Integral Domains Inside Noetherian Power Series Rings: Constructions and Examples
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Abstract. In this monograph the authors gather together results and examples from their work of the past two decades related to power series rings and to completions of Noetherian integral domains.

A major theme is the creation of examples that are appropriate intersections of a field with a homomorphic image of a power series ring over a Noetherian domain. The creation of examples goes back to work of Akizuki and Schmidt in the 1930s and Nagata in the 1950s.

In certain circumstances, the intersection examples may be realized as a directed union, and the Noetherian property for the associated directed union is equivalent to a flatness condition. This flatness criterion simplifies the analysis of several classical examples and yields other examples such as

- A catenary Noetherian local integral domain of any specified dimension of at least two that has geometrically regular formal fibers and is not universally catenary.
- A three-dimensional non-Noetherian unique factorization domain B such that the unique maximal ideal of B has two generators; B has precisely n prime ideals of height two, where n is an arbitrary positive integer; and each prime ideal of B of height two is not finitely generated but all the other prime ideals of B are finitely generated.
- A two-dimensional Noetherian local domain that is a birational extension of a polynomial ring in three variables over a field yet fails to have Cohen-Macaulay formal fibers. This example also demonstrates that Serre's condition S_1 need not lift to the completion; the example is related to an example of Ogoma.

Another theme is an analysis of extensions of integral domains $R \hookrightarrow S$ having trivial generic fiber, that is, every nonzero prime ideal of S has a nonzero intersection with R. Motivated by a question of Hochster and Yao, we present results about

- The height of prime ideals maximal in the generic fiber of certain extensions involving mixed power series/polynomial rings.
- The prime ideal spectrum of a power series ring in one variable over a one-dimensional Noetherian domain.
- The dimension of S if $R \hookrightarrow S$ is a local map of complete local domains having trivial generic fiber.

A third theme relates to the questions:

- What properties of a Noetherian domain extend to a completion?
- What properties of an ideal pass to its extension in a completion?
- What properties extend for a more general multi-adic completion?

We give an example of a three-dimensional regular local domain R having a prime ideal P of height two with the property that the extension of P to the completion of R is not integrally closed.

All of these themes are relevant to the study of prime spectra of Noetherian rings and of the induced spectral maps associated with various extensions of Noetherian rings. We describe the prime spectra of various extensions involving power series.

Contents

Preface

The authors have had as a long-term project the creation of examples using power series to analyze and distinguish several properties of commutative rings and their spectra. This monograph is our attempt to expose the results that have been obtained in this endeavor, to put these results in better perspective and to clarify their proofs. We hope in this way to assist current and future researchers in commutative algebra in utilizing the techniques described here.

Dedication

This monograph is dedicated to Mary Ann Heinzer, to Maria Rotthaus, to Roger Wiegand, and to the past, present and future students of the authors.

William Heinzer, Christel Rotthaus, Sylvia Wiegand

CHAPTER 1

Introduction

When we started to collaborate on this work about twenty years ago, we were inspired by expository talks Judy Sally gave on the following question:

QUESTION 1.1. What rings lie between a Noetherian integral domain and its field of fractions?

We were also inspired by Shreeram Abhyankar's research such as that in his paper [**1**] to ask the following related question:[1]

QUESTION 1.2. Let I be an ideal of a Noetherian integral domain R and let R^* denote the I-adic completion of R. What rings lie between R and R^*? For example, if x and y are indeterminates over a field k, what rings lie between the polynomial ring $k[x, y]$ and the mixed polynomial-power series ring $k[y][[x]]$?

In this book we encounter a wide variety of integral domains fitting the descriptions of Question 1.1 and Question 1.2.

Over the past eighty years, important examples of Noetherian integral domains have been constructed that are an intersection of a field with a homomorphic image of a power series ring. The basic idea is that, starting with a typical Noetherian integral domain R such as a polynomial ring over a field, we look for more unusual Noetherian and non-Noetherian extension rings inside a homomorphic image S of an ideal-adic completion of R. An ideal-adic completion of R is a homomorphic image of a power series ring over R; see Section 3.1 of Chapter 3.[2]

BASIC CONSTRUCTION EQUATION 1.3. This construction features an "intersection" domain A of the form:

$$A \;:=\; L \cap S,$$

where R and S are as in the preceding paragraph, and L is a field between the field of fractions of R and the total quotient ring of S.

We have the following major goals:

(1) To construct new examples of Noetherian rings, continuing a tradition that goes back to Akizuki and Schmidt in the 1930s and Nagata in the 1950s.
(2) To construct new non-Noetherian integral domains that illustrate recent advances in ideal theory.

[1] Ram's work demonstrates the vastness of power series rings; a power series ring in two variables over a field k contains for each positive integer n an isomorphic copy of the power series ring in n variables over k. The authors have fond memories of many pleasant conversations with Ram concerning power series.

[2] Most terminology used in this introduction, such as "ideal-adic completion", "coefficient field", "essentially finitely generated" and "integral closure", are defined in Chapters 2 and 3.

(3) To study birational extensions of Noetherian integral domains as in Question 1.1.

(4) To consider the fibers of an extension $R \hookrightarrow R^*$, where R is a Noetherian domain and R^* is the completion of R with respect to an ideal-adic topology, and to relate these fibers to birational extensions of R.

These objectives form a complete circle, since (4) is used to accomplish (1).

We have been captivated by these topics and have been examining ways to create new rings from well-known ones for a number of years. Several chapters of this monograph, such as Chapters 4, 5, 6, 8, 17 and 21, contain a reorganized development of previous work on this technique.

Basic Construction Equation 1.3 as presented here is *universal* in the following sense: Every Noetherian local domain A having a coefficient field k and with field of fractions L finitely generated over k is an intersection $L \cap S$, as in Basic Construction Equation 1.3, where $S = \widehat{R}/I$ and I is a suitable ideal of the $\mathbf{m}$-adic completion $\widehat{R}$ of a Noetherian local domain $(R, \mathbf{m})$. Furthermore we can choose R so that k is also a coefficient field for the ring R, L is the field of fractions of R and R is essentially finitely generated over k; see Section 5.2 of Chapter 5.

Classical examples of Noetherian integral domains with interesting properties are constructed by Akizuki, Schmidt, and Nagata in [**7**], [**125**], and [**103**]. This work is continued by Brodmann-Rotthaus, Ferrand-Raynaud, Heitmann, Lequain, Nishimura, Ogoma, Rotthaus, Weston and others in [**15**], [**16**], [**35**], [**74**], [**75**], [**76**], [**83**], [**105**], [**110**], [**111**], [**118**], [**119**], and [**140**].

What are the classical examples?

Classical Examples 1.4. Many of the classical examples concern integral closures. Akizuki's 1935 example is a one-dimensional Noetherian local domain R of characteristic zero such that the integral closure of R is not a finitely generated R-module [**7**]. Schmidt's 1936 example is a one-dimensional normal Noetherian local domain R of positive characteristic such that the integral closure of R in a finite purely inseparable extension field is not a finitely generated R-module [**125**, pp. 445-447]. In relation to integral closure, Nagata's classic examples include (1) a two-dimensional Noetherian local domain with a non-Noetherian birational integral extension and (2) a three-dimensional Noetherian local domain such that the integral closure is not Noetherian [**104**, Examples 4 and 5, pp. 205-207].

In Example 4.8 of Chapter 4, we consider another example constructed by Nagata. This is the first occurence of a two-dimensional regular local domain containing a field of characteristic zero that fails to be a Nagata domain, and hence is not excellent. For the definition and information on Nagata rings and excellent rings; see Definitions 2.3.1 and 3.31 in Sections 2.1 and 3.4, and Chapter 14. We describe in Example 4.10 a construction due to Rotthaus of a Nagata domain that is not excellent.

In the foundational work of Akizuki, Nagata and Rotthaus (and indeed in most of the papers cited above) the description of the constructed ring A as the basic intersection domain of Equation 1.3 is not explicitly stated. Instead A is defined as a direct limit or directed union of subrings. In Chapters 4 to 12, we expand the basic construction to include an additional integral domain, also associated to the ideal-adic completion of R with respect to a principal ideal. Our adjusted "Basic Construction" consists of two integral domains that fit with these examples:

Basic Construction 1.5. This construction consists of two integral domains described as follows:

(BC1) The "intersection" integral domain A of Basic Construction Equation 1.3: $A = L \cap S$, the intersection of a field L with a homomorphic image S of the completion of R with respect to a principal ideal, and

(BC2) An "approximation" domain B, that is a directed union inside A that approximates A and is more easily understood; sometimes B is a nested union of localized polynomial rings over R

The details of the construction of B as in (BC2) are given in Chapter 6. Construction Properties Theorems 6.17 and 6.19 describe essential properties of the construction and are used throughout this book.

In certain circumstances the approximation domain B of (BC2) is equal to the intersection domain A of (BC1). In this case, the intersection domain A is a directed union. This yields more information about A. The description of A as an intersection is often unfathomable! In case $A = B$, the critical elements of B that determine L are called *limit-intersecting* over R; see Chapter 6 (Definition 6.5) and Chapters 12, 21 and 22 where we discuss the limit-intersecting condition further.

To see a specific example of the construction, consider the ring $R := \mathbb{Q}[x, y]$, the polynomial ring in the variables x and y over the field $\mathbb{Q}$ of rational numbers. Let S be the formal power series ring $\mathbb{Q}[[x, y]]$ and let L be the field $\mathbb{Q}(x, y, e^x, e^y)$.[3] Then Equation 1.3 yields that

$$\alpha = \frac{e^x - e^y}{x - y} \in A = \mathbb{Q}(x, y, e^x, e^y) \cap \mathbb{Q}[[x, y]], \tag{1.3.a}$$

but $\alpha \notin B$, the approximation domain. In this example, the intersection domain A is Noetherian, whereas the approximation domain B is not Noetherian. More details about this example are given in Chapter 4 (Example 4.4 and Theorem 4.12), Chapter 7 (Theorem 7.2 and Example 7.3) and Chapter 8 (Example 8.11).

A primary task of our study is to determine, for a given Noetherian domain R, whether the ring $A = L \cap S$ of Basic Construction Equation 1.3 is Noetherian. An important observation related to this task is that the Noetherian property for the associated direct limit ring B is equivalent to a flatness condition; see Noetherian Flatness Theorems 8.3 and 8.8. Whereas it took only about a page for Nagata [**104**, page 210] to establish the Noetherian property of his example, the proof of the Noetherian property for the example of Rotthaus took 7 pages [**118**, pages 112-118]. The results presented in Chapter 8 establish the Noetherian property rather quickly for this and other examples.

The construction of B from (BC1) is related to an interesting construction introduced by Ray Heitmann [**74**, page 126]. Let x be a nonzero nonunit in a Noetherian integral domain R, and let R^* denote the (x)-adic completion of R. Heitmann describes a procedure for associating, to each element τ in R^* that is transcendental over R, an extension ring T of $R[\tau]$ having the property that the

[3]This example with power series in two variables does not come from one principal ideal-adic completion of R as in (BC1) above, but it may be realized, for example, by taking first the (x)-adic completion R^* and then taking the (y)-adic completion of R^*, an "iterative" process.

(x)-adic completion of T is R^*.[4] Heitmann uses this technique to construct interesting examples of non-catenary Noetherian rings. In a 1997 article, the present authors adapt the construction of Heitmann to prove a version of Noetherian Flatness Theorem 8.8 of Chapter 8 that applies for one transcendental element τ over a semilocal Noetherian domain R: If the element τ satisfies a flatness condition we call *primarily limit-intersecting*, then the constructed intersection domain A is equal to the approximation domain B and is Noetherian [**55**, Theorem 2.8]; see Remark 8.9.2.

This "primarily limit-intersecting" concept from [**55**] extends to more than one transcendental element τ; see Definition 21.9. This permits the extension of Heitmann's construction to finitely many algebraically independent elements of R^*; see [**55**, Theorem 2.12]. Thus, with Basic Construction Equation 1.3 as presented in Chapter 5, we are able to prove Noetherian Flatness Theorem 8.8 in the case where the base ring R is an arbitrary Noetherian integral domain with field of fractions K, the extension ring S is the (x)-adic completion R^* of R, and the field L is generated over K by a finite set of algebraically independent power series in S.

In the case where S is the ideal-adic completion R^* of R and L is a field between R and the total quotient ring of R^*, the integral domain $A = L \cap R^*$ sometimes inherits nice properties from R^*, for example, the Noetherian property. If the approximation domain B is Noetherian, then B is equal to the intersection domain A. The converse fails however; it is possible for B to be equal to A and not be Noetherian; see Example 13.8. If B is not Noetherian, we can sometimes determine the prime ideals of B that are not finitely generated; see Example 17.1. If a ring has exactly one prime ideal that is not finitely generated, that prime ideal contains all nonfinitely generated ideals of the ring.

In Chapters 7 and 10, we adjust the construction from Chapters 4-9. An "insider" technique is introduced in Chapter 10 and generalized in Chapter 13 for building new examples inside more straightforward examples constructed as above. Using Insider Construction 13.1, the verification of the Noetherian property for the constructed rings is streamlined. Even if one of the constructed rings is not Noetherian, the proof is simplified. We analyze classical examples of Nagata and others from this viewpoint. Chapter 7 contains an investigation of more general rings that involve power series in two variables x and y over a field k, as is the case with the specific example given above in Equation 1.3.a.

In Chapters 17 and 18, we use Insider Construction 13.1 to construct low-dimensional non-Noetherian integral domains that are strangely close to being Noetherian: One example is a three-dimensional local unique factorization domain B inside $k[[x, y]]$; the ring B has maximal ideal $(x, y)B$ and exactly one prime ideal that is not finitely generated; see Example 17.1.

There has been considerable interest in non-Noetherian analogues of Noetherian notions such as the concept of a "regular" ring; see the book by Glaz [**41**]. Rotthaus and Sega in [**124**] show that the approximation domains B constructed in Chapters 17 and 18, even though non-Noetherian, are *coherent regular* local

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[4] Heitmann remarks in [**74**] that this type of extension also occurs in [**104**, page 203]. The ring T is not finitely generated over $R[\tau]$ and no proper $R[\tau]$-subring of T has R^* as its (x)-adic completion. Necessary and sufficient conditions are given in order that T be Noetherian in Theorem 4.1 of [**74**].

rings by showing that every finitely generated submodule of a free module over B has a finite free resolution; see [**124**] and Remark 17.11.[5]

One of our additional goals is to consider the question: "What properties of a ring extend to a completion?" Chapter 15 contains an example of a three-dimensional regular local domain $(A, \mathbf{n})$ with a height-two prime ideal P such that the extension $P\widehat{A}$ to the $\mathbf{n}$-adic completion of A is not integrally closed. In Chapter 16 we prove that the Henselization of a Noetherian local ring having geometrically normal formal fibers is universally catenary; we also present for each integer $n \geq 2$ a catenary Noetherian local integral domain having geometrically normal formal fibers that is not universally catenary.

