.

For item 2, since R has geometrically regular formal fibers, so has R^* by [**93**]. Hence the map $\theta : A^* = R^*/I \longrightarrow \widehat{A} = \widehat{(R^*/I)}$ is regular. By [**71**, Thm. 32.1 (i)] and item 1, it follows that A has geometrically regular formal fibers, that is, the map $A \longrightarrow \widehat{A}$ is regular. □

Theorem 9.13. *Let R be a Noetherian integral domain with field of fractions K. Let z be a nonzero nonunit of R and let R^* denote the (z)-adic completion of R. Assume that I_1 and I_2 are ideals of R^*, such that every prime ideal $P \in \operatorname{Ass}(R^*/I_i)$ satisfies $P \cap R = (0)$, for $i = 1$ and $i = 2$. Also assume*

1. *R is semilocal with geometrically regular formal fibers and z is in the Jacobson radical of R.*
2. *$(R^*/I_1)[1/z]$ and $(R^*/I_2)[1/z]$ are flat R-modules and the ideal $I_1 + I_2$ of R^* contains some power of z.*
3. *For $i = 1, 2$, $A_i := K \cap (R^*/I_i)$ has geometrically regular formal fibers.*

Then $A := K \cap (R^*/(I_1 \cap I_2)) = B$ has geometrically regular formal fibers.

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[3] A generalization of Corollary 9.14 is given in Corollary 16.18

PROOF. Let $I = I_1 \cap I_2$. Since R has geometrically regular formal fibers, by Proposition 9.12.2, it suffices to show for $W \in \operatorname{Spec}(R^*/I)$ with $z \notin W$ that $R_{W_0} \longrightarrow (R^*/I)_W$ is regular, where $W_0 := W \cap R$. As in Theorem 9.10, we have $(R^*/I)[1/z] = (R^*/I_1)[1/z] \oplus (R^*/I_2)[1/z]$. It follows that $(R^*/I)_W$ is a localization of either R^*/I_1 or R^*/I_2. Suppose $(R^*/I)_W = (R^*/I_1)_{W_1}$, where $W_1 \in \operatorname{Spec}(R^*/I_1)$. Then $R_{W_0} = (A_1)_{W_1 \cap A_1}$ and $(A_1)_{W_1 \cap A_1} \longrightarrow (R^*/I_1)_{W_1}$ is regular. A similar argument holds if $(R^*/I)_W$ is a localization of R^*/I_2. Thus, in either case, $R_{W_0} \longrightarrow (R^*/I)_W$ is regular. $\square$

COROLLARY 9.14. *The ring $A = B$ of Example 9.11 has geometrically regular formal fibers, that is, the map $\phi : A \hookrightarrow \widehat{A}$ is regular.*

PROOF. By the definition of R and the observations given in (9.11), the hypotheses of (9.13) are satisfied. $\square$

Exercises

(1) Describe Example 9.9.4 in terms of the Inclusion Construction. In particular, determine the appropriate base ring R for this construction.

(2) For the rings A and A^* of Example 9.11, prove that A^* is universally catenary.

CHAPTER 10

Introduction to the insider construction

We describe in this chapter and in Chapter 13 a version of Inclusion Construction 5.3 that we call the Insider Construction. The general procedure and definition is Insider Construction 13.1 in Chapter 13. We present in this chapter several examples using the technique formalized in Insider Construction 13.1 including two classical examples. The Insider Construction simplifies the creation and verification of examples. Moreover the Insider Construction makes it easier than with Construction 5.3 to determine whether a given example is or is not Noetherian. The Insider Construction is described in more generality and with more details in Chapter 13.

For the examples considered in this chapter, we begin with a base ring that is a localized polynomial ring in two or three variables over a field. We construct two "insider" integral domains inside an "outside" ring A^{out}, where A^{out} is constructed from Localized Polynomial Example Theorem 9.7 with Inclusion Construction 5.3. By Localized Polynomial Example Theorem 9.7, the intersection domain A^{out} is equal to an approximation integral domain B^{out} that is the nested union of localized polynomial rings from Section 6.1. The two insider integral domains contained in A^{out} are: A^{ins}, an intersection of a field with a power series ring as in Construction 5.3, and B^{ins}, a nested union of localized polynomial rings that "approximates" A^{ins} as in Section 6.1.

As we describe in Proposition 10.6, the condition that the insider approximation domain B^{ins} is Noetherian is related to flatness of a map of polynomial rings corresponding to the extension $B^{\text{ins}} \hookrightarrow B^{\text{out}}$. Flatness of this map implies that the insider approximation domain B^{ins} is Noetherian, that the insider intersection domain A^{ins} equals B^{ins} and that A^{ins} is Noetherian. We apply this observation in Proposition 10.2 and Example 10.7 to conclude that certain constructed rings are Noetherian. We use Theorem 13.6 from Chapter 13 to show that other constructed rings are not Noetherian.

We present more details concerning flatness of polynomial extensions in Chapter 11.

10.1. The Nagata example

In Proposition 10.2 we adapt the example of Nagata given as Example 4.8 in Chapter 4, to fit the insider construction.

Setting 10.1. Let k be a field, let x and y be indeterminates over k, and set

$$R := k[x,y]_{(x,y)} \qquad \text{and} \qquad R^* := k[y]_{(y)}[[x]].$$

The power series ring R^* is the xR-adic completion of R. Let $\tau \in xk[[x]]$ be a transcendental element over $k(x)$.[1] We define the *intersection domain* A_τ corresponding to τ by $A_\tau := k(x, y, \tau) \cap R^*$.

Let f be a polynomial in $R[\tau]$ that is algebraically independent over $\mathcal{Q}(R)$, for example, $f = (y + \tau)^2$. We set $A_f := \mathcal{Q}(R[f]) \cap R^*$ to be the *intersection domain* corresponding to f. In the language of the introduction to this chapter, A_τ is the "outsider" intersection domain A^{out} and A_f, which is contained in A_τ, is the "insider" intersection domain A^{ins}.

Recall from Section 6.1 and Remark 6.4 that there are natural "approximating" outsider and insider domains associated to A_τ and A_f, namely,

$$B_\tau := \bigcup_{\mathbf{n}\in\mathbb{N}} k[x, y, \tau_n]_{(x,y,\tau_n)} \quad \text{and} \quad B_f := \bigcup_{\mathbf{n}\in\mathbb{N}} k[x, y, f_n]_{(x,y,\tau_n)}, \tag{10.1.0}$$

where the τ_n are the n^{th} endpieces of τ and the f_n are the n^{th} endpieces of f. The rings B_τ and B_f are nested unions of localized polynomial rings over k in 3 variables. (These approximating domains are the C_n of Remark 6.4.)

By Localized Polynomial Example Theorem 9.7.1, the extension $T := R[\tau] \overset{\psi}{\hookrightarrow} R^*[1/x]$ is flat, where ψ is the inclusion map, and the ring A_τ is Noetherian and is equal to the "computable" nested union B_τ. Let $S := R[f] \subseteq R[\tau]$ and let φ be the embedding $\varphi : S := R[f] \overset{\varphi}{\hookrightarrow} T = R[\tau]$. Put $\alpha := \psi \circ \varphi : S \to R^*[1/x]$. Then we have the following commutative diagram:

(13.1.1)

$$\begin{tikzcd}
 & & R^*[1/x] \\
R \arrow[r, phantom, "\subseteq"] & S := R[f] \arrow[ur, "\alpha:=\psi\circ\varphi"] \arrow[r, "\varphi"] & T := R[\sigma, \tau] \arrow[u, "\psi"']
\end{tikzcd}$$

In Proposition 10.2, we present a different proof of the result of Nagata described in Example 4.8.

PROPOSITION 10.2. *With the notation of Setting 10.1, let $f := (y + \tau)^2$. In the Nagata Example 4.8, the ring $A_f = B_f$ is Noetherian with completion $k[[x, y]]$. Therefore A_f is a two-dimensional regular local domain.*

PROOF. Since $T = R[\tau]$ is a free S-module with free basis $\langle 1, y + \tau\rangle$, the map $S \overset{\varphi}{\hookrightarrow} T$ is flat, by Remark 2.20.2. As mentioned above, the map $T \overset{\psi}{\hookrightarrow} R^*[1/x]$ is flat. Therefore as displayed in Diagram 10.1.1, the map $S \overset{\alpha}{\hookrightarrow} R^*[1/x]$ is flat; see Remark 2.2013. By Noetherian Flatness Theorem 8.8 and Remark 6.4, B_f is Noetherian and $B_f = A_f$. $\square$

REMARK 10.3. In the example of Nagata it is required that the field k have characteristic different from 2. This assumption is not necessary for showing that the domain A_f of Proposition 10.2 is a two-dimensional regular local domain.

---

[1] Since R^* is a localized ring of polynomials and power series over a field, R^* is an integral domain and so every nonzero element of the polynomial ring $R[\tau]$ is a regular element of R^*; thus $k(x, y, \tau) \subseteq \mathcal{Q}(R^*)$.

10.2. Other examples inside a Localized Polynomial Example

To describe other examples, we modify Setting 10.1 so that we use Local Polynomial Example 9.7 with three indeterminates and one or two elements of $k[[x]]$ that are algebraically independent, as follows.

SETTING 10.4. Let k be a field, let x, y, z be indeterminates over k, and set

$$R := k[x,y,z]_{(x,y,z)} \qquad \text{and} \qquad R^* := k[y,z]_{(y,z)}[[x]].$$

The power series ring R^* is the xR-adic completion of R. Let σ and τ in $xk[[x]]$ be algebraically independent over $k(x)$. We define the *intersection domains* A_τ and $A_{\sigma,\tau}$ corresponding to τ and σ, τ as follows:

$$A_\tau := k(x,y,z,\tau) \cap R^* \qquad \text{and} \qquad A_{\sigma,\tau} := k(x,y,z,\sigma,\tau) \cap R^*$$

In the following examples we define f to be an element of $R[\sigma,\tau]$ that is algebraically independent over $\mathcal{Q}(R)$. The *intersection domain* corresponding to f is $A_f = \mathcal{Q}(R[f]) \cap R^*$. In the language of the beginning of this chapter, A_τ or $A_{\sigma,\tau}$ is the "outsider" intersection domain A^{out}, and A_f, which is contained in A_τ or $A_{\sigma,\tau}$, is the "insider" intersection domain A^{ins}.

As in Setting 10.1 for the Nagata Example, there are natural "approximating" domains associated to A_τ, $A_{\sigma,\tau}$ and A_f, respectively, namely, B_τ, $B_{\sigma,\tau}$ and B_f, respectively. The rings B_τ, $B_{\sigma,\tau}$ and B_f are nested unions of localized polynomial rings over k in 4 or 5 variables.

REMARK 10.5. With Setting 10.4, let $T := R[\sigma,\tau]$ and let $S := R[f]$, where f is a polynomial in $R[\sigma,\tau]$ that is algebraically independent over $\mathcal{Q}(R)$. Notice that the inclusion map $\psi : T \hookrightarrow R^*[1/x]$ is flat. Let $\varphi : S \hookrightarrow T$ denote the inclusion map from S to T. Then we have the following commutative diagram with additional maps shown:

(10.5.1)
$$\begin{tikzcd}
& & R^*[1/x] & \\
& & & T[1/x] \arrow[ul, "\psi_x"'] \\
R \arrow[r, phantom, "\subseteq"] & S := R[f] \arrow[uur, "\alpha:=\psi\circ\varphi"] \arrow[urr, "\varphi_x:=i_x\circ\varphi"] \arrow[r, "\varphi"'] & T := R[\tau] \arrow[uu, "\psi"] \arrow[ur, "i_x"'] &
\end{tikzcd}$$

We note the following:

(1) Lemma 11.18 implies the map φ is flat if and only if the nonconstant coefficients of f as a polynomial in $R[\sigma,\tau]$ generate the unit ideal of R.
(2) If φ is flat then α is the composition of two flat homomorphisms and so α is also flat, by Remark 2.2013.
(3) To conclude that α is flat, it suffices to show that $\varphi_x : S \hookrightarrow T[1/x]$ is flat, since the inclusion map $T[1/x] \hookrightarrow R^*[1/x]$ is flat.
(4) Theorem 13.6.1 implies that $\varphi_x : S \hookrightarrow T[1/x]$ is flat if and only if the nonconstant coefficients of f generate the unit ideal of $R[1/x]$.

Proposition 10.6 is a preliminary result regarding flatness; there is a more extensive discussion in Chapter 13:

PROPOSITION 10.6. *With the notation of Remark 10.5, let $\varphi_x : S \hookrightarrow T[1/x]$ denote the composition of $\varphi : S \hookrightarrow T$ with the localization map from T to $T[1/x]$.*

If $\varphi_x : S \hookrightarrow T[1/x]$ *is flat, then the approximating domain* B_f *from Section 6.1 is Noetherian and is equal to the intersection domain* A_f*.*

PROOF. In Diagram‘10.5.1, the map ψ is flat. Thus the map $\psi' : T[1/x] \hookrightarrow R^*[1/x]$ is flat. If $\varphi_x : S \hookrightarrow T[1/x]$ is flat, then $\alpha = \psi' \circ \varphi_x$ in Diagram 10.5.1 is the composition of two flat maps. By Remark 2.20.13, α is flat, and hence by Noetherian Flatness Theorem 8.8, the approximating domain B_f from Section 6.1 is Noetherian and is equal to the intersection domain A_f. ☐

We use Proposition 10.6 to show the Noetherian property for the following example of Rotthaus [**91**], Example 4.10 of Chapters 4.

EXAMPLE 10.7. (Rotthaus) With the Setting 10.4, let $f := (y+\sigma)(z+\tau)$ and consider the insider domain A_f, contained in the outsider domain $\subseteq A_{\sigma,\tau}$. The nonconstant coefficients of $f = yz + \sigma z + \tau y + \sigma\tau$ as a polynomial in $R[\sigma,\tau]$ are $\{1, z, y\}$. They do generate the unit ideal of $R[1/x]$, and so, since we assume Remark 10.5.4 for now, we have φ_x is flat. Thus, by Proposition 10.6, the associated nested union domain B_f is Noetherian and is equal to A_f.

EXAMPLES 10.8. (1) With the Setting 10.4, let $f := y\sigma + z\tau$. We show in Examples 13.7 that the map $R[f] \hookrightarrow R[\sigma,\tau][1/x]$ is not flat and that $A_f = B_f$, i.e., A is “limit-intersecting” as in Definition 6.5, but is not Noetherian. Thus we have a situation where the intersection domain equals the approximation domain, but is not Noetherian. This gives a simpler example of such behavior than the example given in Section 24.1.

(2) The following is a related even simpler example: again with the notation of Setting 10.4, let $f := y\tau + z\tau^2 \in R[\tau] \subseteq A_\tau$. Then the constructed approximation domain B_f is not Noetherian by Theorem 13.6. Moreover, B_f is equal to the intersection domain $A_f := R^* \cap k(x,y,z,f)$ by Corollary 13.5.

In dimension two (the two variable case), an immediate consequence of Valabrega’s Theorem 4.2 is the following.

THEOREM 10.9. (Valabrega) *Let x and y be indeterminates over a field k and let $R = k[x,y]_{(x,y)}$. Then $\widehat{R} = k[[x,y]]$ is the completion of R. If L is a field between the fraction field of R and the fraction field of $k[y]\,[[x]]$, then $A = L \cap \widehat{R}$ is a two-dimensional regular local domain with completion $\widehat{R}$.*

Example 10.8 shows that the dimension three analog to Valabrega’s result fails. With $R = k[x,y,z]_{(x,y,z)}$ the field $L = k(x,y,z,f)$ is between $k(x,y,z)$ and the fraction field of $k[y,z]\,[[x]]$, but $L \cap \widehat{R} = L \cap R^*$ is not Noetherian.

EXAMPLE 10.10. The following example is given in Section 26.4. With the notation of Setting 10.4, let $f = (y+\sigma)^2$ and $g = (y+\sigma)(z+\tau)$. It is shown in Chapter 26 that the intersection domain $A := R^* \cap k(x,y,z,f,g)$ properly contains its associated approximation domain B and that both A and B are non-Noetherian.

CHAPTER 11

The flat locus of a polynomial ring extension

Let R be a Noetherian ring, let n be positive integer and let $z_1, \dots, z_n$ be indeterminates over R. In this chapter we examine the flat locus of a polynomial ring extension φ of the form

$$(11.01) \qquad S := R[f_1, \dots, f_m] \overset{\varphi}{\hookrightarrow} R[z_1, \dots, z_n] =: T,$$

where the f_j are polynomials in $R[z_1, \dots, z_n]$ that are algebraically independent over R.[1] We are motivated to examine the flat locus of the extension φ by the flatness condition in the Insider Construction of Chapter 10.

We discuss in Section 11.1 a general result on flatness. Then in Section 11.2 we consider the Jacobian ideal of the map $\varphi : S \hookrightarrow T$ of (11.01) and describe the nonsmooth and nonflat loci of this map. In Section 11.3 we discuss applications to polynomial extensions. Related results are given in the papers of Picavet [**87**] and Wang [**108**].

11.1. Flatness criteria

Recall that a Noetherian local ring $(R, \mathbf{m})$ of dimension d is Cohen-Macaulay if there exist elements $x_1, \dots, x_d$ in $\mathbf{m}$ that form a regular sequence in the sense that x_1 is a regular element of R, and for i with $2 \le i \le d$, the image of x_i in $R/(x_1, \dots, x_{i-1})R$ is regular; see [**73**, pages 123,134, 136].

Theorem 11.1 is a general result on flatness involving the Cohen-Macaulay property and a trio of Noetherian local rings.

THEOREM 11.1. *Let $(R, \mathbf{m}), (S, \mathbf{n})$ and (T, ℓ) be Noetherian local rings, and assume there exist local maps:*

$$R \longrightarrow S \longrightarrow T,$$

such that

(i) *$R \to T$ is flat and $T/\mathbf{m}T$ is Cohen-Macaulay.*
(ii) *$R \to S$ is flat and $S/\mathbf{m}S$ is a regular local ring.*

Then the following statements are equivalent:

(1) *$S \to T$ is flat.*
(2) *For each prime ideal $\mathbf{w}$ of T, we have $ht(\mathbf{w}) \ge ht(\mathbf{w} \cap S)$.*
(3) *For each prime ideal $\mathbf{w}$ of T such that $\mathbf{w}$ is minimal over $\mathbf{n}T$, we have $ht(\mathbf{w}) \ge ht(\mathbf{n})$.*

PROOF. The implications (2) $\Longrightarrow$ (3) is obvious and the implication (1) $\Longrightarrow$ (2) is clear by Remark 2.20.10. To prove (3) $\Longrightarrow$ (1), we show that (3) implies

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[1] In general for a commutative ring T and a subring R, we say that elements $f_1, \dots, f_m \in T$ are *algebraically independent* over R if, for indeterminates $t_1, \dots, t_m$ over R, the only polynomial $G(t_1, \dots, t_m) \in R[t_1, \dots, t_m]$ with $G(f_1, \dots, f_m) = 0$ is the zero polynomial.

(a) $\mathbf{m}S \otimes_S T \cong \mathbf{m}T$.
(b) The map $S/\mathbf{m}S \longrightarrow T/\mathbf{m}T$ is faithfully flat.

By [**73**, Theorem 22.3], conditions (a) and (b) imply flatness of the map $S \to T$ and thus prove that (3 $\implies$ (1).

Proof of (a): Since $R \hookrightarrow S$ is flat, we have $\mathbf{m}S \cong \mathbf{m}R \otimes_R S$. Therefore

$$\mathbf{m}S \otimes_S T \cong (\mathbf{m} \otimes_R S) \otimes_S T \cong \mathbf{m} \otimes_R T \cong \mathbf{m}T,$$

where the last isomorphism follows because the map $R \to T$ is flat.

Proof of (b): By assumption, $T/\mathbf{m}T$ is Cohen-Macaulay and $S/\mathbf{m}S$ is regular. Thus in view of [**73**, Theorem 23.1], it suffices to show:

$$\dim(T/\mathbf{m}T) \ = \ \dim(S/\mathbf{m}S) + \dim(T/\mathbf{n}T).$$

Let $\mathbf{w}$ be a prime ideal of T such that $\mathbf{n}T \subseteq \mathbf{w}$ and $\mathbf{w}$ is minimal over $\mathbf{n}T$. Since $\ell \cap S = \mathbf{n}$, also $\mathbf{w} \cap S = \mathbf{n}$ and, by [**73**, Theorem 15.1], $\text{ht}(\mathbf{w}) \leq \text{ht}(\mathbf{n})$. By assumption $\text{ht}(\mathbf{w}) \geq \text{ht}(\mathbf{n})$ and therefore $\text{ht}(\mathbf{w}) = \text{ht}(\mathbf{n})$. This equality holds for every minimal prime divisor $\mathbf{w}$ of $\mathbf{n}T$ and therefore $\text{ht}(\mathbf{n}) = \text{ht}(\mathbf{n}T)$. Since $\mathbf{n}$ is generated up to radical by $\text{ht}(\mathbf{n})$ elements, we have

$$\begin{aligned}\dim(T/\mathbf{n}T) &= \dim(T) - \text{ht}(\mathbf{n}T)\\ &= \dim(T) - \text{ht}(\mathbf{w})\\ &= \dim(T) - \text{ht}(\mathbf{n})\end{aligned}$$

Since T and S are flat over R, we have $\text{ht}(\mathbf{m}T) = \text{ht}(\mathbf{m}S) = \text{ht}(\mathbf{m})$ [**73**, Theorem 15.1]. Therefore

$$\begin{aligned}\dim T/\mathbf{n}T &= \dim(T) - \text{ht}(\mathbf{n})\\ &= \dim(T) - \text{ht}(\mathbf{m}T) - (\text{ht}(\mathbf{n}) - \text{ht}(\mathbf{m}S))\\ &= \dim(T/\mathbf{m}T) - \dim(S/\mathbf{m}S)).\end{aligned}$$

Therefore $S/\mathbf{m}S \longrightarrow T/\mathbf{m}T$ is faithfully flat, and hence, by the equivalence of items (1) and (3) of [**73**, Theorem 22.3], $S \longrightarrow T$ is faithfully flat. This completes the proof of Theorem 11.1 ☐

In Theorem 11.2 we present a result closely related to Theorem 11.1 with a Cohen-Macaulay hypothesis on all the fibers of $R \to T$ and a regularity hypothesis on all the fibers of $R \to S$. For more information about the fibers of a map, see Discussion 3.18 and Definition 3.19.

THEOREM 11.2. *Let $(R, \mathbf{m}), (S, \mathbf{n})$ and (T, ℓ) be Noetherian local rings, and assume there exist local maps:*

$$R \longrightarrow S \longrightarrow T,$$

such that

(i) *$R \to T$ is flat with Cohen-Macaulay fibers,*
(ii) *$R \to S$ is flat with regular fibers.*

Then the following statements are equivalent:

(1) *$S \to T$ is flat with Cohen-Macaulay fibers.*
(2) *$S \to T$ is flat.*
(3) *For each prime ideal $\mathbf{w}$ of T, we have $ht(\mathbf{w}) \geq ht(\mathbf{w} \cap S)$.*
(4) *For each prime ideal $\mathbf{w}$ of T such that $\mathbf{w}$ is minimal over $\mathbf{n}T$, we have $ht(\mathbf{w}) \geq ht(\mathbf{n})$.*

Proof. The implications (1) $\implies$ (2) and (3) $\implies$ (4) are obvious and the implication (2) $\implies$ (3) is clear by Remark 2.20.10. By Theorem 11.1, item 4 implies that $S \to T$ is flat.

To show Cohen-Macaulay fibers for $S \to T$, it suffices to show for each $Q \in \operatorname{Spec} T$ with $P := Q \cap S$ that T_Q/PT_Q is Cohen-Macaulay. Let $Q \cap R = q$. By passing to $R/q \subseteq S/qS \subseteq T/qT$, we may assume $Q \cap R = (0)$. Let $\operatorname{ht} P = n$. Since $R \to S_P$ has regular fibers and $P \cap R = (0)$, the ideal PS_P is generated by n elements. Moreover, faithful flatness of the map $S_P \to T_Q$ implies that the ideal PT_Q has height n by Remark 2.20.10. Since T_Q is Cohen-Macaulay, a set of n generators of PS_P forms a regular sequence in T_Q. Hence T_Q/PT_Q is Cohen-Macaulay [**73**, Theorems 17.4 and 17.3]. ☐

Since flatness is a local property by Remark 2.20.4, the following two corollaries are immediate from Theorem 11.2; see also [**87**, Théorème 3.15].

Corollary 11.3. *Let T be a Noetherian ring and let $R \subseteq S$ be Noetherian subrings of T. Assume that $R \to T$ is flat with Cohen-Macaulay fibers and that $R \to S$ is flat with regular fibers. Then $S \to T$ is flat if and only if, for each prime ideal P of T, we have $ht(P) \geq ht(P \cap S)$.*

As a special case of Corollary 11.3, we have:

Corollary 11.4. *Let R be a Noetherian ring and let $z_1, \dots, z_n$ be indeterminates over R. Assume that $f_1, \dots, f_m \in R[z_1, \dots, z_n]$ are algebraically independent over R Then*

1. *$\varphi : S := R[f_1, \dots, f_m] \hookrightarrow T := R[z_1, \dots, z_n]$ is flat if and only if, for each prime ideal P of T, we have $ht(P) \geq ht(P \cap S)$.*
2. *For $Q \in \operatorname{Spec} T$, $\varphi_Q : S \to T_Q$ is flat if and only if for each prime ideal $P \subseteq Q$ of T, we have $ht(P) \geq ht(P \cap S)$.*

Proof. Since S and T are polynomial rings over R, the maps $R \to S$ and $R \to T$ are flat with regular fibers. Hence both assertions follow from Corollary 11.3. ☐

11.2. The Jacobian ideal and the smooth and flat loci

We use the following definitions as in [**105**].

Definition 11.5. Let R be a ring. An R-algebra A is said to be *quasi-smooth* over R if for every R-algebra B and ideal N of B with $N^2 = 0$, every R-algebra homomorphism $g : A \to B/N$ lifts to an R-algebra homomorphims $f : A \to B$. Thus in the commutative diagram below where the maps from $R \to A$ and $R \to B$ are the canonical ring homomorphisms that define A and B as R-algebras and the map $B \to B/N$ is the canonical quotient ring map

$$\begin{tikzcd} R \arrow[r] \arrow[d] & A \arrow[d, "g"'] \\ B \arrow[r] & B/N \end{tikzcd}$$

if A is quasi-smooth over R, then there exists an R-algebra homomorphism $f : A \to B$ that preserves commutativity of the diagram. If A is finitely presented and quasi-smooth over R, then A is said to be *smooth* over R. If A is essentially finitely presented and quasi-smooth over R, then A is said to be *essentially smooth* over R.

Let $\varphi : S \hookrightarrow T$ be as in (11.01).

Definitions and Remarks 11.6. (1) The *Jacobian ideal* J of the extension $S \hookrightarrow T$ is the ideal of T generated by the $m \times m$ minors of the $m \times n$ matrix $\mathcal{J}$ defined as follows:

$$\mathcal{J} := \left(\frac{\partial f_i}{\partial z_j}\right)_{1\leq i\leq m,\ 1\leq j\leq n}.$$

(2) For the extension $\varphi : S \hookrightarrow T$, the *nonflat locus* of φ is the set $\mathcal{F}$, where

$$\mathcal{F} := \{Q \in \operatorname{Spec}(T) \mid \text{the map} \quad \varphi_Q : S \to T_Q \text{ is not flat }\}.$$

We also define the set $\mathcal{F}_{\min}$ and the ideal F of T as follows:

$$\mathcal{F}_{\min} := \{\text{ minimal elements of } \mathcal{F}\} \text{ and } F := \bigcap\{Q \mid Q \in \mathcal{F}\}.$$

By [**73**, Theorem 24.3], the set $\mathcal{F}$ is closed in the Zariski topology on $\operatorname{Spec} T$. Hence

$$\mathcal{F} \;=\; \mathcal{V}(F) \;:=\; \{P \in \operatorname{Spec} T \mid F \subseteq P\ \}.$$

Thus the set $\mathcal{F}_{\min}$ is a finite set and is equal to the set $\operatorname{Min}(F)$ of minimal primes of the ideal F of T.

Since a flat homomorphism satisfies the going-down theorem by Remark 3.7.9, Corollary 11.4 implies that

(i) $\mathcal{F}_{\min} \subseteq \{Q \in \operatorname{Spec} T \mid \operatorname{ht} Q < \operatorname{ht}(Q \cap S)\}$, and

(ii) If $Q \in \mathcal{F}_{\min}$, then every prime ideal $P \subsetneq Q$ satisfies $\operatorname{ht} P \geq \operatorname{ht}(P \cap S)$.

Example and Remarks 11.7. (1) Let k be a field, let x and y be indeterminates over k and set $f = x$, $g = (x-1)y$. Then $k[f,g] \overset{\varphi}{\hookrightarrow} k[x,y]$ is not flat.

Proof. For the prime ideal $P := (x-1) \in \operatorname{Spec}(k[x,y])$, we see that $\operatorname{ht}(P) = 1$, but $\operatorname{ht}(P \cap k[f,g]) = 2$; thus the extension is not flat by Corollary 11.4. $\square$

(2) The Jacobian ideal J of f and g in (1) is given by:

$$J = (\det \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\delta g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}) = (\det \begin{pmatrix} 1 & 0 \\ y & x-1 \end{pmatrix}) = (x-1)k[x,y].$$

(3) In the example of item 1, the nonflat locus is equal to the set of prime ideals Q of $k[x,y]$ that contain the Jacobian ideal $(x-1)k[x,y]$, thus $J = F$.

(4) One can also describe the example of item 1 by taking the base ring R to be the polynomial ring $k[x]$ rather than the field k. Then both $T = R[y]$ and $S = R[g]$ are polynomial rings in one variable over R with $g = (x-1)y$. The Jacobian ideal J is the ideal of T generated by $\frac{\partial g}{\partial y} = x - 1$, so is the same as in item 1.

Remark 11.8. A homomorphism $f : R \to \Lambda$ of Noetherian rings is said to be regular, see Definition 3.22, if f is flat and has geometrically regular fibers. In the case where Λ is a finitely generated R-algebra, the map f is regular if and only if it is smooth as can be seen by taking $\Lambda = A$ in [**105**, Corollary 1.2].

We record in Theorem 11.9 observations about smoothness and flatness that follow from well-known properties of the Jacobian.

Theorem 11.9. *Let R be a Noetherian ring, let $z_1, \ldots, z_n$ be indeterminates over R, and let $f_1, \ldots, f_m \in R[z_1, \ldots, z_n]$ be algebraically independent over R. Consider the embedding $\varphi : S := R[f_1, \ldots, f_m] \hookrightarrow T := R[z_1, \ldots, z_n]$. Let J denote the Jacobian ideal of φ, and let F and $\mathcal{F}_{min}$ be as in* (11.6). *Then*

(1) $Q \in \operatorname{Spec} T$ *does not contain* $J \iff \varphi_Q : S \to T_Q$ *is essentially smooth. Thus* J *defines the nonsmooth locus of* φ.
(2) *If* $Q \in \operatorname{Spec} T$ *does not contain* J, *then* $\varphi_Q : S \to T_Q$ *is flat. Thus* $J \subseteq F$.
(3) $\mathcal{F}_{min} \subseteq \{Q \in \operatorname{Spec} T \mid J \subseteq Q \text{ and } \operatorname{ht}(Q \cap S) > \operatorname{ht} Q\}$.
(4) $\mathcal{F}_{min} \subseteq \{Q \in \operatorname{Spec} T \mid J \subseteq Q,\ \operatorname{ht} Q < \dim S \text{ and } \operatorname{ht}(Q \cap S) > \operatorname{ht} Q\}$.
(5) φ *is flat* $\iff$ *for every* $Q \in \operatorname{Spec}(T)$ *such that* $J \subseteq Q$ *and* $\operatorname{ht}(Q) < \dim S$, *we have* $\operatorname{ht}(Q \cap S) \leq \operatorname{ht}(Q)$.
(6) *If* $\operatorname{ht} J \geq \dim S$, *then* φ *is flat.*

Proof. For item 1, we show that our definition of the Jacobian ideal J given in (11.6) agrees with the description of the smooth locus of an extension given in [**26**], [**105**, Section 4].