We consider excellence in regard to the question: "What properties of the base ring R are preserved by the construction?" Since excellence is an important property satisfied by most of our rings, we present in Chapter 14 a brief exposition of excellent rings. In some cases we determine when the constructed ring is excellent; for example, see Chapter 9 (Polynomial Example Theorems 9.2, 9.5 and 9.7), Chapter 13 and Chapter 27. Assume the ring R is a unique factorization domain (UFD) and R^* is the (a)-adic completion of R with respect to a prime element a of R. We observe that the approximation domain B is then a UFD; see Theorem 6.21 of Chapter 6.

Since the Noetherian property for the approximation domain is equivalent to the flatness of a certain homomorphism, we devote considerable time and space to exploring flat extensions. We present results involving flatness in Chapters 9, 11, 12, 19, 20, 21 and 22.

The application of Basic Construction Equation 1.3 in Chapters 19 and 20 yields "idealwise" examples that are of a different nature from the examples in earlier chapters. Whereas the base ring $(R, \mathbf{m})$ is an excellent normal local domain with $\mathbf{m}$-adic completion $(\widehat{R}, \widehat{\mathbf{m}})$, the field L is more general than in Chapter 5. We take L to be a purely transcendental extension of the field of fractions K of R such that L is contained in the field of fractions of $\widehat{R}$; say $L = K(G)$, where G is a set of elements of $\widehat{\mathbf{m}}$ that are algebraically independent over K. Define $D := L \cap \widehat{R}$, The set G is said to be *idealwise independent* if $K(G) \cap \widehat{R}$ equals the localized polynomial ring $R[G]_{(\mathbf{m},G)}$. The results of Chapters 19 and 20 show that the intersection domain can sometimes be small or large, depending on whether expressions in the power series allow additional prime divisors as denominators. The consideration of idealwise independence leads us to examine other related flatness conditions, namely the concepts of "residual algebraic independence" and "primary independence"; see Definitions 19.24 and 19.12. The analysis and properties related to idealwise independence are summarized in Summaries 19.6 and 20.1.

In Chapters 21 and 22, we consider properties of the constructed rings A and B if R is an excellent normal local domain and $S = R^*$ is an ideal-adic completion of R. In analogy with the concepts of residual algebraic independence and primary independence above, we consider the concepts, "residually limit-intersecting" and "primarily limit-intersecting" for elements of R^* over R; see Definitions 21.9. We

[5]Rotthaus and Sega show more generally that the approximation domains constructed with Insider Construction 13.1 are coherent regular if $R = k[x, y_1 \dots, y_r]_{(x,y_1\dots,y_r)}$ is a localized polynomial ring over a field k, $m = 1$, $r, n \in \mathbb{N}$ and $\tau_1, \dots, \tau_n$ are algebraically independent elements of $xk[[x]]$. The approximation domains used by Rotthaus and Sega can have arbitrarily large Krull dimension, whereas the rings constructed in Chapters 17 and 18 have dimension 3 or 4.

draw connections with Cohen-Macaulay fibers and discuss properties of an example, due to Ogoma, of a three-dimensional normal Nagata local domain whose generic formal fiber is not equidimensional.

In Chapter 26, we study prime ideals and their relations in mixed polynomial-power series extensions of low-dimensional rings. For example, we determine the prime ideal structure of the power series ring $R[[x]]$ over a one-dimensional Noetherian domain R and the prime ideal structure of $k[[x]][y]$, where x and y are indeterminates over a field k. We analyze the generic fibers of mixed polynomial-power series ring extensions in Chapter 24. Motivated by a question of Hochster and Yao, we consider in Chapter **??** extensions of integral domains $S \hookrightarrow T$ having *trivial generic fiber*; that is, every nonzero prime ideal of T intersects S in a nonzero prime ideal.

Let R be a Noetherian ring with Jacobson radical $\mathcal{J}$. In Chapter 27 we consider the multi-ideal-adic completion R^* of R with respect to a filtration $\mathcal{F} = \{Q_n\}_{n\geq 0}$, where $Q_n \subseteq \mathcal{J}^n$ and $Q_{nk} \subseteq Q_n^k$ for each $n, k \in \mathbb{N}$. We prove that R^* is Noetherian. If R is an excellent local ring, we prove that R^* is excellent. If R is a Henselian local ring, we prove that R^* is Henselian.

The topics of this book include the following:

(1) An introduction and glossary for the terms and tools used in the book, Chapters 2 and 3.
(2) The development of the construction of the intersection domain A and the approximation domain B, Chapters 4-10,13.
(3) Flatness properties of maps of rings, Chapters 8, 9, 11, 12, 19-22.
(4) Preservation of properties of rings and ideals under passage to completion, Chapters 15, 27.
(5) The catenary and universally catenary property of Noetherian rings, Chapter 9, 16.
(6) Excellent rings and geometrically regular and geometrically normal formal fibers, Chapters 9, 13, 14, 16.
(7) Examples of non-Noetherian local rings having Noetherian completions, Chapters 7, 13, 17-19.
(8) Examples of Noetherian rings, Chapters 4, 6, 7, 9, 15, 18, **??**.
(9) Prime ideal structure, Chapters 17, 24-**??**.
(10) Approximating a discrete rank-one valuation domain using higher-dimensional regular local rings, Chapter **??**.
(11) Trivial generic fiber extensions, Chapters 24-**??**.
(12) Transfer of excellence, Chapters 9, 13, 27.
(13) Birational extensions of Noetherian domains, Chapters 8, 17, 18, 21, 22.
(14) Completions and multi-ideal-adic completions, Chapters 3, 27.
(15) Exercises to engage the reader in these topics and to lead to further extensions of the material presented here.

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CHAPTER 2

Tools

In this chapter we review conventions and terminology, state several basic theorems and review the concept of flatness of modules and homomorphisms.

2.1. Conventions and terminology

We generally follow the notation of Matsumura [**96**]. Thus by a ring we mean a commutative ring with identity, and a ring homomorphism $R \to S$ maps the identity element of R to the identity element of S. For commutative rings, when we write $R \subseteq S$, we mean that R is a subring of S, and that R contains the identity element of S. We use the words "map", "morphism", and "homomorphism" interchangeably.

The set of prime ideals of a ring R is called the *prime spectrum* of R and is denoted $\operatorname{Spec} R$. The set $\operatorname{Spec} R$ is naturally a partially ordered set with respect to inclusion. For an ideal I of a ring R, let

$$\mathcal{V}(I) \;=\; \{P \in \operatorname{Spec} R \mid I \subseteq P\ \}.$$

The *Zariski topology* on $\operatorname{Spec} R$ is obtained by defining the *closed* subsets to be the sets of the form $\mathcal{V}(I)$ as I varies over all the ideals of R. The *open* subsets are the complements $\operatorname{Spec} R \setminus \mathcal{V}(I)$.

We use $\mathbb{Z}$ to denote the ring of integers, $\mathbb{N}$ for the positive integers, $\mathbb{N}_0$ the nonnegative integers, $\mathbb{Q}$ the rational numbers, $\mathbb{R}$ the real numbers and $\mathbb{C}$ the complex numbers.

Regular elements, regular sequence. An element r of a ring R is said to be a *zerodivisor* if there exists a nonzero element $a \in R$ such that $ar = 0$, and r is a *regular element* if r is not a zerodivisor.

A sequence of elements $x_1, \dots, x_d$ in R is called a *regular sequence* if (i) $(x_1, \dots, x_d)R \neq R$, and (ii) x_1 is a regular element of R, and, for i with $2 \le i \le d$, the image of x_i in $R/(x_1, \dots, x_{i-1})R$ is a regular element; see [**96**, pages 123].

The *total ring of fractions* of the ring R, denoted $\mathcal{Q}(R)$, is the localization of R at the multiplicatively closed set of regular elements, thus $\mathcal{Q}(R) := \{\ a/b \mid a, b \in R$ and b is a regular element$\}$. There is a natural embedding $R \hookrightarrow \mathcal{Q}(R)$ of a ring R into its total ring of fractions $\mathcal{Q}(R)$, where $r \mapsto \frac{r}{1}$ for every $r \in R$.

An *integral domain*, sometimes called a *domain* or an *entire* ring, is a nonzero ring in which every nonzero element is a regular element. If R is a subring of an integral domain S and S is a subring of $\mathcal{Q}(R)$, we say S is *birational over* R, or a *birational extension* of R.

Krull dimension. The *Krull dimension*, or briefly *dimension*, of a ring R, denoted $\dim R$, is n if there exists a chain $P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n$ of prime ideals of R and there is no such chain of length greater than n. We say that $\dim R = \infty$ if there exists a chain of prime ideals of R of length greater than n for each $n \in \mathbb{N}$.

For a prime ideal P of a ring R, the *height* of P, denoted $\operatorname{ht} P$, is $\dim R_P$, where R_P is the localization of R at the multiplicatively closed set $R \setminus P$.

Unique factorization domains. An integral domain R is a *unique factorization domain* (UFD), sometimes called a *factorial ring*, if every nonzero nonunit of R is a finite product of prime elements; an element $p \in R$ is *prime* if pR is a prime ideal.

In a UFD every height-one prime ideal is principal; this is Exercise 2.1.

Local rings. If a ring R (not necessarily Noetherian) has a unique maximal ideal $\mathbf{m}$, we say R is *local* and write $(R, \mathbf{m})$ to denote that R is local with maximal ideal $\mathbf{m}$. If $(R, \mathbf{m})$ and $(S, \mathbf{n})$ are local rings, a ring homomorphism $f : R \to S$ is a *local homomorphism* if $f(\mathbf{m}) \subseteq \mathbf{n}$.

Let $(R, \mathbf{m})$ be a local ring. A subfield k of R is said to be a *coefficient field for* R if the composite map $k \hookrightarrow R \to R/\mathbf{m}$ defines an isomorphism of k onto $R/\mathbf{m}$.

If $(R, \mathbf{m})$ is a subring of a local ring $(S, \mathbf{n})$, then S is said to *dominate* R if $\mathbf{m} = \mathbf{n} \cap R$, or equivalently, if the inclusion map $R \hookrightarrow S$ is a local homomorphism. The local ring $(S, \mathbf{n})$ is said to *birationally dominate* $(R, \mathbf{m})$ if S is an integral domain that dominates R and S is contained in the field of fractions of R.

Nilradical. For an ideal I of a ring R, the *radical* of I, denoted $\operatorname{rad} I$, is the ideal $\operatorname{rad} I = \{a \in R \mid a^n \in I \text{ for some } n \in \mathbb{N}\}$. The ideal I is said to be a *radical ideal* if $\operatorname{rad} I = I$. The *nilradical* of a ring R is $\operatorname{rad}(0)$. The nilradical of R is the intersection of all the prime ideals of R. The ring R is said to be *reduced* if (0) is a radical ideal.

Jacobson radical. The *Jacobson radical* $\mathcal{J}(R)$ of a ring R is the intersection of all maximal ideals of R. An element z of R is in $\mathcal{J}(R)$ if and only if $1 + zr$ is a unit of R for all $r \in R$.

If I is a proper ideal of R, then $1 + I := \{\ 1 + a \mid a \in I\ \}$ is a multiplicatively closed subset of R that does not contain 0. Let $(1+I)^{-1}R$ denote the localization $R_{(1+I)}$ of R at the multiplicatively closed set $1 + I$, [**96**, Section 4]. If P is a prime ideal of R and $P \cap (1 + I) = \emptyset$, then $(P + I) \cap (1 + I) = \emptyset$. Therefore I is contained in every maximal ideal of $(1 + I)^{-1}R$, so $I \subseteq \mathcal{J}((1 + I)^{-1}R)$. In particular for the principal ideal $I = zR$, where z is a nonunit of R, we have $z \in \mathcal{J}((1 + zR)^{-1}R)$.

Finite, finite type, finite presentation. Let R be a ring, let M be an R-module and let S be an R-algebra.

(1) M is said to be a *finite* R-module if M is finitely generated as an R-module.
(2) S is said to be *finite* over R if S is a finitely generated R-module.
(3) S is of *finite type* over R if S is finitely generated as an R-algebra. Equivalently, S is an R-algebra homomorphic image of a polynomial ring in finitely many variables over R.
(4) S is *finitely presented* as an R-algebra if, for some polynomial ring $R[x_1, \ldots, x_n]$ in variables $x_1, \ldots, x_n$ and R-algebra homomorphism $\varphi : R[x_1, \ldots, x_n] \to S$ that is surjective, $\ker \varphi$ is a finitely generated ideal of $R[x_1, \ldots, x_n]$.
(5) S is *essentially finite* over R if S is a localization of a finite R-module.
(6) S is *essentially of finite type* over R if S is a localization of a finitely generated R-algebra. We also say that S is *essentially finitely generated* in this case.

(7) S is *essentially finitely presented* over R if S is a localization of a finitely presented R-algebra.

Symbolic powers. If P is a prime ideal of a ring R and e is a positive integer, the e^{th} *symbolic power of* P, denoted $P^{(e)}$, is defined as

$$P^{(e)} := \{a \in R \mid ab \in P^e \text{ for some } b \in R \setminus P\}.$$

Valuation domains. An integral domain R is a *valuation domain* if for each element $a \in \mathcal{Q}(R) \setminus R$, we have $a^{-1} \in R$. A valuation domain R is called a *discrete rank-one valuation ring* or a *discrete valuation ring* (DVR) if R is Noetherian and not a field; equivalently, R is a local principal ideal domain (PID) and not a field.

REMARKS 2.1. (1) If R is a valuation domain with field of fractions K and F is a subfield of K, then $R \cap F$ is again a valuation domain and has field of fractions F [**104**, (11.5)]. If R is a DVR and the field F is not contained in R, then $R \cap F$ is again a DVR [**104**, (33.7)].

(2) Every valuation domain R has an associated *valuation* v and *value group* G; the valuation v is a function $v : R \to G$ satisfying properties 1 and 2 of Remark 2.2, where the order function $ord_{R,I}$ is replaced by v in the equations of properties 1 and 2. See [**96**, p. 75] for more information about the value group and valuation associated to a valuation domain.

Integral ring extensions, integral closure, normal domains. For commutative rings $R \subseteq S$, we say that S is *integral* over R, or an *integral extension* of R, provided every $a \in S$ is a root of some monic polynomial $f(x)$ in the polynomial ring $R[x]$. An integral domain R is said to be *integrally closed* provided it satisfies the following condition: for every monic polynomial $f(x)$ in the polynomial ring $R[x]$, if $a \in \mathcal{Q}(R)$ is a root of $f(x)$, then $a \in R$. The ring R is a *normal ring* if for each $P \in \operatorname{Spec} R$ the localization R_P is an integrally closed domain [**96**, page 64]. If R is a Noetherian normal ring and $\mathbf{p}_1, \dots, \mathbf{p}_r$ are the minimal primes of R, then R is isomorphic to the direct product $R/\mathbf{p}_1 \times \cdots \times R/\mathbf{p}_r$ and each $R/\mathbf{p}_i$ is an integrally closed domain. Since a nontrivial direct product is not local, a normal Noetherian local ring is a normal domain.

For an extension $R \subseteq S$ of rings, the *integral closure* of R in S is the set of all elements of S that are integral over R. The *integral closure* or *derived normal ring* of an integral domain R is the integral closure of R in its field of fractions.

If R is a normal Noetherian integral domain and L is a finite separable algebraic field extension of $\mathcal{Q}(R)$, then the integral closure of R in L is a finite R-module by [**96**, Lemma 1, page 262] or [**104**, (10.16)]. Thus, if R is a normal Noetherian integral domain of characteristic zero, then the integral closure of R in a finite algebraic field extension is a finite R-module.

The order function associated to an ideal. Let I be a nonzero ideal of a ring R such that $\bigcap_{n=0}^{\infty} I^n = (0)$. Adopt the convention that $I^0 = R$, and for each nonzero element $r \in R$ define

$$\operatorname{ord}_{R,I}(r) := n \quad \text{if} \quad r \in I^n \setminus I^{n+1}.$$

REMARK 2.2. With R, I and $\operatorname{ord}_{R,I}$ as above, consider the following two properties for nonzero elements a, b in R:

(1) If $a + b \neq 0$, then $\operatorname{ord}_{R,I}(a+b) \geq \min\{\operatorname{ord}_{R,I}(a), \operatorname{ord}_{R,I}(b)\}$.
(2) $\operatorname{ord}_{R,I}(ab) = \operatorname{ord}_{R,I}(a) + \operatorname{ord}_{R,I}(b)$.

Clearly the function $\operatorname{ord}_{R,I}$ always satisfies property 1. If $\operatorname{ord}_{R,I}$ satisfies property 2 for all nonzero a, b in R, then the function $\operatorname{ord}_{R,I}$ extends uniquely to a function on $\mathcal{Q}(R) \setminus (0)$ by defining $\operatorname{ord}_{R,I}(\frac{a}{b}) := \operatorname{ord}_{R,I}(a) - \operatorname{ord}_{R,I}(b)$ for nonzero elements $a, b \in R$, and the set

$$V := \{q \in \mathcal{Q}(R) \setminus (0) \mid \operatorname{ord}_{R,I}(q) \geq 0\} \cup \{0\}$$

is a DVR. Therefore if property 2 holds for all nonzero a, b in R, then R is an integral domain and I is a prime ideal of R.

Moreover, in case $\operatorname{ord}_{R,I}$ satisfies property 2 for all nonzero a, b in R, then the function $\operatorname{ord}_{R,I}$ is the valuation on the DVR V described in Remark 2.1.2, and the value group is the ring of integers.

Let A be a commutative ring and let $R := A[[x]] = \{f = \sum_{i=0}^{\infty} f_i x^i \mid f_i \in R\}$, the *formal power series ring* over A in the variable x. With $I := xR$ and f a nonzero element in R, we write $\operatorname{ord} f$ for $\operatorname{ord}_{R,I}(f)$. Thus $\operatorname{ord} f$ is the least integer $i \geq 0$ such that $f_i \neq 0$. The element f_i is called *leading form* of f.

In the case where $(R, \mathbf{m})$ is a local ring, we abbreviate $\operatorname{ord}_{R,\mathbf{m}}$ by ord_R.

Regular local rings. A local ring $(R, \mathbf{m})$ is a *regular local ring*, often abbreviated *RLR*, if R is Noetherian and $\mathbf{m}$ can be generated by $\dim R$ elements. If $(R, \mathbf{m})$ is a regular local ring, then R is an integral domain; thus we may say R is a *regular local domain*. Then also the function ord_R satisfies the properties of Remark 2.2, and the associated valuation domain

$$V := \{q \in \mathcal{Q}(R) \setminus \{0\} \mid \operatorname{ord}_R(q) \geq 0\} \cup \{0\}$$

is a DVR that birationally dominates R. If $x \in \mathbf{m} \setminus \mathbf{m}^2$, then $V = R[\mathbf{m}/x]_{xR[\mathbf{m}/x]}$, where $\mathbf{m}/x = \{y/x \mid y \in \mathbf{m}\}$.

We record the following definitions.

Definitions 2.3. Let R be a commutative ring.

(1) R is called a *Nagata* ring if R is Noetherian and, for every $P \in \operatorname{Spec} R$ and every finite extension field L of $\mathcal{Q}(R/P)$, the integral closure of R/P in L is finitely generated as a module over R/P. In Nagata's book [**104**] a Nagata ring is called *pseudo-geometric*.

It is clear from the definition that a homomorphic image of a Nagata ring is again a Nagata ring. An important result proved by Nagata is that a polynomial ring in finitely many variables over a Nagata ring is again a Nagata ring [**104**, Theorem 36.5, page 132].

(2) An integral domain R is said to be a *Krull domain* if there exists a family $\mathcal{F} = \{V_\lambda\}_{\lambda \in \Lambda}$ of DVRs of its field of fractions $\mathcal{Q}(R)$ such that
- $R = \bigcap_{\lambda \in \Lambda} V_\lambda$, and
- Every nonzero element of $\mathcal{Q}(R)$ is a unit in all but finitely many of the V_λ.

We list the following examples and properties of Krull domains.

(a) A unique factorization domain (UFD) is a Krull domain, and a Noetherian integral domain is a Krull domain if and only if it is integrally closed. An integral domain R is a Krull domain if and only if it satisfies the following three properties:
- R_P is a DVR for each prime ideal P of R of height one.
- $R = \bigcap\{ R_P \mid P \text{ is a height-one prime } \}$.

- Every nonzero element of R is contained in only finitely many height-one primes of R.

(b) If R is a Krull domain, then $\mathcal{F} = \{R_P \mid P \text{ is a height-one prime }\}$ is the unique minimal set of DVRs satisfying the properties in the definition of a Krull domain [**96**, Theorem 12.3]. The family $\mathcal{F}$ is called the family of *essential valuation rings* of R. For each nonzero nonunit a of R the principal ideal aR has no embedded associated prime ideals and a unique irredundant primary decomposition $aR = q_1 \cap \cdots \cap q_t$. If $p_i = \operatorname{rad}(q_i)$, then $R_{p_i} \in \mathcal{F}$ and q_i is a symbolic power of p_i; that is, $q_i = p_i^{(e_i)}$, where $e_i \in \mathbb{N}$; see [**96**, Corollary, page 88].

(3) Let R be a Krull domain and let $R \hookrightarrow S$ be an inclusion map of R into a Krull domain S. The extension $R \hookrightarrow S$ satisfies the **PDE** condition ("pas d'éclatement", or in English "no blowing up") provided that for every height-one prime ideal Q in S, the height of $Q \cap R$ is at most one [**34**, page 30].

(4) A local ring $(R, \mathbf{m})$ is *Henselian* provided the following holds: for every monic polynomial $f(x) \in R[x]$ satisfying $f(x) \equiv g_0(x)h_0(x)$ modulo $\mathbf{m}[x]$, where g_0 and h_0 are monic polynomials in $R[x]$ such that

$$g_0R[x] + h_0R[x] + \mathbf{m}[x] = R[x],$$

there exist monic polynomials $g(x)$ and $h(x)$ in $R[x]$ such that $f(x) = g(x)h(x)$ and such that both

$$g(x) - g_0(x) \quad \text{and} \quad h(x) - h_0(x) \in \mathbf{m}[x].$$

In other words, if $f(x)$ factors modulo $\mathbf{m}[x]$ into two comaximal factors, then this factorization can be lifted back to $R[x]$. Alternatively, Henselian rings are rings for which the conclusion to Hensel's Lemma holds [**96**, Theorem 8.3].

We list results concerning Henselian rings from [**104**], where proofs are given for these results.[1]

(a) Associated with every local ring $(R, \mathbf{m})$, there exists an extension ring that is Henselian and local, called the *Henselization* of R and denoted $(R^h, \mathbf{m}^h)$; see [**104**, Statement 43.5 and the four paragraphs preceding (43.5), p. 180]. By [**104**, (43.3), p. 180, and Theorem 43.5, p. 181], R^h dominates R, R^h has the same residue field as R and $\mathbf{m}R^h = \mathbf{m}^h$. Moreover, by [**104**, page 182], the Henselization R^h of R is unique up to an R-isomomorphism.

(b) The Henselization R^h of a local ring R is faithfully flat over R [**104**, Theorem 43.8]; the concept of faithful flatness is defined in Definitions 2.20.

(c) If $(R, \mathbf{m})$ is a Noetherian local ring, then $(R^h, \mathbf{m}^h)$ is a Noetherian local ring such that with respect to the topologies on R and R^h defined by $\mathbf{m}$ and $\mathbf{m}^h$, respectively, R is a dense subspace of R^h. It follows that $R^h \subseteq \widehat{R}$, where $\widehat{R}$ is the $\mathbf{m}$-adic completion of R [**104**, Theorem 43.10]; the $\mathbf{m}$-adic topology and completion are defined in Definitions 3.1.

[1] The notation in [**104**], in particular the meaning of "local ring" and "finite type", differ from our usage in this book. We have adjusted these results to our terminology.

(d) If R is Henselian, then $R^h = R$ [**104**, (43.11)].
(e) If $(R, \mathbf{m})$ is a local integral domain, then R is Henselian $\iff$ for every integral domain S that is an integral extension of R, S is a local domain [**104**, Theorem 43.12].
(f) If R is a Henselian ring and R' is a local ring that is integral over R, then R' is Henselian [**104**, Corollary 43.16].
(g) If $(R', \mathbf{m}')$ is a local ring that is integral over a local ring $(R, \mathbf{m})$, then $R' \otimes_R R^h = (R')^h$ [**104**, Theorem 43.17].
(h) If $(R', \mathbf{m}')$ is a local ring that dominates the local ring $(R, \mathbf{m})$ and if R' is a localization of a finitely generated integral extension, then $(R')^h$ is a finitely generated module over R^h [**104**, Theorem 43.18].

Definitions 2.4. Let I be an ideal of a ring R.

(1) An element $r \in R$ is *integral over* I if there exists a monic polynomial $f(x) = x^n + \sum_{i=1}^n a_i x^{n-i}$ such that $f(r) = 0$ and such that $a_i \in I^i$ for each i with $1 \le i \le n$.
(2) The *integral closure* $\overline{I}$ of I is the set of elements of R integral over I; $\overline{I}$ is an ideal.
(3) The integral closure of $\overline{I}$ is equal to $\overline{I}$.
(4) If $I = \overline{I}$, then I is said to be *integrally closed*.
(5) The ideal I is said to be *normal* if I^n is integrally closed for every $n \ge 1$.
(6) If J is an ideal contained in I and $JI^{n-1} = I^n$ for some integer $n \ge 1$, then J is said to be a *reduction* of I.

2.2. Basic theorems

Theorem 2.5 is a famous result proved by Krull that is now called the Krull Intersection Theorem.

Theorem 2.5 (Krull [**96**, Theorem 8.10]). *Let I be an ideal of a Noetherian ring R.*

(1) *If I is contained in the Jacobson radical $\mathcal{J}(R)$ of R, then $\bigcap_{n=1}^\infty I^n = 0$, and, for each finite R-module M, we have $\bigcap_{n=1}^\infty I^n M = 0$.*
(2) *If I is a proper ideal of a Noetherian integral domain, then $\bigcap_{n=1}^\infty I^n = 0$.*

Theorem 2.6 is another famous result of Krull that is now called the Krull Altitude Theorem. It involves the concept of a minimal prime divisor of an ideal I of a ring R, where $P \in \operatorname{Spec} R$ is a *minimal prime divisor* of I if $I \subseteq P$ and if $P' \in \operatorname{Spec} R$ and $I \subseteq P' \subseteq P$, then $P' = P$. The *height* of a proper ideal I, denoted $\operatorname{ht} I$, is defined to be $\operatorname{ht} I = \min\{\operatorname{ht} P \mid P \in \operatorname{Spec} R \text{ and } I \subseteq P\}$.

Theorem 2.6 (Krull [**96**, Theorem 13.5]). *Let R be a Noetherian ring and let $I = (a_1, \dots, a_r)R$ be an ideal generated by r elements. If P is a minimal prime divisor of I, then $\operatorname{ht} P \le r$. Hence the height of a proper ideal of R is finite.*

Theorem 2.7 is yet another famous result that is now called the Krull-Akizuki Theorem.

Theorem 2.7 (Krull-Akizuki [**96**, Theorem 11.7]). *Let A be a one-dimensional Noetherian integral domain with field of fractions K, let L be a finite algebraic field extension of K, and let B be a subring of L with $A \subseteq B$. Then*

(1) *The ring B is Noetherian of dimension at most one.*

(2) *If J is a nonzero ideal of B, then B/J is an A-module of finite length.*

To prove that a ring is Noetherian, it suffices by the following well-known result of Cohen to prove that every prime ideal of the ring is finitely generated.

THEOREM 2.8 (Cohen [**24**]). *If each prime ideal of the ring R is finitely generated, then R is Noetherian.*

Theorem 2.9 is another important result proved by Cohen.

THEOREM 2.9 (Cohen [**25**]). *Let R be a Noetherian integral domain and let S be an extension domain of R. For $P \in \operatorname{Spec} S$ and $\mathbf{p} = P \cap R$, we have*

$$\operatorname{ht} P \; + \; \textit{tr.deg.}_{k(\mathbf{p})} k(P) \; \leq \; \operatorname{ht} \mathbf{p} \; + \; \textit{tr.deg.}_R S,$$

where $k(\mathbf{p})$ is the field of fractions of $R/\mathbf{p}$ and $k(P)$ is the field of fractions of S/P.

Theorem 2.10 is a useful result due to Nagata about Krull domains and UFDs.

THEOREM 2.10. [**126**, Theorem 6.3, p. 21] *Let R be a Krull domain. If S is a multiplicatively closed subset of R generated by prime elements and $S^{-1}R$ is a UFD, then R is a UFD.*

We use the following:

FACT 2.11. If D is an integral domain and c is a nonzero element of D such that cD is a prime ideal, then $D = D[1/c] \cap D_{cD}$.

PROOF. Let $\beta \in D[1/c] \cap D_{cD}$. Then $\beta = \frac{b}{c^n} = \frac{b_1}{s}$ for some $b, b_1 \in D$, $s \in D \setminus cD$ and integer $n \geq 0$. If $n > 0$, we have $sb = c^n b_1 \implies b \in cD$. Thus we may reduce to the case where $n = 0$; it follows that $D = D[1/c] \cap D_{cD}$. $\square$

REMARKS 2.12. (1) If R is a Noetherian integral domain and S is a multiplicatively closed subset of R generated by prime elements, then $S^{-1}R$ a UFD implies that R is a UFD [**126**, Theorem 6.3] or [**96**, Theorem 20.2].

(2) If x is a nonzero prime element in an integral domain R such that R_{xR} is a DVR and $R[1/x]$ is a Krull domain, then R is a Krull domain by Fact 2.11; and, by Theorem 2.10, R is a UFD if $R[1/x]$ is a UFD.

(3) Let R be a valuation domain with value group $\mathbb{Z} \oplus \mathbb{Z}$ ordered lexicographically; that is, for every pair $(a, b), (c, d)$ of elements of $\mathbb{Z} \oplus \mathbb{Z}$, $(a, b) > (c, d) \iff a > c$, or $a = c$ and $b > d$. Then the maximal ideal $\mathbf{m}$ of R is principal, say $\mathbf{m} = xR$. It follows that $R[1/x]$ is a DVR; however R is not a Krull domain.

The Eakin-Nagata Theorem is useful for proving descent of the Noetherian property.