To see this, let $u_1, \dots, u_m$ be indeterminates over $R[z_1, \dots, z_n]$ and identify

$$R[z_1, \dots, z_n] \quad \text{with} \quad \frac{R[u_1, \dots, u_m][z_1, \dots, z_n]}{(\{u_i - f_i\}_{i=1,\dots,m})}.$$

Since $u_1, \dots, u_m$ are algebraically independent, the ideal J generated by the minors of $\mathcal{J}$ is the Jacobian ideal of the extension φ by means of this identification. We make this more explicit as follows.

Let $B := R[u_1, \dots, u_m, z_1, \dots, z_n]$ and $I = (\{f_i - u_i\}_{i=1,\dots,m})B$. Consider the following commutative diagram

$$\begin{tikzcd}
S := R[f_1, \dots, f_m] \arrow[r] \arrow[d, "\cong"'] & T := R[z_1, \dots, z_n] \arrow[d, "\cong"'] \\
S_1 := R[u_1, \dots, u_m] \arrow[r] & T_1 := B/I
\end{tikzcd}$$

Define as in [**26**], [**105**, Section 4]

$$H = H_{T_1/S_1} := \text{the radical of } \Sigma\Delta(g_1, \dots, g_s)[(g_1, \dots, g_s) :_B I],$$

where the sum is taken over all s with $0 \leq s \leq m$, for all choices of s polynomials $g, \dots, g_s$ from $I = (\{f_1 - u_1, \dots, f_m - u_m\})B$, where $\Delta := \Delta(g_1, \dots, g_s)$ is the ideal of $T \cong T_1$ generated by the $s \times s$-minors of $\left(\frac{\partial g_i}{\partial z_j}\right)$, and $\Delta = T$ if $s = 0$.

To establish (11.9.1), we show that $H = \operatorname{rad}(J)$. Since u_i is a constant with respect to z_j, we have $\left(\frac{\partial(f_i - u_i)}{\partial z_j}\right) = \left(\frac{\partial f_i}{\partial z_j}\right)$. Thus $J \subseteq H$.

For $g_1, \dots, g_s \in I$, the $s \times s$-minors of $\left(\frac{\partial g_i}{\partial z_j}\right)$ are contained in the $s \times s$-minors of $\left(\frac{\partial f_i}{\partial z_j}\right)$. Thus it suffices to consider s polynomials $g_1, \dots, g_s$ from the set $\{f_1 - u_1, \dots, f_m - u_m\}$. Now $f_1 - u_1, \dots, f_m - u_m$ is a regular sequence in B. Thus for $s < m$, $[(g_1, \dots, g_s) :_B I] = (g_1, \dots, g_s)B$. Thus the $m \times m$-minors of $\left(\frac{\partial f_i}{\partial z_j}\right)$ generate H up to radical, and so $H = \operatorname{rad}(J)$.

Hence by [**26**] or [**105**, Theorem 4.1], T_Q is essentially smooth over S if and only if Q does not contain J.

Item 2 follows from item 1 because essentially smooth maps are flat. In view of Corollary 11.4 and (11.6.2), item 3 follows from item 2.

If $\operatorname{ht} Q \geq \dim S$, then $\operatorname{ht}(Q \cap S) \leq \dim S \leq \operatorname{ht} Q$. Hence the set $\{Q \in \operatorname{Spec} T : J \subseteq Q \text{ and } \operatorname{ht}(Q \cap S) > \operatorname{ht} Q\}$ is equal to the set $\{Q \in \operatorname{Spec} T : J \subseteq Q,\ \operatorname{ht} Q < \dim S \text{ and } \operatorname{ht}(Q \cap S) > \operatorname{ht} Q\}$. Thus item 3 is equivalent to item 4.

The ($\Longrightarrow$) direction of item 5 is clear [**73**, Theorem 9.5]. For the ($\Longleftarrow$) direction of item 5 and for item 6, it suffices to show $\mathcal{F}_{\min}$ is empty, and this holds by item 4. $\square$

Remarks 11.10. (1) For φ as in Theorem 11.9, it would be interesting to identify the set $\mathcal{F}_{\min} = \operatorname{Min}(F)$. In particular we are interested in conditions for $J = F$ and/or conditions for $J \subsetneq F$. Example 11.7 is an example where $J = F$, whereas Examples 11.13 contains several examples where $J \subsetneq F$.
(2) If R is a Noetherian integral domain, then the zero ideal is not in $\mathcal{F}_{\min}$ and so $F \neq \{0\}$.
(3) In view of Theorem 11.9.3, we can describe $\mathcal{F}_{\min}$ precisely as

$$\mathcal{F}_{\min} = \{Q \in \operatorname{Spec} T \mid J \subseteq Q, \operatorname{ht}(Q \cap S) > \operatorname{ht} Q \text{ and } \forall P \subsetneq Q, \operatorname{ht}(P \cap S) \leq \operatorname{ht}(P)\}.$$

(4) Item 3 of Theorem 11.9 implies that for each prime ideal Q of $\mathcal{F}_{\min}$ there exist prime ideals P_1 and P_2 of S with $P_1 \subsetneq P_2$ such that Q is minimal over both P_1T and P_2T.

Corollary 11.11 is immediate from Theorem 11.9.

Corollary 11.11. *Let k be a field, let $z_1, \dots, z_n$ be indeterminates over k and let $f, g \in k[z_1, \dots, z_n]$ be algebraically independent over k. Consider the embedding $\varphi : S := k[f, g] \hookrightarrow T := k[z_1, \dots, z_n]$. Assume that the associated Jacobian ideal J is nonzero.*[2] *Then*

1. $\mathcal{F}_{min} \subseteq \{$*minimal primes Q of J with* $\operatorname{ht}(Q \cap S) > \operatorname{ht} Q = 1$ $\}$.
2. *φ is flat $\iff$ for every height-one prime ideal $Q \in \operatorname{Spec} T$ such that $J \subseteq Q$ we have* $\operatorname{ht}(Q \cap S) \leq 1$.
3. *If* $\operatorname{ht} J \geq 2$, *then φ is flat.*

Remark 11.12. In the case where k is algebraically closed, another argument can be used for Corollary 11.11.3: Each height-one prime ideal $Q \in \operatorname{Spec} T$ has the form $Q = hT$ for some polynomial $h \in T$. If φ is not flat, then there exists a prime ideal Q of T of height one, such that $\operatorname{ht}(Q \cap S) = 2$. Then $Q \cap S$ has the form $(f - a, g - b)S$, where $a, b \in k$. Thus $f - a = f_1 h$ and $g - b = g_1 h$ for some polynomials $f_1, g_1 \in T$. Now the Jacobian ideal J of f, g is the same as the Jacobian ideal of $f - a, g - b$, and an easy computation shows that $J \subseteq hT$. Therefore $\operatorname{ht} J \leq 1$.

Examples 11.13. Let k be a field of characteristic different from 2 and let x, y, z be indeterminates over k.

(1) With $f = x$ and $g = xy^2 - y$, consider $S := k[f, g] \overset{\varphi}{\hookrightarrow} T := k[x, y]$. Then $J = (2xy - 1)T$. Since $\operatorname{ht}((2xy - 1)T \cap S) = 1$, φ is flat by Corollary 11.11.2. But φ is not smooth, since J defines the nonsmooth locus and $J \neq T$; see Theorem 11.9.1. Here we have $J \subsetneq F = T$.

(2) With $f = x$ and $g = yz$, consider $S := k[f, g] \overset{\varphi}{\hookrightarrow} T := k[x, y, z]$. Then $J = (y, z)T$. Since $\operatorname{ht} J \geq 2$, φ is flat by Corollary 11.11.3. Again φ is not smooth since $J \neq T$.

(3) The examples given in items 1 and 2 may also be described by taking $R = k[x]$. In item 1, we then have $S := R[xy^2 - y] \hookrightarrow R[y] =: T$. The Jacobian $J = (2xy - 1)T$ is the same but is computed now as just a derivative. In item 2,

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[2] This is automatic if the field k has characteristic zero.

we have $S := R[yz] \hookrightarrow R[y,z] =: T$. The Jacobian $J = (y,z)T$ is now computed by taking the partial derivatives $\frac{\partial(yz)}{\partial y}$ and $\frac{\partial(yz)}{\partial z}$.

(4) Let $R = k[x]$ and $S = R[xyz] \hookrightarrow R[y,z] =: T$. Then $J = (xz, xy)T$. Thus J has two minimal primes xT and $(y,z)T$. Notice that $xT \cap S = (x, xyz)S$ is a prime ideal of S of height two, while $(y,z)T \cap S$ has height one. Therefore $J \subsetneq F = xT$.

(5) Let $R = k[x]$ and $S = R[xy + xz] \hookrightarrow R[y,z] =: T$. Then $J = xT$.

(6) Let $R = k[x]$ and $S = R[xy + z^2] \hookrightarrow R[y,z] =: T$. Then $J = (y,z)T$. Hence $S \hookrightarrow T$ is flat but not regular.

(7) Let $R = k[x]$ and $S = R[xy + z] \hookrightarrow R[y,z] =: T$. Then $J = T$. Hence $S \hookrightarrow T$ is a regular map.

Corollary 11.14. *With the notation of Theorem 11.9, we have*

1. *If $Q \in \mathcal{F}_{min}$, then Q is a nonmaximal prime of T.*
2. *$\mathcal{F}_{min} \subseteq \{Q \in \operatorname{Spec} T : J \subseteq Q,\ \dim(T/Q) \geq 1$ and $\operatorname{ht}(Q \cap S) > \operatorname{ht} Q\}$.*
3. *φ is flat $\iff$ $\operatorname{ht}(Q \cap S) \leq \operatorname{ht}(Q)$ for every nonmaximal $Q \in \operatorname{Spec}(T)$ with $J \subseteq Q$.*
4. *If $\dim R = d$ and $\operatorname{ht} J \geq d + m$, then φ is flat.*

Proof. For item 1, suppose $Q \in \mathcal{F}_{\min}$ is a maximal ideal of T. Then $\operatorname{ht} Q < \operatorname{ht}(Q \cap S)$ by Theorem 11.9.3. By localizing at $R \setminus (R \cap Q)$, we may assume that R is local with maximal ideal $Q \cap R := \mathbf{m}$. Since Q is maximal, T/Q is a field finitely generated over $R/\mathbf{m}$. By the Hilbert Nullstellensatz [**73**, Theorem 5.3], T/Q is algebraic over $R/\mathbf{m}$ and $\operatorname{ht} Q = \operatorname{ht}(\mathbf{m}) + n$. It follows that $Q \cap S = P$ is maximal in S and $\operatorname{ht} P = \operatorname{ht}(\mathbf{m}) + m$. The algebraic independence hypothesis for the f_i implies that $m \leq n$, and therefore that $\operatorname{ht} P \leq \operatorname{ht} Q$. This contradiction proves item 1. Item 2 follows from Theorem 11.9.3 and item 1.

Item 3 follows from Theorem 11.9.5 and item 1, and item 4 follows from Theorem 11.9.6. □

As an immediate corollary to Theorem 11.9 and Corollary 11.14, we have:

Corollary 11.15. *Let R be a Noetherian ring, let $z_1, \dots, z_n$ be indeterminates over R and let $f_1, \dots, f_m \in R[z_1, \dots, z_n]$ be algebraically independent over R. Consider the embedding $\varphi : S := R[f_1, \dots, f_m] \hookrightarrow T := R[z_1, \dots, z_n]$, let J be the Jacobian ideal of φ and let F be the (reduced) ideal that describes the nonflat locus of φ as in (11.6.2). Then $J \subseteq F$ and either $F = T$, that is, φ is flat, or $\dim(T/Q) \geq 1$, for each $Q \in \operatorname{Spec}(T)$ that is minimal over F.*

11.3. Applications to polynomial extensions

Proposition 11.16 considers behavior of the extension $\varphi : S \hookrightarrow T$ with respect to prime ideals of R.

Proposition 11.16. *Let R be a commutative ring, let $z_1, \dots, z_n$ be indeterminates over R, and let $f_1, \dots, f_m \in R[z_1, \dots, z_n]$ be algebraically independent over R. Consider the embedding $\varphi : S := R[f_1, \dots, f_m] \hookrightarrow T := R[z_1, \dots, z_n]$.*

1. *If $\mathbf{p} \in \operatorname{Spec} R$ and $\varphi_{\mathbf{p}T} : S \to T_{\mathbf{p}T}$ is flat, then $\mathbf{p}S = \mathbf{p}T \cap S$ and the images $\overline{f_i}$ of the f_i in $T/\mathbf{p}T \cong (R/\mathbf{p})[z_1, \dots, z_n]$ are algebraically independent over $R/\mathbf{p}$.*

(2) *If* φ *is flat, then for each* $\mathbf{p} \in \operatorname{Spec}(R)$ *we have* $\mathbf{p}S = \mathbf{p}T \cap S$ *and the images* $\overline{f_i}$ *of the* f_i *in* $T/\mathbf{p}T \cong (R/\mathbf{p})[z_1, \dots, z_n]$ *are algebraically independent over* $R/\mathbf{p}$.

Proof. Item 2 follows from item 1, so it suffices to prove item 1. Assume that $T_{\mathbf{p}T}$ is flat over S. Then $\mathbf{p}T \neq T$ and it follows from [**73**, Theorem 9.5] that $\mathbf{p}T \cap S = \mathbf{p}S$. If the $\overline{f_i}$ were algebraically dependent over $R/\mathbf{p}$, then there exist indeterminates $t_1, \dots, t_m$ and a polynomial $G \in R[t_1, \dots, t_m] \setminus \mathbf{p}R[t_1, \dots, t_m]$ such that $G(f_1, \dots, f_m) \in \mathbf{p}T$. This implies $G(f_1, \dots, f_m) \in \mathbf{p}T \cap S$. But $f_1, \dots, f_m$ are algebraically independent over R and $G(t_1, \dots, t_m) \notin \mathbf{p}R[t_1, \dots, t_m]$ implies $G(f_1, \dots, f_m) \notin \mathbf{p}S = \mathbf{p}T \cap S$, a contradiction. □

Proposition 11.17. *Let* R *be a Noetherian integral domain containing a field of characteristic zero. Let* $z_1, \dots, z_n$ *be indeterminates over* R *and let* $f_1, \dots, f_m \in R[z_1, \dots, z_n]$ *be algebraically independent over* R. *Consider the embedding* $\varphi : S := R[f_1, \dots, f_m] \hookrightarrow T := R[z_1, \dots, z_n]$. *Let* J *be the associated Jacobian ideal and let* F *be the reduced ideal of* T *defining the nonflat locus of* φ . *Then*

(1) *If* $\mathbf{p} \in \operatorname{Spec} R$ *and* $J \subseteq \mathbf{p}T$, *then* $\varphi_{\mathbf{p}T} : S \to T_{\mathbf{p}T}$ *is not flat. Thus we also have* $F \subseteq \mathbf{p}T$.

(2) *If the embedding* $\varphi : S \hookrightarrow T$ *is flat, then for every* $\mathbf{p} \in \operatorname{Spec}(R)$ *we have* $J \nsubseteq \mathbf{p}T$.

Proof. Item 2 follows from item 1, so it suffices to prove item 1. Let $\mathbf{p} \in \operatorname{Spec} R$ with $J \subseteq \mathbf{p}T$, and suppose $\varphi_{\mathbf{p}T}$ is flat. Let $\overline{f_i}$ denote the image of f_i in $T/\mathbf{p}T$. Consider

$$\overline{\varphi} : \overline{S} := (R/\mathbf{p})[\overline{f_1}, \dots, \overline{f_m}] \to \overline{T} := (R/\mathbf{p})[z_1, \dots, z_n].$$

By Proposition 11.16, $\overline{f_1}, \dots, \overline{f_m}$ are algebraically independent over $\overline{R} := R/\mathbf{p}$. Since the Jacobian ideal commutes with homomorphic images, the Jacobian ideal of $\overline{\varphi}$ is zero. Thus for each $Q \in \operatorname{Spec} \overline{T}$ the map $\overline{\varphi}_Q : \overline{S} \to \overline{T_Q}$ is not smooth. But taking $Q = (0)$ gives $\overline{T_Q}$ is a field separable over the field of fractions of $\overline{S}$ and hence $\overline{\varphi_Q}$ is a smooth map. This contradiction completes the proof. □

Theorem 11.18 follows from [**87**, Proposition 2.1] in the case of one indeterminate z, so in the case where $T = R[z]$.

Theorem 11.18. *Let* R *be a Noetherian integral domain, let* $z_1, \dots, z_n$ *be indeterminates over* R, *and let* $T = R[z_1, \dots z_n]$. *Suppose* $f \in T \setminus R$. *Then the following are equivalent:*

(1) $R[f] \to T$ *is flat.*
(2) *For each prime ideal* q *of* R, *we have* $qT \cap R[f] = qR[f]$.
(3) *For each maximal ideal* q *of* R, *we have* $qT \cap R[f] = qR[f]$.
(4) *The nonconstant coefficients of* f *generate the unit ideal of* R.
(5) $R[f] \to T$ *is faithfully flat.*

Proof. Since $f \in T \setminus R$ and R is an integral domain, the ring $R[f]$ is a polynomial ring in the indeterminate f over R. Thus the map $R \to R[f]$ is flat with regular fibers. Hence Corollary 11.4 implies that $R[f] \hookrightarrow T$ is flat $\iff$ for each $Q \in \operatorname{Spec} T$ we have $\operatorname{ht} Q \geq \operatorname{ht}(Q \cap R[f]$. Let $q := Q \cap R$. We have $\operatorname{ht} q = \operatorname{ht} qR[f] = \operatorname{ht} qT$. Thus $R[f] \hookrightarrow T$ is flat implies for each $q \in \operatorname{Spec} R$ that $qT \cap R[f] = qR[f]$. Moreover, if $P := Q \cap R[f]$ properly contains $qR[f]$, then

$\operatorname{ht} P = 1 + \operatorname{ht} q$, while if Q properly contains qT, then $\operatorname{ht} Q \geq 1 + \operatorname{ht} q$. Therefore (1) $\iff$ (2) follows from Corollary 11.4. It is obvious that (2) $\implies$ (3).

(3) $\implies$ (4): Let $a \in R$ be the constant term of f. If the nonconstant coefficients of f are contained in a maximal ideal q of R, then $f - a \in qT \cap R[f]$. Since R is an integral domain, the element $f - a$ is transcendental over R and $f - a \notin qR[f]$ since $R[f]/qR[f]$ is isomorphic to the polynomial ring $(R/q)[x]$. Therefore $qT \cap R[f] \neq qR[f]$ if the nonconstant coefficients of f are in q.

(4) $\implies$ (2): Let $q \in \operatorname{Spec} R$ and consider the map

$$R[f] \otimes_R R/q = R[f]/qR[f] \xrightarrow{\varphi} T \otimes_R R/q = T/qT \cong (R/q)[z_1, \ldots, z_n]. \tag{11.1}$$

Since the nonconstant coefficients of f generated the unit ideal of R, the image of f in $(R/q)[z_1, \ldots, z_n]$ has positive degree. This implies that φ is injective and $qT \cap R[f] = qR[f]$.

This completes a proof that items (1), (2), (3) and (4) are equivalent. To show that these equivalent statements imply (5), it suffices to show for $P \in \operatorname{Spec}(R[f])$ that $PT \neq T$. Let $q = P \cap R$, and let $\kappa(q)$ denote the field of fractions of R/q. Let $\overline{f}$ denote the image of f in $R[f]/qR[f]$. Then $R[f]/qR[f] \cong (R/q)[\overline{f}]$, a polynomial ring in one variable over R/q. Tensoring the map φ of equation 11.1 with $\kappa(q)$ gives an embedding of the polynomial ring $\kappa(q)[\overline{f}]$ into $\kappa(q)[z_1, \ldots, z_n]$. The image of P in $\kappa(q)[\overline{f}]$ is either zero or a maximal ideal of $\kappa(q)[\overline{f}]$. In either case, its extension to $\kappa(q)[z_1, \ldots, z_n]$ is a proper ideal. Therefore $PT \neq T$. It is obvious that (5) $\implies$ (1), so this completes the proof of Theorem 11.18. $\square$

Remark 11.19. A different proof that (4) $\implies$ (1) in Theorem 11.18 is as follows: Let v be another indeterminate and consider the commutative diagram

$$\begin{tikzcd}
R[v] \arrow[r] \arrow[d, "\pi"'] & T[v] = R[z_1, \ldots, z_n, v] \arrow[d, "\pi'"'] \\
R[f] \arrow[r, "\varphi"] & \frac{R[z_1, \ldots, z_n, v]}{(v - f(z_1, \ldots, z_n))}.
\end{tikzcd}$$

where π maps $v \to f$ and π' is the canonical quotient homomorphism. By [**71**, Corollary 2, p. 152] or [**73**, Theorem 22.6 and its Corollary, p. 177], φ is flat if the coefficients of $f - v$ generate the unit ideal of $R[v]$. Moreover, the coefficients of $f - v$ as a polynomial in $z_1, \ldots, z_n$ with coefficients in $R[v]$ generate the unit ideal of $R[v]$ if and only if the nonconstant coefficients of f generate the unit ideal of R. For if $a \in R$ is the constant term of f and $a_1, \ldots, a_r$ are the nonconstant coefficients of f, then $(a_1, \ldots, a_r)R = R$ clearly implies that $(a - v, a_1, \ldots, a_r)R[v] = R[v]$. On the other hand, if $(a - v, a_1, \ldots, a_r)R[v] = R[v]$, then setting $v = a$ implies that $(a_1, \ldots, a_r)R = R$.

We observe in Proposition 11.20 that one direction of Theorem 11.18 holds for more than one polynomial: see also [**87**, Theorem 3.8] for a related result concerning flatness.

Proposition 11.20. *Let $z_1, \ldots, z_n$ be indeterminates over an integral domain R. Let $f_1, \ldots, f_m$ be polynomials in $R[z_1, \ldots, z_n] := T$ that are algebraically independent over $\mathcal{Q}(R)$. If the inclusion map $\varphi : S := R[f_1, \ldots f_m] \to T$ is flat, then the nonconstant coefficients of each of the f_i generate the unit ideal of R.*

Proof. The algebraic independence of the f_i implies that the inclusion map $R[f_i] \hookrightarrow R[f_1, \dots, f_m]$ is flat, for each i with $1 \leq i \leq m$. If $S \longrightarrow T$ is flat, then so is the composition $R[f_i] \longrightarrow S \longrightarrow T$, and the statement follows from Theorem 11.18 $\square$

Exercises

(1) Let k be an algebraically closed field of characteristic zero and let T denote the polynomial ring $k[x]$. Let $f \in T$ be a polynomial of degree $d \geq 2$ and let $S := k[f]$.

(i) Prove that the map $S \hookrightarrow T$ is free and hence flat.

(ii) Prove that the prime ideals $Q \in \operatorname{Spec} T$ for which $S \to T_Q$ is not a regular map are precisely the primes Q such that the derivative $\frac{df}{dx} \in Q$.

(iii) Deduce that $S \hookrightarrow T$ is not smooth.

(2) With $S = k[x, xy^2 - y] \hookrightarrow T = k[x, y]$ and $J = (2xy - 1)T$ as in Examples 11.13.1, prove that $\operatorname{ht}(J \cap S) = 1$.

Suggestion. Show that $J \cap S \cap k[x] = (0)$ and use that, for A an integral domain, prime ideals of the polynomial ring $A[y]$ that intersect A in (0) are in one-to-one correspondence with prime ideals of $K[y]$, where $K = (A \setminus \{0\})^{-1}A$ is the field of fractions of A.

(3) Let $z_1, \dots, z_n$ be indeterminates over a ring R, and let $T = R[z_1, \dots z_n]$. Suppose $f \in T \setminus R$. Modify the proof of (3) implies (4) of Theorem 11.18 to prove that $qT \cap R[f] = qR[f]$ for each maximal ideal q of R implies that the nonconstant coefficients of f generate the unit ideal of R without the assumption that the ring R is an integral domain.

Suggestion. Assume that the nonconstant coefficients of f are contained in a maximal ideal q of R. Observe that one may assume that f as a polynomial in $R[z_1, \dots, z_n]$ has zero as its constant term and that the ring R is local with maximal ideal q. Let M be a monomial in the support of f of minimal total degree and let $b \in R$ denote the coefficient of M for f. Then b is nonzero, but $f \in qR[f]$ implies that $b \in qb$ and this implies, by Nakayama's lemma, that $b = 0$.

(4) Let k be a field and let $T = k[[u, v, w, z]]$ be the formal power series ring over k in the variables u, v, w, z. Define a k-algebra homomorphism φ of T into the formal power series ring $k[[x, y]]$ by defining

$$\varphi(u) \; = \; x^4, \quad \varphi(v) \; = \; x^3y, \quad \varphi(w) \; = \; xy^3, \quad \varphi(z) \; = \; y^4.$$

Let $P = \ker(\varphi)$ and let $I = (v^3 - u^2w,\ w^3 - z^2v)T$. Notice that $I \subset P$, and that the ring $\varphi(T) = k[[x^4, x^3y, xy^3, y^4]]$ is not Cohen-Macaulay. Let $S = T/I$, and let $R = k[[u, z]] \subset T$.

(a) Prove that $P \cap R = (0)$.

(b) Prove that the ring S is Cohen-Macaulay and a finite free R-module.

(c) Prove that PS is a minimal prime of S and S/PS is not flat over R.

Suggestion. To see that S is module finite over R, observe that

$$\frac{S}{(u, z)S} \;=\; \frac{T}{(u, z, v^3 - u^2w, w^3 - z^2v)T},$$

and the ideal $(u, z, v^3 - u^2w, w^3 - z^2v)T$ is primary for the maximal ideal of T. Hence by Theorem 3.8, S is a finite R-module.

CHAPTER 12

Height-one primes and limit-intersecting elements

Let z be a nonzero nonunit of a normal Noetherian integral domain R and let R^* denote the (z)-adic completion of R. As in Construction 5.3, we consider in this chapter the structure of a subring A of R^* of the form $A := \mathcal{Q}(R)(\tau_1, \tau_2, \dots \tau_s) \cap \widehat{R}$, where $\tau_1, \tau_2, \dots, \tau_s \in zR^*$ are algebraically independent elements over R and every nonzero element of $R[\tau_1, \tau_2, \dots, \tau_s]$ is regular on R^*.

If the intersection ring A can be expressed as a directed union B of localized polynomial extension rings of R as in Section 6.1, then the computation of A is easier. Recall that $\tau_1, \tau_2, \dots, \tau_s$ are called limit-intersecting for A if the ring A is such a directed union; see Definitions 6.5 and 6.12. In Weak Flatness Theorem 12.5 (Inclusion Version) we give criteria for $\tau_1, \tau_2, \dots, \tau_s$ to be limit-intersecting for A. We present a version of this result for Homomorphic Image Construction 5.4 in Weak Flatness Theorem 12.6 (Homomorphic Image Version).

We use Weak Flatness Theorem 12.5 to establish in Examples 13.7 a family of examples where the approximating ring B is equal to the intersection ring A and is not Noetherian. In Chapter 22 we consider stronger forms of the limit-intersecting condition that are useful for constructing examples. In Chapter 23 we consider conditions for A to be excellent.

12.1. The limit-intersecting condition

We consider the following two properties of an extension $S \hookrightarrow T$ of commutative rings involving height-one prime ideals.

Definitions 12.1. Let $S \hookrightarrow T$ be an extension of commutative rings.

(1) We say that the extension $S \hookrightarrow T$ is *weakly flat*, or that T is *weakly flat* over S, if every height-one prime ideal P of S with $PT \neq T$ satisfies $PT \cap S = P$.

(2) We say that the extension $S \hookrightarrow T$ is *height-one preserving*, or that T is a *height-one preserving* extension of S, if for every height-one prime ideal P of S with $PT \neq T$ there exists a height-one prime ideal Q of T with $PT \subseteq Q$.

Proposition 12.2. *Let $S \hookrightarrow T$ be an extension of commutative rings where S is a Krull domain.*

(1) *If every nonzero element of S is regular on T and each height-one prime ideal of S is contracted from T, then $S = T \cap \mathcal{Q}(S)$.*

(2) *If $S \hookrightarrow T$ is a birational extension and each height-one prime of S is contracted from T, then $S = T$.*

(3) *If T is a Krull domain and $T \cap \mathcal{Q}(S) = S$, then each height-one prime of S is the contraction of a height-one prime of T, and the extension $S \hookrightarrow T$ is height-one preserving and weakly flat.*

PROOF. Item 1 follows from item 2. For item 2, recall from Definition 2.3.2 that $S = \cap\{S_{\mathbf{p}} \,|\, \mathbf{p}$ is a height-one prime ideal of $S\}$. We show that $T \subseteq S_{\mathbf{p}}$, for each height-one prime ideal of S. Since $\mathbf{p}$ is contracted from T, there exists a prime ideal $\mathbf{q}$ of T such that $\mathbf{q} \cap S = \mathbf{p}$; see Exercise 12.7. Then $S_{\mathbf{p}} \subseteq T_{\mathbf{q}}$ and $T_{\mathbf{q}}$ birationally dominates $S_{\mathbf{p}}$. Since $S_{\mathbf{p}}$ is a DVR, we have $S_{\mathbf{p}} = T_{\mathbf{q}}$. Therefore $T \subseteq S_{\mathbf{p}}$, for each $\mathbf{p}$. It follows that $T = S$.

For item 3, since T is a Krull domain, Definition 2.3.2 implies that

$$T = \bigcap\{T_{\mathbf{q}} \,|\, \mathbf{q} \text{ is a height-one prime ideal of } T\}.$$

Hence

$$S = T \cap \mathcal{Q}(S) = \bigcap\{T_{\mathbf{q}} \cap \mathcal{Q}(S) \,|\, \mathbf{q} \text{ is a height-one prime ideal of } T\}.$$

Since each $T_{\mathbf{q}}$ is a DVR, Remark 2.1 implies that $T_{\mathbf{q}} \cap \mathcal{Q}(S)$ is either the field $\mathcal{Q}(S)$ or a DVR birational over S. By the discussion in Definition 2.3.2, for each height-one prime $\mathbf{p}$ of S, the localization $S_{\mathbf{p}}$ is a DVR of the form $T_{\mathbf{q}} \cap \mathcal{Q}(S)$. It follows that each height-one prime ideal $\mathbf{p}$ of S is contracted from a height-one prime ideal $\mathbf{q}$ of T, and that T is height-one preserving and weakly flat over S. □

Corollary 12.3 demonstrates the relevance of the weakly flat property for an extension of a Krull domain.

COROLLARY 12.3. *Let $S \hookrightarrow T$ be an extension of commutative rings where S is a Krull domain such that every nonzero element of S is regular on T and $PT \neq T$ for every height-one prime ideal P of S. If $S \hookrightarrow T$ is weakly flat, then $S = \mathcal{Q}(S) \cap T$.*

PROOF. Assume that T is weakly flat over S and that $PT \neq T$ for each height-one prime ideal P of S. Then each height-one prime ideal of S is contracted from T. Thus by Proposition 12.2.1, $S = \mathcal{Q}(S) \cap T$. □

REMARKS 12.4. Let $S \hookrightarrow T$ be an extension of Krull domains.

(a If $S \hookrightarrow T$ is flat, then $S \hookrightarrow T$ is height-one preserving, weakly flat and satisfies PDE. See Definition 2.3.3 and [**10**, Chapitre 7, Proposition 15, page 19].

(b) If U is a multiplicative system in S consisting of units of T, then $S \hookrightarrow T$ is height-one preserving (respectively weakly flat, respectively satisfies PDE) $\iff$ $U^{-1}S \hookrightarrow T$ is height-one preserving (respectively weakly flat, respectively satisfies PDE). This follows since $S \hookrightarrow U^{-1}S$ is flat.