THEOREM 2.13 (Eakin-Nagata [**96**, Theorem 3.7(i)]). *If B is a Noetherian ring and A is a subring of B such that B is a finitely generated A-module, then A is Noetherian.*

Krull domains have an approximation property with respect to the family of DVRs and valuations (as in Remarks 2.1.2) obtained by localizing at height-one primes.

THEOREM 2.14. *(Approximation Theorem* [**96**, Theorem 12.6]*) For A a Krull domain with field of fractions K, let $P_1, \ldots, P_r$ be height-one primes of A, and let*

v_i denote the valuation with value group $\mathbb{Z}$ associated to the DVR A_{P_i}, for each i with $1 \le i \le r$. For arbitrary integers $e_1, \dots, e_r$, there exists $x \in K$ such that

$$v_i(x) \;=\; e_i \quad \textit{for} \quad 1 \;\le\; i \;\le\; r \quad \textit{and} \quad v(x) \;\ge\; 0,$$

for every valuation v associated to a height-one prime ideal of A that is not in the set $\{P_1, \dots, P_r\}$.

An interesting result proved by Nishimura is

THEOREM 2.15 (Nishimura [**105**, Theorem, page 397], or [**96**, Theorem 12.7]). *Let R be a Krull domain. If R/P is Noetherian for every height-one prime ideal P of R, then R is Noetherian.*

REMARK 2.16. It is observed in [**53**, Lemma 1.5] that the conclusion of Theorem 2.15 still holds if it is assumed that R/P is Noetherian for all but at most finitely many of the height-one primes P of R.

Theorem 2.17 is useful for describing the maximal ideals of a power series ring $R[[x]]$. It is related to the fact that an element $f = a_0 + a_1x + a_2x^2 + \cdots \in R[[x]]$ with the $a_i \in R$ is a unit of $R[[x]]$ if and only if a_0 is a unit of R.

THEOREM 2.17 ([**104**, Theorem 15.1]). *Let $R[[x]]$ be the formal power series ring in a variable x over a commutative ring R. There is a one-to-one correspondence between the maximal ideals $\mathbf{m}$ of R and the maximal ideals $\mathbf{m}^*$ of $R[[x]]$ where $\mathbf{m}^*$ corresponds to $\mathbf{m}$ if and only if $\mathbf{m}^*$ is generated by $\mathbf{m}$ and x.*

As an immediate corollary of Theorem 2.17, we have

COROLLARY 2.18. *The element x is in the Jacobson radical $\mathcal{J}(R[[x]])$ of the power series ring $R[[x]]$. In the formal power series ring $S := R[[x_1, \dots, x_n]]$, the ideal $(x_1, \dots, x_n)S$ is contained in the Jacobson radical $\mathcal{J}(S)$ of S.*

Theorem 2.19 is an important result first proved by Chevalley.

THEOREM 2.19 (Chevalley [**22**]). *Let $(R, \mathbf{m})$ be a Noetherian local domain. There exists a DVR that birationally dominates R.*

More generally, let P be a prime ideal of a Noetherian integral domain R. There exists a DVR V that birationally contains R and has center P on R, that is, the maximal ideal of V intersects R in P.

2.3. Flatness

The concept of flatness was introduced by Serre in the 1950's in an appendix to his paper [**127**]. Mumford writes in [**98**, page 424]: "The concept of flatness is a riddle that comes out of algebra, but which technically is the answer to many prayers."

DEFINITIONS 2.20. A module M over a ring R is *flat* over R if tensoring with M preserves exactness of every exact sequence of R-modules. The R-module M is said to be *faithfully flat* over R if, for every sequence $\mathcal{S}$ of R-modules,

$$\mathcal{S}: \quad 0 \longrightarrow M_1 \longrightarrow M_2,$$

the sequence $\mathcal{S}$ is exact if and only if its tensor product with M, $\mathcal{S} \otimes_R M$, is exact.

A ring homomorphism $\phi : R \to S$ is said to be a *flat homomorphism* if S is flat as an R-module.

Flatness is preserved by several standard ring constructions as we record in Remarks 2.21. There is an interesting elementwise criterion for flatness that is stated as item 2 of Remarks 2.21.

REMARKS 2.21. The following facts are useful for understanding flatness. We use these facts to obtain the results in Chapters 8 and 17.

(1) Since localization at prime ideals commutes with tensor products, the module M is flat as an R-module $\iff$ M_Q is flat as an R_Q-module, for every prime ideal Q of R.
(2) An R-module M is flat over R if and only if for every $m_1, \dots, m_n \in M$ and $a_1, \dots, a_n \in R$ such that $\sum a_i m_i = 0$, there exist a positive integer k, a subset $\{b_{ij}\}_{i=1,}^{n}{}_{j=1}^{k} \subseteq R$, and elements $m'_1, \dots, m'_k \in M$ such that $m_i = \sum_{j=1}^{k} b_{ij} m'_j$ for each i and $\sum_{i=1}^{n} a_i b_{ij} = 0$ for each j; see [**96**, Theorem 7.6] or [**94**, Theorem 1]. Thus every free module is flat.
(3) A finitely generated module over a local ring is flat if and only if it is free [**94**, Proposition 3.G].
(4) If the ring S is a localization of R, then S is flat as an R-module [**94**, (3.D), page 19].
(5) Let S be a flat R-algebra. Then S is faithfully flat over R $\iff$ one has $JS \neq S$ for every proper ideal J of R; see [**94**, Theorem 3, page 28] or [**96**, Theorem 7.2].
(6) If the ring S is a flat R-algebra, then every regular element of R is regular on S [**94**, (3.F)].
(7) Let S be a faithfully flat R-algebra and let I be an ideal of R. Then $IS \cap R = I$ [**96**, Theorem 7.5].
(8) Let R be a subring of a ring S. If S is Noetherian and faithfully flat over R, then R is Noetherian; see Exercise 8 at the end of this chapter.
(9) Let R be an integral domain with field of fractions K and let S be a faithfully flat R-algebra. By item 6, every nonzero element of R is regular on S and so K naturally embeds in the total quotient ring $\mathcal{Q}(S)$ of S. By item 7, all ideals in R extend and contract to themselves with respect to S, and thus $R = K \cap S$. In particular, if $S \subseteq K$, then $R = S$ [**94**, page 31].
(10) If $\phi : R \to S$ is a flat homomorphism of rings, then ϕ satisfies the Going-down Theorem [**94**, (5.D), page 33]. This implies for each $P \in \operatorname{Spec} S$ that the height of P in $S$is greater than or equal to the height of $\phi^{-1}(P)$ in R.
(11) Let $R \to S$ be a flat homomorphism of rings and let I and J be ideals of R. Then $(I \cap J)S = IS \cap JS$. If J is finitely generated, then $(I :_R J)S = IS :_S JS$; see [**96**, Theorem 7.4] or [**94**, (3.H) page 23].
(12) Consider the following short exact sequence of R-modules:

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0\,.$$

If A and C are flat over R, then so is B.
(13) If S is a flat R-algebra and M is a flat S-module, then M is a flat R-module [**96**, page 46].
(14) If S is an R-algebra and M is faithfully flat over both R and S, then S is faithfully flat over R [**96**, page 46].

The following standard result about flatness follows from what Matsumura calls "change of coefficient ring". It is convenient to refer to both the module and homomorphism versions.

FACT 2.22. Let C be a commutative ring, let D, E and F be C-algebras.

(1) If $\psi : D \to E$ is a flat C-algebra homomorphism, then $\psi \otimes_C 1_F : D \otimes_C F \to E \otimes_C F$ is a flat C-algebra homomorphism.
(2) If E is a flat D-module via the C-algebra homomorphism ψ, then $E \otimes_C F$ is a flat $D \otimes_C F$-module via the C-algebra homomorphism $\psi \otimes_C 1_F$.

PROOF. By the definition of flat homomorphism in Definitions 2.20, the two statements are equivalent. Since E is a flat D-module, $E \otimes_D (D \otimes_C F)$ is a flat $(D \otimes_C F)$-module by [**96**, p. 46, Change of coefficient ring]. Since $E \otimes_D (D \otimes_C F) = E \otimes_C F$, Fact 2.22 follows. $\square$

We use Remark 2.23.3 in Chapter 7.

REMARKS 2.23. Let R be an integral domain.

(1) Every flat R-module M is torsionfree, i.e., if $r \in R, x \in M$ and $rx = 0$, then $r = 0$ or $x = 0$; see [**94**, (3.F), page 21]
(2) Every finitely generated torsionfree module over a PID is free; see for example [**30**, Theorem 5, page 462].
(3) Every torsionfree module over a PID is flat. This follows from item 2 and Remark 2.21.3.
(4) Every injective homomorphism of R into a field is flat. This follows from Remarks 2.21.13 and 2.21.4.

In Chapter 3 we discuss other tools we will be using involving ideal-adic completions and properties of excellent rings .

Exercises

(1) Prove that every height-one prime ideal of a UFD is principal.
(2) Let V be a local domain with nonzero principal maximal ideal yV. Prove that V is a DVR if $\bigcap_{n=1}^{\infty} y^n V = (0)$.

 Comment: It is not being assumed that V is Noetherian, so it needs to be established that V has dimension one.
(3) Prove as stated in Remark 2.1 that if R is a valuation domain with field of fractions K and F is a subfield of K, then $R \cap F$ is again a valuation domain and has field of fractions F; also prove that if R is a DVR and the field F is not contained in R, then $R \cap F$ is again a DVR.
(4) Prove that a unique factorization domain is a Krull domain.
(5) Let R be a Noetherian ring. Let $P_1 \subset P_2$ be prime ideals of R. If there exists a prime ideal Q of R with Q distinct from P_1 and P_2 such that $P_1 \subset Q \subset P_2$, prove that there exist infinitely many such prime ideals Q.

 Suggestion: Apply Krull's Altitude Theorem 2.6, and use the fact that an ideal contained in a finite union of primes is contained in one of them; see for example [**9**, Proposition 1.11, page 8].

(6) Prove as asserted in Remark 2.2 that, if $\operatorname{ord}_{R,I}(ab) = \operatorname{ord}_{R,I}(a) + \operatorname{ord}_{R,I}(b)$, for all nonzero a, b in R, and if we define $\operatorname{ord}_{R,I}(\frac{a}{b}) := \operatorname{ord}_{R,I}(a) - \operatorname{ord}_{R,I}(b)$ for nonzero elements $a, b \in R$, then:
   (a) The function $\operatorname{ord}_{R,I}$ extends uniquely to a function on $\mathcal{Q}(R) \setminus (0)$ with this definition.
   (b) $V := \{q \in \mathcal{Q}(R) \setminus (0) \mid \operatorname{ord}_{R,I}(q) \geq 0\} \cup \{0\}$ is a DVR, and
   (c) R is an integral domain and I is a prime ideal.

(7) Let $R[[x]]$ be the formal power series ring in a variable x over a commutative ring R.
   (i) Prove that $a_0 + a_1x + a_2x^2 + \cdots \in R[[x]]$, where the $a_i \in R$, is a unit of $R[[x]]$ if and only if a_0 is a unit of R.
   (ii) Prove that x is contained in every maximal ideal of $R[[x]]$.
   (iii) Prove Theorem 2.17 that the maximal ideals $\mathbf{m}$ of R are in one-to-one correspondence with the maximal ideals $\mathbf{m}^*$ of $R[[x]]$, where $\mathbf{m}^*$ corresponds to $\mathbf{m}$ if and only if $\mathbf{m}^*$ is generated by $\mathbf{m}$ and x.

(8) Prove items 4-8 of Remarks 2.21.

   Suggestion: For the proof of item 8, use item 7.

(9) Let $f : A \to B$ be a ring homomorphism and let P be a prime ideal of A. Prove that there exists a prime ideal Q in B that contracts in A to P if and only if the extended ideal $f(P)B$ contracts to P in A, i.e., $P = f(P)B \cap A$. (Here we are using the symbol $\cap$ as in Matsumura [**96**, item (3), page xiii].)

(10) Let $f : A \hookrightarrow B$ be an injective ring homomorphism and let P be a minimal prime of A.
   (i) Prove that there exists a prime ideal Q of B that contracts in A to P.
   (ii) Deduce that there exists a minimal prime Q of B that contracts in A to P.

   Suggestion: Consider the multiplicatively closed set $A \setminus P$ in B.

(11) Let P be a height-one prime of a Krull domain A and let v denote the valuation with value group $\mathbb{Z}$ associated to the DVR A_P. If A/P is Noetherian, prove that $A/P^{(e)}$ is Noetherian for every positive integer e.

   Suggestion: Using Theorem 2.14, show there exists $x \in \mathcal{Q}(A)$ such that $v(x) = 1$ and $1/x \in A_Q$ for every height-one prime Q of A different from P. Let $B = A[x]$.
   (i) Show that $P = xB \cap A$ and $B = A + xB$.
   (ii) Show that $A/P \cong B/xB \cong x^iB/x^{i+1}B$ for every positive integer i.
   (iii) Deduce that B/x^eB is a Noetherian B-module and thus a Noetherian ring.
   (iv) Prove that $x^eB \cap A \subseteq x^eA_P \cap A = P^{(e)}$ and B/x^eB is a finite $A/(x^eB \cap A)$-module generated by the images of $1, x, \ldots, x^{e-1}$.
   (v) Apply Theorem 2.13 to conclude that $A/(x^eB \cap A)$ and hence $A/P^{(e)}$ is Noetherian.

(12) Let A be a Krull domain having the property that A/P is Noetherian for all but at most finitely many of the $P \in \operatorname{Spec} A$ with $\operatorname{ht} P = 1$. Prove that A is Noetherian.

   Suggestion: By Nishimura's result Theorem 2.15, and Cohen's result Theorem 2.8, it suffices to prove each prime ideal of A of height greater than one is finitely generated. Let $P_1, \ldots, P_n$ be the height-one prime ideals of A for which

A/P_i may fail to be Noetherian. For each nonunit $a \in A \setminus (P_1 \cup \cdots \cup P_n)$, observe that $aA = Q_1^{(e_1)} \cap \cdots \cap Q_s^{(e_s)}$, where $Q_1, \ldots, Q_s$ are height-one prime ideals of A not in the set $\{P_1, \ldots, P_n\}$. Consider the embedding $A/aA \hookrightarrow \prod(A/Q_i^{(e_i)})$. By Exercise 11, each $A/Q_i^{(e_i)}$ is Noetherian. Apply Theorem 2.13 to conclude that A/aA is Noetherian. Deduce that every prime ideal of A of height greater than one is finitely generated.

(13) Let R be a two-dimensional Noetherian integral domain. Prove that every Krull domain that birationally dominates R is Noetherian.

Comment: It is known that the integral closure of a two-dimensional Noetherian integral domain is Noetherian [**104**, (33.12)]. A proof of Exercise 13 is given in [**47**, Theorem 9]. An easier proof may be obtained using Nishimura's result Theorem 2.15.

CHAPTER 3

More tools

In this chapter we discuss ideal-adic completions. We describe several results concerning complete local rings. We review the definitions of catenary and excellent rings and record several results about these rings.

3.1. Introduction to ideal-adic completions

Definitions 3.1. Let R be a commutative ring with identity. A *filtration* on R is a descending sequence $\{I_n\}_{n=0}^{\infty}$ of ideals of R. Associated to a filtration there is a well-defined completion R^* that may be defined to be the inverse limit [1]

$$R^* = \varprojlim_n \; R/I_n \tag{3.1.1}$$

and a canonical homomorphism $\psi : R \to R^*$ [**108**, Chapter 9]. The map ψ induces a map $R \to R^*/I^n R^*$, and it follows that

$$R^*/I_nR^* \;\cong\; R/I_n; \tag{3.1.2}$$

see [**108**, page 412] or [**96**, page 55] for more details.

We assume that $\bigcap_{n=0}^{\infty} I_n = (0)$. Then the family $\{I_n\}$ determines a metric on R: For $x \neq y \in R$, the distance from x to y is $d(x,y) = 2^{-n}$, where n is the largest n such that $x-y \in I_n$. This metric gives rise to a Hausdorff topology [**96**, page 55]. In particular, the map ψ is injective, and R may be regarded as a subring of R^*.

In the terminology of Northcott, a filtration $\{I_n\}_{n=0}^{\infty}$ is said to be *multiplicative* if $I_0 = R$ and $I_nI_m \subseteq I_{n+m}$, for all $m \geq 0$, $n \geq 0$ [**108**, page 408]. A well-known example of a multiplicative filtration on R is the I-adic filtration $\{I^n\}_{n=0}^{\infty}$, where I is a fixed ideal of R such that $\bigcap_{n=0}^{\infty} I^n = (0)$. In this case we say $R^* := \varprojlim_n \; R/I^n$ is the *I-adic completion* of R. If the canonical map $R \to R^*$ is an isomorphism, we say that R is *I-adically complete.* An ideal L of R is *closed in the* I-adic topology on R if $\bigcap_{n=1}^{\infty}(L + I^n) = L$.

We reserve the notation $\widehat{R}$ for the situation where R is a local ring with maximal ideal $\mathbf{m}$ such that $\bigcap_{n=0}^{\infty} \mathbf{m}^n = (0)$ and $\widehat{R}$ is the $\mathbf{m}$-adic completion of R. For a local ring $(R, \mathbf{m})$, we say that $\widehat{R}$ is "the" *completion* of R. If $\mathbf{m}$ is generated by elements $a_1, \ldots, a_n$, then $\widehat{R}$ is realizable by taking the a_1-adic completion R_1^* of R, then the a_2-adic completion R_2^* of R_1^*, ..., and then the a_n-adic completion of R_{n-1}^*.

We record the following results about ideal-adic completions.

Remarks 3.2. Let I be an ideal of a ring R.

[1] We refer to Appendix A of [**96**] for the definition of direct and inverse limits. Also see the discussion of inverse limits in [**9**, page 103].

(1) If R is I-adically complete, then I is contained in the Jacobson radical $\mathfrak{J}(R)$; see [**96**, Theorem 8.2] or [**94**, 24.B, pages 73-74].
(2) If R is a Noetherian ring, then the I-adic completion R^* of R is flat over R [**94**, Corollary 1, page 170].
(3) If R is Noetherian, then the I-adic completion R^* of R is faithfully flat over $R \iff$ for each proper ideal J of R we have $JR^* \neq R^*$.
(4) If R is a Noetherian ring and $I \subseteq \mathfrak{J}(R)$, then the I-adic completion R^* is faithfully flat over R [**94**, Theorem 56, page 172]. Moreover, we have $\dim R = \dim R^*$ [**94**, pages 173-175].
(5) If $I = (a_1, \dots, a_n)R$ is an ideal of a Noetherian ring R, then the I-adic completion R^* of R is isomorphic to a quotient of the formal power series ring $R[[x_1, \dots, x_n]]$; namely,

$$R^* \;=\; \frac{R[[x_1,\dots,x_n]]}{(x_1 - a_1, \dots, x_n - a_n)R[[x_1,\dots,x_n]]}$$

[**96**, Theorem 8.12].

REMARK 3.3. Suppose that $z \in R$ and $\bigcap_{n=1}^{\infty} z^n R = (0)$. Then the *$z$-adic completion* R^* of R is the inverse limit $R^* := \varprojlim_n R/z^n R$.

(1) Let y be an indeterminate over R. If the ideal $(y-z)R[[y]]$ is closed in the J-adic topology on $R[[y]]$, where $J := (y,z)R[[y]]$, then the J-adic completion R^* also has the form

$$R^* \;=\; \frac{R[[y]]}{(y-z)R[[y]]}. \tag{3.3.1}$$

This follows from [**104**, (17.5)].
(2) If R is Noetherian, then the ideal $(y-z)R[[y]]$ is closed in the J-adic topology on $R[[y]]$ and the representation of R^* as in Equation 3.3.1 holds by Remark 3.2.5. A direct proof of this statement may also be given as follows: let $^-$ denote image in $R[[y]]/(y-z)R[[y]]$. It suffices to show that $\bigcap_{n=1}^{\infty} \overline{(y,z)^n R[[y]]} = (\overline{0})$.

We have $\overline{(y,z)^n R[[y]]} = \overline{y^n R[[y]]}$, for every $n \in \mathbb{N}$. By Corollary 2.18, the element y is in the Jacobson radical of $R[[y]]$. Hence $\overline{y}$ is in the Jacobson radical of $\overline{R[[y]]}$, a Noetherian ring. We have

$$\bigcap_{n=1}^{\infty} \overline{(y,z)^n R[[y]]} = \bigcap_{n=1}^{\infty} \overline{y^n R[[y]]} = (\overline{0}).$$

The second equality follows from Theorem 2.5.1. Therefore $(y-z)R[[y]]$ is closed in the J-adic topology. Thus if R is Noetherian then R^* has the form of Equation 3.3.1.
(3) In general, if R^* has the form of Equation 3.3.1, then the elements of R^* are power series in z with coefficients in R, but without the uniqueness of expression as power series that occurs in the formal power series ring $R[[y]]$. If R is already complete in its (z)-adic topology, then $R = R^*$, but often it is the case for a Noetherian integral domain R that there exist elements of R^* that are transcendental over the field of fractions of R; see Section 3.2.

We use the following definitions.

DEFINITION 3.4. A Noetherian local ring R is said to be

(1) *analytically unramified* if the completion $\widehat{R}$ is reduced, i.e., has no nonzero nilpotent elements;
(2) *analytically irreducible* if the completion $\widehat{R}$ is an integral domain;
(3) *analytically normal* if the completion $\widehat{R}$ is an integrally closed (i.e., normal) domain.

If R has any one of the properties of Definition 3.4, then R is reduced. If R has either of the last two properties, then R is an integral domain. If $\widehat{R}$ is not reduced, then R is *analytically ramified*; if $\widehat{R}$ is not an integral domain, then R is *analytically reducible*.

3.2. Uncountable transcendence degree for a completion

In this section, we make a small excursion to consider some cases where the transcendence degree of completions and power series rings are uncountable over the base ring. We call these results "facts", because they appear to be well known. We include brief proofs here to make the results more accessible.

We begin with a useful fact about uncountable Noetherian commutative rings.

FACT 3.5. If R is an uncountable Noetherian commutative ring, then there exists a prime ideal P of R such that R/P is uncountable. Hence there exists a minimal prime P_0 of R such that R/P_0 is uncountable.

PROOF. The ring R contains a finite chain of submodules

$$0 = M_0 \subset M_1 \subset \cdots \subset M_\ell = R$$

such that each quotient $M_i/M_{i-1} = R/P_i$, for some prime ideal P_i of R, [**96**, Theorem 6.4]. If each of the quotients were countable then R would be countable. Thus R/P is uncountable for some prime ideal P of R, and hence R/P_0 is uncountable, for each minimal prime P_0 contained in P. □

FACT 3.6. If R is a countable Noetherian integral domain and z is a nonzero nonunit of R, then the (z)-adic completion R^* of R contains an uncountable subset that is algebraically independent over R. That is, R^* has uncountable transcendence degree over R.

PROOF. The (z)-adic completion R^* of R is uncountable. By Fact 3.5 there exists a minimal prime P_0 of R^* such that R^*/P_0 is uncountable. Since R is a Noetherian integral domain, R^* is flat over R by Remark 3.2.2. Thus, by Remark 2.21.9, $P_0 \cap R = 0$. In R^*/P_0 there exists an uncountable subset of algebraically independent elements over R. By taking preimages in R^*, we get an uncountable subset of the (z)-adic completion R^* of R. This set is algebraically independent over R since the algebraic closure of the field of fractions of the countable ring R is countable. □

In relation to transcendence degree and filtrations, Joe Lipman brought Remark 3.7 and Fact 3.8 to our attention; he also indicated the proofs sketched below.

REMARK 3.7. Let k be a field and let R be a ring containing k. Let $(I_a)_{a\in A}$ be a family of ideals of R with index set A such that the family is closed under finite intersection and the intersection of all of the I_a is (0). If $m \in \mathbb{N}$ and $v_1, \dots, v_m \in R$

are linearly independent vectors over k, then for some a their images in R/I_a are linearly independent. Otherwise, if V is the vector space generated by the v_i, then $(V \cap I_a)_{a \in A}$ would be an infinite family of nonzero vector subspaces of the finite-dimensional vector space V that is closed under finite intersection and such that the intersection of all of them is (0), a contradiction.

Fact 3.8. Let y be an indeterminate over a field k. Then the power series ring $k[[y]]$ has uncountable transcendence degree over k.

Proof. We show the k-vector space dimension of $k[[y]]$ is uncountable. For this, let k_0 be the prime subfield of k. We consider the family $\{I_n := y^n k_0[[y]]\}_{n \in \mathbb{N}}$ of ideals of $k_0[[y]]$ and the corresponding family $\{I'_n := y^n k[[y]]\}_{n \in \mathbb{N}}$ of ideals of $k[[y]]$. For every $n \in \mathbb{N}$, the k-homomorphism $\varphi : k \otimes_{k_0} k_0[[y]] \to k[[y]]$ induces a map $\overline{\varphi} : k \otimes_{k_0} (k_0[[y]]/y^n k_0[[y]]) \to k[[y]]/y^n k[[y]]$ that is an isomorphism of two n-dimensional vector spaces over k.

Since $k_0[[y]]$ is uncountable and k_0 is countable, the k_0-vector space dimension of $k_0[[y]]$ is uncountable, and so there is an uncountable subset $\mathcal{B}$ of $k_0[[y]]$ that is linearly independent over k_0. Let $v_1, \dots v_m$ be a finite subset of $\mathcal{B}$. Then by Remark 3.7 the images of $v_1, \dots, v_m$ in $k_0[[y]]/(y^n k_0[[y]])$ are linearly independent over k_0, for some n. Since $\overline{\varphi}$ is a k-isomorphism, the images of $v_1, \dots, v_m$ in $k[[y]]/(y^n k[[y]])$ are linearly independent over k. Thus $v_1, \dots, v_m$ must be linearly independent over k. Therefore $\mathcal{B}$ is linearly independent over k. □

3.3. Basic results about completions

In Proposition 3.9 we give conditions for an ideal to be closed with respect to an I-adic topology.

Proposition 3.9. *Let I be an ideal in a ring R and let R^* denote the I-adic completion of R.*

(1) *Let L be an ideal of R such that LR^* is closed in the I-adic topology on R^*. Then L is closed in the I-adic topology on R if and only if $LR^* \cap R = L$.* [2]
(2) *If R is Noetherian and I is contained in the Jacobson radical of R, then every ideal L of R is closed in the I-adic topology on R.*
(3) *If R^* is Noetherian, then every ideal $\mathfrak{A}$ of R^* is closed in the I-adic topology on R^*.*

Proof. For item 1, we have $LR^* = \bigcap_{n=1}^{\infty}(L + I^n)R^*$, since the ideal LR^* is closed in R^*. By Equation 3.1.2, $R/I^n \cong R^*/I^n R^*$, for each $n \in \mathbb{N}$. It follows that

$$(3.9.0) \qquad R/(L + I^n) \cong R^*/(L + I^n)R^*, \quad \text{and} \quad L + I^n = (L + I^n)R^* \cap R,$$

for each $n \in \mathbb{N}$. By Equation 3.9.0, L is closed in R if and only if $LR^* \cap R = L$. This proves item 1. Item 2 now follows from statements 3 and 4 of Remark 3.2.

Item 3 follows from item 2, since IR^* is contained in the Jacobson radical of R^* by Remark 3.2.1. □

In Theorem 8 of Cohen's famous paper [**23**] on the structure and ideal theory of complete local rings a result similar to Nakayama's lemma is obtained without the

[2] Here, as in [**96**, page xiii], we interpret $LR^* \cap R$ to be the preimage $\psi^{-1}(LR^*)$, where $\psi : R \to R^*$ is the canonical map of R to its I-adic completion R^*.

usual finiteness condition of Nakayama's lemma [**96**, Theorem 2.2]. As formulated in [**96**, Theorem 8.4], the result is:

THEOREM 3.10. (A version of Cohen's Theorem 8) *Let I be an ideal of a ring R and let M be an R-module. Assume that R is complete in the I-adic topology and $\bigcap_{n=1}^{\infty} I^n M = (0)$. If M/I is generated over R/I by elements $\overline{w}_1, \ldots, \overline{w}_s$ and w_i is a preimage in M of $\overline{w}_i$ for $1 \leq i \leq s$, then M is generated over R by $w_1, \ldots, w_s$.*

Let K be a field and let $R = K[[x_1, \ldots, x_n]]$ be a formal power series ring in n variables over K. It is well-known that there exists a K-algebra embedding of R into the formal power series ring $K[[y, z]]$ in two variables over K [**149**, page 219]. We observe in Corollary 3.11 restrictions on such an embedding.

COROLLARY 3.11. *Let $(R, \mathbf{m})$ be a complete local ring and assume that the map $\varphi : (R, \mathbf{m}) \to (S, \mathbf{n})$ is a local homomorphism.*

(1) *If $\mathbf{m}S$ is $\mathbf{n}$-primary and $S/\mathbf{n}$ is finite over $R/\mathbf{m}$, then S is a finitely generated R-module.*
(2) *If $\mathbf{m}S = \mathbf{n}$ and $R/\mathbf{m} = S/\mathbf{n}$, then φ is surjective.*
(3) *If $R = K[[x_1, \ldots, x_n]]$ is a formal power series ring in $n > 2$ variables over the field K and $S = K[[y, z]]$ is a formal power series ring in two variables over K, then $\varphi(\mathbf{m})S$ is not $\mathbf{n}$-primary.*

We record in Remarks 3.13 several consequences of Cohen's structure theorems for complete local rings. We use the following definitions.

DEFINITIONS 3.12. Let $(R, \mathbf{m})$ be a local ring.

(1) $(R, \mathbf{m})$ is said to be *equicharacteristic* if R has the same characteristic as its residue field $R/\mathbf{m}$.
(2) A subfield k of R is a *coefficient field* of R if the canonical map of $R \to R/\mathbf{m}$ restricts to an isomorphism of k onto $R/\mathbf{m}$.

REMARKS 3.13.