(c) If each height-one prime of S is the radical of a principal ideal, in particular, if S is a UFD, then the extension $S \hookrightarrow T$ is height-one preserving. To see this, let P be a height-one prime of S and suppose that P is the radical of the principal ideal xS. Then $PT \neq T$ if and only if xT is a proper principal ideal of T and every proper principal ideal in a Krull domain is contained in a height-one prime. Hence if $PT \neq T$, then PT is contained in a height-one prime of T.

With these results and remarks in hand, we return to the investigation of the structure of the intersection domain A mentioned in the introduction to this chapter: When does A equal the approximation domain B? We first consider the intersection domain A of Inclusion Construction 5.3 and the approximation ring B of Section 6.1. We show in Weak Flatness Theorem 12.5 that, if the base ring R of the construction is a normal Noetherian domain and the extension

$$R[\tau_1,\dots,\tau_s]\hookrightarrow R^*[1/z]$$

is weakly flat, then the intersection domain A is equal to the approximation domain B; that is, $\tau_1,\dots,\tau_s$ are limit-intersecting in the sense of Definition 6.5.

Weak Flatness Theorem 12.5. (Inclusion Version) *Let R be a normal Noetherian integral domain and let $z\in R$ be a nonzero nonunit. Let R^* denote the (z)-adic completion of R and let $\tau_1,\dots,\tau_s\in R^*$ be algebraically independent over R. Assume that every nonzero element of the polynomial ring $R[\tau_1,\dots,\tau_s]$ is regular on R^*. Let $A=\mathcal{Q}(R)(\tau_1,\dots,\tau_s)\cap R^*$ and let B be the approximation domain defined in Section 6.1. Consider the following statements:*

1. *$A=B$.*
2. *The extension $R[\tau_1,\dots,\tau_s]\hookrightarrow R^*[1/z]$ is weakly flat.*
3. *The extension $B\hookrightarrow R^*[1/z]$ is weakly flat.*
4. *The extension $B\hookrightarrow R^*$ is weakly flat.*

Then

- (a) *Items 2, 3, and 4 are equivalent.*
- (b) *Item 2 $\implies$ item 1.*
- (c) *If R^* is normal, then the four items are equivalent.*

Proof. For (a), we show item 4 $\implies$ item 3 $\implies$ item 2 $\implies$ item 4. To see that item 4 $\implies$ item 3, we have

$$B\overset{\text{w.f.}}{\hookrightarrow}R^*\overset{\text{flat}}{\hookrightarrow}R^*[1/z].$$

Thus, for a height-one prime ideal P of B with $PR^*[1/z]\neq R^*[1/z]$, we have $PR^*\neq R^*$ and $z\notin P$, and so $PR^*[1/z]\cap B=PR^*\cap B=P$, where the last equality uses $B\overset{\text{w.f.}}{\hookrightarrow}R^*$. Thus item 3 holds.

Item 3 $\implies$ item 2: We have $B\overset{\text{w.f.}}{\hookrightarrow}R^*[1/z]$ implies $B[1/z]\overset{\text{w.f.}}{\hookrightarrow}R^*[1/z]$, by Remarks 12.4.b. By Construction Properties Theorem 6.19.2, $B[1/z]$ is a localization of $R[\tau_1,\dots,\tau_s]$. Thus, by Remark 12.4.b, we have $R[\tau_1,\dots,\tau_s]\overset{\text{w.f.}}{\hookrightarrow}R^*[1/z]$.

To see that item 2 $\implies$ item 4, let $P\in\operatorname{Spec}B$ have height one and suppose $PR^*\neq R^*$. If $z\in P$, then, by Construction Properties Theorem 6.19.3, we have $P/zB=PR^*/zR^*$, and so $PR^*\cap B=P$ in this case. Thus we assume $z\notin P$; then $PB[1/z]\cap B=P$.

By assumption, $R[\tau_1,\dots,\tau_s]\hookrightarrow R^*[1/z]$ is weakly flat. Since $B[1/z]$ is a localization of $R[\tau_1,\dots,\tau_s]$ and of B, Remark 12.4.b implies that $B[1/z]\hookrightarrow R^*[1/z]$ is weakly flat. Since $PR^*[1/z]\neq R^*[1/z]$, we have $PR^*[1/z]\cap B[1/z]=PB[1/z]$. Thus $PR^*\cap B=P$ and so $B\hookrightarrow R$ is weakly flat, as desired.

We show item 2 $\implies$ item 1: Since B is a Krull domain and the extension $B\hookrightarrow A$ is birational, by Proposition 12.2.2, it suffices to show that every height-one prime ideal $\mathbf{p}$ of B is contracted from A. As in the proof of item 2 $\implies$ item 4, Construction Properties Theorem 6.19.3 implies that each height-one prime of B containing zB is contracted from A.

Let $\mathbf{p}$ be a height-one prime of B that does not contain zB. Consider the prime ideal $\mathbf{q}=R[\tau_1,\dots,\tau_s]\cap\mathbf{p}$. Since $B[1/z]$ is a localization of the ring $R[\tau_1,\dots,\tau_s]$, we see that $B_{\mathbf{p}} = R[\tau_1, \dots, \tau_s]_{\mathbf{q}}$ and so $\mathbf{q}$ has height one in $R[\tau_1, \dots, \tau_s]$. The weakly flat hypothesis implies $\mathbf{q}R^* \cap R[\tau_1, \dots, \tau_s] = \mathbf{q}$. Hence there exists a prime ideal $\mathbf{w}$ of R^* with $\mathbf{w} \cap R[\tau_1, \dots, \tau_s] = \mathbf{q}$. This implies that $\mathbf{w} \cap B = \mathbf{p}$ and thus also $(\mathbf{w} \cap A) \cap B = \mathbf{p}$. Hence every height-one prime ideal of B is the contraction of a prime ideal of A. Thus $A = B$ as desired.

To prove (c), we assume R^* is a normal Noetherian domain. Thus R^* is a Krull domain; see Definition 2.3.2. We prove item 1 $\implies$ item 4: Since $B = A = \mathcal{Q}(B) \cap R^*$, Proposition 12.2 implies the extension $B \hookrightarrow R^*$ is weakly flat. □

In Theorem 12.6, we present a version of Weak Flatness Theorem 12.5 that applies to Homomorphic Image Construction 5.4.

Weak Flatness Theorem 12.6. (Homomorphic Image Version) *Let R be a normal Noetherian integral domain and let $z \in R$ be a nonzero nonunit. Let R^* denote the (z)-adic completion of R and let I be an ideal of R^* having the property that $P \cap R = (0)$ for each associated prime ideal P of I. Let the rings A and B be as defined in Section 6.12. Assume that B is a Krull domain. Then*

1. *If the extension $R \hookrightarrow (R^*/I)[1/z]$ is weakly flat, then $A = B$, that is, the construction is limit-intersecting as in Definition 6.12.*
2. *If R^*/I is a normal integral domain, then the following statements are equivalent:*
   - (a) *$A = B$.*
   - (b) *$R \hookrightarrow (R^*/I)[1/z]$ is weakly flat.*
   - (c) *The extension $B \hookrightarrow (R^*/I)[1/z]$ is weakly flat.*
   - (d) *The extension $B \hookrightarrow R^*/I$ is weakly flat.*

Proof. Theorem 6.17.3 implies that each height-one prime of B containing zB is contracted from R^*/I. Since $B[1/z]$ is a localization of $R[1/z] = U[1/z]$ by Theorem 6.17, and $R \hookrightarrow (R^*/I)[1/z]$ is weakly flat, it follows that $B \hookrightarrow (R^*/I)[1/z]$ is weakly flat. Therefore $B \hookrightarrow R^*/I$ is weakly flat, and by Proposition 12.2.1, we have $B = (R^*/I) \cap \mathcal{Q}(B) = A$. This proves item 1.

For item 2, since R^*/I is a normal integral domain, $A = (R^*/I) \cap \mathcal{Q}(R)$ is a Krull domain. As noted in the proof of item 1, Theorem 6.17 implies that each height-one prime of B containing zB is contracted from R^*/I and $B[1/z]$ is a localization of $R[1/z] = U[1/z]$. It follows that (b), (c) and (d) are equivalent. By Proposition 12.2.3, (a) $\implies$ (d), and by Proposition 12.2.1, (d) $\implies$ (a). □

12.2. Height-one primes in extensions of integral domains

We observe in Proposition 12.7 that a weakly flat extension of Krull domains is height-one preserving.

Proposition 12.7. *If $\phi : S \hookrightarrow T$ is a weakly flat extension of Krull domains, then ϕ is height-one preserving. Moreover, for every height-one prime ideal P of S with $PT \neq T$ there is a height-one prime ideal Q of T with $Q \cap S = P$.*

Proof. Let $P \in \operatorname{Spec} S$ with $\operatorname{ht} P = 1$. By assumption $PT \cap S = P$. Then $S \setminus P$ is a multiplicatively closed subset of T and $PT \cap (S \setminus P) = \emptyset$. Let Q' be an ideal of T that is maximal with respect to $Q' \cap (S \setminus P) = \emptyset$. Then $P \subseteq Q'$, Q' is a prime ideal of T and $Q' \cap S = P$. Let a be a nonzero element of P and let $Q \subseteq Q'$ be a minimal prime divisor of aT. Then Q has height one and $(0) \neq Q \cap S \subseteq P$. Thus $Q \cap S = P$. □

The height-one preserving condition does *not* imply weakly flat as we demonstrate in Example 12.8.

Example 12.8. Let x and y be variables over a field k, let $R = k[[x]][y]_{(x,y)}$ and let $C = k[[x, y]]$. There exists an element $\tau \in \mathbf{n} = (x, y)C$ that is algebraically independent over $\mathcal{Q}(R)$. For any such element τ, let $S = R[\tau]_{(\mathbf{m},\tau)}$. Since R is a UFD, the ring S is also a UFD and the local inclusion map $\varphi : S \hookrightarrow C$ is height-one preserving. There exists a height-one prime ideal P of S such that $P \cap R = 0$. Since the map $S \hookrightarrow C$ is a local map, we have $PC \neq C$. Because φ is height-one preserving, there exists a height-one prime ideal Q of C such that $PC \subseteq Q$. Since C is the $\mathbf{m}$-adic completion $\widehat{R}$ of R and the generic formal fiber of R is zero-dimensional, $\dim(C \otimes_R \mathcal{Q}(R)) = 0$. Hence $Q \cap R \neq 0$. We have $P \subseteq Q \cap S$ and $P \cap R = (0)$. It follows that P is strictly smaller than $Q \cap S$, so $Q \cap S$ has height greater than one. Therefore the extension $\varphi : S \hookrightarrow C$ is not weakly flat.

Remarks 12.9. (1) By Proposition 12.7, an injective map of Krull domains that is weakly flat is also height-one preserving. Thus the equivalent conditions of Theorem 12.5 imply that $B \hookrightarrow R^*$ is height-one preserving.

(2) If the ring B in Theorem 12.5 is Noetherian, then $A = B$ and the equivalent conclusions of Theorem 12.5 hold.

(3) In Chapter 23, (22.12), we present an example of a three-dimensional regular local domain R that dominates $\mathbb{Q}[x, y, z]_{(x,y,z)}$ and has completion $\mathbb{Q}[[x, y, z]]$ and is such that there exists an element τ in the (y)-adic completion of R that is residually limit-intersecting in y over R but fails to be primarily limit-intersecting in y over R. In particular, the rings A and B constructed using τ are equal, yet A and B are not Noetherian. We show in Chapter 23, (23.12) that if R is a Noetherian semilocal domain, then $\tau_1, \ldots, \tau_s \in yR^*$ are primarily limit-intersecting in y over R if and only if B is Noetherian. If this holds, we also have $B = A$.

Question 12.10. Let $(C, \mathbf{n})$ be a complete Noetherian local domain that dominates a quasilocal Krull domain $(D, \mathbf{m})$. Assume that the inclusion map $D \hookrightarrow C$ is height-one preserving, and that $\tau \in \mathbf{n}$ is algebraically independent over D. Does it follow that the local inclusion map $\varphi : S := D[\tau]_{(\mathbf{m},\tau)} \hookrightarrow C$ is height-one preserving?

Discussion 12.11. If D has torsion divisor class group, then S also has torsion divisor class group and by item c of Remark 12.4, the extension $S \hookrightarrow C$ is height-one preserving, and so the answer to Question 12.10 is affirmative in this case. To consider the general case, let P be a height-one prime ideal of S that is not assumed to be the radical of a principal ideal. One may then consider the following cases:

Case (i): If $\operatorname{ht}(P \cap D) = 1$, then $P = (P \cap D)S$. Since $D \hookrightarrow C$ is height-one preserving, $(P \cap D)C \subseteq Q$, for some height-one prime ideal Q of C. Then $PC = (P \cap D)SC \subseteq Q$ as desired.

Case (ii): Suppose $P \cap D = (0)$. Let U denote the multiplicative set of nonzero elements of D. Let t be an indeterminate over D and let $S_1 = D[t]_{(\mathbf{m},t)}$. Consider the following commutative diagram where the map from S_1 to S is the D-algebra isomorphism taking t to τ and λ is the extension mapping $C[[t]]$ onto C.

$$\begin{tikzcd}
 & U^{-1}S_1 \arrow[r, "\subseteq"] & U^{-1}C[t]_{(\mathbf{n},t)} & \\
D \arrow[r, "\subseteq"] \arrow[d, "="'] & S_1 = D[t]_{(\mathbf{m},t)} \arrow[u, "\cup"'] \arrow[r, "\subseteq"] \arrow[d, "\cong"'] & C[t]_{(\mathbf{n},t)} \arrow[u, "\cup"'] \arrow[r, "\subseteq"] & C[[t]] \arrow[d, "\lambda"'] \\
D \arrow[r, "\subseteq"] & S = D[\tau]_{(\mathbf{m},\tau)} & {} \arrow[r, "\varphi"] & C.
\end{tikzcd}$$

Under the above isomorphism of S with S_1, the prime ideal P correponds to a height-one prime ideal P_1 of S_1 such that $P_1 \cap D = (0)$. Since $U^{-1}S_1$ is a localization of a polynomial ring in one variable over a field, the extended ideal $P_1U^{-1}S_1$ is a principal prime ideal. Therefore P_1 is contained in a proper principal ideal of $U^{-1}C[t]_{(\mathbf{n},t)}$.

However, in the above diagram it can happen that the inclusion map

$$U^{-1}S_1 \hookrightarrow U^{-1}C[t]_{(\mathbf{n},t)}$$

may fail to be faithfully flat. As an example to illustrate this, let

$$D := k[x, y = e^x - 1]_{(x,y)} \hookrightarrow k[[x]] =: C$$

and let $P = (xt - y)D[t]$. Then P extends to the whole ring in $U^{-1}C[t]_{(\mathbf{n},t)}$ since $t - \frac{y}{x}$ is a unit of $U^{-1}C[t]_{(\mathbf{n},t)}$.

Exercises

Let $T = k[x, y, z]$ be a polynomial ring in the 3 variables x, y, z over a field k, and consider the subring $S = k[xy, xz, yz]$ of T.

(1) Prove that the field extension $\mathcal{Q}(T)/\mathcal{Q}(S)$ is algebraic with $[\mathcal{Q}(T) : \mathcal{Q}(S)] = 2$.
(2) Deduce that $xy,\ xz,\ yz$ are algebraically independent over k, so S is a polynomial ring in 3 variables over k.
(3) Prove that the extension $S \hookrightarrow T$ is height-one preserving, but is not weakly flat.
(4) Prove that $T \cap \mathcal{Q}(S) = S[x^2, y^2, z^2]$ is a Krull domain that properly contains S.
(5) Prove that the map $S \hookrightarrow T[\frac{1}{xyz}]$ is flat.
(6) Prove that $S[\frac{1}{xyz}] = T[\frac{1}{xyz}]$. (Notice that $S[\frac{1}{xyz}]$ is not a localization of S since xyz is not in $\mathcal{Q}(S)$.)
(7) Give a direct proof of Proposition 12.3 using primary decomposition in the case where T is also a Krull domain.

Suggestion Let $\mathbf{p}$ be a height-one prime ideal of S, let $0 \neq a \in \mathbf{p}$ and consider an irredundant primary decomposition

$$aT = Q_1 \cap \cdots \cap Q_s$$

of the principal ideal aT.

(a) Why does a primary decomposition exist? Why is each Q_i primary for a height-one prime ideal P_i of T?

(b) Show that $aS = \mathcal{Q}(S) \cap aT$.

(c) Show that there exists an integer $t \in \{1, ldots, s\}$ such that the ideal $!_1 \cap \ldots \cap Q_t \cap S$ is the P-primary component of aS. Show that $P_i \cap S = P$, for some i. (Or: Show that T is weakly flat over S.)

For the converse, add ...

CHAPTER 13

Insider construction details,

In this chapter we continue the development of the Insider Construction begun in Chapter 10. In Section 13.1 we expand the notation for the Insider Construction to the case where the base ring R is a Noetherian domain that is not necessarily a polynomial ring over a field. As before we first construct an "outside" Noetherian domain A^{out} using Inclusion Construction 5.3. We require that this intersection domain A^{out} is equal to its corresponding approximation integral domain B^{out} that is the nested union of localized polynomial rings from Section 6.1. Then we construct inside $A^{\text{out}} = B^{\text{out}}$ two "insider" integral domains: A^{ins}, an intersection of a field with a power series ring as in Construction 5.3, and B^{ins}, a nested union of localized polynomial rings that "approximates" A^{ins} as in Section 6.1. We show that B^{ins} is Noetherian and equal to A^{ins} if a certain map of polynomial rings over R is flat.

In Section 13.1, we describe background and notation for the construction. Theorem 13.3 in Section 13.2 gives necessary and sufficient conditions for the integral domains constructed with the insider construction to be Noetherian and equal. In Section 13.3, we use the analysis of flatness for polynomial extensions from Chapter 11 to obtain a general flatness criterion for the Insider Construction. This yields examples where the constructed domains A and B are equal and are not Noetherian.

In Chapter 15, we use the Insider Construction to establish the existence for each integer $d \geq 3$ and each integer h with $2 \leq h \leq d-1$ of a d-dimensional regular local domain $(A, \mathbf{n})$ having a prime ideal P of height h with the property that the extension $P\widehat{A}$ is not integrally closed. In Chapter 17, we use the Insider Construction to obtain, for each positive integer n, an explicit example of a 3-dimensional quasilocal unique factorization domain B such that the maximal ideal of B is 2-generated, B has precisely n prime ideals of height two, and each prime ideal of B of height two is not finitely generated.

13.1. Describing the construction

We use the following setting and details for the Insider Construction in this chapter. This setting includes Noetherian domains that are not necessarily local and thus generalizes Settings 10.1 and 10.4.

Insider Construction 13.1. (Generalized version) Let R be a Noetherian integral domain. Let z be a nonzero nonunit of R and let R^* be the (z)-adic completion of R. The intersection domain of Construction 5.3 and the corresponding approximation domain of Section 6.1 are inside R^*. Assume that $\tau_1, \ldots, \tau_n \in zR^*$ are algebraically independent over R and are such that nonzero elements of $R[\tau_1, \ldots, \tau_n]$ are regular on R^*. Let $\underline{\tau}$ abbreviate the list $\tau_1, \ldots, \tau_n$. We define the *intersection* *domain corresponding to* $\underline{\tau}$ to be the ring $A_{\underline{\tau}} := K(\tau_1, \dots, \tau_n) \cap R^*$. The Noetherian Example Theorem 8.8 implies that $A_{\underline{\tau}}$ is simultaneously Noetherian and computable as a nested union $B_{\underline{\tau}}$ of certain associated localized polynomial rings over R using $\underline{\tau}$ if and only if the extension $T := R[\underline{\tau}] := R[\tau_1, \dots, \tau_n] \overset{\psi}{\hookrightarrow} R^*[1/z]$ is flat. Moreover, if this flatness condition holds, then $A_{\underline{\tau}}$ is a localization of a subring of $T[1/z]$ and $A_{\underline{\tau}}[1/z]$ is a localization of T.

We assume that $\psi : T \hookrightarrow R^*[1/z]$ is flat, so that the intersection domain $A_{\underline{\tau}}$ is Noetherian and computable, and we take the outside domain to be $A^{\text{out}} := A_{\underline{\tau}} = B_{\underline{\tau}}$ so that $B^{\text{out}} = A^{\text{out}}$. Then we construct new "insider" examples inside $A^{\text{out}} = A_{\underline{\tau}}$ as follows: We choose elements $f_1, \dots, f_m$ of $T := R[\underline{\tau}]$, considered as polynomials in the τ_i with coefficients in R and abbreviated by $\underline{f}$. Assume that $f_1, \dots, f_m$ are algebraically independent over K; thus $m \leq n$. As above we define $A_{\underline{f}} := K(f_1, \dots, f_m) \cap R^*$ to be the *intersection domain corresponding to* $\underline{f}$. We let $B_{\underline{f}}$ be the *approximation domain corresponding to* $\underline{f}$ that approximates $A_{\underline{f}}$, obtained using the f_i as in Section 6.1. Sometimes we refer to $B_{\underline{f}}$ as the *nested union domain corresponding to* $A_{\underline{f}}$. We define $A^{\text{ins}} := A_{\underline{f}}$, and $B^{\text{ins}} := B_{\underline{f}}$.

Recall that the nested union domains $B_{\underline{\tau}}$ and $B_{\underline{f}}$ are localizations of $R[\underline{\tau}]$ and $R[\underline{f}]$ respectively by Construction Properties Theorem 6.19.2. Clearly $B_{\underline{f}} \subseteq B_{\underline{\tau}}$ are quasilocal domains with $B_{\underline{\tau}}$ dominating $B_{\underline{f}}$.

Set $S := R[\underline{f}] := R[f_1, \dots, f_m]$, let φ be the embedding

$$\varphi : S := R[\underline{f}] \overset{\varphi}{\hookrightarrow} T := R[\underline{\tau}], \tag{13.1.1}$$

and let ψ be the inclusion map: $R[\underline{\tau}] \hookrightarrow R^*[1/z]$. Put $\alpha := \psi \circ \varphi : S \to R^*[1/z]$. Then we have

(13.1.2)
$$\begin{tikzcd}
 & & R^*[1/x] \\
R \arrow[r, phantom, "\subseteq"] & S := R[f] \arrow[ur, "\alpha:=\psi\varphi"] \arrow[r, "\varphi"] & T := R[\tau] \arrow[u, "\psi"']
\end{tikzcd}$$

We show in Proposition 10.6 of Chapter 10 for the special case where R is a localized polynomial ring over a field in two or three variables that, if $\varphi_x : S \hookrightarrow T[1/x]$ is flat, then $A_{\underline{f}}$ is Noetherian and is equal to the corresponding approximating domain $B_{\underline{f}}$. In Section 13.2 we make a more thorough analysis of conditions for $A_{\underline{f}}$ to be Noetherian and equal to $B_{\underline{f}}$. In Section 13.3, we present examples where $B_{\underline{f}} = A_{\underline{f}}$ is not Noetherian.

REMARK 13.2. If R is a Noetherian local domain, then R^* is local and hence the intersection domains $A_{\underline{\tau}}$ and $A_{\underline{f}}$ are also local with $A_{\underline{f}}$ possibly non-Noetherian. By Remark 6.4.1 the approximating domain $B_{\underline{f}}$ is also local.

13.2. The flat locus of the insider construction

We assume the notation of Insider Construction 13.1 for this discussion and refer the reader to Section 6.1 for details concerning the approximation domains $B_{\underline{\tau}}$ and $B_{\underline{f}}$ corresponding to the intersection domains $A_{\underline{\tau}}$ and $A_{\underline{f}}$, respectively, of (13.1).

Theorem 8.8 is the basis for our construction of examples.

Let

$$F := \cap\{P \in \operatorname{Spec}(T) \mid \varphi_P : S \to T_P \text{ is not flat }\}. \tag{13.3.0}$$

Thus, as in (11.6.2), the ideal F defines the nonflat locus of the map $\varphi : S \to T$.

For $Q^* \in \operatorname{Spec}(R^*[1/z])$, we consider flatness of the localization $\varphi_{Q^*\cap T}$ of the map φ in Equation 13.1.1:

$$\varphi_{Q^*\cap T} : S \longrightarrow T_{Q^*\cap T} \tag{13.3.1}$$

Theorem 13.3 enables us to recover information about the flatness of α in Diagram 13.1.2 from the map $\varphi : S \to T$.

THEOREM 13.3. *Let R be Noetherian domain, let z be a nonzero nonunit of R and let R^* be the z-adic completion of R. With the notation of Diagram 13.1.2 and Equations 13.3.0 and 13.3.1, we have*

1. *For $Q^* \in \operatorname{Spec}(R^*[1/z])$, the map $\alpha_{Q^*} : S \to (R^*[1/z])_{Q^*}$ is flat if and only if the map $\varphi_{Q^*\cap T}$ in (13.3.1) is flat.*
2. *The following are equivalent:*
   - (i) *The ring A is Noetherian and $A = B$.*
   - (ii) *The ring B is Noetherian.*
   - (iii) *For every maximal $Q^* \in \operatorname{Spec}(R^*[1/z])$, the map $\varphi_{Q^*\cap T}$ in (13.3.1) is flat.*
   - (iv) $FR^*[1/z] = R^*[1/z]$.
3. *The map $\varphi_z : S \hookrightarrow T[1/z]$ is flat if and only if $FT[1/z] = T[1/z]$. Moreover, either of these conditions implies B is Noetherian and $B = A$. It then follows that $A[1/z]$ is a localization of S.*

PROOF. For item 1, we have $\alpha_{Q^*} = \psi_{Q^*} \circ \varphi_{Q^*\cap T} : S \to T_{Q^*\cap T} \to (R^*[1/z])_{Q^*}$. Since the map ψ_{Q^*} is faithfully flat, the composition α_{Q^*} is flat if and only if $\varphi_{Q^*\cap T}$ is flat [**71**, page 27]. For item 2, the equivalence of (i) and (ii) is part of Theorem 8.8. The equivalence of (ii) and (iii) follows from item 1 and Theorem 8.8. For the equivalence of (iii) and (iv), we use $FR^* \neq R^* \iff F \subseteq Q^* \cap T$, for some Q^* maximal in $\operatorname{Spec}(R^*)[1/z]) \iff$ the map in (13.3.1) fails to be flat. The first statement of Item (3) follows from the definition of F and the fact that the nonflat locus of $\varphi : S \to T$ is closed. Theorem 8.8 implies the final statement of item (3). □

13.3. The non-flat locus of the insider construction

To examine the map $\alpha : S \to R^*[1/z]$ in more detail, we use the following terminology.

DEFINITION 13.4. For an extension of Noetherian rings $\varphi : B' \hookrightarrow A'$ and for $d \in \mathbb{N}$, we say that $\varphi : B' \hookrightarrow A'$, *satisfies* LF_d if for each $P \in \operatorname{Spec}(A')$ with $\operatorname{ht}(P) \le d$, the composite map $B' \to A' \to A'_P$ is flat.

COROLLARY 13.5. *Let R be a normal Noetherian domain. With the notation of (13.1) and (13.3.0), we have* $\operatorname{ht}(FR^*[1/z]) > 1 \iff \varphi : S \to R^*[1/z]$ *satisfies* LF_1. *These equivalent conditions imply $B = A$.*

PROOF. The first equivalence follows from the definition of LF_1. Since LF_1 implies weakly flat, Theorem 12.5 implies that $B = A$. □

Theorem 13.6. *Let R be Noetherian domain, let z be a nonzero nonunit of R and let R^* be the z-adic completion of R. With the notation of (13.1), if $m = 1$, that is, there is only one polynomial $f_1 = f$, $S := R[f]$ and $T := R[\tau_1, \ldots, \tau_n]$, then*

1. *The map $S \hookrightarrow T[1/z]$ is flat $\iff$ the nonconstant coefficients of f as a polynomial in $R[\tau_1, \ldots, \tau_n]$ with coefficients in R generate the unit ideal in $R[1/z]$,*
2. *Either of the conditions in (1) implies the insider approximation domain B_f is Noetherian and is equal to the insider intersection domain A_f.*
3. *B_f is Noetherian and $B_f = A_f \iff$ for every prime ideal Q^* in R^* with $z \notin Q^*$, the nonconstant coefficients of f generate the unit ideal in R_q, where $q := Q^* \cap R$.*
4. *If the nonconstant coefficients of f generate an ideal L of $R[1/z]$ of height d, then the map $S \hookrightarrow R^*[1/z]$ satisfies LF_{d-1}, but not LF_d.*

Proof. Item (1) follows from Theorem 11.18 for the ring $R[1/z]$ with $z_i = \tau_i$. By Theorems 8.8 and 13.3, the first condition in item 1 implies item (2).

For item 3, suppose the nonconstant coefficients of f generate the unit ideal of R_q. Then by Theorem 11.18, $R_q[f] \hookrightarrow R_q[\tau_1, \ldots, \tau_n]$ is flat. Since $R_q[\tau_1, \ldots, \tau_n] \to R^*_{Q^*}$ is flat, the map $R_q[f] \hookrightarrow R^*_{Q^*}$ is also flat. For the other direction, we may assume that f has constant term zero. Suppose there exists $Q^* \in \operatorname{Spec} R^*$ with $z \notin Q^*$ such that the coefficients of f are in qR_q, where $q = Q^* \cap R$. Then $f \in qR^*_{Q^*}$ and $qR^*_{Q^*} \neq R^*_{Q^*}$. If $R[f] \hookrightarrow R^*_{Q^*}$ is flat, then we have $qR^*_{Q^*} \cap R[f] = qR[f]$. This implies $f \in qR[f]$, a contradiction.

For item 4, if $Q^* \in \operatorname{Spec}(R^*[1/z])$, then the map $S \hookrightarrow (R^*[1/z])_{Q^*}$ is not flat if and only if $L \subseteq Q^*$. By hypothesis there exists a prime ideal Q^* of height d containing L, but no such prime ideal of height less than d. □

We return to Example 10.8 and establish a more general result.

Examples 13.7. Let k be a field and let $x, y_1, \ldots, y_d$ be indeterminates over k. Let R be either

1. The polynomial ring $R := k[x, y_1, \ldots, y_d]$ with (x)-adic completion $R^* = k[y_1, \ldots, y_d][[x]]$, or
2. The localized polynomial ring $R := k[x, y_1, \ldots, y_d]_{(x, y_1, \ldots, y_d)}$ with (x)-adic completion R^*.

With the notation of (13.1), and setting $z = x$ and $m = 1$, assume that n and s are each greater than or equal to d. Let $f_1 = f := y_1\tau_1 + \cdots + y_d\tau_d$. By Theorem 13.6.4, the map $\varphi_x : S := R[f] \hookrightarrow T[1/x]$ satisfies LF_{d-1}, but fails to satisfy LF_d because the ideal $L = (y_1, \ldots, y_d)R[1/x]$ of nonconstant coefficients of f has height d. For $d \geq 2$, the map $\varphi_x : S \hookrightarrow T[1/x]$ satisfies LF_1 and hence by Corollary 13.5, we have $A_f = B_f$, that is A_f is "limit-intersecting". However, since φ_x does not satisfy LF_d, the map φ_x is not flat and thus A_f is not Noetherian.

CHAPTER 14

Excellent rings: motivation and explanation

In the 1950s, M. Nagata constructed an example of a normal Noetherian local domain $(R, \mathbf{m})$ such that the $\mathbf{m}$-adic completion $\widehat{R}$ is not reduced [**79**, Example 6, p.208], [**77**]. He constructed another example of a normal Noetherian local domain $(R, \mathbf{m})$ that contains a field of characteristic zero and has the property that $\widehat{R}$ is not an integral domain [**79**, Example 7, p.209]. The existence of these examples motivated the search for appropriate conditions to impose on a Noetherian local ring R that will ensure that R does not exhibit the bad behavior of these examples. We consider the following questions:

Questions 14.1.