(1) Every equicharacteristic complete Noetherian local ring has a coefficient field; see [**23**], [**96**, Theorem 28.3], [**104**, (31.1)].
(2) If k is a coefficient field of a complete Noetherian local ring $(R, \mathbf{m})$ and $x_1, \ldots, x_n$ are generators of $\mathbf{m}$, then every element of R can be expanded as a power series in $x_1, \ldots, x_n$ with coefficients in k; see [**104**, (31.1)]. Thus R is a homomorphic image of a formal power series ring in n variables over k.
(3) (i) Every complete Noetherian local ring, whether equicharacteristic or not, is Henselian (defined in Definition 2.3.4) by [**104**, (30.3)].
(ii) Every complete Noetherian local ring is a homomorphic image of a complete regular local ring.
(iii) Every complete regular local ring is a power series ring over either a field or a complete discrete valuation ring [**23**], [**104**, (31.1)].
(4) If $(R, \mathbf{m})$ is a complete Noetherian local domain, then R is a finite integral extension of a complete regular local domain [**104**, (31.6)] and the integral closure of R in a finite algebraic field extension is a finite R-module [**104**, (32.1)].
(5) If a Noetherian local ring R is analytically unramified, then the integral closure of R is a finite R-module [**104**, (32.2)].

(6) Let $(R, \mathbf{m})$ be a one-dimensional Noetherian local domain. The following two statements then hold [**104**, Ex. 1 on page 122] and [**79**].
   (i) The integral closure $\overline{R}$ of R is a finite R-module if and only if the completion $\widehat{R}$ of R is reduced, i.e., if and only if R is analytically unramified.
   (ii) The minimal primes of $\widehat{R}$ are in one-to-one correspondence with the maximal ideals of $\overline{R}$.

A classical result of Rees describes necessary and sufficient conditions in order that a Noetherian local ring $(R, \mathbf{m})$ be analytically unramified.

THEOREM 3.14. (Rees) [**117**] *Let $(R, \mathbf{m})$ be a reduced Noetherian local ring with total ring of fractions $\mathcal{Q}(R)$. Then the following are equivalent.*

(1) *The ring R is analytically unramified.*
(2) *For every choice of finitely many elements $\lambda_1, \ldots, \lambda_n$ in $\mathcal{Q}(R)$, the integral closure of $R[\lambda_1, \ldots, \lambda_n]$ is a finite $R[\lambda_1, \ldots, \lambda_n]$-module.*

The following is an immediate corollary of Theorem 3.14.

COROLLARY 3.15. (Rees) [**117**] *Let $(R, \mathbf{m})$ be an analytically unramified Noetherian local ring and let $\lambda_1, \ldots, \lambda_n$ be elements of $\mathcal{Q}(R)$. For every prime ideal P of $A = R[\lambda_1, \ldots, \lambda_n]$, the local ring A_P is also analytically unramified.*

3.4. Chains of prime ideals, fibers of maps and excellence

We begin by discussing chains of prime ideals.

DEFINITIONS 3.16. Let P and Q be prime ideals of a ring A.

(1) If $P \subsetneq Q$, we say that the inclusion $P \subsetneq Q$ is *saturated* if there is no prime ideal of A strictly between P and Q.
(2) A possibly infinite chain of prime ideals $\cdots \subsetneq P_i \subsetneq P_{i+1} \subsetneq \cdots$ is called *saturated* if every inclusion $P_i \subsetneq P_{i+1}$ is saturated.
(3) A ring A is *catenary* provided for every pair of prime ideals $P \subsetneq Q$ of A, every chain of prime ideals from P to Q can be extended to a saturated chain and every two saturated chains from P to Q have the same number of inclusions.
(4) A ring A is *universally catenary* provided every finitely generated A-algebra is catenary.
(5) A ring A is said to be *equidimensional* if $\dim A = \dim A/P$ for every minimal prime P of A.

Theorem 3.17 is a well-known result of Ratliff [**96**, Theorem 31.7].

THEOREM 3.17. *A Noetherian local domain A is universally catenary if and only if its completion $\widehat{A}$ is equidimensional.*

A sharper result also due to Ratliff relating the universally catenary property to properties of the completion is Theorem 3.18.

THEOREM 3.18. [**114**, Theorem 2.6] *A Noetherian local ring $(R, \mathbf{m})$ is universally catenary if and only if the completion of $R/\mathbf{p}$ is equidimensional for every minimal prime ideal $\mathbf{p}$ of R.*

REMARK 3.19. Every Noetherian local ring that is a homomorphic image of a regular local ring, or even a homomorphic image of a Cohen-Macaulay local ring, is universally catenary [**96**, Theorem 17.9, page 137].

We record in Proposition 3.20 an implication of the Krull Altitude Theorem 2.6.

PROPOSITION 3.20. *Let* $(R, \mathbf{m})$ *be a catenary Noetherian local domain and let* $P \in \operatorname{Spec} R$ *with* $\dim R/P = n \geq 1$. *Let* d *be an integer with* $1 \leq d \leq n$, *and let*

$$\mathcal{A} \ := \ \{Q \in \operatorname{Spec} R \mid P \subseteq Q \text{ and } \dim R/Q \ = \ d \ \}.$$

Then $P = \bigcap_{Q \in \mathcal{A}} Q$.

PROOF. If $d = n$, then $P \in \mathcal{A}$ and the statement is true. To prove the assertion for d with $1 \leq d < n$, it suffices to prove it in the case where $\dim R/P = d + 1$; for if the statement holds in the case where $n = d + 1$, then by an iterative procedure on intersections of prime ideals, the statement also holds for $n = d + 2, \cdots$.

Thus we assume $n = d + 1$. Since $\operatorname{ht}(\mathbf{m}/P) \geq 2$, Krull's Altitude Theorem 2.6 implies that there exist infinitely many prime ideals properly between P and $\mathbf{m}$; see Exercise 5 in Chapter 2. Theorem 2.6 also implies that for each element $a \in \mathbf{m} \setminus P$ and each minimal prime Q of $P + aR$, we have $\operatorname{ht}(Q/P) = 1$. Since R is catenary, it follows that $\dim(R/Q) = \dim(R/P) - 1 = d$. Therefore the set $\mathcal{A}$ is infinite. Since an ideal in a Noetherian ring has only finitely many minimal primes, we have $P = \bigcap_{Q \in \mathcal{A}} Q$. $\square$

DISCUSSION 3.21. Let $f : A \to B$ be a ring homomorphism. The map f can always be factored as the composite of the surjective map $A \to f(A)$ followed by the inclusion map $f(A) \hookrightarrow B$. This is often helpful for understanding the relationship of A and B. If J is an ideal of B, then $f^{-1}(J)$ is an ideal of A called the *contraction* of J to A with respect to f. If Q is a prime ideal of B, then $P := f^{-1}(Q)$ is a prime ideal of A. Thus associated with the ring homomorphism $f : A \to B$, there is a well-defined *spectral map* $f^* : \operatorname{Spec} B \to \operatorname{Spec} A$ of topological spaces, where for $Q \in \operatorname{Spec} B$ we define $f^*(Q) = f^{-1}(Q) = P \in \operatorname{Spec} A$.

Let A be a ring and let $P \in \operatorname{Spec}(A)$. The *residue field of* A at P, denoted $k(P)$, is the field of fractions $\mathcal{Q}(A/P)$ of A/P. By permutability of localization and residue class formation we have $k(P) = A_P/PA_P$.

Given a ring homomorphism $f : A \to B$ and an ideal I of A, the ideal $f(I)B$ is called the *extension* of I to B with respect to f. For $P \in \operatorname{Spec} A$, the extension ideal $f(P)B$ is, in general, not a prime ideal of B. The *fiber* over P in $\operatorname{Spec} B$ is the set of all $Q \in \operatorname{Spec} B$ such that $f^*(Q) = P$. Exercise 7 of Chapter 2 asserts that the fiber over P is nonempty if and only if P is the contraction of the extended ideal $f(P)B$. In general, the fiber over P in $\operatorname{Spec} B$ is the spectrum of the ring

$$C \ := \ B \otimes_A k(P) = S^{-1}(B/f(P)B) = (S^{-1}B)/(S^{-1}f(P)B), \tag{3.21.0}$$

where S is the multiplicatively closed set $A \setminus P$; see [**96**, last paragraph, p. 47]. Notice that a prime ideal Q of B contracts to P in A if and only if $f(P) \subseteq Q$ and $Q \cap S = \emptyset$. This describes exactly the prime ideals of C as in Equation 3.21.0.

For $Q^* \in \operatorname{Spec} C$, and $Q = Q^* \cap B$, we have $P = Q \cap A$ and

$$Q^* \ = \ QC, \quad \text{and} \quad C_{Q^*} \ = \ B_Q/PB_Q \ = \ B_Q \otimes_A k(P); \tag{3.21.1}$$

see [**96**, top, p. 48].

Definitions 3.22. Let $f : A \to B$ be a ring homomorphism of Noetherian rings, let $P \in \operatorname{Spec} A$, and let $k(P)$ be as in Discussion 3.21.

(1) The fiber over P with respect to the map f is said to be *regular* if the ring $B \otimes_A k(P)$ is a Noetherian regular ring, i.e., $B \otimes_A k(P)$ is a Noetherian ring with the property that its localization at every prime ideal is a regular local ring.
(2) The fiber over P with respect to the map f is said to be *normal* if the ring $B \otimes_A k(P)$ is a normal Noetherian ring, i.e., $B \otimes_A k(P)$ is a Noetherian ring with the property that its localization at every prime ideal is a normal Noetherian local domain.

Definitions 3.23. Let $f : A \to B$ be a ring homomorphism of Noetherian rings, and let $P \in \operatorname{Spec} A$.

(1) The fiber over P with respect to the map f is said to be *geometrically regular* if for every finite extension field F of $k(P)$ the ring $B \otimes_A F$ is a Noetherian regular ring. The map $f : A \to B$ is said to have *geometrically regular* fibers if for each $P \in \operatorname{Spec} A$ the fiber over P is geometrically regular.
(2) The fiber over P with respect to the map f is said to be *geometrically normal* if for every finite extension field F of $k(P)$ the ring $B \otimes_A F$ is a Noetherian normal ring. The map $f : A \to B$ is said to have *geometrically normal* fibers if for each $P \in \operatorname{Spec} A$ the fiber over P is geometrically normal.

Remark 3.24. Let $f : A \to B$ be a ring homomorphism with A and B Noetherian rings and let $P \in \operatorname{Spec} A$. To check that the fiber of f over P is geometrically regular as in Definition 3.23, it suffices to show that $B \otimes_A F$ is a Noetherian regular ring for every finite purely inseparable field extension F of $k(P)$, [**44**, Théorème (22.5.8)]. Thus, if the characteristic of the field $k(P) = A_P/PA_P$ is zero, then, for every ring homomorphism $f : A \to B$ with B Noetherian, the fiber over P is geometrically regular if and only if it is regular.

Definitions 3.25. Let $f : A \to B$ be a ring homomorphism, where A and B are Noetherian rings.

(1) The homomorphism f is said to be *regular* if it is flat with geometrically regular fibers. See Definition 2.20 for the definition of flat.
(2) The homomorphism f is said to be *normal* if it is flat with geometrically normal fibers.

Remark 3.26. Since a regular local ring is a normal Noetherian local domain, every regular homomorphism of Noetherian rings is a normal homomorphism.

Example 3.27. Let x be an indeterminate over a field k of characteristic zero, and let

$$A \ := \ k[x(x-1), x^2(x-1)]_{(x(x-1),x^2(x-1))} \ \subset \ k[x]_{(x)} \ =: \ B.$$

Then $(A, \mathbf{m}_A)$ and $(B, \mathbf{m}_B)$ are one-dimensional local domains with the same field of fractions $k(x)$ and with $\mathbf{m}_A B = \mathbf{m}_B$. Hence the inclusion map $f : A \hookrightarrow B$ has geometrically regular fibers. Since $A \neq B$, the map f is not flat by Remark 2.21.8. Hence f is not a regular morphism.

We present in Chapter 11 examples of maps of Noetherian rings that are regular, and other examples of maps that are flat but fail to be regular.

The formal fibers of a Noetherian local ring as in Definition 3.28 play an important role in the concept of excellence of a Noetherian ring.

DEFINITION 3.28. Let $(R, \mathbf{m})$ be a Noetherian local ring and let $\widehat{R}$ be the $\mathbf{m}$-adic completion of R. The *formal fibers* of R are the fibers of the canonical inclusion map $R \hookrightarrow \widehat{R}$.

DEFINITION 3.29. A Noetherian ring A is called a *G-ring* if for each prime ideal P of A the map of A_P to its PA_P-adic completion is regular, or, equivalently, the formal fibers of A_P are geometrically regular for each prime ideal P of A.

REMARK 3.30. In Definition 3.29 it suffices that for every maximal ideal $\mathbf{m}$ of A, the map from $A_{\mathbf{m}}$ to its $\mathbf{m}A_{\mathbf{m}}$-adic completion is regular, by [**96**, Theorem 32.4]

DEFINITION 3.31. A Noetherian ring A is *excellent* if

(i) A is universally catenary,
(ii) A is a G-ring, and
(iii) for every finitely generated A-algebra B, the set $\operatorname{Reg}(B)$ of primes P of B for which B_P is a regular local ring is an open subset of $\operatorname{Spec} B$.

REMARKS 3.32. The class of excellent rings includes the ring of integers as well as all fields and all complete Noetherian local rings [**96**, page 260]. All Dedekind domains of characteristic zero are excellent [**94**, (34.B)].

The usefulness of the concept of excellent rings is enhanced by the fact that the class of excellent rings is stable under the ring-theoretic operations of localization and passage to a finitely generated algebra [**44**, Chap. IV], [**94**, (33.G) and (34.A)]. Therefore excellence is preserved under homomorphic images. An excellent ring is a Nagata ring [**94**, Theorem 78, page 257].

We give further motivation and explanations for the definition of excellence in Chapter 14.

REMARK 3.33. In Corollary 9.14 of Chapter 9, we prove that the 2-dimensional Noetherian local ring B constructed in Example 9.11 has the property that the map $f : B \to \widehat{B}$ has geometrically regular fibers. This ring B of Example 9.11 is also an example of a catenary ring that is not universally catenary. Thus the property of having geometrically regular formal fibers does not imply that a Noetherian local ring is excellent.

Exercises

(1) ([**31**]) Let R be a commutative ring and let P be a prime ideal of the power series ring $R[[x]]$. Let $P(0)$ denote the ideal in R of constant terms of elements of P.
   (i) If $x \notin P$ and $P(0)$ is generated by n elements of R, prove that P is generated by n elements of $R[[x]]$.
   (ii) If $x \in P$ and $P(0)$ is generated by n elements of R, prove that P is generated by $n+1$ elements of $R[[x]]$.
   (iii) If R is a PID, prove that every prime ideal of $R[[x]]$ of height one is principal.

(2) Let R be a DVR with maximal ideal yR and let $S = R[[x]]$ be the formal power series ring over R in the variable x. Let $f \in S$. Recall that f is a unit in S if and only if the constant term of f is a unit in R by Exercise 4 of Chapter 2.
 (a) Show that S is a 2-dimensional RLR with maximal ideal $(x, y)S$.
 (b) If g is a factor of f and S/fS is a finite R-module, then S/gS is a finite R-module.
 (c) If n is a positive integer and $f := x^n + y$, then S/fS is a DVR. Moreover, S/fS is a finite R-module if and only if $R = \widehat{R}$, i.e., R is complete.
 (d) If f is irreducible and $fS \neq xS$, then S/fS is a finite R-module implies that R is complete.
 (e) If R is complete, then S/fS is a finite R-module for each nonzero f in S.

 Suggestion: For item (d) use that if R is not complete, then by Nakayama's lemma, the completion of R is not a finite R-module. For item (e) use Theorem 3.10.

 Let f be a monic polynomial in x with coefficients in R.

 What are necessary and sufficient conditions in order that the residue class ring S/fS is a finite R-module?

(3) (Related to Dumitrescu's article [**29**]) Let R be an integral domain and let $f \in R[[x]]$ be a nonzero nonunit of the formal power series ring $R[[x]]$. Prove that the principal ideal $fR[[x]]$ is closed in the (x)-adic topology, that is, $fR[[x]] = \bigcap_{m \geq 0}(f, x^m)R[[x]]$.

 Suggestion: Reduce to the case where $c = f(0)$ is nonzero. Then f is a unit in the formal power series ring $R[\frac{1}{c}][[x]]$. If $g \in \bigcap_{m \geq 0}(f, x^m)R[[x]]$, then $g = fh$ for some $h \in R[\frac{1}{c}][[x]]$, say $h = \sum_{n \geq 0} h_n x^n$, with $h_n \in R[\frac{1}{c}]$. Let $m \geq 1$. As $g \in (f, x^m)R[[x]]$, $g = fq + x^m r$, for some $q, r \in R[[x]]$. Thus $g = fh = fq + x^m r$, hence $f(h - q) = x^m r$. As $f(0) \neq 0$, $h - q = x^m s$, for some $s \in R[\frac{1}{c}][[x]]$. Hence $h_0, h_1, \ldots, h_{m-1} \in R$.

(4) Let R be a commutative ring and let $f = \sum_{n \geq 0} f_n x^n \in R[[x]]$ be a power series having the property that its leading form f_r is a regular element of R, that is, $\operatorname{ord} f = r$, so $f_0 = f_1 = \cdots = f_{r-1} = 0$, and f_r is a regular element of R. As in the previous exercise, prove that the principal ideal $fR[[x]]$ is closed in the (x)-adic topology.

(5) Let $f : A \hookrightarrow B$ be as in Example 3.27.
 (i) Prove as asserted in the text that f has geometrically regular fibers but is not flat.
 (ii) Prove that the inclusion map of $C := k[x(x-1)]_{(x(x-1))} \hookrightarrow k[x]_{(x)} = B$ is flat and has geometrically regular fibers. Deduce that the map $C \hookrightarrow B$ is a regular map.

(6) Let $\phi : (R, \mathbf{m}) \hookrightarrow (S, \mathbf{n})$ be an injective local map of the Noetherian local ring $(R, \mathbf{m})$ into the Noetherian local ring $(S, \mathbf{n})$. Let $\widehat{R} = \varprojlim_n R/\mathbf{m}^n$ denote the $\mathbf{m}$-adic completion of R and let $\widehat{S} = \varprojlim_n S/\mathbf{n}^n$ denote the $\mathbf{n}$-adic completion of S.
 (i) Prove that there exists a map $\widehat{\phi} : \widehat{R} \to \widehat{S}$ that extends the map $\phi : R \hookrightarrow S$.

(ii) Prove that $\widehat{\phi}$ is injective if and only if for each positive integer n there exists a positive integer s_n such that $\mathbf{n}^{s_n} \cap R \subseteq \mathbf{m}^n$.

(iii) Prove that $\widehat{\phi}$ is injective if and only if for each positive integer n the ideal $\mathbf{m}^n$ is closed in the topology on R defined by the ideals $\{\mathbf{n}^n \cap R\}_{n \in \mathbb{N}}$, i.e., the topology on R that defines R as a subspace of S.

Suggestion: For each $n \in \mathbb{N}$, we have $\mathbf{m}^n \subseteq \mathbf{n}^n \cap R$. Hence there exists a map $\phi_n : R/\mathbf{m}^n \to R/(\mathbf{n}^n \cap R) \hookrightarrow S/\mathbf{n}^n$, for each $n \in N$. The family of maps $\{\phi_n\}_{n \in \mathbb{N}}$ determines a map $\widehat{\phi} : \widehat{R} \to \widehat{S}$. Since $R/\mathbf{m}^n$ is Artinian, the descending chain of ideals $\{\mathbf{m}^n + (\mathbf{n}^s \cap R)\}_{s \in \mathbb{N}}$ stabilizes, and $\mathbf{m}^n$ is closed in the subspace topology if and only if there exists a positive integer s_n such that $\mathbf{n}^{s_n} \cap R \subseteq \mathbf{m}^n$. This holds for each $n \in \mathbb{N}$ if and only if the $\mathbf{m}$-adic topology on R is the subspace topology from S.

(7) Let $(R, \mathbf{m})$, $(S, \mathbf{n})$ and $(T, \mathbf{q})$ be Noetherian local rings. Assume there exist injective local maps $f : R \hookrightarrow S$ and $g : S \hookrightarrow T$, and let $h := gf : R \hookrightarrow T$ be the composite map. For $\widehat{f} : \widehat{R} \to \widehat{S}$ and $\widehat{g} : \widehat{S} \to \widehat{T}$ and $\widehat{h} : \widehat{R} \to \widehat{T}$ as in the previous exercise, prove that $\widehat{h} = \widehat{g}\widehat{f}$.

(8) Let $(R, \mathbf{m})$ and $(S, \mathbf{n})$ be Noetherian local rings such that S dominates R and the $\mathbf{m}$-adic completion $\widehat{R}$ of R dominates S.

(i) Prove that R is a subspace of S.

(ii) Prove that $\widehat{R}$ is an algebraic retract of $\widehat{S}$, i.e., $\widehat{R} \hookrightarrow \widehat{S}$ and there exists a surjective map $\pi : \widehat{S} \to \widehat{R}$ such that π restricts to the identity map on the subring $\widehat{R}$ of $\widehat{S}$.

(9) Let k be a field and let R be the localized polynomial ring $k[x]_{xk[x]}$, and thus $\widehat{R} = k[[x]]$. Let $n \geq 2$ be a positive integer. If $\operatorname{char} k = p > 0$, assume that n is not a multiple of p.

(i) Prove that there exists $y \in k[[x]]$ such that $y^n = 1 + x$.

(ii) For y as in (i), let $S := R[yx] \hookrightarrow k[[x]]$. Prove that S is a local ring integral over R with maximal ideal $(x, yx)S$. By the previous exercise, $\widehat{R} = k[[x]]$ is an algebraic retract of $\widehat{S}$.

(iii) Prove that the integral closure $\overline{S}$ of S is not local. Indeed, if the field k contains a primitive n-th root of unity, then $\overline{S}$ has n distinct maximal ideals. Deduce that $\widehat{R} \neq \widehat{S}$, so $\widehat{R}$ is a nontrivial algebraic retract of $\widehat{S}$.

Suggestion: Use Remark 3.13 parts (3) and (6ii).

(10) (Cohen) Let $(B, \mathbf{n})$ be a local ring that is not necessarily Noetherian. If the maximal ideal $\mathbf{n}$ is finitely generated and $\bigcap_{n=1}^{\infty} \mathbf{n}^n = (0)$, prove that the completion $\widehat{B}$ of B is Noetherian [**23**] or [**104**, (31.7)]

Suggestion: Use Theorem 3.10.

Comment: In [**23**, page 56] Cohen defines $(B, \mathbf{n})$ to be a *generalized local ring* if $\mathbf{n}$ is finitely generated and $\bigcap_{n=1}^{\infty} \mathbf{n}^n = (0)$. He proves that the completion of a generalized local ring is Noetherian, and that a complete generalized local ring is Noetherian [**23**, Theorems 2 and 3]. Cohen mentions that he does not know whether there exists a generalized local ring that is not Noetherian. Nagata in [**99**] gives such an example of a non-Noetherian generalized local ring $(B, \mathbf{n})$. In Nagata's example $\widehat{B} = k[[x, y]]$ is a formal power series ring in two variables over a field. Heinzer and Roitman in [**50**] survey properties of generalized local rings including this example of Nagata.

CHAPTER 4

First examples of the construction

The basic idea of the Inclusion Construction 5.3 defined in the next chapter is: Start with a well understood Noetherian domain R, then take an ideal-adic completion R^* of R and intersect R^* with an appropriate field L between R and the total quotient ring of R^*. Define $A := L \cap R^*$. This is made more explicit in Section 5.1 of Chapter 5.

In this chapter we illustrate the construction with several examples.

4.1. Elementary examples

We first consider examples where R is a polynomial ring over a field k In the case of one variable the situation is well understood:

EXAMPLE 4.1. Let y be a variable over a field k, let $R := k[y]$, and let L be a subfield of the field of fractions of $k[[y]]$ such that $k(y) \subseteq L$. Then the intersection domain $A := L \cap k[[y]]$ is a rank-one discrete valuation domain (DVR) with field of fractions L (see Remark 2.1), maximal ideal yA and y-adic completion $A^* = k[[y]]$. For example, if we work with the field $\mathbb{Q}$ of rational numbers and our favorite transcendental function e^y, and we put $L = \mathbb{Q}(y, e^y)$, then A is a DVR having residue field $\mathbb{Q}$ and field of fractions L of transcendence degree 2 over $\mathbb{Q}$.

The integral domain A of Example 4.1 with $k = \mathbb{Q}$ is perhaps the simplest example of a Noetherian local domain on an algebraic function field $L/\mathbb{Q}$ of two variables that is not essentially finitely generated over its ground field $\mathbb{Q}$, i.e., A is not the localization of a finitely generated $\mathbb{Q}$-algebra. However A does have a nice description as an infinite nested union of localized polynomial rings in 2 variables over $\mathbb{Q}$; see Example 6.6. Thus in a certain sense there is a good description of the elements of the intersection domain A in this case.

The case where the base ring R involves two variables is more interesting. The following theorem of Valabrega [**138**] is useful in considering this case.

THEOREM 4.2. (Valabrega) *Let C be a DVR, let y be an indeterminate over C, and let L be a subfield of $\mathcal{Q}(C[[y]])$ such that $C[y] \subset L$. Then the integral domain $D = L \cap C[[y]]$ is a two-dimensional regular local domain having completion $\widehat{D} = \widehat{C}[[y]]$, where $\widehat{C}$ is the completion of C.*

Exercise 4 of this chapter outlines a proof for Theorem 4.2. Applying Valabrega's Theorem 4.2, we see that the intersection domain is a two-dimensional regular local domain with the "right" completion in the following two examples:

EXAMPLE 4.3. Let x and y be indeterminates over $\mathbb{Q}$ and let C be the DVR $\mathbb{Q}(x, e^x) \cap \mathbb{Q}[[x]]$. Then $A_1 := \mathbb{Q}(x, e^x, y) \cap C[[y]] = C[y]_{(x,y)}$ is a two-dimensional regular local domain with maximal ideal $(x, y)A_1$ and completion $\mathbb{Q}[[x, y]]$.

EXAMPLE 4.4. (This example is mentioned in Chapter 1 and is generalized in Theorem 9.5; see Remark 9.8.2.) Let x and y be indeterminates over $\mathbb{Q}$ and let C be the DVR $\mathbb{Q}(x, e^x) \cap \mathbb{Q}[[x]]$ as in Example 4.3. Then $A_2 := \mathbb{Q}(x, y, e^x, e^y) \cap C[[y]]$ is a two-dimensional regular local domain with maximal ideal $(x, y)A_2$ and completion $\mathbb{Q}[[x, y]]$. See Theorem 4.12.

REMARKS 4.5. (1) There is a significant difference between the integral domains A_1 of Example 4.3 and A_2 of Example 4.4. As is shown in Theorem 9.5, the two-dimensional regular local domain A_1 of Example 4.3 is, in a natural way, a nested union of three-dimensional regular local domains. It is possible therefore to describe A_1 rather explicitly. On the other hand, the two-dimensional regular local domain A_2 of Example 4.4 contains, for example, the element $\frac{e^x - e^y}{x - y}$. As discussed in Example 7.3, the associated nested union domain B naturally associated with A_2 is a nested union of four-dimensional RLRs, is three-dimensional and is not Noetherian. Notice that the two-dimensional regular local ring A_1 is a subring of an algebraic function field in three variables over $\mathbb{Q}$, while A_2 is a subring of an algebraic function field in four variables over $\mathbb{Q}$. Since the field $\mathbb{Q}(x, e^x, y)$ is contained in the field $\mathbb{Q}(x, e^x, y, e^y)$, the local ring A_1 is dominated by the local ring A_2.

(2) It is shown in Theorem 19.20 and Corollary 19.23 of Chapter 19 that if we go outside the range of Valabrega's theorem, that is, if we take more general subfields L of the field of fractions of $\mathbb{Q}[[x, y]]$ such that $\mathbb{Q}(x, y) \subseteq L$, then the intersection domain $A = L \cap \mathbb{Q}[[x, y]]$ can be, depending on L, a localized polynomial ring in $n \geq 3$ variables over $\mathbb{Q}$ or even a localized polynomial ring in infinitely many variables over $\mathbb{Q}$. In particular, $A = L \cap \mathbb{Q}[[x, y]]$ need not be Noetherian. Theorem 4.12 describes possibilities for the intersection domain A in this setting.

4.2. Historical examples

There are classical examples, related to singularities of algebraic curves, of one-dimensional Noetherian local domains $(R, \mathbf{m})$ such that the $\mathbf{m}$-adic completion $\widehat{R}$ is not an integral domain, that is, R is analytically reducible; see Section 2.2, Remarks 3.13, and Theorem 3.14. We demonstrate this in Example 4.6.