1. What properties should a "nice" Noetherian ring have?
2. What properties on a Noetherian local ring ensure good behavior with respect to completion?
3. What properties on a Noetherian ring ensure "nice" properties of finitely generated algebras over the given ring?

In the 1960s, A. Grothendieck systematically investigated Noetherian rings that are exceptionally well behaved. He called these rings "excellent". The intent of his definition of excellent rings is that these rings should have the same nice properties as the rings in classical algebraic geometry. The rings in classical algebraic geometry are the affine rings

$$A = k[x_1, \dots, x_n]/I,$$

where k is a field and I is a radical ideal of the polynomial ring $S_n := k[x_1, \dots, x_n]$. That is,

$$I = \sqrt{I} = \{f \in S_n \mid f^t \in I \text{ for some positive integer } t\}.$$

There are three fundamental properties of affine rings that are relevant for the definition of excellent rings. The first two are straightforward:

Property 1: Every algebra of finite type over an affine ring is again an affine ring.

Property 2: Every affine ring is universally catenary.

We recall for Property 3 the following definition.

Definition 14.2. Let A be a Noetherian ring. The *singular locus* of A is:

$$\operatorname{Sing}(A) = \{P \in \operatorname{Spec}(A) \mid A_P \text{ is not a regular local ring}\}.$$

Let k be an algebraically closed field. An *affine algebraic variety* V is a subset of affine n-space k^n for some positive integer n such that V is the zero set of an ideal I of the polynomial ring $S_n = k[x_1, \dots, x_n]$:

$$V = \mathcal{Z}(I) = \{a \in k^n \mid f(a) = 0, \text{ for all } f \in I \}.$$

It is clear that $V = \mathcal{Z}(\sqrt{I})$. We define the *singular locus* of V to be $\operatorname{Sing}(S_n/\sqrt{I})$.

It is a basic fact of algebraic geometry that the singular locus of an affine algebraic variety is again an affine algebraic variety [**33**, Theorem 5.3] This motivates Property 3:

Property 3: For every affine ring A over an arbitrary field k, the singular locus $\operatorname{Sing}(A)$ is closed in the Zariski topology of $\operatorname{Spec}(A)$, that is, there is an ideal $J \subseteq A$ such that $\operatorname{Sing}(A) = \mathcal{V}(J)$.

Property 3 is the key to the definition of excellent rings. Let $A = S/I$, where $S = k[x_1, \dots, x_n]$ is a polynomial ring over a perfect field k. Let P be a prime ideal of S with $I \subseteq P$, let $p = P/I$, and let r be the height of I_P in S_P. Assume that $I = (f_1, \dots, f_s)S$. Then the *Jacobian criterion for regularity* asserts that

$$\begin{aligned} A_p \ \text{ is regular over } k &\iff \quad \operatorname{rank}(\partial f_i/\partial x_j)\operatorname{mod}(P) \ = \ r. \\ &\iff \quad P \not\supseteq \text{ the ideal generated by the} \\ &\qquad\qquad r \times r \text{ minors of } (\partial f_i/\partial x_j). \end{aligned} \tag{14.2.0}$$

The statement that A_p is regular over k means that the map $\psi : k \hookrightarrow A_p$ is a regular map in the sense of Definition 3.22; that is, ψ is flat and has geometrically regular fibers. Equivalently, ψ is flat and, for each prime ideal P of A and each finite algebraic field extension L of k, the ring $A_P \otimes_k L$ is a regular local ring. Since k is a field, A_P is a free k-module and so the extension ψ is flat by Remark 2.20.2. If k is a perfect field, every algebraic extension is separable algebraic; this implies that $A_P \otimes_k L$ is a regular local ring when A_P is. Thus, in the case that k is a perfect field, the map $k \hookrightarrow A_p$ is regular if and only if A_p is a regular local ring. For a proof of the Jacobian criterion for regularity see [**73**, Theorem 30.3 and Remark 2, p. 235].

If k is not a perfect field the Jacobian criterion 14.2.0 as stated above does not imply an equivalence of the condition that A_p is a regular local ring with the conditions on the righthand side. Instead, in the case where k is a non-perfect field, the condition on the righthand side of Criterion 14.2.0 is equivalent to the statement that A_p is smooth[1] over k, a condition that is stronger than the statement that A_p is a regular local ring[**73**, Theorem 30.3].

Example 14.3 is an example of a local ring over a non-perfect field k that is a regular local ring, but is not smooth over k.

Example 14.3. Let k be a field of characteristic $p > 0$ such that k is not perfect, that is, k^p is properly contained in k. Let $a \in k \setminus k^p$ and let $f = x^p - a$. Then $L = k[x]/(f)$ is a proper purely inseparable extension field of k. Since $\partial f/\partial x = 0$, the Jacobian criterion for smoothness shows that L fails to be smooth over k. However, L is a field and thus a regular local ring. There exists a k^p-derivation $D : k[x] \to k[x]$ with $D(f) \neq 0$. This reflects the basic idea of Nagata's Jacobian criterion for regularity [**73**, Theorem 30.10].

If k is a field of characteristic $p > 0$, the main idea of Nagata's Jacobian criterion for regularity is to include the k^{p^n}-derivations of k for all $n \in \mathbb{N}$ in addition to the partial derivatives $\partial f_i/\partial x_j$.

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[1] For the definition of smoothness see Definition 11.5.

A first approach towards enlarging the class of affine rings might be to consider those rings that admit "Jacobian criteria", but this class is rather small. The following example from [**96**, p. 319] illustrates an example of an excellent Noetherian local ring that fails to satisfy Jacobian criteria:

Example 14.4. Let $\sigma = e^{(e^x - 1)} \in \mathbb{Q}[x]$. Notice that σ and $\partial\sigma/\partial x$ are algebriacally independent over $\mathbb{Q}(x)$. As in Example 4.1, consider the intersection domain

$$A \ := \ \mathbb{Q}(x, \sigma) \cap \mathbb{Q}[[x]].$$

By Remark 2.1, A is a DVR with maximal ideal xA and field of fractions $\mathbb{Q}(x, \sigma)$. We have $\mathbb{Q}[x]_{(x)} \subset A \subset \mathbb{Q}[[x]]$. If $d : A \hookrightarrow \mathbb{Q}[[x]]$ is a derivation, then $d(\sigma) = d(x)\partial\sigma/\partial x$. Since σ and $\partial\sigma/\partial x$ are algebraically independent over $\mathbb{Q}(x)$ it follows that $d(\sigma) \notin A$ whenever $d(x) \neq 0$. Hence there is only the trivial derivation from A into itself. Since every DVR containing a field of characteristic 0 is excellent [**32**, Chap IV], [**91**], the ring A is excellent.

There is, however, an important class of Noetherian local rings that admit Jacobian and regularity criteria, namely, the class of complete Noetherian local rings. These criteria were established by Nagata and Grothendieck and are similar to the above mentioned criterion. The main objective of the theory of excellent rings is to exploit the Jacobian criteria on the completion $\widehat{A}$ of an excellent local ring A in order to describe certain properties of A, even though the ring A itself may fail to admit Jacobian criteria. This theory requires considerable theoretical background. For example, Grothendieck's theory of formal smoothness and regularity was developed to work out the connection between a local ring A and its completion $\widehat{A}$.

In the following discussion we sketch the main ideas in the definition of excellent rings.

Discussion 14.5. Let (A, m) be a Noetherian local ring. By Cohen's structure theorems the m-adic completion $\widehat{A}$ of A is the homomorphic image of a formal power series ring over a ring K, where K is either a field or a complete discrete valuation ring, that is, $\widehat{A} \cong K[[x_1, \dots, x_n]]/I$; see Remarks 3.11.3. The singular locus $\operatorname{Sing} \widehat{A}$ of $\widehat{A}$ is closed by the Jacobian criterion on complete Noetherian local rings [**73**, Corollary to Theorem 30.10]. One way to guarantee closure of the singular locus of A is to require that the singular loci of A and $\widehat{A}$ be generated by an ideal J of A and the extension $J\widehat{A}$ of J to $\widehat{A}$, that is, to require:

$$\text{(14.5.a)} \quad \operatorname{Sing}(A) \ = \ \mathcal{V}(J) \quad \text{and} \quad \operatorname{Sing}(\widehat{A}) \ = \ \mathcal{V}(J\widehat{A}), \quad \text{for some ideal } J \text{ of } A.$$

We show in Proposition 14.6 below that Condition 14.5.a is equivalent to Condition 14.5.b.

$$\text{(14.5.b)} \qquad \text{for all } Q \in \operatorname{Spec}(\widehat{A}), \quad A_{Q \cap A} \ \text{ is regular } \ \iff \ \widehat{A}_Q \ \text{ is regular.}$$

We first make some observations about Condition 14.5.b. The backward direction " $\Leftarrow$ " of Condition 14.5.b is always satisfied: By Remark 3.7.2, $\widehat{A}$ is flat over A for every Noetherian local ring A. Thus the induced morphism $A_{Q \cap A} \longrightarrow \widehat{A}_Q$ is faithfully flat. Since flatness descends regularity [**73**, Theorem 23.7(i)], if $\widehat{A}_Q$ is regular, then $A_{Q \cap A}$ is regular.

The forward direction " $\Rightarrow$ " of Condition 14.5.b requires an additional assumption about the map $f : A \to \widehat{A}$. If we assume that the fiber of f over $P = Q \cap A$ is regular as in Definition 3.19, it follows that the ring $\widehat{A}_Q/P\widehat{A}_Q$ is regular. By [**73**, Theorem 23.7(ii)], the ring $\widehat{A}_Q$ is regular, if A_P and $\widehat{A}_Q/P\widehat{A}_Q$ are both regular.

Proposition 14.6. *Let $(A, \mathbf{m})$ be a Noetherian local ring and let $\widehat{A}$ be its* $\mathbf{m}$*-adic completion. Condition 14.5.a is equivalent to Condition 14.5.b.*

Proof. It is clear that Condition 14.5.a implies Condition 14.5.b. Assume Condition 14.5.b and let I be the radical ideal of $\widehat{A}$ such that $\operatorname{Sing}(\widehat{A}) = \mathcal{V}(I)$. Then $I = \bigcap_{i=1}^n Q_i$, where the Q_i are prime ideals of $\widehat{A}$. Let $P_i = Q_i \cap A$ for each i and let $I \cap A = J$. Then $J = \bigcap_{i=1}^n P_i$. We observe that $\operatorname{Sing}(A) = \mathcal{V}(J)$. Since $\widehat{A}_{Q_i}$ is not regular, Condition 14.5.b implies that A_{P_i} is not regular. Let $P \in \operatorname{Spec} A$. If $J \subseteq P$, then $P_i \subseteq P$ for some i, and $P_i \subseteq P$ implies that A_{P_i} is a localization of A_P. Therefore A_P is not regular.

Assume that $J \not\subseteq P$. There exists $Q \in \operatorname{Spec} \widehat{A}$ such that $Q \cap A = P$, and it is clear that $I \not\subseteq Q$. Hence $\widehat{A}_Q$ is regular, and thus by Condition 14.5.b the ring A_P is regular. Therefore $\operatorname{Sing} A = \mathcal{V}(J)$.

It remains to observe that $\sqrt{J\widehat{A}} = I$. Clearly $\sqrt{J\widehat{A}} \subseteq I$. Let $Q \in \operatorname{Spec} \widehat{A}$ with $J\widehat{A} \subseteq Q$. Then $J \subseteq Q \cap A := P$ and A_P is not regular. By Condition 14.5.b, the ring $\widehat{A}_Q$ is not regular, so $I \subseteq Q$. $\square$

As in Chapter 3, if A is a ring and $p \in \operatorname{Spec}(A)$, then $k(p)$ denotes the field of fractions $\mathcal{Q}(A/p)$ of A/p. By permutability of localization and residue class formation $k(p) = A_p/pA_p$.

Let $(A, \mathbf{m})$ be a Noetherian local ring and let $\widehat{A}$ be its $\mathbf{m}$-adic completion. Recall from Definition 3.24 that the formal fibers of A are the rings $\widehat{A} \otimes_A k(p) = (A \setminus p)^{-1}(\widehat{A}/p\widehat{A})$ where $p \in \operatorname{Spec}(A)$.

As we observe in the paragraph above Proposition 14.6, Condition 14.5.b holds for a local ring A if the formal fibers of A are regular. In this case by Proposition 14.6 the singular locus of A is closed. Thus we have:

Corollary 14.7. *Let A be a Noetherian local ring such that the formal fibers of A are regular. Then the singular locus* $\operatorname{Sing} A$ *is closed.*

In order to obtain that every algebra essentially of finite type over A also has the property that its singular locus is closed, the stronger condition that the formal fibers of A are geometrically regular as in Definition 3.20 is needed.

Remark 14.8. Let $(A, \mathbf{m})$ be a Noetherian local ring and let $f : A \to B$ be a map to a Noetherian A-algebra. For $P \in \operatorname{Spec} A$, let $C = (A \setminus P)^{-1}(B/f(P)B$ denote the fiber over P in B.

In Definition 3.20, a prime ideal P in A is said to be geometrically regular if for every finite extension field F of $k(P)$ the condition for the fiber over P to be regular, given in Definition 3.19, holds true for every finite purely inseparable field extension $k \subseteq L$. That is, suppose that $B \otimes_k L$ is regular for every finite purely inseparable field extension $k \subseteq L$. Then the ring $B \otimes L$ is regular for every finite field extension $k \subseteq L$.

In the case of a Noetherian local ring, Grothendieck defined excellence as follows:

Definition 14.9. Let A be a Noetherian local ring. Then A is *excellent* if

(a) The formal fibers of A are geometrically regular, that is, for all $P \in \operatorname{Spec}(A)$, the ring $\widehat{A} \otimes_A L$ is regular for all finite purely inseparable field extensions $k(P) \subseteq L$.

(b) A is universally catenary.

Notice that Condition (iii) of Definition 3.27 about closedness of the singular locus of finitely generated A-algebras is not included in Definition 14.9. For the case of the local ring A, this follows from Corollary 14.7.

If A is an excellent local ring then its completion $\widehat{A}$ inherits many properties from A. In particular, we have :

Theorem 14.10. [**32**, Section on excellence] *Let (A, m) be an excellent local ring, $Q \in Spec(A)$, and $P = Q \cap A$ its contraction to A. Then the ring A_P is regular (normal, reduced, Cohen-Macaulay, Gorenstein, respectively) if and only if the ring $\widehat{A}_Q$ is regular (normal, reduced, Cohen-Macaulay, Gorenstein, respectively).*

Corollary 14.11. *If (A, m) is an excellent local ring then $Sing(A)$ is closed in $Spec(A)$.*

If A is not a local ring the formal fibers of A are the formal fibers of the local rings A_m where m is a maximal ideal of A. We say that A has geometrically regular formal fibers if the local rings A_m for all maximal ideals $m \subseteq A$ have geometrically regular formal fibers. If A is a semilocal ring with geometrically regular formal fibers then $\operatorname{Sing}(A)$ is again closed in $\operatorname{Spec}(A)$. If A is a non-semilocal ring with geometrically regular formal fibers then it is possible that $\operatorname{Sing}(A)$ is no longer closed in $\operatorname{Spec}(A)$, as in an example by Nishimura, [**81**]. Therefore the definition of a general (non semilocal) excellent ring requires an additional condition that guarantees that the singular locus of A and of all algebras of finite type over A are closed in their respective spectra. A definition of execllence equivalent to that given in Definition 3.27 is the following:

Definition 14.12. A Noetherian ring A is excellent, if

(1) The formal fibers of A are geometrically regular.
(2) For every finitely generated A-algebra B the singular locus $\operatorname{Sing}(B)$ of B is closed in $\operatorname{Spec}(B)$.
(3) A is univerally catenary.

In working with excellent rings it is often helpful to make use of the notion of a regular morphism. Recall that in Definition 3.22 a homomorphism $f : A \longrightarrow B$ of Noetherian rings is defined to be regular if f is flat and if for every prime ideal $P \in \operatorname{Spec}(A)$ the ring $B \otimes_A k(P)$ is geometrically regular over $k(P)$.

Following Matsumura, in Definition 3.25 we have defined a Noetherian ring A to be a G-ring if the natural map $A_P \longrightarrow \widehat{A_P}$ is regular for all $P \in \operatorname{Spec} A$. If this condition is satisfied for all maximal ideals of A, then it also holds for all prime ideals of A.

Let A be a Noetherian ring and assume that $f : A \to B$ defines B as an A-algebra of finite type. Assume that $g : B \to C$ is a map of Noetherian rings. By Matsumura [**73**, Theorem 32.1] the composition $gf : A \to C$ is regular if g is regular. Since B is an A-algebra of finite type, the following theorem applies:

Theorem 14.13. [**105**, Corollary 1.2] *Let $f : A \longrightarrow B$ be a morphism of Noetherian rings with B of finite type over A. Then the following are equivalent:*

(1) f *is regular.*
(2) f *is smooth, i.e, B is a smooth A-algebra.*

Since B is of finite type over A, the non-smooth locus of B over A, that is, the set

$$T = \{P \in \operatorname{Spec}(B) \mid B_P \text{ is not smooth over } A\},$$

is closed in $\operatorname{Spec}(B)$. Thus there is an ideal $J \subseteq B$ with $T = V(J)$. By Elkik [**26**], the ideal J can be computed by using the Jacobian matrix of B over A. For the above situation $A \xrightarrow{f} B \xrightarrow{g} C$, one can then conclude that gf is a regular morphism if and only if $JC = C$.

CHAPTER 15

Integral closure under extension to the completion

Using the insider construction described in Chapter 13, we present in this chapter an example of a 3-dimensional regular local domain $(A, \mathbf{n})$ having a height-two prime ideal P with the property that the extension $P\widehat{A}$ of P to the $\mathbf{n}$-adic completion $\widehat{A}$ of A is not integrally closed. The ring A in the example is a 3-dimensional regular local domain that is a nested union of 5-dimensional regular local domains.

More generally, we use this same technique to establish the existence for each integer $d \geq 3$ and each integer h with $2 \leq h \leq d-1$ of a d-dimensional regular local domain $(A, \mathbf{n})$ having a prime ideal P of height h with the property that the extension $P\widehat{A}$ is not integrally closed, where $\widehat{A}$ is the $\mathbf{n}$-adic completion of A. A regular local domain having a prime ideal with this property is necessarily not excellent, see item 7 of Remark 15.6.

We discuss in Section 15.1 conditions in order that integrally closed ideals of a ring R extend to integrally closed ideals of R', where R' is an R-algebra. In particular, we consider under what conditions integrally closed ideals of a Noetherian local ring A extend to integrally closed ideals of the completion $\widehat{A}$ of A.

15.1. Integral closure under ring extension

For properties of integral closure of ideals, rings and modules we refer to [**106**]. In particular, we use the following definition.

Definitions 15.1. Let I be an ideal of a ring R.

(1) An element $r \in R$ is *integral over* I if there exists a positive integer n and a monic polynomial $f(x) = x^n + \sum_{i=1}^{n} a_i x^{n-i}$ such that $f(r) = 0$ and such that $a_i \in I^i$ for each i with $1 \leq i \leq n$.
(2) If J is an ideal contained in I and $JI^{n-1} = I^n$ for some positive integer n, then J is said to be a *reduction* of I.
(3) The *integral closure* $\overline{I}$ of I is the set of elements of R integral over I.
(4) If $I = \overline{I}$, then I is said to be *integrally closed*.
(5) The ideal I is said to be *normal* if I^n is integrally closed for every $n \geq 1$.

Remarks 15.2. We record the following facts about an ideal I of a ring R.

(1) An element $r \in R$ is integral over I if and only if I is a reduction of the ideal $L = (I, r)R$. To see this equivalence, observe that for a monic

polynomial $f(x)$ as in Definition 15.1.1, we have

$$f(r) = 0 \iff r^n = -\sum_{i=1}^{n} a_i r^{n-i} \in IL^{n-1} \iff L^n = IL^{n-1}.$$

(2) It is well known that $\overline{I}$ is an integrally closed ideal [**106**, Corollary 1.3.1].
(3) An ideal is integrally closed if and only if it is not a reduction of a properly bigger ideal.
(4) A prime ideal is always integrally closed. More generally, a radical ideal is always integrally closed.
(5) Let a, b be elements in a Noetherian ring R and let $I := (a^2, b^2)R$. The element ab is integral over I. If a, b form a regular sequence, then $ab \notin I$ and the ideal I is not integrally closed. More generally, if $h \geq 2$ and $a_1, \ldots, a_h$ form a regular sequence in R and $I := (a_1^h, \ldots, a_h^h)R$, then I is not integrally closed.

Our work in this chapter is motivated by the following questions:

QUESTIONS 15.3.

1. Craig Huneke: "Does there exist an analytically unramified Noetherian local ring $(A, \mathbf{n})$ that has an integrally closed ideal I for which the extension $I\widehat{A}$ to the $\mathbf{n}$-adic completion $\widehat{A}$ is not integrally closed?"
2. Sam Huckaba: "If there is such an example, can the ideal of the example be chosen to be a normal ideal?" See Definition 15.1.6.

Related to Question 15.3.1, we construct in Example 15.7 a 3-dimensional regular local domain A having a height-two prime ideal $I = P = (f, g)A$ such that $I\widehat{A}$ is not integrally closed. Thus the answer to Question 15.3.1 is "yes". This example also shows that the answer to Question 15.3.2 is again "yes". Since f, g form a regular sequence and A is Cohen-Macaulay, the powers P^n of P have no embedded associated primes and therefore are P-primary [**71**, (16.F), p. 112], [**73**, Ex. 17.4, p. 139]. Since the powers of the maximal ideal of a regular local domain are integrally closed, the powers of P are integrally closed. Thus the Rees algebra $A[Pt] = A[ft, gt]$ is a normal domain while the Rees algebra $\widehat{A}[ft, gt]$ is not integrally closed.

REMARKS 15.4. Without the assumption that A is analytically unramified, there exist examples even in dimension one where an integrally closed ideal of a Noetherian local domain A fails to extend to an integrally closed ideal in $\widehat{A}$. If A is reduced but analytically ramified, then the zero ideal of A is integrally closed, but its extension to $\widehat{A}$ is not integrally closed.

As we remark in Chapter 1, examples exhibiting this behavior have been known for a long time. An example in characteristic zero of a one-dimensional Noetherian local domain that is analytically ramified is given by Akizuki in his 1935 paper [**7**]. An example in positive characteristic is given by F.K. Schmidt [**98**, pp. 445-447]. Another example due to Nagata is given in [**79**, Example 3, pp. 205-207]. (See also [**79**, (32.2), p. 114].)

Let R be a commutative ring and let R' be an R-algebra. In Remark 15.6 we list cases where extensions to R' of integrally closed ideals of R are again integrally closed. In this connection we use the following definition.

Definition 15.5. The R-algebra R' is said to be *quasi-normal* if R' is flat over R and the following condition $(N_{R,R'})$ holds: If C is any R-algebra and D is a C-algebra in which C is integrally closed, then also $C \otimes_R R'$ is integrally closed in $D \otimes_R R'$.

Remarks 15.6. Let R be a commutative ring and let R' be an R-algebra.

(1) By [**67**, Lemma 2.4], if R' satisfies $(N_{R,R'})$ and I is an integrally closed ideal of R, then IR' is integrally closed in R'.

(2) Assume that R and R' are Noetherian rings and that R' is a flat R-algebra. Let I be an integrally closed ideal of R. The flatness of R' over R implies every $P' \in \operatorname{Ass}(R'/IR')$ contracts in R to some $P \in \operatorname{Ass}(R/I)$ [**73**, Theorem 23.2]. Since a regular map is quasi-normal, if the map $R \to R'_{P'}$ is regular for each $P' \in \operatorname{Ass}(R'/IR')$, then IR' is integrally closed.

(3) Principal ideals of an integrally closed domain are integrally closed.

(4) In general, integral closedness of ideals is a local condition. If R' is an R-algebra that is *locally normal* in the sense that for every prime ideal P' of R', the local ring $R'_{P'}$ is an integrally closed domain, then the extension to R' of every principal ideal of R is integrally closed by item 3. In particular, if $(A, \mathbf{n})$ is an analytically normal Noetherian local domain, then every principal ideal of A extends to an integrally closed ideal of $\widehat{A}$.

(5) Let $(A, \mathbf{n})$ be a Noetherian local ring and let $\widehat{A}$ be the $\mathbf{n}$-adic completion of A. Since $A/\mathbf{q} \cong \widehat{A}/\mathbf{q}\widehat{A}$ for every $\mathbf{n}$-primary ideal $\mathbf{q}$ of A, the $\mathbf{n}$-primary ideals of A are in one-to-one inclusion preserving correspondence with the $\widehat{\mathbf{n}}$-primary ideals of $\widehat{A}$. It follows that an $\mathbf{n}$-primary ideal I of A is a reduction of a properly larger ideal of A if and only if $I\widehat{A}$ is a reduction of a properly larger ideal of $\widehat{A}$. Therefore an $\mathbf{n}$-primary ideal I of A is integrally closed if and only if $I\widehat{A}$ is integrally closed.

(6) If R is an integrally closed domain, then for every ideal I and element x of R we have $\overline{xI} = x\overline{I}$. If $(A, \mathbf{n})$ is analytically normal and also a UFD, then every height-one prime ideal of A extends to an integrally closed ideal of $\widehat{A}$. In particular if A is a regular local domain, then $P\widehat{A}$ is integrally closed for every height-one prime P of A. If $(A, \mathbf{n})$ is a 2-dimensional local UFD, then every nonprincipal integrally closed ideal of A has the form xI, where I is an $\mathbf{n}$-primary integrally closed ideal and $x \in A$. In particular, this is the case if $(A, \mathbf{n})$ is a 2-dimensional regular local domain. In view of item 5, it follows that every integrally closed ideal of A extends to an integrally closed ideal of $\widehat{A}$ in the case where A is a 2-dimensional regular local domain.

(7) If $(A, \mathbf{n})$ is an excellent local ring, then the map $A \hookrightarrow \widehat{A}$ is quasi-normal by [**32**, (7.4.6) and (6.14.5)], and in this case every integrally closed ideal of A extends to an integrally closed ideal of $\widehat{A}$.

(8) Let $(A, \mathbf{n})$ be a Noetherian local domain and let A^h denote the Henselization of A. Every integrally closed ideal of A extends to an integrally closed ideal of A^h. This follows because A^h is a filtered direct limit of étale A-algebras [**67**, (iii), (i), (vii) and (ix), pp. 800- 801]. Since the map from

A to its completion $\widehat{A}$ factors through A^h, every integrally closed ideal of A extends to an integrally closed ideal of $\widehat{A}$ if and only every integrally closed ideal of A^h extends to an integrally closed ideal of $\widehat{A}$.

15.2. Extension to the completion

We use results from Chapters 8, 9 and 13 In the construction of the following example.

Construction of Example 15.7. Let k be a field of characteristic zero and let x, y and z be indeterminates over k. Let $R := k[x,y,z]_{(x,y,z)}$ and let R^* be the (x)-adic completion of R. Thus $R^* = k[y,z]_{(y,z)}[[x]]$, the formal power series ring in x over the 2-dimensional regular local ring $k[y,z]_{(y,z)}$.

Let α and β be elements of $xk[[x]]$ that are algebraically independent over $k(x)$. Set

$$f \ = \ (y-\alpha)^2, \quad g \ = \ (z-\beta)^2, \quad \text{ and } A \ = \ k(x,y,z,f,g) \ \cap \ R^*.$$

Let $T := R[\alpha,\beta]$ and $S := R[f,g]$. Notice that S and T are both polynomial rings in 2 variables over R and that S is a subring of T. Indeed, with $u := y-\alpha$ and $v := z-\beta$, we have $T = R[u,v]$ and $S = R[u^2,v^2]$. Let

$$A^{\text{out}} \ := \ R^* \ \cap \ k(x,y,z,\alpha,\beta).$$

By Corollary 9.7, the map $T \hookrightarrow R^*[1/x]$ is flat and A^{out} is a 3-dimensional regular local domain that is a directed union of 5-dimensional regular local domains. Also A^{out} has (x)-adic completion R^*

By Theorem 13.3, the ring A is Noetherian and equal to its approximation domain B provided that the map $S \hookrightarrow T[1/x]$ is flat. In our situation, the ring T is a free S-module with $\{1, \ y-\alpha, \ z-\beta, \ (y-\alpha)(z-\beta)\}$ as a free module basis. Therefore the map $S \hookrightarrow T[1/x]$ is flat. The following commutative diagram where all the labeled maps are the natural inclusions displays this situation:

$$\begin{tikzcd}
B = A = R^* \cap \mathcal{Q}(S) \arrow[r, "\gamma_1"] & A^{\text{out}} = R^* \cap \mathcal{Q}(T) \arrow[r, "\gamma_2"] & R^* = A^* \\
S = R[,f,g] \arrow[u, "\delta_1"] \arrow[r, "\varphi"] & T = R[\alpha,\beta] \arrow[u, "\delta_2"] \arrow[r, equal] & T \arrow[u, "\psi"]
\end{tikzcd} \tag{15.1}$$

In order to better understand the structure of A, we recall some of the details of the approximation domain B associated to A.

Approximation Technique 15.8. With k, x, y, z, f, g, R and R^* as in Construction 15.7, we have

$$f = y^2 + \sum_{j=1}^{\infty} b_j x^j, \qquad g = z^2 + \sum_{j=1}^{\infty} c_j x^j,$$

where $b_j, \in k[y]$ and $c_j \in k[z]$. The r^{th} *endpiece*s for f and g are the sequences $\{f_r\}_{r=1}^{\infty}$, $\{g_r\}_{r=1}^{\infty}$ of elements in R^* defined for each $r \geq 1$ by:

$$f_r := \sum_{j=r}^{\infty} \frac{b_j x^j}{x^r} \qquad \text{and} \qquad g_r := \sum_{j=r}^{\infty} \frac{c_j x^j}{x^r}.$$

For each integer $r \geq 1$, the ring B_r is the polynomial ring $k[x, y, z, f_r, g_r]$ localized at the maximal ideal generated by $(x, y, z, f_r - b_r, g_r - c_r)$, and the approximation domain $B = \bigcup_{r=1}^{\infty} B_r$.

THEOREM 15.9. *Let A be the ring constructed in (15.7) and let $P = (f, g)A$, where f and g are as in (15.7). Then*

1. *The ring $A = B$ is a 3-dimensional regular local domain that is a nested union of 5-dimensional regular local domains.*
2. *The ideal P is a height-two prime ideal of A.*
3. *The (xA)-adic completion A^* of A is $R^* = k[y, z]_{(y,z)}[[x]]$ and PA^* is not integrally closed.*
4. *The completion $\widehat{A}$ of A is $\widehat{R} = k[[x, y, z]]$ and $P\widehat{A}$ is not integrally closed.*

PROOF. Notice that the polynomial ring $T = R[\alpha, \beta] = R[y - \alpha, z - \beta]$ is a free module of rank 4 over the polynomial subring $R[f, g]$ since $f = (y - \alpha)^2$ and $g = (z - \beta)^2$. Hence the extension

$$R[f, g] \hookrightarrow R[\alpha, \beta][1/x]$$

is flat. Thus item 1 follows from Theorem 13.3.

For item 2, it suffices to observe that P has height two and that, for each positive integer r, $P_r := (f, g)B_r$ is a prime ideal of B_r. We have $f = xf_1 + y^2$ and $g = xg_1 + z^2$. It is clear that $(f, g)k[x, y, z, f, g]$ is a height-two prime ideal. Since B_1 is a localized polynomial ring over k in the variables $x, y, z, f_1 - b_1, g_1 - c_1$, we see that

$$P_1 B_1[1/x] = (xf_1 + y^2, xg_1 + z^2)B_1[1/x]$$

is a height-two prime ideal of $B_1[1/x]$. Indeed, setting $f = g = 0$ is equivalent to setting $f_1 = -y^2/x$ and $g_1 = -z^2/x$. Therefore the residue class ring $(B_1/P_1)[1/x]$ is isomorphic to a localization of the integral domain $k[x, y, z][1/x]$. Since B_1 is Cohen-Macaulay and f, g form a regular sequence, and since $(x, f, g)B_1 = (x, y^2, z^2)B_1$ is an ideal of height three, we see that x is in no associated prime of $(f, g)B_1$ (see, for example [**73**, Theorem 17.6]). Therefore $P_1 = (f, g)B_1$ is a height-two prime ideal.

For $r > 1$, there exist elements $u_r \in k[x, y]$ and $v_r \in k[x, z]$ such that $f = x^r f_r + u_r x + y^2$ and $g = x^r g_r + v_r x + z^2$. An argument similar to that given above shows that $P_r = (f, g)B_r$ is a height-two prime of B_r. Therefore $(f, g)B$ is a height-two prime of $B = A$.

For items 3 and 4, $R^* = B^* = A^*$ by Construction 15.7 and it follows that $\widehat{A} = k[[x, y, z]]$. To see that $PA^* = (f, g)A^*$ and $P\widehat{A} = (f, g)\widehat{A}$ are not integrally closed, observe that $\xi := (y - \alpha)(z - \beta)$ is integral over PA^* and $P\widehat{A}$ since $\xi^2 = fg \in P^2$. On the other hand, $y - \alpha$ and $z - \beta$ are nonassociate prime elements in the local unique factorization domains A^* and $\widehat{A}$. An easy computation shows that $\xi \notin P\widehat{A}$. Since $PA^* \subseteq P\widehat{A}$, this completes the proof. $\square$

In Example 15.10, we observe that this same technique may be used to construct a d-dimensional regular local domain $(A, \mathbf{n})$ having a prime ideal P of height h with the property that the extension $P\widehat{A}$ is not integrally closed for each integer $d \geq 3$ and each integer h with $2 \leq h \leq d - 1$

Example 15.10. Let k be a field of characteristic zero and for an integer $n \geq 2$ let $x, y_1, \dots, y_n$ be indeterminates over k. Let R denote the $d := n+1$ dimensional regular local ring obtained by localizing the polynomial ring $k[x, y_1, \dots, y_n]$ at the maximal ideal generated by $(x, y_1, \dots, y_n)$. Let h be an integer with $2 \leq h \leq n$ and let $\tau_1, \dots, \tau_h \in xk[[x]]$ be algebraically independent over $k(x)$. For each i with $1 \leq i \leq h$, define $f_i = (y_i - \tau_i)^h$, and set $u_i = y_i - \tau_i$. Consider the rings

$$S := R[f_1, \dots, f_h] \quad \text{and} \quad T := R[\tau_1, \dots, \tau_h] = R[u_1, \dots, u_h].$$

Notice that S and T are polynomial rings in h variables over R and that T is a finite free integral extension of S. Let R^* denote the (x)-adic completion of R, and define

$$A := R^* \cap \mathcal{Q}(S) \quad \text{and} \quad A^{\text{out}} := R^* \cap \mathcal{Q}(T).$$

The following commutative diagram where the labeled maps are the natural inclusions displays this situation:

$$\begin{tikzcd}
B = A = R^* \cap \mathcal{Q}(S) \arrow[r, "\gamma_1"] & A^{\text{out}} = R^* \cap \mathcal{Q}(T) \arrow[r, "\gamma_2"] & R^* = A^* \\
S = R[, f_1, \dots, f_h] \arrow[u, "\delta_1"] \arrow[r, "\varphi"] & T = R[\tau_1, \dots, \tau_h] \arrow[u, "\delta_2"] \arrow[r, equal] & T \arrow[u, "\psi"]
\end{tikzcd} \tag{15.2}$$

Since the map $S \hookrightarrow T$ is flat, Theorem 13.3 implies that A is a d-dimensional regular local ring and is equal to its approximation domain B. Let $P := (f_1, \dots, f_h)A$. An argument similar to that given in Theorem 15.9 shows that P is a prime ideal of A of height h. We have $y_i - \tau_i \in A^*$. Let $\xi = \prod_{i=1}^{h}(y_i - \tau_i)$. Then $\xi^h = f_1 \cdots f_h \in P^h$ implies ξ is integral over PA^*. Since $y_1 - \tau_1, \dots, y_h - \tau_h$ is a regular sequence in A^*, we see that $\xi \notin PA^*$. Therefore the extended ideals PA^* and $P\widehat{A}$ are not integrally closed

15.3. Comments and Questions

In connection with Theorem 15.9 it is natural to ask the following question.

Question 15.11. For P and A as in Theorem 15.9, is P the only prime of A that does not extend to an integrally closed ideal of $\widehat{A}$?

Comments 15.12. In relation to Example 15.7 and to Question 15.11, consider the following commutative diagram, where the labeled maps are the natural inclusions:

$$\begin{tikzcd}
B = A = R^* \cap \mathcal{Q}(S) \arrow[r, "\gamma_1"] & A^{\text{out}} = R^* \cap \mathcal{Q}(T) \arrow[r, "\gamma_2"] & R^* = A^* \\
S = R[, f, g] \arrow[u, "\delta_1"] \arrow[r, "\varphi"] & T = R[\alpha, \beta] \arrow[u, "\delta_2"] \arrow[r, equal] & T \arrow[u, "\psi"]
\end{tikzcd} \tag{15.3}$$

Let $\gamma = \gamma_2 \cdot \gamma_1$. Referring to the diagram above, we observe the following:

(1) Theorem 13.3 implies that $A[1/x]$ is a localization of S. Furthermore, by Proposition 9.5 of Chapter 8, A^{out} is excellent. Notice, however, that A is not excellent since there exists a prime ideal P of A such that $P\widehat{A}$ is

not integrally closed. The excellence of A^{out} implies that if $Q^* \in \operatorname{Spec} A^*$ and $x \notin Q^*$, then the map $\psi_{Q^*} : T \to A^*_{Q^*}$ is regular [**32**, (7.8.3 v)].

(2) Let $Q^* \in \operatorname{Spec} A^*$ be such that $x \notin Q^*$ and let $\mathbf{q}' = Q^* \cap T$. By [**73**, Theorem 32.1] and Item 1 above, if $\varphi_{\mathbf{q}'} : S \to T_{\mathbf{q}'}$ is regular, then $\gamma_{Q^*} : A \to A^*_{Q^*}$ is regular.

(3) Let I be an ideal of A. Since A' and A^* are excellent and both have completion $\widehat{A}$, Remark 15.6.3 shows that the ideals IA', IA^* and $I\widehat{A}$ are either all integrally closed or all fail to be integrally closed.

(4) The Jacobian ideal of $\varphi : S := k[x, y, z, f, g] \hookrightarrow T := k[x, y, z, \alpha, \beta]$ is the ideal of T generated by the determinant of the matrix

$$\mathcal{J} := \begin{pmatrix} \frac{\partial f}{\partial \alpha} & \frac{\partial g}{\partial \alpha} \\ \frac{\partial f}{\partial \beta} & \frac{\partial g}{\partial \beta} \end{pmatrix}.$$

Since the characteristic of the field k is zero, this ideal is $(y - \alpha)(z - \beta)T$.

In Proposition 15.13, we relate the behavior of integrally closed ideals in the extension $\varphi : S \to T$ to the behavior of integrally closed ideals in the extension $\gamma : A \to A^*$.

PROPOSITION 15.13. *With the setting of Theorem 15.9 and Comment 15.12.2, let I be an integrally closed ideal of A such that $x \notin Q$ for each $Q \in \operatorname{Ass}(A/I)$. Let $J = I \cap S$. If JT is integrally closed (resp. a radical ideal) then IA^* is integrally closed (resp. a radical ideal).*

PROOF. Since the map $A \to A^*$ is flat, x is not in any associated prime of IA^*. Therefore IA^* is contracted from $A^*[1/x]$ and it suffices to show $IA^*[1/x]$ is integrally closed (resp. a radical ideal). Our hypothesis implies $I = IA[1/x] \cap A$. By Comment 15.12.1, $A[1/x]$ is a localization of S. Thus every ideal of $A[1/x]$ is the extension of its contraction to S. It follows that $IA[1/x] = JA[1/x]$. Thus $IA^*[1/x] = JA^*[1/x]$.

Also by Comment 15.12.1, the map $T \to A^*[1/x]$ is regular. If JT is integrally closed, then Remark 15.6.7 implies that $JA^*[1/x]$ is integrally closed. If JT is a radical ideal, then the regularity of the map $T \to A^*[1/x]$ implies the $JA^*[1/x]$ is a radical ideal. □

PROPOSITION 15.14. *With the setting of Theorem 15.9 and Comment 15.12, let $Q \in \operatorname{Spec} A$ be such that $Q\widehat{A}$ (or equivalently QA^*) is not integrally closed. Then*

(1) *Q has height two and $x \notin Q$.*
(2) *There exists a minimal prime Q^* of QA^* such that with $\mathbf{q}' = Q^* \cap T$, the map $\varphi_{\mathbf{q}'} : S \to T_{\mathbf{q}'}$ is not regular.*
(3) *Q contains $f = (y - \alpha)^2$ or $g = (z - \beta)^2$.*
(4) *Q contains no element that is a regular parameter of A.*

PROOF. By Remark 15.6.6, the height of Q is two. Since $A^*/xA^* = A/xA = R/xR$, we see that $x \notin Q$. This proves item 1.

By Remark 15.6.7, there exists a minimal prime Q^* of QA^* such that $\gamma_{Q^*} : A \to A^*_{Q^*}$ is not regular. Thus item 2 follows from Comment 15.12.2.

For item 3, let Q^* and $\mathbf{q}'$ be as in item 2. Since γ_{Q^*} is not regular it is not essentially smooth [**32**, 6.8.1]. By [**49**, (2.7)], $(y-\alpha)(z-\beta) \in \mathbf{q}'$. Hence $f = (y-\alpha)^2$ or $g = (z - \beta)^2$ is in $\mathbf{q}'$ and thus in Q. This proves item 3.

Suppose $w \in Q$ is a regular parameter for A. Then A/wA and A^*/wA^* are two-dimensional regular local domains. By Remark 15.6.6, QA^*/wA^* is integrally closed, but this implies that QA^* is integrally closed, which contradicts our hypothesis that QA^* is not integrally closed. This proves item 4. $\square$

Question 15.15. In the setting of Theorem 15.9 and Comment 15.12, let $Q \in \operatorname{Spec} A$ with $x \notin Q$ and let $\mathbf{q} = Q \cap S$. If QA^* is integrally closed, does it follow that $\mathbf{q}T$ is integrally closed?

Question 15.16. In the setting of Theorem 15.9 and Comment 15.12, if a prime ideal Q of A contains f or g, but not both, and does not contain a regular parameter of A, does it follow that QA^* is integrally closed ?

In Example 15.7, the three-dimensional regular local domain A contains height-one prime ideals P such that $\widehat{A}/P\widehat{A}$ is not reduced. This motivates us to ask:

Question 15.17. Let $(A, \mathbf{n})$ be a three-dimensional regular local domain and let $\widehat{A}$ denote the $\mathbf{n}$-adic completion of A. If for each height-one prime P of A, the extension $P\widehat{A}$ is a radical ideal, i.e., the ring $\widehat{A}/P\widehat{A}$ is reduced, does it follow that $P\widehat{A}$ is integrally closed for each $P \in \operatorname{Spec} A$?

Remark 15.18. A problem analogous to that considered here in the sense that it also deals with the behavior of ideals under extension to completion is addressed by Loepp and Rotthaus in [**69**]. They construct nonexcellent Noetherian local domains to demonstrate that tight closure need not commute with completion.

CHAPTER 16

Catenary local rings with geometrically normal formal fibers

In this chapter[1] we consider the catenary property in a Noetherian local ring $(R, \mathbf{m})$ having geometrically normal formal fibers. We prove that the Henselization R^h of R is universally catenary, and relate the catenary and universally catenary properties of R to the fibers of the map $R \hookrightarrow R^h$. We present for each integer $n \geq 2$ an example of a catenary Noetherian local integral domain of dimension n that has geometrically regular formal fibers and is not universally catenary.

16.1. Introduction

Recall that a ring R is *catenary* if, for every pair of comparable prime ideals $P \subset Q$ of R, every saturated chain of prime ideals from P to Q has the same length, see Definitions 3.14. The ring R is *universally catenary* if every finitely generated R-algebra is catenary.

Krull proved [**64**] that every integral domain that is finitely generated as an extension ring of a field is catenary. Cohen proved [**18**] that every complete Noetherian local ring is catenary. These results motivated the question of whether every Noetherian ring (or equivalently every Noetherian local integral domain) is catenary. Using power series rings, Nagata in [**76**], see also [**79**, Example 2, pages 203-205], answered this question by giving an example of a non-catenary family of Noetherian local domains. Each domain in this family is not integrally closed and its integral closure is catenary.

These examples of Nagata motivated the question of whether the integral closure of a Noetherian local domain is catenary. Work on this question continued for over 20 years with Ratliff [**88**], [**89**] a leading researcher in the area. In 1980 T. Ogoma [**84**] resolved the question by establishing the existence of a 3-dimensional Henselian Nagata local domain that is integrally closed but not catenary. Heitmann in [**61**] gave a simplified presentation of Ogoma's example.

Heitmann in [**59**] obtained the following notable characterization of the complete Noetherian local rings that are the completion of a UFD. He proved that every complete Noetherian local ring $(R, \mathbf{m})$ that has depth at least two and has the property that no element in the prime subring of R is a zero-divisor on R is the completion of a Noetherian local UFD. Heitmann used this result to establish the existence of a 3-dimensional Noetherian local UFD $(A, \mathbf{n})$ that is not universally catenary.

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[1] Material in this chapter comes from a paper we wrote that is included in a volume dedicated to Shreeram S. Abhyankar in celebration of his seventieth birthday. In his mathematical work Ram has opened up many avenues. In this chapter we are pursuing one of these related to power series and completions.

We recall that a Noetherian local ring $(R, \mathbf{m})$ with $\mathbf{m}$-adic completion $\widehat{R}$ has *geometrically normal* (respectively *geometrically regular) formal fibers* if for each prime P of R and for each finite algebraic extension k' of the field $k(P) := R_P/PR_P$, the ring $\widehat{R} \otimes_R k(P) \otimes_{k(P)} k'$ is normal (respectively regular), see Definition 3.20.

In this chapter we investigate the catenary property in Noetherian local rings having geometrically normal formal fibers. In Example 16.11 we apply a technique developed in previous chapters to construct, for each integer $n \geq 2$, an example of a catenary Noetherian local integral domain of dimension n with geometrically regular formal fibers that is not universally catenary.

Let $(R, \mathbf{m})$ be a Noetherian local ring. We denote the Henselization of R by R^h. Recall that $(R, \mathbf{m})$ is *formally equidimensional*, or in other terminology *quasi-unmixed*, provided all the minimal primes of the $\mathbf{m}$-adic completion $\widehat{R}$ have the same dimension. A theorem of Ratliff relating the universal catenary property to properties of the completion, see Theorem 3.16 of Chapter 3, is crucial for our work.

In Section 16.2 we present results concerning conditions for a Noetherian local ring $(R, \mathbf{m})$ to be universally catenary. The theorem of Ratliff mentioned above leads to our observation in Proposition 16.1 that a Henselian Noetherian local ring having geometrically normal formal fibers is universally catenary.

Assume that $(R, \mathbf{m})$ is a Noetherian local integral domain having geometrically normal formal fibers. Corollary 16.4 implies that the Henselization R^h of R is universally catenary; moreover, if the integral closure $\overline{R}$ of R is local, then R is universally catenary. Theorem 16.6 asserts that R is universally catenary if and only if the set Γ is empty, where

$$\Gamma := \{P \in \operatorname{Spec}(R^h) \mid \dim(R^h/P) < \dim(R/P \cap R))\}.$$

We also prove that the subset Γ of $\operatorname{Spec} R^h$ is stable under generalization in the sense that if $Q \in \Gamma$ and $P \in \operatorname{Spec} R^h$ is such that $P \subseteq Q$, then $P \in \Gamma$. Thus Γ satisfies a strong "going down" property.

In Theorem 16.7 we prove for R as above that R is catenary but is not universally catenary if and only if the set Γ is nonempty and each prime ideal P in Γ has dimension one. It follows in this case that Γ is a finite subset of the minimal primes of R^h. Thus, as we observe in Corollary 16.8, if R is catenary but not universally catenary, this is signaled by the existence of dimension one minimal primes of the $\mathbf{m}$-adic completion $\widehat{R}$ of R. If R is catenary, each minimal prime of $\widehat{R}$ having dimension different from the dimension of R must have dimension one.

In Section 16.3 we present examples to illustrate the results of Section 16.2. We apply a construction involving power series, homomorphic images and intersections developed in previous chapters, see Chapter 8.

The construction begins with a Noetherian integral domain that may be taken to be a polynomial ring in several indeterminates over a field. In Theorem 16.9 we extend this construction by proving that in certain circumstances it is possible to transfer the flatness, Noetherian and computability properties of integral domains associated with ideals $I_1, \ldots, I_n$ of R to the integral domain associated with their intersection $I = I_1 \cap \cdots \cap I_n$.

We apply these concepts in Examples 16.10 - 16.12 to obtain Noetherian local domains that are not universally catenary. In Remark 16.13, we specifiy precisely which of these rings are catenary. These domains illustrate the results of Section

16.2, because in Section 16.5 we prove that they have geometrically regular formal fibers.

We would like to thank M. Brodmann and R. Sharp for raising a question on catenary/universally catenary rings that motivated our work in this chapter.

16.2. Geometrically normal formal fibers

Throughout this section $(R, \mathbf{m})$ is a Noetherian local ring. We use Theorem 3.16, the result of Ratliff mentioned above, that relates the universally catenary property to properties of the completion in order to prove:

Proposition 16.1. *Let $(R, \mathbf{m})$ be a Henselian Noetherian local ring having geometrically normal formal fibers.*

1. *For each prime ideal P of R, the extension $P\widehat{R}$ to the $\mathbf{m}$-adic completion of R is also a prime ideal.*
2. *The ring R is universally catenary.*

Proof. Item 2 follows from item 1 and Theorem 3.16. In order to prove item 1, observe that the completion of R/P is $\widehat{R}/P\widehat{R}$, and R/P is a Noetherian Henselian local integral domain having geometrically normal formal fibers. Hence, by passing from R to R/P, to prove item 1, it suffices to prove that if R is a Henselian Noetherian local integral domain having geometrically normal formal fibers, then the completion $\widehat{R}$ of R is an integral domain.

Since R has normal formal fibers, the completion $\widehat{R}$ of R is reduced. Hence the integral closure $\overline{R}$ of R is a finitely generated R-module [**79**, (32.2)]. Moreover, since R is Henselian, $\overline{R}$ is local [**79**, (43.12)].

The completion $\widehat{\overline{R}}$ of $\overline{R}$ is $\widehat{R} \otimes_R \overline{R}$ [**79**, (17.8)]. Since the formal fibers of R are geometrically normal, the formal fibers of $\overline{R}$ are also geometrically normal. It follows that $\widehat{\overline{R}}$ is normal [**73**, Corollary, page 184], and hence an integral domain because $\overline{R}$ is local. Since $\widehat{R}$ is a flat R-module, $\widehat{R}$ is a subring of $\widehat{\overline{R}}$. Therefore $\widehat{R}$ is an integral domain. $\square$

Remark 16.2. Let $(R, \mathbf{m})$ be a Noetherian local domain. An interesting result proved by Nagata [**79**, (43.20)] establishes the existence of a one-to-one correspondence between the minimal primes of the Henselization R^h of R and the maximal ideals of the integral closure $\overline{R}$ of R. Moreover, if a maximal ideal $\overline{\mathbf{m}}$ of $\overline{R}$ corresponds to a minimal prime q of R^h, then the integral closure of the Henselian local domain R^h/q is the Henselization of $\overline{R}_{\overline{\mathbf{m}}}$ [**79**, Ex. 2, page 188], [**75**]. Therefore $\operatorname{ht}(\overline{\mathbf{m}}) = \dim(R^h/q)$.

Remark 16.3. Let $(R, \mathbf{m})$ be a Noetherian local ring, let $\widehat{R}$ denote the $\mathbf{m}$-adic completion of R, and let R^h denote the Henselization of R. It is well known that the canonical map $R \hookrightarrow R^h$ is a regular map with zero-dimensional fibers, and that $\widehat{R}$ is also the completion of R^h with respect to its unique maximal ideal $\mathbf{m}^h = \mathbf{m}R^h$. The following statements are equivalent:

1. The map $R \hookrightarrow \widehat{R}$ has (geometrically) normal fibers.
2. The map $R^h \hookrightarrow \widehat{R}$ has (geometrically) normal fibers.

Let P be a prime ideal of R and let $U = R \setminus P$. Then $PR^h = P_1 \cap \cdots \cap P_n$, where the P_i are the minimal prime ideals of PR^h. Then $P\widehat{R} = (\cap_{i=1}^n P_i)\widehat{R}$. Since $\widehat{R}$ is faithfully flat over R^h, finite intersections distribute over this extension, and $P\widehat{R} = \cap_{i=1}^n (P_i\widehat{R})$. Let $S = U^{-1}(\widehat{R}/P\widehat{R})$ denote the fiber over P in $\widehat{R}$ and let $q_i = P_iS$. The ideals $q_1, \ldots, q_n$ of S intersect in (0) and are pairwise comaximal because for $i \neq j$, $(P_i + P_j) \cap U \neq \emptyset$. Therefore $S \cong \prod_{i=1}^n (S/q_i)$. Since a Noetherian ring is normal if and only if it is a finite product of normal Noetherian domains, the fiber over P in $\widehat{R}$ is normal if and only if the the fiber over each of the P_i in $\widehat{R}$ is normal.

Corollary 16.4. *Let R be a Noetherian local domain having geometrically normal formal fibers. Then*

1. *The Henselization R^h of R is universally catenary.*
2. *If the integral closure $\overline{R}$ of R is again local, then R is universally catenary.*

In particular, if R is a normal Noetherian local domain having geometrically normal formal fibers, then R is universally catenary.

Proof. For item 1, the Henselization R^h of R is a Noetherian local ring having geometrically normal formal fibers, so Proposition 16.1 implies that R^h is universally catenary. For item 2, if the integral closure of R is local, then, by Remark 16.2, the Henselization R^h has a unique minimal prime. Since R^h is universally catenary, the completion $\widehat{R}$ is equidimensional, and hence R is universally catenary. □

Theorem 16.5 relates the catenary property of R to the height of maximal ideals in the integral closure of R.

Theorem 16.5. *Let $(R, \mathbf{m})$ be a Noetherian local domain of dimension d and let $\overline{R}$ denote the integral closure of R. If $\overline{R}$ contains a maximal ideal $\overline{\mathbf{m}}$ with* $\mathrm{ht}(\overline{\mathbf{m}}) = r \notin \{1, d\}$, *then there exists a saturated chain of prime ideals in R of length $\leq r$. Hence in this case R is not catenary.*

Proof. Since $\overline{R}$ has only finitely many maximal ideals [**79**, (33.10)], there exists $b \in \overline{\mathbf{m}}$ such that b is in no other maximal ideal of $\overline{R}$. Let $R' = R[b]$ and let $\mathbf{m}' = \overline{\mathbf{m}} \cap R'$. Notice that $\overline{\mathbf{m}}$ is the unique prime ideal of $\overline{R}$ that contains $\mathbf{m}'$. By the Going Up Theorem[**73**, (9.3)], $\mathrm{ht}\,\mathbf{m}' = r$. Since R' is a finitely generated R-module and is birational over R, there exists a nonzero element $a \in \mathbf{m}$ such that $aR' \subseteq R$. It follows that $R[1/a] = R'[1/a]$. The maximal ideals of $R[1/a]$ have the form $PR[1/a]$, where $P \in \mathrm{Spec}(R)$ is maximal with respect to not containing a. Since there are no prime ideals strictly between P and $\mathbf{m}$ [**73**, (13.5)], if $\mathrm{ht}\,P = h$, then there exists in R a saturated chain of prime ideals through P of length $h + 1$. Thus to show R is not catenary, it suffices to establish the existence of a maximal ideal of $R[1/a]$ having height different from $d - 1$. Since $R[1/a] = R'[1/a]$, the maximal ideals of $R[1/a]$ correspond to the prime ideals P' in R' maximal with respect to not containing a. Since $\mathrm{ht}\,\mathbf{m}' > 1$, there exists $c \in \mathbf{m}'$ such that c is not in any minimal prime of aR' nor in any maximal ideal of R' other than $\mathbf{m}'$. Hence there exist prime ideals of R' containing c and not containing a. Let $P' \in \mathrm{Spec}(R')$ be maximal with respect to $c \in P'$ and $a \notin P'$. Then $P' \subset \mathbf{m}'$, so $\mathrm{ht}\,P' \leq r - 1 < d - 1$. It follows that there exists a saturated chain of prime ideals of R of length $\leq r$, and hence R is not catenary. □

Theorem 16.6. *Let* $(R, \mathbf{m})$ *be a Noetherian local integral domain having geometrically normal formal fibers and let* R^h *denote the Henselization of* R*. Consider the set*

$$\Gamma := \{P \in \operatorname{Spec}(R^h) \mid \dim(R^h/P) < \dim(R/(P \cap R))\}.$$

The following statements then hold.

(1) *For* $\mathbf{p} \in \operatorname{Spec}(R)$*, the ring* $R/\mathbf{p}$ *is not universally catenary if and only if there exists* $P \in \Gamma$ *such that* $\mathbf{p} = P \cap R$*.*
(2) *The set* Γ *is empty if and only if* R *is universally catenary.*
(3) *If* $\mathbf{p} \subset \mathbf{q}$ *are prime ideals in* R *and if there exists* $Q \in \Gamma$ *with* $Q \cap R = \mathbf{q}$*, then there also exists* $P \in \Gamma$ *with* $P \cap R = \mathbf{p}$*.*
(4) *If* $Q \in \Gamma$*, then each prime ideal* P *of* R^h *such that* $P \subseteq Q$ *is also in* Γ*, that is, the subset* Γ *of* $\operatorname{Spec} R^h$ *is stable under generalization.*

Proof. The map of $R/\mathbf{p}$ to its $\mathbf{m}$-adic completion $\widehat{R}/\mathbf{p}\widehat{R}$ factors through $R^h/\mathbf{p}R^h$. Since $R \hookrightarrow \widehat{R}$ has geometrically normal fibers, so does the map $R^h \hookrightarrow \widehat{R}$. Proposition 16.1 implies that each prime ideal P of R^h extends to a prime ideal $P\widehat{R}$. Therefore, by Theorem 3.16, the ring $R/\mathbf{p}$ is universally catenary if and only if $R^h/\mathbf{p}R^h$ is equidimensional if and only if there does not exist $P \in \Gamma$ with $P \cap R = \mathbf{p}$. This proves items 1 and (2). To prove item 3, observe that if $R/\mathbf{p}$ is universally catenary, then $R/\mathbf{q}$ is also universally catenary [**73**, Theorem 31.6].

It remains to prove item 4. Let $P \in \operatorname{Spec} R^h$ be such that $P \subset Q$, and let $\operatorname{ht}(Q/P) = n$. Since the fibers of the map $R \hookrightarrow R^h$ are zero-dimensional, the contraction to R of an ascending chain of primes

$$P \;=\; P_0 \;\subsetneq\; P_1 \;\subsetneq\; \cdots \;\subsetneq P_n \;=\; Q$$

of R^h is a strictly ascending chain of primes from $\mathbf{p} := P \cap R$ to $\mathbf{q} := Q \cap R$. Hence $\operatorname{ht}(\mathbf{q}/\mathbf{p}) \geq n$. Since R^h is catenary, we have

$$\dim(R^h/P) \;=\; n + \dim(R^h/Q) \;<\; n + \dim(R/\mathbf{q}) \;\leq\; \dim(R/\mathbf{p}),$$

where the strict inequality is because $Q \in \Gamma$. Therefore $P \in \Gamma$. □

Theorem 16.7. *Let* $(R, \mathbf{m})$ *be a Noetherian local integral domain having geometrically normal formal fibers and let* Γ *be defined as in Theorem 16.6. The ring* R *is catenary but not universally catenary if and only if*

(i) *the set* Γ *is nonempty, and*
(ii) $\dim R^h/P = 1$*, for each prime ideal* $P \in \Gamma$*.*

If these conditions holds, then each $P \in \Gamma$ *is a minimal prime of* R^h*, and* Γ *is a finite nonempty open subset of* $\operatorname{Spec} R^h$*.*

Proof. Assume that R is catenary but not universally catenary. By Theorem 16.6, the set Γ is nonempty and there exist minimal primes P of R^h such that $\dim(R^h/P) < \dim(R^h)$. By Remark 16.2, if a maximal ideal $\overline{\mathbf{m}}$ of $\overline{R}$ corresponds to a minimal prime P of R^h, then $\operatorname{ht}(\overline{\mathbf{m}}) = \dim(R^h/P)$. Since R is catenary, Theorem 16.5 implies that the height of each maximal ideal of the integral closure $\overline{R}$ of R is either one or $\dim(R)$. Therefore $\dim(R^h/P) = 1$ for each minimal prime P of R^h for which $\dim(R^h/P) \neq \dim(R^h)$. Item 4 of Theorem 16.6 implies each $P \in \Gamma$ is a minimal prime of R^h and $\dim R^h/P = 1$.

For the converse, assume that Γ is nonempty and each prime $P \in \Gamma$ has dimension one. Then R is not universally catenary by item 2 of Theorem 16.6 and by item 4 of Theorem 16.6, each prime ideal in Γ is a minimal prime of R^h and therefore lies over (0) in R. To show R is catenary, it suffices to show for each nonzero nonmaximal prime ideal $\mathbf{p}$ of R that $\operatorname{ht}(\mathbf{p}) + \dim(R/\mathbf{p}) = \dim(R)$ [**73**, Theorem 31.4]. Let $P \in \operatorname{Spec}(R^h)$ be a minimal prime of $\mathbf{p}R^h$. Since R^h is flat over R with zero-dimensional fibers, $\operatorname{ht}(\mathbf{p}) = \operatorname{ht}(P)$. Let Q be a minimal prime of R^h with $Q \subseteq P$. Then $Q \notin \Gamma$. For by assumption every prime of Γ has dimension one, so if Q were in Γ, then $Q = P$. But $P \cap R = \mathbf{p}$, which is nonzero, and $Q \cap R = (0)$. Therefore $Q \notin \Gamma$ and hence $\dim(R^h/Q) = \dim(R^h)$. Since R^h is catenary, it follows that $\operatorname{ht}(P) + \dim(R^h/P) = \dim(R^h)$. Since $P \notin \Gamma$, we have $\dim(R/\mathbf{p}) = \dim(R^h/P)$. Therefore $\operatorname{ht}(\mathbf{p}) + \dim(R/\mathbf{p}) = \dim(R)$ and R is catenary. $\square$

Corollary 16.8. *If R has geometrically normal formal fibers and is catenary but not universally catenary, then there exist in the $\mathbf{m}$-adic completion $\widehat{R}$ of R minimal prime ideals q such that $\dim(\widehat{R}/q) = 1$.*

Proof. By Theorem 16.7, each prime ideal $Q \in \Gamma$ has dimension one and is a minimal prime of R^h. Moreover, $Q\widehat{R} := q$ is a minimal prime of $\widehat{R}$. Since $\dim(R^h/Q) = 1$, we have $\dim(\widehat{R}/q) = 1$. $\square$

16.3. A method for constructing examples

In Theorem 16.9 we show in certain circumstances that the flatness, Noetherian and computability properties associated with ideals $I_1, \ldots, I_n$ of the (z)-adic completion R^* of R as described in Theorem 8.3 transfer to the integral domain associated with their intersection. In Section 16.5, we show that the property of regularity of formal fibers also transfers in certain cases to the integral domain associated with their intersection. Theorem 16.9 is a generalization of Theorem 9.10.

Theorem 16.9. *Assume that R is a Noetherian integral domain with field of fractions K, $z \in R$ is a nonzero nonunit, R^* is the (z)-adic completion of R, and $I_1, \ldots, I_n$ are ideals of R^* such that, for each $i \in \{1, \ldots, n\}$, each associated prime of R^*/I_i intersects R in (0). Also assume the map $R \hookrightarrow (R^*/I_i)[1/z]$ is flat for each i and that the localizations at z of the I_i are pairwise comaximal; that is, for all $i \neq j$, $(I_i + I_j)R^*[1/z] = R^*[1/z]$. Let $I := I_1 \cap \cdots \cap I_n$, $A := K \cap (R^*/I)$ and, for $i \in \{1, 2, \ldots n\}$, let $A_i := K \cap (R^*/I_i)$. Then*

1. *Each associated prime of R^*/I intersects R in (0).*
2. *The map $R \hookrightarrow (R^*/I))[1/z]$ is flat, so the ring A is Noetherian and is equal to its associated approximation ring B. The (z)-adic completion A^* of A is R^*/I, and the (z)-adic completion A_i^* of A_i is R^*/I_i, for $i \in \{1, \ldots, n\}$.*
3. *The ring $A^*[1/z] \cong (A_1^*)[1/z] \oplus \cdots \oplus (A_n^*)[1/z]$. If $Q \in \operatorname{Spec}(A^*)$ and $z \notin Q$, then $(A^*)_Q$ is a localization of one of the A_i^*.*
4. *We have $A \subseteq A_1 \cap \cdots \cap A_n$ and $\cap_{i=1}^n (A_i)[1/z] \subseteq A_P$ for each $P \in \operatorname{Spec} A$ with $z \notin P$. Thus we have $A[1/z] = \cap_{i=1}^n (A_i)[1/z]$.*

Proof. By Theorem 8.3, the (z)-adic completion A_i^* of A_i is R^*/I_i. Since $\operatorname{Ass}(R^*/I) \subseteq \bigcup_{i=1}^n \operatorname{Ass}(R^*/I_i)$, the condition on assassinators of Theorem 8.3 holds for the ideal I. The natural R-algebra homomorphism $\pi : R^* \to \bigoplus_{i=1}^n (R^*/I_i)$ has kernel I. Further, the localization of π at z is onto because $(I_i + I_j)R^*[1/z] =$

$R^*[1/z]$ for all $i \neq j$. Thus $(R^*/I)[1/z] \cong \bigoplus_{i=1}^n (R^*/I_i)[1/z] = \bigoplus_{i=1}^n (A_i^*)[1/z]$ is flat over R. Therefore A is Noetherian and is equal to its associated approximation ring B, and $A^* = R^*/I$ is the (z)-adic completion of A.

If $Q \in \operatorname{Spec}(A^*)$ and $z \notin Q$, then A^*_Q is a localization of

$$A^*[1/z] \cong (A_1^*)[1/z] \oplus \ \cdots \ \oplus \ (A_n^*)[1/z].$$

Every prime ideal of $\bigoplus_{i=1}^n (A_i^*)[1/z]$ has the form $(Q_i)A_i^*[1/z] \oplus \bigoplus_{j\neq i}(A_j^*)[1/z]$, where $Q_i \in \operatorname{Spec}(A_i^*)$. It follows that A^*_Q is a localization of A_i^* for some $i \in \{1, \ldots, n\}$.

Since R^*/I_i is a homomorphic image of R^*/I, the intersection ring $A \subseteq A_i$ for each i. Let $P \in \operatorname{Spec} A$ with $z \notin P$. Since $A^* = R^*/I$ is faithfully flat over A, there exists $P^* \in \operatorname{Spec}(A^*)$ with $P^* \cap A = P$. Then $z \notin P^*$ implies $A^*_{P^*} = (A_i^*)_{P_i^*}$, where $P_i^* \in \operatorname{Spec}(A_i^*)$ for some $i \in \{1, \ldots, n\}$. Let $P_i = P_i^* \cap A_i$. Since $A_P \hookrightarrow A^*_{P^*}$ and $(A_i)_{P_i} \hookrightarrow (A_i^*)_{P_i^*}$ are faithfully flat, we have

$$A_P = A^*_{P^*} \cap K = (A_i^*)_{P_i^*} \cap K = (A_i)_{P_i} \supseteq (A_i)[1/z].$$

It follows that $\bigcap_{i=1}^n (A_i)[1/z] \subseteq A_P$. Thus we have

$$\bigcap_{i=1}^n (A_i)[1/z] \ \subseteq \ \bigcap\{A_P \mid P \in \operatorname{Spec} A \text{ and } z \notin P\} \ = \ A[1/z].$$

Since $A[1/z] \subseteq (A_i)[1/z]$, for each i, we have $A[1/z] = \bigcap_{i=1}^n (A_i)[1/z]$. $\square$

16.4. Examples that are not universally catenary

Example 9.11 is an example of a two-dimensional Noetherian local domain A having geometrically regular formal fibers such that the completion $\widehat{A}$ of A has two minimal primes one of dimension one and one of dimension two. Thus A is not universally catenary. We generalize this example in the following.

We construct in Example 16.10 a two-dimensional Noetherian local domain having geometrically normal formal fibers such that the completion has any desired finite number of minimal primes of dimensions one and two.

EXAMPLE 16.10. Let r and s be positive integers and let R be the localized polynomial ring in three variables $R := k[x, y, z]_{(x,y,z)}$, where k is a field of characteristic zero and the field of fractions of R is $K := k(x, y, z)$. Then the (x)-adic completion of R is $R^* := k[y, z]_{(y,z)}[[x]]$. Let $\tau_1, \ldots, \tau_r, \beta_1, \beta_2, \ldots, \beta_s, \gamma \in xk[[x]]$ be algebraically independent power series over $k(x)$. Define

$$Q_i \ := \ (z-\tau_i, y-\gamma)R^*, \ \text{ for } \ i \in \{1, \ldots, r\} \ \text{ and } \ P_j \ := \ (z-\beta_j)R^*, \ \text{ for } \ j \in \{1, \ldots, s\}.$$

We apply Theorem 16.9 with $I_i = Q_i$ for $1 \leq i \leq r$, and $I_{r+j} = P_j$ for $1 \leq j \leq s$. Then the I_i satisfy the comaximality condition at the localization at x. Let $I := I_1 \cap \cdots \cap I_{r+s}$ and let $A := K \cap (R^*/I)$. For J an ideal of R^* containing I, let $\bar{J}$ denote the image of J in R^*/I. Then, for each i with $1 \leq i \leq r$, $\dim(\,(R^*/I)/\bar{Q}_i) = \dim(R^*/Q_i) = 1$ and, for each j with $1 \leq j \leq s$, $\dim(\,(R^*/I)/\bar{P}_j) = 2$. Thus A^* contains r minimal primes of dimension one and s minimal primes of dimension two. Since A^* modulo each of its minimal primes is a regular local ring, the completion $\widehat{A}$ of A also has precisely r minimal primes of dimension one and s minimal primes of dimension two.

The integral domain A birationally dominates R and is birationally dominated by each of the A_i. Corollary 16.18 implies that A has geometrically regular formal fibers. Since $\dim(A) = 2$, A is catenary.

We show in Example 16.11 that for every integer $n \geq 2$ there is a Noetherian local domain $(A, \mathbf{m})$ of dimension n that has geometrically regular formal fibers and is catenary but not universally catenary.

EXAMPLE 16.11. Let $R = k[x, y_1, \dots, y_n]_{(x,y_1,\dots,y_n)}$ be a localized polynomial ring of dimension $n+1$ where k is a field of characteristic zero. Let $\sigma, \tau_1, \dots, \tau_n \in xk[[x]]$ be $n+1$ algebraically independent elements over $k(x)$ and consider in $R^* = k[y_1, \dots, y_n]_{(y_1,\dots,y_n)}[[x]]$ the ideals

$$I_1 = (y_1 - \sigma)R^* \quad \text{and} \quad I_2 = (y_1 - \tau_1, \dots, y_n - \tau_n)R^*.$$

Then the ring

$$A = k(x, y_1, \dots, y_n) \cap (R^*/(I_1 \cap I_2))$$

is the desired example. The completion $\widehat{A}$ of A has two minimal primes $I_1\widehat{A}$ having dimension n and $I_2\widehat{A}$ having dimension one. By Corollary 16.18, A has geometrically regular formal fibers. Therefore the Henselization A^h has precisely two minimal prime ideals P, Q which may be labeled so that $P\widehat{A} = I_1\widehat{A}$ and $Q\widehat{A} = I_2\widehat{A}$. Thus $\dim(A^h/P) = n$ and $\dim(A^h/Q) = 1$. By Theorem 16.7, A is catenary but not universally catenary.

In Example 16.12 we construct for each positive integer t and specified nonnegative integers $n_1, \dots, n_t$ with $n_1 \geq 1$, a t-dimensional Noetherian local domain A that has geometrically regular formal fibers and birationally dominates a $t+1$-dimensional regular local domain such that the completion $\widehat{A}$ of A has, for each r with $1 \leq r \leq t$, exactly n_r minimal primes $\mathbf{p}_{rj}$ of dimension $t+1-r$. Moreover, each $\widehat{A}/\mathbf{p}_{rj}$ is a regular local ring of dimension $t+1-r$. If $n_i > 0$ for some $i \neq 1$, then A is not universally catenary and is not a homomorphic image of a regular local domain. It follows from Remark 16.2 that the derived normal ring $\overline{A}$ of A has exactly n_r maximal ideals of height $t+1-r$ for each r with $1 \leq r \leq t$.

EXAMPLE 16.12. Let t be a positive integer and let n_r be a nonnegative integer for each r with $1 \leq r \leq t$. Assume that $n_1 \geq 1$. We construct a t-dimensional Noetherian local domain A that has geometrically regular formal fibers such that $\widehat{A}$ has exactly n_r minimal primes of dimension $t+1-r$ for each r. Let $x, y_1 \dots, y_t$ be indeterminates over a field k of characteristic zero.

Let $R = k[x, y_1, \dots, y_t]_{(x,y_1,\dots,y_t)}$, let $R^* = k[y_1, \dots, y_t]\,[[x]]_{(x,y_1,\dots,y_t)}$ denote the (x)-adic completion of R and let K denote the field of fractions of R. For every $r, j, i \in \mathbb{N}$ such that $1 \leq r \leq t$, $1 \leq j \leq n_r$ and $1 \leq i \leq r$, choose elements $\{\tau_{rji}\}$ of $xk[[x]]$ so that the set $\bigcup\{\tau_{rji}\}$ is algebraically independent over $k(x)$.

For each r, j with $1 \leq r \leq t$ and $1 \leq j \leq n_r$, define the prime ideal $P_{rj} := (y_1 - \tau_{rj1}, \dots, y_r - \tau_{rjr})$ of height r in R^*. Notice that R^*/P_{rj} is a regular local ring of dimension $t+1-r$. Theorem 9.5 implies that the extension $R \hookrightarrow (R^*/P_{rj})[1/x]$ is flat, and that the intersection domain $A_{rj} := K \cap (R^*/P_{rj})$ is a regular local ring of dimension $t+1-r$ that has (x)-adic completion R^*/P_{rj}.

Let $I := \bigcap P_{rj}$ be the intersection of all the prime ideals P_{rj}. Since the $\tau_{rji} \in xk[[x]]$ are algebraically independent over $k(x)$, the sum of any two of these ideals P_{rj} and P_{mi}, where we may assume $r \leq m$, has radical $(x, y_1, \dots, y_m)R^*$, and thus $(P_{rj} + P_{mi})R^*[1/x] = R^*[1/x]$. It follows that the representation of I as the intersection of the P_{rj} is irredundant and $\operatorname{Ass}(R^*/I) = \{P_{rj} \mid 1 \le r \le t, 1 \le j \le n_r\}$. Since each $P_{rj} \cap R = (0)$, we have $R \hookrightarrow R^*/I$, and the intersection domain $A := K \cap (R^*/I)$ is well defined.

By Theorem 16.9, the map $R \hookrightarrow (R^*/I)[1/x]$ is flat, A is Noetherian and A is a localization of a subring of $R[1/x]$. Since R^*/P_{rj} is a regular local ring of dimension $t+1-r$, the minimal primes of $\widehat{A}$ all have the form $\mathbf{p}_{rj} := P_{rj}\widehat{A}$ and each $\widehat{A}/\mathbf{p}_{rj}$ is a regular local ring of dimension $t+1-r$. The ring A birationally dominates the $(t+1)$-dimensional regular local domain R and all the stated properties hold once we prove Corollary 16.18.

Remark 16.13. By Theorem 16.7, the ring A constructed in Example 16.12 is catenary if and only if each minimal prime of $\widehat{A}$ has dimension either one or t. By taking $n_r = 0$ for $r \notin \{1, t\}$ in Example 16.12, we obtain additional examples of catenary Noetherian local domains A of dimension t having geometrically regular formal fibers for which the completion $\widehat{A}$ has precisely n_t minimal primes of dimension one and n_1 minimal primes of dimension t.

Remark 16.14. Let $(A, \mathbf{n})$ be a Noetherian local domain constructed as in Example 16.12 and let A^h denote the Henselization of A. Since each minimal prime of $\widehat{A}$ is the extension of a minimal prime of A^h and also the extension of a minimal prime of A^*, the minimal primes of A^h and A^* are in a natural one-to-one correspondence. Let P be the minimal prime of A^h corresponding to a minimal prime $\mathbf{p}$ of A^*. Since the minimal primes of A^* extend to pairwise comaximal prime ideals of $A^*[1/x]$, for each prime ideal $Q \supset P$ of A^h with $x \notin Q$, the prime ideal P is the unique minimal prime of A^h contained in Q. Let $\mathbf{q} := Q \cap A$. We have $\operatorname{ht}\mathbf{q} = \operatorname{ht} Q$, and either $\dim(A/\mathbf{q}) > \dim(A^h/Q)$ or else every saturated chain of prime ideals of A containing $\mathbf{q}$ has length less than $\dim A$.

In this connection, we ask:

Question 16.15. Let $(A, \mathbf{n})$ be a Noetherian local domain constructed as in Example 16.12. If A is not catenary, does it follow that the set

$$\Gamma \;:=\; \{P \in \operatorname{Spec}(A^h) \mid \dim(A^h/P) \;<\; \dim(A/(P \cap A))\}$$

is infinite?

Remark 16.16. We thank L. Avramov for suggesting we consider the depth of the rings constructed in Example 16.12. The catenary rings that arise from this construction all have depth one. However, Example 16.12 can be used to construct, for each integer $t \ge 3$ and integer d with $2 \le d \le t-1$, an example of a noncatenary Noetherian local domain A of dimension t and depth d having geometrically regular formal fibers. The (x)-adic completion A^* of A has precisely two minimal primes, one of dimension t and one of dimension d. To establish the existence of such an example, with notation as in Example 16.12, we set $m = t - d + 1$ and take $n_r = 0$ for $r \notin \{1, m\}$ and $n_1 = n_m = 1$. Let

$$P_1 := P_{11} = (y_1 - \tau_{111})R^* \text{ and } P_m := P_{m1} = (y_1 - \tau_{m11}, \ldots, y_m - \tau_{m1m})R^*.$$

Consider $A^* = R^*/(P_1 \cap P_m)$ and the short exact sequence

$$0 \longrightarrow \frac{P_1}{P_1 \cap P_m} \longrightarrow \frac{R^*}{P_1 \cap P_m} \longrightarrow \frac{R^*}{P_1} \longrightarrow 0.$$

Since P_1 is principal and not contained in P_m, we have $P_1 \cap P_m = P_1 P_m$ and $P_1/(P_1 \cap P_m) \cong R^*/P_m$. It follows that $\operatorname{depth} A = \operatorname{depth} A^* = \operatorname{depth}(R^*/P_m) = d$; [**62**, page 103, ex 14] or [**14**, Prop. 1.2.9, page 11]. By Remark 16.2, the derived normal ring $\overline{A}$ of A has precisely two maximal ideals one of height t and one of height d.

16.5. Regular maps and geometrically regular formal fibers

We show in Corollary 16.18 that each of the rings A constructed in Examples 16.10, 16.11 and 16.12 has geometrically regular formal fibers.

Theorem 16.17. *Assume that R, K, z and R^* are as in Noetherian Flatness Theorem 8.3. For a positive integer n, let $I_1, \dots, I_n$ be ideals of R^* such that each associated prime of R^*/I_i intersects R in (0), for $i = 1, \dots, n$. Let $I := I_1 \cap \cdots \cap I_n$. Assume that*

1. *R is semilocal with geometrically regular formal fibers and z is in the Jacobson radical of R.*
2. *Each $(R^*/I_i)[1/z]$ is a flat R-module and, for each $i \neq j$, the ideals $I_i(R^*)[1/z]$ and $I_j R^*[1/z]$ are comaximal in $(R^*)[1/z]$.*
3. *For $i = 1, \dots, n$, $A_i := K \cap (R^*/I_i)$ has geometrically regular formal fibers.*

Then $A := K \cap (R^*/I)$ is equal to its approximation domain B, and has geometrically regular formal fibers.

Proof. Since R has geometrically regular formal fibers, by (9.12.2), it suffices to show for $W \in \operatorname{Spec}(R^*/I)$ with $z \notin W$ that $R_{W_0} \longrightarrow (R^*/I)_W$ is regular, where $W_0 := W \cap R$. As in Theorem 16.9, we have

$$(R^*/I)[1/z] \ = \ (R^*/I_1)[1/z] \ \oplus \cdots \oplus \ (R^*/I_n)[1/z].$$

It follows that $(R^*/I)_W$ is a localization of R^*/I_i for some $i \in \{1, \dots, n\}$. If $(R^*/I)_W = (R^*/I_i)_{W_i}$, where $W_i \in \operatorname{Spec}(R^*/I_i)$, then $R_{W_0} = (A_i)_{W_i \cap A_i}$ and $(A_i)_{W_i \cap A_i} \longrightarrow (R^*/I_i)_{W_i}$ is regular. Thus $R_{W_0} \longrightarrow (R^*/I)_W$ is regular. □

Corollary 16.18. *The rings A of Examples 16.10, 16.11 and 16.12 have geometrically regular formal fibers, that is, the map $\phi : A \to \widehat{A}$ is regular.*

Proof. By the definition of R and the observations given in (9.12), the hypotheses of (16.17) are satisfied. □

Exercises

1. Let $(R, \mathbf{m})$ be a catenary Noetherian local domain having geometrically normal formal fibers. If R is not universally catenary, prove that R has depth one.

   Suggestion: Use Theorem 16.7 and [**14**, Prop. 1.2.13].

CHAPTER 17

Non-Noetherian insider examples of dimension 3,

17.1. Introduction

In this chapter we use Insider Construction 13.1 of Section 13.1 to construct examples where the insider approximation domain B is local and non-Noetherian, but is very close to being Noetherian. The localizations of B at all nonmaximal prime ideals are Noetherian, and most prime ideals of B are finitely generated. Sometimes just one prime ideal is not finitely generated.

In Section 17.2 we describe, for each positive integer m, a three-dimensional local unique factorization domain B such that the maximal ideal of B is two-generated, B has precisely m prime ideals of height two, each prime ideal of B of height two is not finitely generated and all the other prime ideals of B are finitely generated. We give more details about a specific case where there is precisely one nonfinitely generated prime ideal. Section 17.3 contains the verification of the properties of the three-dimensional examples. In Chapter 18 we present a generalization to dimension four.

17.2. A family of examples in dimension 3

In this section we construct examples as described in Examples 17.1. In Discussion 17.5 we give more details for a special case of the example with exactly one nonfinitely generated prime ideal. We display the prime spectrum for this special case in Diagram 17.3.2.

Examples 17.1. For each positive integer m, we construct an example of a non-Noetherian local integral domain $(B, \mathbf{n})$ such that:

(1) $\dim B = 3$.
(2) The ring B is a UFD that is not catenary, as defined in (3.14.3).
(3) The maximal ideal $\mathbf{n}$ of B is generated by two elements.
(4) The $\mathbf{n}$-adic completion of B is a two-dimensional regular local domain.
(5) For every non-maximal prime ideal P of B, the ring B_P is Noetherian.
(6) The ring B has precisely m prime ideals of height two.
(7) Every prime ideal of B of height two is not finitely generated; all other prime ideals of B are finitely generated.

To establish the existence of the examples in Examples 17.1, we use the following notation: Let k be a field, let x and y be indeterminates over k, and set

$$R := k[x,y]_{(x,y)}, \quad K := k(x,y) \quad \text{and} \quad R^* := k[y]_{(y)}[[x]].$$

The power series ring R^* is the xR-adic completion of R. Let $\tau \in xk[[x]]$ be transcendental over $k(x)$. For each integer i with $1 \le i \le m$, let $p_i \in R \setminus xR$ be such that $p_1R^*, \dots, p_mR^*$ are m prime ideals. For example, if each $p_i \in R \setminus (x,y)^2R$, then each p_iR^* is prime in R^*. In particular one could take $p_i = y - x^i$. Let $p := p_1 \cdots p_m$. We set $f := p\tau$ and consider the injective R-algebra homomorphism $S := R[f] \hookrightarrow R[\tau] =: T$. In this construction the polynomial rings S and T have the same field of fractions $K(f) = K(\tau)$. Hence the intersection domain

$$A \;:=\; A_f \; R^* \;\cap\; K(f) \;=\; R^* \;\cap\; K(\tau) := \; A_\tau. \tag{17.1.0}$$

By Valabrega's Theorem 4.2, A is a two-dimensional regular local domain with maximal ideal $(x,y)A$ and the $(x,y)A$-adic completion of A is $k[[x,y]]$.

Let $\tau := c_1x + c_2x^2 + \cdots + c_ix^i + \cdots \in xk[[x]]$, where the $c_i \in k$ and define the "n^{th} endpieces" τ_n of τ by

$$\tau_n \;:=\; \sum_{i=n+1}^{\infty} c_ix^{i-n} \;=\; \frac{\tau - \sum_{i=1}^{n} c_ix^i}{x^n}. \tag{17.1.a}$$

As in Equation 6.1.2 we have the following relation between the n^{th} and $(n+1)^{\text{st}}$ endpieces τ_n and τ_{n+1}:

$$\tau_n \;=\; c_{n+1}x \;+\; x\tau_{n+1}. \tag{17.1.b}$$

Define $f_n := p\tau_n$, set $U_n = k[x,y][f_n] = k[x,y,f_n]$, a three-dimensional polynomial ring over R, and set $B_n = (U_n)_{(x,y,f_n)} = k[x,y,f_n]_{(x,y,f_n)}$, a three-dimensional localized polynomial ring. Similarly set $U_{\tau n} = k[x,y,\tau_n]$, a three-dimensional polynomial ring containing U_n, and $B_{\tau n} = k[x,y,\tau_n]_{(x,y,\tau_n)}$, a localized polynomial ring containing $U_{\tau n}$ and B_n. Let U, B, U_τ and B_τ be the nested union domains defined as follows:

$$U \;:=\; \bigcup_{n=0}^{\infty} U_n \;\subseteq\; U_\tau \;:=\; \bigcup_{n=0}^{\infty} U_{\tau n}; \qquad B \;:=\; \bigcup_{n=0}^{\infty} B_n \;\subseteq\; B_\tau \;:=\; \bigcup_{n=0}^{\infty} B_{\tau n} \;\subseteq\; A.$$

REMARK 17.2. By Construction Properties Theorem 6.19.2, with adjustments using Remark 6.4, we have
(17.2.0)

$$U[1/x] = U_0[1/x] = k[x,y,f][1/x]; \quad U_\tau[1/x] = U_{\tau,0}[1/x] = k[x,y,\tau][1/x],$$

$B[1/x]$ is a localization of $S = R[f]$ and $B[1/x]$ is a localization of B_n. Similarly, $B_\tau[1/x]$ is a localization of $T = R[\tau]$.

We establish in Theorem 17.9 of Section 17.3 that the rings B of Examples 17.1 have properties 1 through 7 and also some additonal properties.

Assuming properties 1 through 7 of Examples 17.1, we describe the ring B of Examples 17.1 in the case where $m = 1$ and $p = p_1 = y$ as follows:

EXAMPLE 17.3. Let the notation be as in Examples 17.1. Thus

$$R = k[x,y]_{(x,y)}, \quad f = y\tau, \quad f_n = y\tau_n, \quad B_n = R[y\tau_n]_{(x,y,y\tau_n)}, \quad B = \bigcup_{n=0}^{\infty} B_n.$$

As we show in Section 17.3, the ideal $Q := (y, \{y\tau_n\}_{n=0}^{\infty})B$ is the unique prime ideal of B of height 2. Moreover, Q is not finitely generated and is the only prime ideal of B that is not finitely generated. We also have $Q = yA \cap B$, and $Q \cap B_n = (y, y\tau_n)B_n$ for each $n \geq 0$.

To identify the ring B up to isomorphism, we include the following details: By Equation 17.1.b, we have $\tau_n = c_{n+1}x + x\tau_{n+1}$. Thus we have

$$f_n \;=\; xf_{n+1} \;+\; yxc_{n+1}. \tag{17.3.1}$$

The family of equations (17.3.1) uniquely determines B as a nested union of the three-dimensional RLRs $B_n = k[x, y, f_n]_{(x,y,f_n)}$.

We recall the following terminology of [**117**, page 325].

Definition 17.4. If a ring C is a subring of a ring D, a prime ideal P of C is *lost* in D if $PD \cap C \neq P$.

Discussion 17.5. Assuming properties 1 through 7 of Examples 17.1, if q is a height-one prime of B, then B/q is Noetherian if and only if q is not contained in Q. This is clear since q is principal, Q is the unique prime of B that is not finitely generated, and, by Cohen's Theorem 2.7, a ring is Noetherian if each prime ideal of the ring is finitely generated.

The height-one primes q of B may be separated into several types as follows:
Type I. The primes $q \not\subseteq Q$ have the property that B/q is a one-dimensional Noetherian local domain. These primes are contracted from A, i.e., they are not lost in A. To see this, consider $q = gB$ where $g \notin Q$. Then gA is contained in a height one prime P of A. Hence $g \in (P \cap B) \setminus Q$, and so $P \cap B \neq Q$. Since $\mathbf{m}_B A = \mathbf{m}_A$, we have $P \cap B \neq \mathbf{m}_B$. Therefore $P \cap B$ is a height-one prime containing q, so $q = P \cap B$ and $B_q = A_P$.

There are infinitely many primes q of type I, because every element of $\mathbf{m}_B \setminus Q$ is contained in a prime q of type I. Thus $\mathbf{m}_B \subseteq Q \cup \bigcup\{q \text{ of Type I}\}$. Since $\mathbf{m}_B$ is not the union of finitely many strictly smaller prime ideals, there are infinitely many primes q of Type I.

Type I*. Among the primes of Type I, we label the prime ideal xB as Type I*. The prime ideal xB is special since it is the unique height-one prime q of B for which R^*/qR^* is not complete. If q is a height-one prime of B such that $x \notin qR^*$, then $x \notin q$ by Proposition 6.20.3. Thus R^*/qR^* is complete with respect to the powers of the nonzero principal ideal generated by the image of x mod qR^*. Notice that $R^*/xR^* \cong k[y]_{yk[y]}$.

If q is a height-one prime of B not of Type I, then $\overline{B} = B/q$ has precisely three prime ideals. These prime ideals form a chain: $(\overline{0}) \subset \overline{Q} \subset \overline{(x,y)B} = \overline{\mathbf{m}_B}$.

Type II. We define the primes of Type II to be the primes $q \subset Q$ such that q has height one and is contracted from a prime p of $A = k(x, y, f) \cap R^*$, i.e., q is not lost in A. For example, the prime $y(y + \tau)B$ is of Type II by Lemma 17.13. For q of this type, B/q is dominated by the one-dimensional Noetherian local domain A/p. Thus B/q is a non-Noetherian generalized local ring in the sense of Cohen, [**18**].

For q of Type II, the maximal ideal of B/q is not principal. This follows because a generalized local domain having a principal maximal ideal is a DVR [**79**, (31.5)].

There are infinitely many height-one primes of Type II, for example, $y(y+x^t\tau)B$ for each $t \in \mathbb{N}$ by Lemma 17.12. For q of Type II, the DVR B_q is birationally dominated by A_p. Hence $B_q = A_p$ and the ideal $\sqrt{qA} = p \cap yA$.

That each element $y(y + x^t\tau)$ is irreducible and thus generates a height-one prime ideal, is done in greater generality in Lemma 17.12.

Type III. The primes of Type III are the primes $q \subset Q$ such that q has height one and is not contracted from A, i.e., q is lost in A. For example, the prime yB and the prime $(y + x^t y\tau)B$ for $t \in \mathbb{N}$ are of Type III by Lemma 17.13. Since the elements y and $y + x^t y\tau$ are in $\mathbf{m}_B$ and are not in $\mathbf{m}_B^2$ and since B is a UFD, these elements are necessarily prime. There are infinitely many such prime ideals by Lemma 17.12. For q of Type III, we have $\sqrt{qA} = yA$.

If $q = yB$ or $q = (y + x^t y\tau)B$, then the image $\overline{\mathbf{m}_B}$ of $\mathbf{m}_B$ in B/q is principal. It follows that the intersection of the powers of $\overline{\mathbf{m}_B}$ is Q/q and B/q is not a generalized local ring. For if P is a principal prime ideal of a ring and P' is a prime ideal properly contained in P, then P' is contained in the intersection of the powers of P; see [**62**, page 7, ex. 5].

The picture of $\operatorname{Spec}(B)$ is shown below.

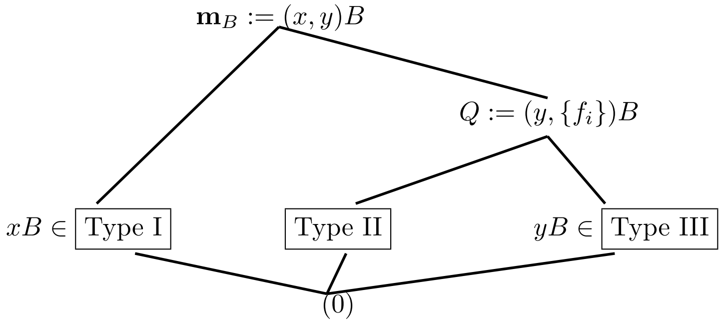

Diagram 17.3.2

In Remarks 17.6 we examine the height-one primes of B from a different perspective.

Remarks 17.6. (1) Assume the notation of Example 17.3. If w is a nonzero prime element of B such that $w \notin Q$, then wA is a prime ideal in A and is the unique prime ideal of A lying over wB. To see this, observe that $w \notin yA$ since $w \notin Q = yA \cap B$. It follows that $y \notin p$, for every prime ideal $p \in \operatorname{Spec} A$ that is a minimal prime of wA. Thus $p \cap B \neq Q$. Since we assume the properties of Examples 17.1 hold, $p \cap B$ has height one. Therefore $p \cap B = wB$. Hence the DVR B_{wB} is birationally dominated by A_p, and thus $B_{wB} = A_p$. This implies that p is the unique prime of A lying over wB. We also have $wB_{wB} = pA_p$. Since A is a UFD and p is the unique minimal prime of wA, it follows that $wA = p$. In particular, q is not lost in A, Definition 17.4.

If q is a height-one prime ideal of B that is contained in Q, then yA is a minimal prime of qA, and q is of Type II or III depending on whether or not qA has other minimal prime divisors.

To see this, observe that if yA is the only prime divisor of qA, then qA has radical yA and $yA \cap B = Q$ implies that Q is the radical of $qA \cap B$. Thus q is lost in A and q is of Type III.

On the other hand, if there is a minimal prime ideal $p \in \operatorname{Spec} A$ of qA that is different from yA, then y is not in $p \cap B$ and hence $p \cap B \neq Q$. Since Q is the only prime ideal of B of height two, it follows that $p \cap B$ is a height-one prime and thus $p \cap B = q$. Thus q is not lost in A and q is of Type II.

We observe that for every Type II prime q there are exactly two minimal primes of qA; one of these is yA and the other is a height-one prime p of A such that $p \cap B = q$. For every height-one prime ideal p of A such that $p \cap B = q$, we have B_q is a DVR that is birationally dominated by A_p and hence $B_q = A_p$. The uniqueness of B_q implies that there is precisely one such prime ideal p of A.

An example of a height-one prime ideal q of Type II is $q := (y^2 + y\tau)B$. The prime ideal $qA = (y^2 + y\tau)A$ has the two minimal primes yA and $(y + \tau)A$.

(2) The ring B/yB is a rank 2 valuation domain. This can be seen directly or else one may apply [**56**, Prop. 3.5(iv)]; see Exercise 17.3. For other prime elements g of B with $g \in Q$, it need not be true that B/gB is a valuation domain. If g is a prime element contained in $\mathbf{m}_B^2$, then the maximal ideal of B/gB is 2-generated but not principal and thus B/gB cannot be a valuation domain. For a specific example over the field $\mathbb{Q}$, let $g = x^2 + y^2\tau$.

17.3. Verification of the three-dimensional examples

In Theorem 17.9 we record and establish the properties asserted in Examples 17.1 and other properties of the ring B. We make some preliminary remarks:

REMARKS 17.7. (1) Assume the setting of Noetherian Flatness Theorem 8.8 with the additional assumption that R is a Noetherian local domain with maximal ideal $\mathbf{m}$. By Remark 6.4, B as defined in Theorem 8.8 is the directed union of the localized polynomial rings $B_r := U_{P_r}$, where $P_r := (\mathbf{m}, \tau_{1r}, \dots, \tau_{sr})U_{\underline{\tau}r}$, with notation as in Theorem 8.8.

(2) With the notation of Examples 17.1, where R is the localized polynomial ring $k[x,y]_{(x,y)}$ over a field k, $R^* = k[y]_{(y)}[[x]]$ is the (x)-adic completion of R and $f \in xR^*$ is transcendental over K, Remark 6.4 shows that $B = \bigcup B_r$, where $B_r = (U_r)_{P_r}$, $U_r = k[x, y, f_r]$ and $P_r = (x, y, f_r)U_r$, is also the same ring B as that described in Theorem 8.8. A similar remark applies to B_τ with appropriate modifications to $B_{\tau,r}$, $U_{\tau,r}$ and $P_{\tau,r}$. The corresponding rings called B, B_r etc. in Example 18.1 of Chapter 18 also satisfy the corresponding relations. Furthermore B_τ is the same ring B as in the setting in Localized Polynomial Example Theorem 9.7 where $r = 1 = s$.

(3) Thus the results of Noetherian Flatness Theorem 8.8, Construction Properties Theorem 6.19, and Propositions 6.20 and 6.21 hold for the rings B and B_τ of Examples 17.1 and for the rings of Example 18.1 in Chapter 18. Also Localized Polynomial Example Theorem 9.7holds for B_τ. (With the rings U_r defined as non-localized polynomial rings as in Examples 17.1, we do not have the same conclusions for $U = \bigcup U_r$ as for the domains called U or U_r in those results.)

In order to examine more closely the prime ideal structure of the ring B of Examples 17.1, we establish in Proposition 17.8 some properties of its overring A and of the map $\operatorname{Spec} A \to \operatorname{Spec} B$.

PROPOSITION 17.8. *With the notation of Examples 17.1, we have*

(1) $A = B_\tau$ *and* $A[1/x]$ *is a localization of* $R[\tau]$.

(2) *For* $P \in \operatorname{Spec} A$ *with* $x \notin P$, *the following are equivalent:*

(**a**) $A_P = B_{P\cap B}$ (**b**) $\tau \in B_{P\cap B}$ (**c**) $p \notin P$.

PROOF. By Localized Polynomial Example Theorem 9.7 with $r = 1, y = y_1, s = 1$, and $\tau = \tau_1$, as discussed in Remarks 17.7.3, $A = B_\tau$ and is Noetherian. By Remark 17.2, the ring $A[1/x]$ is a localization of $R[\tau]$. Thus item 1 holds.

For item 2, since $\tau \in A$, (a) $\implies$ (b) is clear. For (b) $\implies$ (c) we show that $p \in P \implies \tau \notin B_{P\cap B}$. By Remark 17.2, $B[1/x]$ is a localization of $R[f]$. Since

$x \notin P$, the ring $B_{P\cap B}$ is a localization of $R[f]$, and thus $B_{P\cap B} = R[f]_{P\cap R[f]}$. The assumption that $p \in P$ implies that some $p_i \in P$, and so $R[f]_{P\cap R[f]}$ is contained in $V := R[f]_{p_iR[f]}$, a DVR. Since $R[f]$ is a polynomial ring over R, f is a unit in V. Hence $\tau = f/p \notin V$ and thus $\tau \notin R[f]_{P\cap R[f]}$. This shows that (b) $\implies$ (c).

For (c) $\implies$ (a), notice that $f = p\tau$ implies that $R[f][1/xp] = R[\tau][1/xp]$. By item 1, $A[1/x]$ is a localization of $R[\tau][1/x]$ and so $A[1/xp]$ is a localization of $R[\tau][1/xp] = R[f][1/xp]$. Thus $A[1/xp]$ is a localization of $R[f]$. By Remark 17.2, $B[1/x]$ is a localization of $R[f]$. Since $xp \notin P$ and $x \notin P\cap B$, we have that A_P and $B_{P\cap B}$ are both localizations of $R[f]$. Thus we have

$$A_P = R[f]_{PA_P\cap R[f]} = R[f]_{(P\cap B)B_{P\cap B}\cap R[f]} = B_{P\cap B}.$$

This completes the proof of Proposition 17.8. □

Theorem 17.9. *As in the notation of Examples 17.1, let* $R := k[x,y]_{(x,y)}$, *where* k *is a field, and* x *and* y *are indeterminates. Set* $R^* = k[y]_{(y)}[[x]]$, *let* $\tau \in xk[[x]]$ *be transcendental over* $k(x)$, *and, for each integer* i *with* $1 \le i \le m$, *let* $p_i \in R \setminus xR$ *be such that* $p_1R^*, \dots, p_mR^*$ *are* m *prime ideals. Let* $p := p_1 \cdots p_m$ *and set* $f := p\tau$. *With the approximation domain* B *and the intersection domain* A *defined as in Examples 17.1,* $A := A_f = A_\tau$. *Set* $Q_i := p_iR^* \cap B$, *for each* i *with* $1 \le i \le m$. *Then:*

1. *The ring* B *is a three-dimensional non-Noetherian local UFD with maximal ideal* $\mathbf{n} = (x,y)B$, *and the* $\mathbf{n}$*-adic completion of* B *is the two-dimensional regular local ring* $k[[x,y]]$.
2. *The rings* $B[1/x]$ *and* B_P, *for each nonmaximal prime ideal* P *of* B, *are regular Noetherian UFDs, and the ring* B/xB *is a DVR.*
3. *The ring* A *is a two-dimensional regular local domain with maximal ideal* $\mathbf{m}_A := (x,y)A$, *and* $A = B_\tau$. *The ring* A *is excellent if the field* k *has characteristic zero. If* k *is a perfect field of characteristic* p, *then* A *is not excellent*
4. *The ideal* $\mathbf{m}_A$ *is the only prime ideal of* A *lying over* $\mathbf{n}$.
5. *The ideals* Q_i *are the only height-two prime ideals of* B.
6. *The ideals* Q_i *are not finitely generated and they are the only nonfinitely generated prime ideals of* B.
7. *The ring* B *has saturated chains of prime ideals from* (0) *to* $\mathbf{n}$ *of length two and of length three, and hence is not catenary.*

Proof. For item 1, since B is a directed union of three-dimensional regular local domains, $\dim B \le 3$. By Proposition 6.20, B is local with maximal ideal $(x,y)B$, xB and p_iB are prime ideals, and, by Construction Properties Theorem 6.19.4, the (x)-adic completion of B is equal to R^*, the (x)-adic completion of R. Thus the $\mathbf{n}$-adic completion of B is $k[[x,y]]$. Since each $Q_i = \bigcup_{i=1}^{\infty} Q_{in}$, where $Q_{in} = p_iR^* \cap B_n$, we see that each Q_i is a prime ideal of B with $p_i, f \in Q_i$ and $x \notin Q_i$. Since $p_iB = \bigcup p_iB_n$, we have $f \notin p_iB$. Thus

$$(0) \subsetneq p_iB \subsetneq Q_i \subsetneq (x,y)B.$$

This chain of prime ideals of length at least three yields that $\dim B = 3$ and that the height of each Q_i is 2.

The prime ideal $p_iR^*[1/x]$ has height one, whereas $p_iR^*[1/x] \cap S = (p_i, f)S$ has height two. Since flat extensions satisfy the going-down property, by Remark 2.20.10, the map $S = R[f] \to R^*[1/x]$ is not flat. Therefore Noetherian FlatnessTheorem 8.8 implies that the ring B is not Noetherian. By Proposition 6.21, B is a UFD, and so item 1 holds.

For item 2, by Theorem 6.19.3, $B/xB = R/xR$, and so B/xB is a DVR. By Proposition 6.21, $B[1/x]$ is a regular Noetherian UFD. If $x \in P$ and P is nonmaximal, then, again by Theorem 6.19.3, $P = xB$ and so B_P is a DVR and a regular Noetherian UFD. If $x \notin P$, the ring B_P is a localization of $B[1/x]$ and so is a regular Noetherian UFD. Thus item 2 holds.

The statements in item 3 that A is a two-dimensional regular local domain with maximal ideal $\mathbf{m}_A = (x, y)A$ and $A = B_\tau$ follow from Localized Polynomial Example Theorem 9.7. If the field k has characteristic zero, then A is also excellent by Theorem 9.2 (if the non-localized ring is excellent, so is the localization).

If the field k is perfect with characteristic $p > 0$, then the ring A is not excellent by Remark‘9.4. This completes the proof of item 3.

By Theorem 6.19.3, $A/xA = R/xR$, and so $\mathbf{m}_A = (x, y)A$ is the unique prime ideal of A lying over $\mathbf{n} = (x, y)B$. Thus item 4 holds and for item 5 we see that x is not in any height-two prime ideal of B.

To complete the proof of item 5, it remains to consider $P \in \operatorname{Spec} B$ with $x \notin P$ and $\operatorname{ht} P > 1$. By Proposition 6.20.3, we have $x^n \notin PR^*$ for each $n \in \mathbb{N}$. Thus $\operatorname{ht}(PR^*) \le 1$. Since $A \hookrightarrow R^*$ is faithfully flat, $\operatorname{ht}(PA) \le 1$. Let P' be a height-one prime ideal of A containing PA. Since $\dim B = 3$, $\operatorname{ht} P > 1$ and $x \notin P' \cap B$, it follows that $P = P' \cap B$. If $p \notin P$, then Proposition 17.8 implies that $A_{P'} = B_P$. Since P' is a height-one prime ideal of A, it follows that P is a height-one prime ideal of B in case $x \notin P$ and $\mathbf{p} \notin P$.

Now suppose that $p_i \in P$ for some i. Then p_iR^* is a height-one prime ideal contained in PR^* and so $p_iR^* = PR^*$. Hence P is squeezed between p_iB and $Q_i = p_iR^* \cap B \neq (x, y)B$. Since $\dim B = 3$, either P has height one or $P = Q_i$ for some i. This completes the proof of item 5.

For item 6, we show that each Q_i is not finitely generated by showing that $f_{n+1} \notin (p_i, f_n)B$ for each $n \ge 0$. By Equation 17.3.1, we have $f_n = xf_{n+1} + pxc_{n+1}$. Assume that $f_{n+1} \in (p_i, f_n)B$. Then

$$(p_i, f_n)B = (p_i, xf_{n+1} + pxc_{n+1})B \implies f_{n+1} = ap_i + b(xf_{n+1} + pxc_{n+1}),$$

for some $a, b \in B$. Thus $f_{n+1}(1 - xb) \in p_iB$. Since $1 - xb$ is a unit of B, it follows that $f_{n+1} \in p_iB$, and thus $f_{n+1} \in p_iB_{n+r}$, for some $r \ge 1$. The relations $f_t = xf_{t+1} + pxc_{t+1}$, for each $t \in \mathbb{N}$, imply that

$$f_{n+1} = xf_{n+2} + pxc_{n+2} = x^2f_{n+3} + px^2c_{n+3} + pxc_{n+2} = \cdots = x^{r-1}f_{n+r} + p\alpha,$$

where $\alpha \in R$. Thus $x^{r-1}f_{n+r} \in (p, f_{n+1})B_{n+r}$. Since $f_{n+1} \in p_iB_{n+r}$, we have $x^{r-1}f_{n+r} \in p_iB_{n+r}$. This implies $f_{n+r} \in p_iB_{n+r}$, a contradiction because the ideal $(p_i, f_{n+r})B_{n+r}$ has height two. We conclude that Q_i is not finitely generated.

Since B is a UFD, the height-one primes of B are principal and since the maximal ideal of B is two-generated, every nonfinitely generated prime ideal of B has height two and thus is in the set $\{Q_1, \dots, Q_m\}$. This completes the proof of item 6.

For item 7, the chain $(0) \subset xB \subset (x, y)B = \mathbf{m}_B$ is saturated and has length two, while the chain $(0) \subset p_1B \subset Q_1 \subset \mathbf{m}_B$ is saturated and has length three. □

Remark 17.10. With the notation of Examples 17.1 and Theorem 17.9, we obtain the following additional details about the prime ideals of B.

1. If $P \in \operatorname{Spec} B$, $P \neq (0)$ and $P \neq \mathbf{m}_B$, then $\operatorname{ht}(PR^*) = 1$ and $\operatorname{ht}(PA) = 1$. Thus every nonmaximal prime ideal of B is contained in a nonmaximal prime ideal of A.
2. If $P \in \operatorname{Spec} B$ is such that $P \cap R = (0)$, then $\operatorname{ht}(P) \leq 1$ and P is principal.
3. If $P \in \operatorname{Spec} B$, $\operatorname{ht} P = 1$ and $P \cap R \neq 0$, then $P = (P \cap R)B$.
4. Let p_i be one of the prime factors of p. Then p_iB is prime in B. Moreover the ideals p_iB and $Q_i := p_iA \cap B = (p_i, f_1, f_2, \ldots)B$ are the only nonmaximal prime ideals of B that contain p_i. Thus they are the only prime ideals of B that lie over p_iR in R.
5. The constructed ring B has Noetherian spectrum.

Proof. For the proof of item 1, if $P = Q_i$ for some i, then $PR^* \subseteq p_iR^*$ and $\operatorname{ht} PR^* = 1$. If P is not one of the Q_i, then P is a principal height-one prime and $\operatorname{ht} PR^* = 1$ by Theorem 17.9 parts 5 and 1. Since A is Noetherian and local, R^* is faithfully flat over A and hence $\operatorname{ht} PA = 1$. The proof that $\operatorname{ht}(PR^*) \leq 1$ is contained in the proof of item 5 of Theorem 17.9.

For item 2, $\operatorname{ht} P \leq 1$ because the field of fractions $K(f)$ of B has transcendence degree one over the field of fractions K of R; see Cohen's Theorem 2.8. Since B is a UFD, P is principal.

For item 3, if $x \in P$, then $P = xB$ and the statement is clear. Assume $x \notin P$. By Remark 17.2, $B[1/x]$ is a localization of B_n, and so $\operatorname{ht}(P \cap B_n) = 1$ for all integers $n \geq 0$. Thus $(P \cap R)B_n = P \cap B_n$, for each n, and so $P = (P \cap R)B$.

For item 4, p_iB is prime by Proposition 6.20.2. By Theorem 17.9, $\dim B = 3$ and the Q_i are the only height-two primes of B. Since the ideal $p_iR + p_jR$ is $\mathbf{m}_R$-primary for $i \neq j$, it follows that $p_iB + p_jB$ is $\mathbf{n}$-primary, and hence p_iB and Q_i are the only nonmaximal prime ideals of B that contain p_i.

Item 5 follows from Theorem 17.9, since the prime spectrum is Noetherian if it satisfies the ascending chain condition and if, for each finite set in the spectrum, there are only finitely many points minimal with respect to containing all of them. Thus the proof is complete. □

Remark 17.11. Rotthaus and Sega prove that the rings B in the setting of Theorems 17.9 and 18.5 are coherent and regular in the sense that every finitely generated submodule of a free module has a finite free resolution [**97**]. For the ring $B = \bigcup_{n=1}^{\infty} B_n$ of these constructions, it is stated in [**97**] that $B_n[1/x] = B_{n+k}[1/x] = B[1/x]$ and that B_{n+k} is generated over B_n by a single element for all positive integers n and k. This is not correct for the local rings B_n. However, if instead of using the localized polynomial rings B_n and their union B of the construction for these theorems, one uses the underlying polynomial rings U_n and their union U defined in Equation 6.1.5, or those defined in Examples 17.1, then one does have that $U_n[1/x] = U_{n+k}[1/x] = U[1/x]$ and that U_{n+k} is generated over U_n by a single element for all positive integers n and k; see Remark 17.2.

We use the following lemma.

Lemma 17.12. *Let the notation be as in Examples 17.1 and Theorem 17.9.*

1. *For every element $c \in \mathbf{m}_R \setminus xR$ and every $t \in \mathbb{N}$, the element $c + x^t f$ is a prime element of the UFD B.*

(2) *For every fixed element $c \in \mathbf{m}_R \setminus xR$, the set $\{c+x^t f\}_{t\in\mathbb{N}}$ consists of infinitely many nonassociate prime elements of B, and so there exist infinitely many distinct height-one primes of B of the form $(c+x^t f)B$.*

Proof. For the first item, since $f = p\tau$, Equation 17.3.1 implies that

$$f_r = pc_{r+1}x + xf_{r+1}$$

for each $r \geq 0$. In $B_0 = k[x,y,f]_{(x,y,f)}$, the polynomial $c + x^t f$ is linear in the variable $f = f_0$ and the coefficient x^t of f is relatively prime to the constant term c. Thus $c+x^t f$ is irreducible in B_0. Since $f = f_0 = pc_1x+xf_1$ in $B_1 = k[x,y,f_1]_{(x,y,f_1)}$, the polynomial $c + x^t f = c + x^t pc_1 x + x^{t+1}f_1$ is linear in the variable f_1 and the coefficient x^{t+1} of f_1 is relatively prime to the constant term c. Thus $c + x^t f$ is irreducible in B_1. To see that this pattern continues, observe that in B_2, we have

$$\begin{aligned} f \ &= pc_1x + xf_1 \ = \ pc_1x + pc_2x^2 + x^2 f_2 \implies \\ c + x^t f \ &= \ c + pc_1x^{t+1} + pc_2x^{t+2} + x^{t+2}f_2, \end{aligned}$$

a linear polynomial in the variable f_2. Thus $c + x^t f$ is irreducible in B_2 and a similar argument shows that $c+x^t f$ is irreducible in B_r for each positive integer r. Therefore for each $t \in \mathbb{N}$, the element $c + x^t f$ is prime in B.

For item 2, we prove that $(c+x^t f)B \neq (c+x^m f)B$, for positive integers $t > m$. Assume that $\mathbf{q} := (c+x^t f)B = (c+x^m f)B$ is a height-one prime ideal of B. Then

$$(x^t - x^m)f \ = \ x^m(x^{t-m} - 1)f \ \in \ \mathbf{q}.$$

Since $c \notin xB$ we have $\mathbf{q} \neq xB$. Thus $x^m \notin \mathbf{q}$. Since B is local, $x^{t-m} - 1$ is a unit of B. It follows that $f \in \mathbf{q}$ and thus $(c,f)B \subseteq \mathbf{q}$. By Remark 17.2, $B[1/x]$ is a localization of $R[f] = S$, and $x \notin \mathbf{q}$ implies that $B_{\mathbf{q}} = S_{\mathbf{q}\cap S}$. This is a contradiction since the ideal $(c,f)S$ has height two.

We conclude that there exist infinitely many distinct height-one primes of the form $(c+x^t f)B$. $\square$

Lemma 17.13 is useful for giving a more precise description of Spec B for B as in Examples 17.1. For each nonempty finite subset H of $\{Q_1,\dots,Q_m\}$, we show there exist infinitely many height-one prime ideals contained in each $Q_i \in H$, but not contained in Q_j if $Q_j \notin H$. Recall that "lost" is defined in Definition 17.4.

Lemma 17.13. *With the notation of Example 17.1 and Theorem 17.9, let G be a nonempty subset of $\{1,\dots,m\}$, let $H = \{Q_i \mid i \in G\}$, and let $p_G = \prod\{p_i \mid i \in G\}$. Then we have, for each $t \in \mathbb{N}$:*

(1) *$(p_G + x^t f)B$ is a prime ideal of B that is lost in A.*

(2) *$(p_G^2 + x^t f)B$ is a prime ideal of B that is not lost in A; see Definition 17.4.*

The sets $\{(p_G + x^t f)B\}_{t\in\mathbb{N}}$ and $\{(p_G^2 + x^t f)B\}_{t\in\mathbb{N}}$ are both infinite. Moreover, the prime ideals in both item 1 and item 2 are contained in each Q_i such that $Q_i \in H$, but are not contained in Q_j if $Q_j \notin H$.

Proof. For item 1, we have

$$(p_G + x^t f)A \cap B = p_G(1 + x^t\tau \prod_{j\notin G} p_j)A \cap B = p_G A \cap B = \bigcap_{i\in G} Q_i. \tag{17.13.1}$$

Thus each prime ideal of B of the form $(p_G + x^t f)B$ is lost in A and R^*. By the second item of Lemma 17.12, there exist infinitely many height-one primes $(p_G + x^t f)B$ of B that are lost in A and R^*.

For item 2, we have

$$
\begin{aligned}
(p_G^2 + x^t f)A \cap B &= (p_G^2 + x^t p_G (\prod_{j \notin G} p_j)\tau)A \cap B \\
&= p_G(p_G + x^t(\prod_{j \notin G} p_j)\tau)A \cap B \subsetneq p_G A \cap B = \bigcap_{i \in G} Q_i.
\end{aligned}
\tag{17.13.2}
$$

The strict inclusion is because $p_G + x^t(\prod_{j \notin G} p_j)\tau \in \mathbf{m}_A$. This implies that prime ideals of B of form $(p_G^2 + x^t f)B$ are not lost. By Lemma 17.12 there are infinitely many distinct prime ideals of that form.

The "moreover" statement for the prime ideals in item 1 follows from Equation 17.13.1. Equation 17.13.2 implies that the prime ideals in item 2 are contained in each $Q_i \in H$. For $j \notin G$, if $p_G^2 + x^t f \in Q_j$, then $p_j + x^t f \in Q_j$ implies that $p_G^2 - p_j \in Q_j$ by subtraction. Since $p_j \in Q_j$, this would imply that $p_G^2 \in Q_j$, a contradiction. This completes the proof of Lemma 17.13. □

REMARK 17.14. With the notation of Examples 17.1, consider the birational inclusion $B \hookrightarrow A$ and the faithfully flat map $A \hookrightarrow R^*$. The following statements hold concerning the inclusion maps $R \hookrightarrow B \hookrightarrow A \hookrightarrow R^*$, and the associated maps in the opposite direction of their spectra: (See Discussion 3.18 for information concerning the spectral maps.)

1. The map $\operatorname{Spec} R^* \to \operatorname{Spec} A$ is surjective, since every prime ideal of A is contracted from a prime ideal of R^*, while the maps $\operatorname{Spec} R^* \to \operatorname{Spec} B$ and $\operatorname{Spec} A \to \operatorname{Spec} B$ are not surjective. All the induced maps to $\operatorname{Spec} R$ are surjective since the map $\operatorname{Spec} R^* \to \operatorname{Spec} R$ is surjecive.
2. By Lemma 17.13, each of the prime ideals Q_i of B contains infinitely many height-one primes of B that are the contraction of prime ideals of A and infinitely many that are not.

   An ideal contained in a finite union of prime ideals is contained in one of the prime ideals; see [**5**, Prop. 1.11, page 8] or [**73**, Ex. 1.6, page 6]. Thus there are infinitely many non-associate prime elements of the UFD B that are not contained in the union $\bigcup_{i=1}^m Q_i$. We observe that for each prime element q of B with $q \notin \bigcup_{i=1}^m Q_i$ the ideal qA is contained in a height-one prime $\mathbf{q}$ of A and $\mathbf{q} \cap B$ is properly contained in $\mathbf{m}_B$ since $\mathbf{m}_A$ is the unique prime ideal of A lying over $\mathbf{m}_B$. Hence $\mathbf{q} \cap B = qB$. Thus each qB is contracted from A and R^*.

   In the four-dimensional example B of Theorem 18.5, each height-one prime of B is contracted from R^*, but there are infinitely many height-two primes of B that are lost in R^*, in the sense of (17.4); see Section 18.3.
3. Among the prime ideals of the domain B of Examples 17.1 that are not contracted from A are the p_iB. Since $p_iA \cap B = Q_i$ properly contains p_iB, the prime ideal p_iB is lost in A.
4. Since x and y generate the maximal ideals of B and A, and since B is integrally closed, a version of Zariski's Main Theorem [**86**], [**27**], implies that A is not essentially finitely generated as a B-algebra. ("Essentially finitely generated" is defined in Section 2.1.)

Using the information above, we display below a picture of Spec(B) in the case $m = 2$.

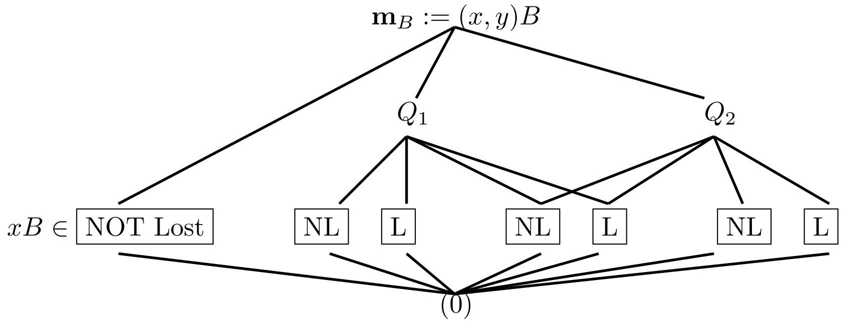

Diagram 17.14.0

Comments on Diagram 17.14.0. Here we have $Q_1 = p_1R^* \cap B$ and $Q_2 = p_2R^* \cap B$, and each box represents an infinite set of height-one prime ideals. We label a box "NL" for "not lost" and "L" for "lost". An argument similar to that given for the Type I primes in Example 17.3 shows that the height-one primes q such that $q \notin Q_1 \cup Q_2$ are not lost. That the other boxes are infinite follows from Lemma 17.13.

Exercises

(1) Let $R = k[x, y]_{(x,y)}$ be the localized polynomial ring in the variables x, y over a field k. Consider the local quadratic transformation $S := R[\frac{y}{x}]_{x,\frac{y}{x})R[\frac{y}{x}]}$ of the 2-dimensional RLR R. Using the terminology of Definition 17.4
  (a) Prove that there are infinitely many height-one primes of R that are lost in S.
  (b) Prove that there are infinitely many height-one primes of R that are not lost in S.
  (c) Describe precisely the height-one primes of R that are lost in S, and the prime ideals of R that are not lost in S.

(2) Prove the assertion in Remark 17.14 that each of the prime ideals Q_i of B contains infinitely many height-one primes of B that are the contraction of prime ideals of A and infinitely many that are not, i.e., there exist infinitely many height-one primes of B contained in Q_i that are lost in A and infinitely many that are not lost in A.

  Suggestion: A solution for this exercise can be patterned along the lines of the arguments given in Example 17.3. Since $A[1/x]$ is a localization of the polynomial ring $R[\tau]$, for every nonzero element $c \in (x, y)R$, the ideal $(\tau - c)A$ is a height-one prime in A, and $a\tau - ac$ is a nonzero element in each of the prime ideals Q_i of B. Since p_iA is the only prime ideal of A lying over Q_i in B, the ideal $(\tau - c)A \cap B$ is a height-one prime of B. Also consider elements of the form $p_i + x^n f \in B$.

(3) In connection with Remarks 17.6.2, let $(R, \mathbf{m})$ be a local domain with principal maximal ideal $\mathbf{m} = aR$
  (a) Prove that $\bigcap_{n=1}^{\infty} \mathbf{m}^n = P$, where P is a prime ideal properly contained in $\mathbf{m}$.
  (b) Prove that R/P is a DVR.

(c) [**56**, Prop. 3.5(iv)] Prove that R is a valuation domain if and only if R_P is a valuation domain.

(d) Construct an example of a local domain $(R, \mathbf{m})$ with principal maximal ideal $\mathbf{m}$ such that R is not a valuation domain.

Suggestion: To construct an example for part d, let x, y be indeterminates over a field k, let $U = k(x)[y]$, let W be the DVR U_{yU}, and let $P = yW$, the maximal ideal of W. Let $R = k[x^2]_{(x^2k[x^2])} + P$.

CHAPTER 18

Non-Noetherian insider examples of dimension 4,

18.1. Introduction

In this chapter we extend the methods of Chapter 17 to construct a four-dimensional local domain that is not Noetherian, but is very close to being Noetherian. We use Insider Construction 13.1 of Section 13.1. This four-dimensional non-Noetherian local unique factorization domain has exactly one prime ideal Q of height three; the ideal Q is not finitely generated.

Section 18.2 contains a description of the example. In Section 18.3 we verify that the example has the properties.

18.2. A 4-dimensional prime spectrum

In Example 18.1, we present a four-dimensional example analogous to Example 17.3.

Example 18.1. Let k be a field, let x, y and z be indeterminates over k. Set

$$R :=\ k[x,y,z]_{(x,y,z)} \qquad \text{and} \qquad R^* :=\ k[y,z]_{(y,z)}[[x]],$$

and let $\mathbf{m}_R$ and $\mathbf{m}_{R^*}$ denote the maximal ideals of R and R^*, respectively. The power series ring R^* is the xR-adic completion of R. Consider τ and σ in $xk[[x]]$

$$\tau \ :=\ \sum_{n=1}^{\infty} c_n x^n \quad \text{and} \quad \sigma \ :=\ \sum_{n=1}^{\infty} d_n x^n,$$

where the c_n and d_n are in k and τ and σ are algebraically independent over $k(x)$. Define

$$f \ :=\ y\tau + z\sigma \quad \text{and} \quad A \ :=\ A_f = R^* \cap k(x,y,z,f),$$

that is, A is the intersection domain associated with f. For each integer $n \geq 0$, let τ_n and σ_n be the n^{th} endpieces of τ and σ as in Equation 17.1.a. Then the n^{th} endpiece of f is $f_n = y\tau_n + z\sigma_n$. As in Equation 17.1.b, we have

$$\tau_n \ =\ x\tau_{n+1}\ +\ xc_{n+1} \quad \text{and} \quad \sigma_n \ =\ x\sigma_{n+1}\ +\ xd_{n+1},$$

where c_{n+1} and d_{n+1} are in the field k. Therefore

$$\begin{aligned} f_n \ &=\ y\tau_n\ +\ z\sigma_n\ =\ yx\tau_{n+1} + yxc_{n+1}\ +\ zx\sigma_{n+1}\ +\ zxd_{n+1} \\ &=\ xf_{n+1}\ + yxc_{n+1}\ +\ zxd_{n+1}. \end{aligned} \tag{18.1.1}$$

The approximation domains U_n, B_n, U and B for A are as follows:

$$\begin{aligned} &\text{For } n\ \geq\ 0, \quad U_n\ :=\ k[x,y,z,f_n] \qquad B_n\ :=\ k[x,y,z,f_n]_{(x,y,z,f_n)} \\ &U\ :=\ \bigcup_{n=0}^{\infty} U_n \quad \text{and} \qquad B\ :=\ B_f\ =\ \bigcup_{n=0}^{\infty} B_n. \end{aligned} \tag{18.1.2}$$

Thus B is the directed union of 4-dimensional localized polynomial rings. It follows that $\dim B \leq 4$.

The rings A and B are constructed inside the intersection domain $A_{\tau,\sigma} := R^* \cap k(x,y,z,\tau,\sigma)$. By Corollary 9.7, the domain $A_{\tau,\sigma}$ is Noetherian and equals the approximation domain $B_{\tau,\sigma}$ associated to τ,σ and is a three-dimensional RLR that is a directed union of 5-dimensional RLRs and the extension $T := R[\tau,\sigma] \hookrightarrow R^*[1/x]$ is flat.

Before we list and establish the other properties of Example 18.1 in Theorem 18.5, we prove the following proposition concerning the Jacobian ideal and flatness in Example 18.1. The Jacobian ideal is defined and discussed in Definition and Remarks 11.6.1.

Proposition 18.2. *With the notation of Example 18.1, we have*

1. *For the extension $\varphi : S = R[f] \hookrightarrow T = R[\tau,\sigma]$, the Jacobian ideal J is the ideal $(y,z)T$. Thus the nonflat locus F of φ contains J.*
2. *For every $P \in \operatorname{Spec}(R^*[1/x])$, the ideal $(y,z)R^*[1/x] \nsubseteq P \iff$ the map $B_{P\cap B} \hookrightarrow (R^*[1/x])_P$ is flat. Thus the ideal $F_1 := (y,z)R^*[1/x]$ defines the nonflat locus of the map $B \hookrightarrow R^*[1/x]$.*
3. *For every height-one prime ideal $\mathbf{p}$ of R^*, we have $\operatorname{ht}(\mathbf{p} \cap B) \leq 1$.*
4. *For every prime element w of B, $wR^* \cap B = wB$.*

Proof. For item 1, the Jacobian ideal is the ideal of T generated by the 1×1 minors of the matrix $(y \quad z)$ by (11.6.1), and so $J = (y,z)T$. By Theorem 11.9.2, $(y,z)T \subseteq F$.

For item 2, the two statements are equivalent by the definition of nonflat locus in Definition and Remarks 11.6.2. To compute the nonflat locus of $B \hookrightarrow R^*[1/x]$, we use that $T := R[\tau,\sigma] \hookrightarrow R^*[1/x]$ is flat as noted in Example 18.1. Let $P \in \operatorname{Spec}(R^*[1/x])$ and let $Q := P \cap T$. The map $B \hookrightarrow R^*[1/x]_P$ is flat $\iff$ the composition

$$k[x,y,z,f] \hookrightarrow k[x,y,z,\tau,\sigma] \hookrightarrow R^*[1/x]_P \text{ is flat } \iff$$

$$S := k[x,y,z,f] \overset{\varphi}{\hookrightarrow} T_Q = k[x,y,z,\tau,\sigma]_Q \text{ is flat.}$$

By item 1, the Jacobian ideal of φ is the ideal $J = (y,z)T$. Since $(y,z)T \cap S = (y,z,f)S$ has height 3, φ_Q is not flat for every $Q \in \operatorname{Spec}(T)$ such that $(y,z)T \subseteq Q$. Thus the nonflat locus of $B \hookrightarrow R^*[1/x]$ is defined by $F_1 = (y,z)R^*[1/x]$ as stated in item 2.

Item 3 is clear if $\mathbf{p} = xR^*$. Let $\mathbf{p}$ be a height-one prime of R^* other than xR^*. Since $\mathbf{p}$ does not contain $(y,z)R^*$, the map $B_{\mathbf{p}\cap B} \hookrightarrow (R^*)_{\mathbf{p}}$ is faithfully flat. Thus $\operatorname{ht}(\mathbf{p} \cap B) \leq 1$. This establishes item 3.

Item 4 is clear if $wB = xB$. Assume that $wB \neq xB$ and let $\mathbf{p}$ be a height-one prime ideal of R^* that contains wR^*. Then $\mathbf{p}R^*[1/x] \cap R^* = \mathbf{p}$, and by item 3, $\mathbf{p} \cap B$ has height at most one. We have $\mathbf{p} \cap B \supseteq wR^* \cap B \supseteq wB$. Thus item 4 follows. □

Next we prove a proposition about homomorphic images of the constructed ring B. This result enables us in Corollary 18.4 to relate the ring B of Example 18.1 to the ring B of Example 17.3.

Proposition 18.3. *Assume the notation of Example 18.1, and let w be a prime element of $R = k[x,y,z]_{(x,y,z)}$ with $wR \neq xR$. Let $\pi : R^* \to R^*/wR^*$ be the natural* *homomorphism, and let* $^-$ *denote image in* R^*/wR^*. *Let* B' *be the approximation domain formed by considering* $\overline{R}$ *and the endpieces* $\overline{f}_n$ *of* $\overline{f}$, *defined analogously to Equation 17.1.a. That is,* B' *is defined by setting*

$$U'_n \ = \ \overline{k[x,y]}[\overline{f}_n], \ B'_n \ = \ (U'_n)_{\mathbf{n}'_n}, \ U' \ = \ \bigcup_{n=1}^{\infty} U'_n, \ \text{and} \ B' \ = \ \bigcup_{n=1}^{\infty} B'_n,$$

where $\mathbf{n}'_n$ *is the maximal ideal of* U'_n *that contains* $\overline{f}_n$ *and the image of* $\mathbf{m}_R$. *Then* $B' = \overline{B}$.

Proof. By Proposition 6.20.2, wB is a prime ideal of B. By Proposition 18.2.4, $wR^* \cap B = wB$. Hence $\overline{B} = B/(wR^* \cap B) = B/wB$. We have

$$\overline{R}/x\overline{R} \ = \ \overline{B}/x\overline{B} \ = \ \overline{R^*}/x\overline{R^*},$$

and the ring $\overline{R^*}$ is the $(\overline{x})$-adic completion of $\overline{R}$. Since the ideal $(y,z)R$ has height 2 and the kernel of π has height 1, at least one of $\overline{y}$ and $\overline{z}$ is nonzero. Since τ and σ are algebraically independent over $k(x,y,z)$, the element $\overline{f} = \overline{y} \cdot \overline{\tau} + \overline{z} \cdot \overline{\sigma}$ of the integral domain $\overline{B}$ is transcendental over $\overline{R}$. Similarly the endpieces $\overline{f}_n$ are transcendental over $\overline{R}$. The fact that $\overline{R^*}$ may fail to be an integral domain does not affect the algebraic independence of these elements that are inside the integral domain $\overline{B}$.

By Remark 6.4 we have $U_n[1/x] = U[1/x]$, and thus $wU \cap U_n = wU_n$ for each $n \in \mathbb{N}$. Since B_n is a localization of U_n, we also have $wB \cap B_n = wB_n$. Since $wR^* \cap B = wB$, it follows that $wR^* \cap B_n = wB_n$. Thus we have

$$\overline{R} \ \subseteq \ \overline{B_n} \ = \ B_n/wB_n \ \subseteq \ \overline{B} \ = \ B/wB \ \subseteq \ \overline{R^*} \ = \ R^*/wR^*.$$

We conclude that $\overline{B} = \bigcup_{n=0}^{\infty} \overline{B_n}$. Since $B'_n = \overline{B}_n$, we have $B' = \overline{B}$. □

Corollary 18.4. *The homomorphic image* B/zB *of the ring* B *of Example 18.1 is isomorphic to the three-dimensional ring* B *of Example 17.3.*

Proof. Assume the notation of Example 18.1 and Proposition 18.3 and let $w = z$. We show that the ring $B/zB \cong C$, where C is the ring called B in Example 17.3. By Proposition18.3, we have $B' = \overline{B/zB}$, where B' is the approximation domain over $\overline{R} = R/zR$ using the element $\overline{f}$, transcendental over $\overline{R}$. Let R_C denote the base ring $k[x,y]_{(x,y)}$ for C in Example 17.3, and let $\psi_0 : \overline{R} \to R_C$ denote the k-isomorphism defined by $\overline{x} \mapsto x$ and $\overline{y} \mapsto y$. Then, as in the proof of Proposition 18.3, $\overline{R^*}$ is the $(\overline{x})$-adic completion of $\overline{R}$. Thus ψ_0 extends to an isomorphism $\psi : \overline{R^*} \to (R_C)^*$ that agrees with ψ_0 on $\overline{R}$ and such that $\psi(\overline{\tau}) = \tau$. Furthermore $\psi(\overline{f}) = \psi(\overline{y} \cdot \overline{\tau} + \overline{z} \cdot \overline{\sigma}) = y\tau$, which is the transcendental element f used in the construction of C. Thus ψ is an isomorphism from $\overline{B} = B/zB$ to C, the ring constructed in Example 17.3. □

18.3. Verification of the example

We record in Theorem 18.5 properties of the ring B and its prime spectrum.

Theorem 18.5. *As in Example 18.1,* $R := k[x,y,z]_{(x,y,z)}$ *with* k *a field and* x, y *and* z *indeterminates, and* $R^* := k[y,z]_{(y,z)}[[x]]$, *the* xR*-adic completion of* R. *Let* τ *and* $\sigma \in xk[[x]]$ *be algebraically independent over* $k(x)$. *Set* $f := y\tau + z\sigma$, $A := R^* \cap k(x,y,z,f)$, *and* $B := \bigcup_{n=0}^{\infty} B_n = \bigcup_{n=0}^{\infty} k[x,y,z,f_n]_{(x,y,z,f_n)}$ *as in (18.1.2). Let* $Q := (y,z)R^* \cap B$. *Then*

(1) *The rings* A *and* B *are equal.*
(2) *The ring* B *is a four-dimensional non-Noetherian local UFD with maximal ideal* $\mathbf{m}_B = (x, y, z)B$*, and the* $\mathbf{m}_B$*-adic completion of* B *is the three-dimensional RLR* $k[[x, y, z]]$*.*
(3) *The ring* $B[1/x]$ *is a Noetherian regular UFD, the ring* B/xB *is a two-dimensional RLR, and, for every nonmaximal prime ideal* P *of* B*, the ring* B_P *is an RLR.*
(4) *The ideal* Q *is the unique prime ideal of* B *of height 3.*
(5) *The ideal* Q *equals* $\bigcup_{n=0}^{\infty} Q_n$ *where* $Q_n := (y, z, f_n)B_n$*,* Q *is a nonfinitely generated prime ideal, and* $QB_Q = (y, z, f)B_Q$*.*
(6) *There exist infinitely many height-two prime ideals of* B *not contained in* Q *and each of these prime ideals is contracted from* R^**.*
(7) *For certain height-one primes* p *contained in* Q*, there exist infinitely many height-two primes between* p *and* Q *that are contracted from* R^**, and infinitely many that are not contracted from* R^**. Hence the map* $\operatorname{Spec} R^* \to \operatorname{Spec} B$ *is not surjective.*
(8) *Every saturated chain of prime ideals of* B *has length either 3 or 4, and there exist saturated chains of prime ideals of lengths both 3 and 4. Thus* B *is not catenary.*
(9) *Each height-one prime ideal of* B *is the contraction of a height-one prime ideal of* R^**.*
(10) B *has Noetherian spectrum.*

We prove Theorem 18.5 below. First, assuming Theorem 18.5, we display a picture of $\operatorname{Spec}(B)$ and make comments about the diagram.

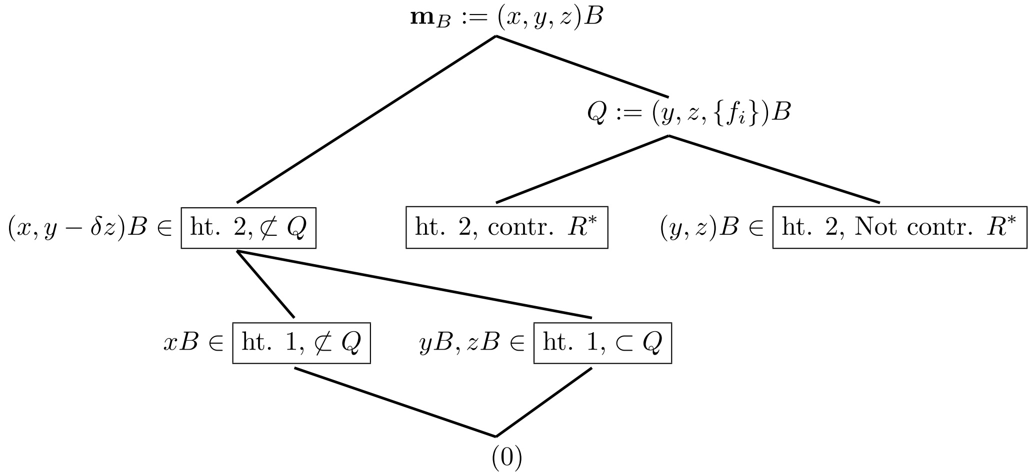

Diagram 18.5.0

Comments on Diagram 18.5.0. A line going from a box at one level to a box at a higher level indicates that every prime ideal in the lower level box is contained in at least one prime ideal in the higher level box. Thus as indicated in the diagram, every height-one prime gB of B is contained in a height-two prime of B that contains x and so is not contained in Q. This is obvious if $gB = xB$ and can be seen by considering minimal primes of $(g, x)B$ otherwise. Thus B has no maximal saturated chain of length 2. We have not drawn any lines from the lower level righthand box to higher boxes that are contained in Q because we are uncertain about what inclusion relations exist for these primes. We discuss this situation in Remarks 18.12.

Proof. (of Theorem 18.5.) By Proposition 18.2.1, $(y,z)T \subseteq F$, for the nonflat locus F of the extension $S \hookrightarrow T$. By Corollary 13.5, $\operatorname{ht}(FR^*[1/x]) > 1$ implies $A = B$ for the approximation and intersection domains B and A corresponding to the element f of R^*. This completes item 1. (Alternatively notice that Proposition 18.2.4 implies that the extension $R^* \to B$ is weakly flat, and that implies $A = B$, by Weak Flatness Theorem 12.5.)

For item 2, since B is a directed union of four-dimensional RLRs, we have $\dim B \leq 4$. By Corollary 18.4 and Theorem 17.9, $\dim(B/zB) = 3$. Thus $\dim B \geq 4$, and so $\dim B = 4$. By Proposition 6.20.5, the ring B is local with maximal ideal $\mathbf{m}_B = (x,y,z)B$. By Krull's Altitude Theorem 2.6, B is not Noetherian. The ring B is a UFD by Proposition 6.21. Since the (x)-adic completion of B is R^*, the $\mathbf{m}_B$-adic completion of B is $k[[x,y,z]]$.

For item 3, by Proposition 6.21, the ring $B[1/x]$ is a Noetherian regular UFD. By Construction Properties Theorem 6.19.3, we have $R/xR = B/xB$. Thus B/xB is a two-dimensional RLR.

For the last part of item 3, if $x \notin P$, then B_P is a localization of $B[1/x]$, which is Noetherian and regular, and so B_P is a regular local ring. In particular, this proves that B_Q is a regular local ring. If $x \in P$ and $\operatorname{ht} P = 1$, then $P = (x)$ and B_{xB} is a DVR. If $x \in P$ and $\operatorname{ht}(P) = 2$, the ideal P is finitely generated since B/xB is an RLR. Since B is a UFD from item 2, it follows that B_P is a local UFD of dimension 2 with finitely generated maximal ideal. Thus B_P is Noetherian by Cohen's Theorem 2.7. This, combined with B/xB a regular local ring, implies that B_P is a regular local ring. Since $\operatorname{ht} P \leq 2$ for every nonmaximal prime ideal P of R with $x \in P$, this completes the proof of item 3.

For item 4, since $(y,z)R^*$ is a prime ideal of R^*, the ideal $Q = (y,z)R^* \cap B$ is prime. By Proposition 6.20.2, the ideals yB and $(y,z)B$ are prime. Consider the chain of prime ideals

$$(0) \subset yB \subset (y,z)B \subset Q \subset \mathbf{m}_B.$$

The list y,z,f,x shows that each of the inclusions is strict; for example, we have $f \in Q \setminus (y,z)B$ since $f \notin (y,z)B_n$ for every $n \in \mathbb{N}$. By item 2 we have $\operatorname{ht} \mathbf{m}_B = 4$. Thus $\operatorname{ht} Q = 3$. This also implies that $(y,z)B$ is a height-two prime ideal of B.

For the uniqueness in item 4, let P be a nonmaximal prime ideal of B. If $x \in P$, then $B/xB = R/xR$ implies that $\operatorname{ht} P \leq 2$. If $x \notin P$, then, by Proposition 6.20.3, $x^n \notin PR^*$ for each positive integer n. Hence $PR^*[1/x] \neq R^*[1/x]$. Let P_1 be a prime ideal of $R^*[1/x]$ such that $P \subseteq P_1$. If both y and z are in P_1, then $(y,z)R^*[1/x] \subseteq P_1$. Since $(y,z)R^*[1/x]$ is maximal, we have $(y,z)R^*[1/x] = P_1$. Therefore, $P \subseteq (y,z)R^*[1/x] \cap B = Q$, and so either $\operatorname{ht}(P) \leq 2$ or $P = Q$.

Finally suppose that $x \notin P$ and y or z is not in P_1. Then the map $\psi : B \to R^*[1/x]_{P_1}$ is flat by Proposition 18.2.2. Since $\dim R^*[1/x] = 2$ we have $\operatorname{ht}(P_1) \leq 2$. Flatness of ψ implies $\operatorname{ht}(P_1 \cap B) \leq 2$, by Remark 2.20flgd. Hence $\operatorname{ht} P \leq 2$. This proves item 4.

For item 5, let $Q' = \bigcup_{n=0}^{\infty} Q_n$, where each $Q_n = (y,z,f_n)B_n$. Each Q_n is a prime ideal of height 3 in the 4-dimensional RLR B_n. Therefore Q' is a prime ideal of B of height ≤ 3 that is contained in Q. The ideal $(y,z)B$ is a prime ideal of height 2 strictly contained in Q by the proof of item 3. Hence $\mathrm{ht}(Q') = 3$ and we have $Q' = Q$.

To show the ideal Q is not finitely generated, we show for each positive integer n that $f_{n+1} \notin (y, z, f_n)B$. By Equation 18.1.1, $f_n = xf_{n+1} + yxc_{n+1} + zxd_{n+1}$. If $f_{n+1} \in (y, z, f_n)B$, then $f_{n+1} = ay + bz + c(xf_{n+1} + yxc_{n+1} + zxd_{n+1})$, where $a, b, c \in B$. This implies $f_{n+1}(1 - cx)$ is in the ideal $(y, z)B$. By Proposition 6.20, $x \in \mathcal{J}(B)$, and so $1 - cx$ is a unit of B. This implies $f_{n+1} \in (y, z)B \cap B_{n+1}$.

For each positive integer j, we show that $(y, z)B \cap B_j = (y, z)B_j$. It is clear that $(y, z)B_j \subseteq (y, z)B \cap B_j$. To show the reverse inclusion, it suffices to show for each integer $j \geq 0$ that $(y, z)B_{j+1} \cap B_j \subseteq (y, z)B_j$. We have $B_j[f_{j+1}] \subseteq (B_j)_{(y,z)B_j}$ since $f_{j+1} = \frac{f_j}{x} + yc_{j+1} + zd_{j+1}$ by (18.1.1). The center of the 2-dimensional RLR $(B_j)_{(y,z)B_j}$ on $B_j[f_{j+1}]$ is the prime ideal $(y, z)B_j[f_{j+1}]$. This prime ideal is contained in the maximal ideal $(x, y, z, f_{j+1})B_j[f_{j+1}]$; it follows that $B_{j+1} \subseteq (B_j)_{(y,z)B_j}$ and so $(y, z)B_{j+1} \cap B_j \subseteq (y, z)B_j$.

Thus $(y, z)B \cap B_{n+1} = (y, z)B_{n+1}$, and $f_{n+1} \in (y, z)B_{n+1}$. Since the ring B_{n+1} is Noetherian and the ideal $(y, z, f_{n+1})B_{n+1}$ has height three, this contradicts Krull's Altitude Theorem 2.6. We conclude that Q is not finitely generated.

We show above for item 3 that B_Q is a three-dimensional regular local ring. Since $Q = (y, z, f, f_1, f_2, \dots)B$ and, since x is a unit of B_Q, it follows from Remark 17.2 that $QB_Q = (y, z, f)B_Q$. This establishes item 5.

For item 6, since $x \notin Q$ and $B/xB \cong R/xR$ is a Noetherian ring of dimension two, there are infinitely many height-two primes of B containing xB; see Exercise 1. This proves there are infinitely many height-two primes of B not contained in Q. If P is a height-two prime of B not contained in Q, then $\mathrm{ht}(\mathbf{m}_B/P) = 1$, by item 4 above, and so, by Proposition 6.20.5, P is contracted from R^*. This completes the proof of item 6.

For item 7 we show that $p = zB$ has the stated properties. By Corollary 18.4, the ring B/zB is isomorphic to the ring called B in Example 17.3. For convenience we relabel the ring of Example 17.3 as B'. By Theorem 17.9, B' has exactly one non-finitely generated prime ideal, which we label Q', and $\mathrm{ht}\, Q' = 2$. It follows that $Q/zB = Q'$. By Discussion 17.5, there are infinitely many height-one primes contained in Q' of Type II (that is, primes that are contracted from R^*/zR^*) and infinitely many height-one primes contained in Q' of Type III (that is, primes that are not contracted from R^*/zR^*). The preimages in R^* of these primes are height-two primes of B that are contained in Q and contain zB. It follows that there are infinitely many contracted from R^* and there are infinitely many not contracted from R^*, as desired for item 7.

For item 8, we have a saturated chain of prime ideals

$$(0) \subset xB \subset (x, y)B \subset (x, y, z)B = \mathbf{m}_B$$

of length 3 since $B/xB = R/xR$ by Theorem 6.19.3. We have a saturated chain of prime ideals

$$(0) \subset yB \subset (y, z)B \subset Q \subset \mathbf{m}_B$$

of length 4 from the proof of item 4. Hence B is not catenary. By item 2, $\dim B = 4$, and so there is no saturated chain of prime ideals of B of length greater than 4. By Comments 18.5.0, there is no saturated chain of prime ideals of B of length less than 3.

For item 9, since R^* is a Krull domain and $B = A = \mathcal{Q}(B) \cap R^*$, it follows that B is a Krull domain and each height-one prime of B is the contraction of a height-one prime of R^*.

Item 10 follows since B/xB and $B[1/x]$ are Noetherian [**35**]. $\square$

REMARKS 18.6. Let the notation be as in Theorem 18.5.

(1) It follows from Theorem 18.5 that the localization $B[1/x]$ has a unique maximal ideal $QB[1/x] = (y, z, f)B[1/x]$ of height three and has infinitely many maximal ideals of height two. We observe that $B[1/x]$ has no maximal ideal of height one. To show this last statement it suffices to show for each irreducible element p of B with $pB \neq xB$ there exists $P \in \operatorname{Spec} B$ with $pB \subsetneq P$ and $x \notin P$. Assume there does not exist such a prime ideal P. Consider the ideal $(p, x)B$. This ideal has height two and has only finitely many minimal primes since B/xB is Noetherian. Let g be an element of $\mathbf{m}_B$ not contained in any of the minimal primes of $(p, x)B$. Every prime ideal of B that contains $(g, p)B$ also contains x and hence has height greater than two. Since $x \notin Q$, it follows that $(g, p)B$ is $\mathbf{m}_B$-primary, and hence that $(g, p)R^*$ is $\mathbf{m}_{R^*}$-primary. Since R^* is Noetherian and $\operatorname{ht} \mathbf{m}_{R^*} = 3$, this contradicts Krull's Altitude Theorem 2.6.

(2) Every ideal I of B such that IR^* is $\mathbf{m}_{R^*}$-primary is $\mathbf{m}_B$-primary by Proposition 6.20.5.

(3) Define

$$C_n := \frac{B_n}{(y, z)B_n} \quad \text{and} \quad C := \frac{B}{(y, z)B}.$$

We have $C = \bigcup_{n=0}^{\infty} C_n$ by item 1. We show that C is a rank 2 valuation domain with principal maximal ideal generated by the image of x. For each positive integer n, let $g_n \in C_n$ denote the image in C_n of the element $f_n \in B_n$ and let x denote the image of x. Then $C_n = k[x, g_n]_{(x, g_n)}$ is a 2-dimensional RLR. By (18.1.1), $f_n = xf_{n+1} + x(c_n y + d_n z)$. It follows that $g_n = xg_{n+1}$ for each $n \in \mathbb{N}$. Thus C is an infinite directed union of quadratric transformations of 2-dimensional regular local rings. Thus C is a valuation domain of dimension at most 2 by [**2**]. By items 2 and 4 of Theorem 18.5, $\dim C \geq 2$, and therefore C is a valuation domain of rank 2. The maximal ideal of C is xC.

We note that by Corollary 18.4, $B/zB \cong D$, where D is the ring B of Example 17.3. By an argument similar to that of Proposition 18.3 and Corollary 18.4 we see that the above ring C is isomorphic to D/yD.

QUESTION 18.7. For the ring B constructed as in Example 18.1, we ask: Is Q the only prime ideal of B that is not finitely generated?

Theorem 18.5 implies that the only possible nonfinitely generated prime ideals of B other than Q have height two. We do not know whether every height-two prime ideal of B is finitely generated. We show in Corollary 18.10 and Theorem 18.11 that certain of the height-two primes of B are finitely generated.

We recall Lemma 8.2, which was the key to the proof of Theorem 8.3. For convenience we repeat two parts of the lemma that are useful in this chapter:

LEMMA 18.8. *Let S be a subring of a ring T and let $b \in S$ be a regular element of both S and T. Assume that $bS = bT \cap S$ and $S/bS = T/bT$. Then*

(1) *$T[1/b]$ is flat over S $\iff$ T is flat over S.*

(2) *If T and $S[1/b]$ are both Noetherian and T is flat over S, then S is Noetherian*

THEOREM 18.9. *Assume the notation of Noetherian Flatness Theorem 8.8, and assume that F is an ideal of $R^*[1/a]$ that defines the nonflat locus of the map $\varphi : B \to R^*[1/a]$. Let I be an ideal in B such that $IR^* \cap B = I$ and a is regular on R^*/IR^*.*

(1) *If $IR^*[1/a] + F = R^*[1/a]$, then $\varphi \otimes_B (B/I)$ is flat.*
(2) *If $R^*[1/a]/IR^*[1/a]$ is flat over B/I, then R^*/IR^* is flat over B/I.*
(3) *If $\varphi \otimes_B (B/I)$ is flat, then B/I is Noetherian.*

PROOF. The hypothesis of item 1 implies that φ_P is flat for each $P \in \operatorname{Spec} R^*[1/a]$ with $I \subseteq P$. Hence for each such P we have $\varphi_P \otimes_B (B/I)$ is flat. Since flatness is a local property, it follows that $\varphi \otimes_B (B/I)$ is flat.

For items 2 and 3, apply Lemma 18.8 with $S = B/I$ and $T = R^*/IR^*$; the element b of Lemma 18.8 is the image in B/IB of the element a from the setting of Theorem 8.8. Since $IR^* \cap B = I$, the ring B/I embeds into R^*/IR^*, and since $B/aB = R^*/aR^*$, the ideal $a(R^*/IR^*) \cap (B/I) = a(B/I)$. Thus item 2 and item 3 of Theorem 18.9 follow from item 1 and item 2, respectively, of Lemma 18.8. ☐

COROLLARY 18.10. *Assume the notation of Example 18.1. Let w be a prime element of B. Then B/wB is Noetherian if and only if $w \notin Q$. Thus every every nonfinitely generated ideal of B is contained in Q.*

PROOF. If $w \in Q$, then B/wB is not Noetherian since Q is not finitely generated. Assume $w \notin Q$. Since B/xB is known to be Noetherian, we may assume that $wB \neq xB$. By Proposition 18.2.1, $QR^*[1/x] = (y,z)R^*[1/x]$ defines the nonflat locus of $\varphi : B \to R^*[1/x]$. Since $wR^*[1/x] + (y,z)R^*[1/x] = R^*[1/x]$, Theorem 18.9 with $I = wB$ and $a = x$ implies that B/wB is Noetherian.

For the second statement, we use that every nonfinitely generated ideal is contained in an ideal maximal with respect to not being finitely generated and the latter ideal is prime. Thus it suffices to show every prime ideal P not contained in Q is finitely generated. If $P \not\subseteq Q$, then, since B is a UFD, there exists a prime element $w \in P \setminus Q$. By the first statement, B/wB is Noetherian, and so P is finitely generated. ☐

THEOREM 18.11. *Assume the notation of Example 18.1. Let w be a prime element of R with $w \in (y,z)k[x,y,z]$. If w is linear in either y or z, then Q/wB is the unique nonfinitely generated prime ideal of B/wB. Thus Q is the unique nonfinitely generated prime ideal of B that contains w.*

PROOF. Let $\overline{}$ denote image under the canonical map $\pi : R^* \to R^*/wR^*$. We may assume that w is linear in z, that the coefficient of z is 1 and therefore that $w = z - yg(x,y)$, where $g(x,y) \in k[x,y]$. Thus $\overline{R} \cong k[x,y]_{(x,y)}$. By Proposition 18.3, $\overline{B}$ is the approximation domain over $\overline{R}$ with respect to the transcendental element

$$\overline{f} = \overline{y} \cdot \overline{\tau} + \overline{z} \cdot \overline{\sigma} = \overline{y} \cdot \overline{\tau} + \overline{y} \cdot \overline{g(x,y)} \cdot \overline{\sigma}.$$

The setting of Proposition 6.21 applies with B replaced by $\overline{B}$, the underlying ring R replaced by $\overline{R}$, and $z = \overline{x}$. Thus the ring $\overline{B}$ is a UFD, and so every height-one prime ideal of $\overline{B}$ is principal. Since $w \in Q$ and Q is not finitely generated, it follows that $\operatorname{ht}(\overline{Q}) = 2$ and that $\overline{Q}$ is the unique nonfinitely generated prime ideal of $\overline{B}$. Hence the theorem holds. ☐

REMARKS 18.12. It follows from Proposition 6.20.5 that every height two prime of B that is not contained in Q is contracted from a prime ideal of R^*. As we state in item 7 of Theorem 18.5, there are infinitely many height-two prime ideals of B that are contained in Q and are contracted from R^* and there are infinitely many height-two prime ideals of B that are contained in Q and are *not* contracted from R^*. In particular infinitely many of each type exist between zB and Q by Corollary 18.4, and similarly infinitely many of each type exist between yB and Q.

Since B_Q is a three-dimensional regular local ring, for each height-one prime p of B with $p \subset Q$, the set

$$\mathcal{S}_p \;=\; \{P \in \operatorname{Spec} B \mid p \subset P \subset Q \text{ and } \operatorname{ht} P = 2\}$$

is infinite. The infinite set $\mathcal{S}_p$ is the disjoint union of the sets $\mathcal{S}_{pc}$ and $\mathcal{S}_{pn}$, where the elements of $\mathcal{S}_{pc}$ are contracted from R^* and the elements of $\mathcal{S}_{pn}$ are not contracted from R^*.

We do not know whether there exists a height-one prime p contained in Q having the property that one of the sets $\mathcal{S}_{pc}$ or $\mathcal{S}_{pn}$ is empty. Furthermore if one of these sets is empty, which one is empty? If there are some such height-one primes p with one of the sets $\mathcal{S}_{pc}$ or $\mathcal{S}_{pn}$ empty, which height-one primes are they? It would be interesting to know the answers to these questions.

Exercises

(1) Let R be a Noetherian ring. Let $P_1 \subset P_2$ be prime ideals of R. If there exists a prime ideal Q of R with Q distinct from P_1 and P_2 such that $P_1 \subset Q \subset P_2$, prove that there exist infinitely many such prime ideals Q.

Suggestion: Apply Krull's Altitude Theorem 2.6, and use the fact that an ideal contained in a finite union of primes is contained in one of them [**5**, Prop. 1.11, page 8].

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