EXAMPLE 4.6. Let X and Y be variables over $\mathbb{Q}$ and consider the localized polynomial ring

$$S := \mathbb{Q}[X, Y]_{(X,Y)} \text{ and the quotient ring } R := \frac{S}{(X^2 - Y^2 - Y^3)S}.$$

Since the polynomial $X^2 - Y^2 - Y^3$ is irreducible in the polynomial ring $\mathbb{Q}[X, Y]$, the ring R is a one-dimensional Noetherian local domain. Let x and y denote the images in R of X and Y, respectively. The principal ideal yR is primary for the maximal ideal $\mathbf{m} = (x, y)R$, and so the $\mathbf{m}$-adic completion $\widehat{R}$ is also the y-adic completion of R. Thus

$$\widehat{R} = \frac{\mathbb{Q}[X][[Y]]}{(X^2 - Y^2(1 + Y))}.$$

Since $1 + Y$ has a square root $(1 + Y)^{1/2} \in \mathbb{Q}[[Y]]$, we see that $X^2 - Y^2(1 + Y)$ factors in $\mathbb{Q}[X][[Y]]$ as

$$X^2 - Y^2(1 + Y) = (X - Y(1 + Y)^{1/2}) \cdot (X + Y(1 + Y)^{1/2}).$$

Thus $\widehat{R}$ is not an integral domain. Since the polynomial $Z^2 - (1+y) \in R[Z]$ has x/y as a root and $x/y \notin R$, the integral domain R is not normal; see Section 2.1. The birational integral extension $\overline{R} := R[\frac{x}{y}]$ has two maximal ideals,

$$\mathbf{m}_1 \ := \ (\mathbf{m}, \frac{x}{y} - 1)\overline{R} \ = \ (\frac{x-y}{y})\overline{R} \quad \text{and} \quad \mathbf{m}_2 \ := \ (\mathbf{m}, \frac{x}{y} + 1)\overline{R} \ = \ (\frac{x+y}{y})\overline{R}.$$

To see, for example, that $\mathbf{m}_1 = (\frac{x-y}{y})\overline{R}$, it suffices to show that $\mathbf{m} \subset (\frac{x-y}{y})\overline{R}$. It is obvious that $x - y \in (\frac{x-y}{y})\overline{R}$. We also clearly have $\frac{x^2-y^2}{y^2} \in (\frac{x-y}{y})\overline{R}$, and $x^2 - y^2 = y^3$. Hence $\frac{y^3}{y^2} = y \in (\frac{x-y}{y})\overline{R}$, and so $\mathbf{m}_1$ is principal and generated by $\frac{x-y}{y}$. Similarly, the maximal ideal $\mathbf{m}_2$ is principal and is generated by $\frac{x+y}{y}$. Thus $\overline{R} = R[\frac{x}{y}]$ is a PID, and hence is integrally closed. To better understand the structure of R and $\overline{R}$, it is instructive to extend the homomorphism

$$\varphi: \ S \ \longrightarrow \ \frac{S}{(X^2 - Y^2 - Y^3)S} \ = \ R.$$

Let $X_1 := X/Y$ and $S' := S[X_1]$. Then S' is a regular integral domain and the map φ can be extended to a map $\psi : S' \to R[\frac{x}{y}]$ such that $\psi(X_1) = \frac{x}{y}$. The kernel of ψ is a prime ideal of S' that contains $X^2 - Y^2 - Y^3$. Since $X = YX_1$, and Y^2 is not in ker ψ, we see that ker $\psi = (X_1^2 - 1 - Y)S'$. Thus

$$\psi: \ S' \ \longrightarrow \ \frac{S'}{(X_1^2 - 1 - Y)S'} \ = \ R[\frac{x}{y}] \ = \ \overline{R}.$$

Notice that $X_1^2 - 1 - Y$ is contained in exactly two maximal ideals of S', namely

$$\mathbf{n}_1 : = \ (X_1 - 1, \ Y)S' \quad \text{and} \quad \mathbf{n}_2 : = \ (X_1 + 1, \ Y)S'.$$

The rings $S_1 := S'_{\mathbf{n}_1}$ and $S_2 := S'_{\mathbf{n}_2}$ are two-dimensional RLRs that are local quadratic transformations [1] of S, and the map ψ localizes to define maps

$$\psi_{\mathbf{n}_1} : \ S_1 \ \to \ \frac{S_1}{(X_1^2 - 1 - Y)S_1} \ = \ \overline{R}_{\mathbf{m}_1} \ \text{ and } \ \psi_{\mathbf{n}_2} : \ S_2 \ \to \ \frac{S_2}{(X_1^2 - 1 - Y)S_2} \ = \ \overline{R}_{\mathbf{m}_2}.$$

Thus the integral closure $\overline{R}$ of R is a homomorphic image of a regular domain of dimension two with precisely two maximal ideals.

REMARK 4.7. Examples given by Akizuki [**7**] and Schmidt [**125**], provide one-dimensional Noetherian local domains R such that the integral closure $\overline{R}$ is not finitely generated as an R-module; equivalently, the completion $\widehat{R}$ of R has nonzero nilpotents; see [**104**, (32.2) and Ex. 1, page 122] and the paper of Katz [**79**, Corollary 5].

If R is a normal one-dimensional Noetherian local domain, then R is a rank-one discrete valuation domain (DVR) and it is well-known that the completion of R is again a DVR. Thus R is analytically irreducible. Zariski showed that the normal Noetherian local domains that occur in algebraic geometry are analytically normal; see [**149**, pages 313-320] and Section 3.4. In particular, the normal local domains occurring in algebraic geometry are analytically irreducible.

This motivated the question of whether there exists a normal Noetherian local domain for which the completion is not a domain. Nagata produced such examples

[1] Chapter **??** contains more information about local quadratic transformations; see Definitions 23.1.

in [**103**]. He also pinpointed sufficient conditions for a normal Noetherian local domain to be analytically irreducible [**104**, (37.8)].

In Example 4.8, we present a special case of a construction of Nagata [**103**], [**104**, Example 7, pages 209-211] of a two-dimensional regular local domain A that is not excellent and a two-dimensional normal Noetherian local domain D that is analytically reducible. These concepts are defined in Sections 3.4 and 3.1. Nagata proves that A is Noetherian with completion $\widehat{A} = \mathbb{Q}[[x, y]]$. Although Nagata constructs A as a nested union of subrings, we show in Example 4.8 that it is also possible to describe A as an intersection.

EXAMPLE 4.8. (Nagata) [**104**, Example 7, pages 209-211] Let x and y be algebraically independent over $\mathbb{Q}$ and let R be the localized polynomial ring $R = \mathbb{Q}[x, y]_{(x,y)}$. Then the completion of R is $\widehat{R} = \mathbb{Q}[[x, y]]$. Let $\tau \in x\mathbb{Q}[[x]]$ be an element that is transcendental over $\mathbb{Q}(x, y)$, e.g., $\tau = e^x - 1$. Let $\rho := y + \tau$ and $f := \rho^2 = (y + \tau)^2$. Now define

$$A := \mathbb{Q}(x, y, f) \cap \mathbb{Q}[[x, y]] \quad \text{and} \quad D := \frac{A[z]}{(z^2 - f)A[z]},$$

where z is an indeterminate. It is clear that the intersection ring A is a Krull domain having a unique maximal ideal. Nagata proves that f is a prime element of A and that A is a two-dimensional regular local domain with completion $\widehat{A} = \mathbb{Q}[[x, y]]$; see Proposition 10.2. He also shows that D is a normal Noetherian local domain. We show other properties of the integral domains A and D in Remarks 4.9. We return to this example in Section 10.1.

REMARKS 4.9. (1) The integral domain D in Example 4.8 is analytically reducible. This is because the element f factors as a square in the completion $\widehat{A}$ of A. Thus

$$\widehat{D} = \frac{\mathbb{Q}[[x, y, z]]}{(z - (y + \tau))(z + (y + \tau))},$$

which is not an integral domain.

(2) The two-dimensional regular local domain A of Example 4.8 is not a Nagata ring and therefore is not excellent. For the definition of a Nagata ring, see Definition 2.3.1, and for the definition of excellence, see Section 3.4, Definition 3.31. To see that A is not a Nagata ring, notice that A has a principal prime ideal generated by f that factors as a square in $\widehat{A} = \mathbb{Q}[[x, y]]$; namely f is the square of the prime element ρ of $\widehat{A}$. Therefore the one-dimensional local domain A/fA has the property that its completion $\widehat{A}/f\widehat{A}$ has a nonzero nilpotent element. This implies that the integral closure of the one-dimensional Noetherian domain A/fA is not finitely generated over A/fA by Remark 3.13.6(i). Hence A is not a Nagata ring. Moreover, the map $A \hookrightarrow \widehat{A} = \mathbb{Q}[[x, y]]$ is not a regular morphism; see Section 3.4.

The existence of examples such as the normal Noetherian local domain D of Example 4.8 naturally motivated the question: Is a Nagata domain necessarily excellent? Rotthaus shows in [**118**] that the answer is "no" as described below.

In Example 4.10, we present a special case of the construction of Rotthaus. In [**118**] the ring A is constructed as a direct limit. We show in Example 4.10 that A can also be described as an intersection. For this we use that A is Noetherian implies that its completion $\widehat{A}$ is a faithfully flat extension, and then we apply Remark 2.21.9.

EXAMPLE 4.10. (Rotthaus) Let x, y, z be algebraically independent over $\mathbb{Q}$ and let R be the localized polynomial ring $R = \mathbb{Q}[x, y, z]_{(x,y,z)}$. Let $\tau_1 = \sum_{i=1}^{\infty} a_i y^i \in \mathbb{Q}[[y]]$ and $\tau_2 = \sum_{i=1}^{\infty} b_i y^i \in \mathbb{Q}[[y]]$ be power series such that y, τ_1, τ_2 are algebraically independent over $\mathbb{Q}$, for example, $\tau_1 = e^y - 1$ and $\tau_2 = e^{y^2} - 1$. Let $u := x + \tau_1$ and $v := z + \tau_2$. Define

$$A \ := \ \mathbb{Q}(x, y, z, uv) \ \cap \ (\mathbb{Q}[x, z]_{(x,z))}[[y]]).$$

We demonstrate the properties of the ring A in Remark 4.11.

REMARK 4.11. The integral domain A of Example 4.10 is a Nagata domain that is not excellent. Rotthaus shows in [**118**] that A is Noetherian and that the completion $\widehat{A}$ of A is $\mathbb{Q}[[x, y, z]]$, so A is a 3-dimensional regular local domain. Moreover she shows the formal fibers of A are reduced, but are not regular. Since u, v are part of a regular system of parameters of $\widehat{A}$, it is clear that $(u, v)\widehat{A}$ is a prime ideal of height two. It is shown in [**118**], that $(u, v)\widehat{A} \cap A = uvA$. Thus uvA is a prime ideal and $\widehat{A}_{(u,v)\widehat{A}}/uv\widehat{A}_{(u,v)\widehat{A}}$ is a non-regular formal fiber of A. Therefore A is not excellent.

Since A contains a field of characteristic zero, to see that A is a Nagata domain it suffices to show for each prime ideal P of A that the integral closure of A/P is a finite A/P-module. Since the formal fibers of A are reduced, the integral closure of A/P is a finite A/P-module; see Remark 3.13.5.

4.3. Iterative examples

We present a family of examples contained in $k[[x, y]]$, where k is a field and x and y are indeterminates. We show that for certain values of the parameters that occur in the examples, one obtains an example of a 3-dimensional local Krull domain $(B, \mathbf{n})$ such that B is not Noetherian, $\mathbf{n} = (x, y)B$ is 2-generated and the $\mathbf{n}$-adic completion $\widehat{B}$ of B is a two-dimensional regular local domain; see Examples 7.3 and 8.11.

Let R be the localized polynomial ring $R := k[x, y]_{(x,y)}$. If $\sigma, \tau \in \widehat{R} = k[[x, y]]$ are algebraically independent over R, then the polynomial ring $R[t_1, t_2]$ in two variables t_1, t_2 over R, can be identified with a subring of $\widehat{R}$ by means of an R-algebra isomorphism mapping $t_1 \to \sigma$ and $t_2 \to \tau$. The structure of the local domain $A = k(x, y, \sigma, \tau) \cap \widehat{R}$ depends on the residual behavior of σ and τ with respect to prime ideals of $\widehat{R}$. Theorem 4.12 illustrates this in the special case where $\sigma \in k[[x]]$ and $\tau \in k[[y]]$. A special case of this is given in Example 4.4. Theorem 4.12 introduces a technique that is further developed in Chapters 5, 6, 7 and 8.

THEOREM 4.12. *Let k be a field, let x and y be indeterminates over k, and let*

$$\sigma := \sum_{i=0}^{\infty} a_i x^i \in k[[x]] \qquad \text{and} \qquad \tau := \sum_{i=0}^{\infty} b_i y^i \in k[[y]]$$

be formal power series that are algebraically independent over the fields $k(x)$ and $k(y)$, respectively. Then $A := k(x, y, \sigma, \tau) \cap k[[x, y]]$ is a two-dimensional regular local domain with maximal ideal $(x, y)A$ and completion $\widehat{A} = k[[x, y]]$. The intersection ring A has a subring B with the following properties:

- *The local ring A birationally dominates the local ring B.*

- *The ring B is the directed union of a chain of four-dimensional regular local domains that are localized polynomial rings in four variables over k and are birationally dominated by both A and B.*
- *The local ring B is a Krull domain with maximal ideal $\mathbf{n} = (x, y)B$.*
- *The dimension of B is either 2 or 3, depending on the choice of σ and τ.*
- *The local ring B is Hausdorff in the topology defined by the powers of $\mathbf{n}$.*
- *The $\mathbf{n}$-adic completion $\widehat{B}$ of B is canonically isomorphic to $k[[x, y]]$.*

Moreover the following statements are equivalent:

(1) *The ring B is Noetherian.*
(2) *The dimension of B is 2.*
(3) *The rings B and A are equal.*
(4) *Every finitely generated ideal of B is closed in the $\mathbf{n}$-adic topology on B.*
(5) *Every principal ideal of B is closed in the $\mathbf{n}$-adic topology on B.*

Depending on the choice of σ and τ, B may or may not be Noetherian. In particular there exist certain values for σ and τ such that $B \neq A$ and other values such that $B = A$.

To establish the asserted properties of the ring A of Theorem 4.12, we use the following consequence of the useful result of Valabrega stated as Theorem 4.2 above. Since the construction of A can be realized by a succession of two principal ideal-adic completions, first using power series in x, then using power series in y, we consider it an "iterative" example.

Proposition 4.13. *With the notation of Theorem 4.12, let $C = k(x, \sigma) \cap k[[x]]$ and let L be the field of fractions of $C[y, \tau]$. Then the ring $A = L \cap C[[y]]$ is a two-dimensional regular local domain with maximal ideal $(x, y)A$ and completion $\widehat{A} = k[[x, y]]$.*

Proof. The ring C is a rank-one discrete valuation domain with completion $k[[x]]$, and the field $k(x, y, \sigma, \tau) = L$ is an intermediate field between the fields of fractions of the rings $C[y]$ and $C[[y]]$. Hence, by Theorem 4.2, $A = L \cap C[[y]]$ is a regular local domain with completion $k[[x, y]]$. $\square$

Remark 4.14. In examining properties of subrings of the formal power series ring $k[[x, y]]$ over the field k, we use that the subfields $k((x))$ and $k((y))$ of the field $\mathcal{Q}(k[[x, y]])$ are linearly disjoint over k as defined for example in [**148**, page 109]. It follows that if $\alpha_1, \dots, \alpha_n \in k[[x]]$ are algebraically independent over $k(x)$ and $\beta_1, \dots, \beta_m \in k[[y]]$ are algebraically independent over $k(y)$, then the elements $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m$ are algebraically independent over $k(x, y)$.

We return to the proof of Theorem 4.12 in Chapter 7.

Exercises

(1) Prove that the intersection domain A of Example 4.1 is a DVR with field of fractions L and (y)-adic completion $A^* = \mathbb{Q}[[y]]$.

    Comment. Exercise 2 of Chapter 2 implies that A is a DVR. With the additional hypothesis of Example 4.1, it is true that the (y)-adic completion of A is $\mathbb{Q}[[y]]$.

(2) Let R be an integral domain with field of fractions K.

(i) Let F be a subfield of K and let $S := F \cap R$. For each principal ideal aS of S, prove that $aS = aR \cap S$.

(ii) Assume that S is a subring of R with the same field of fractions K. Prove that $aS = aR \cap S$ for each $a \in S \iff S = R$.

(3) Let R be a local domain with maximal ideal $\mathbf{m}$ and field of fractions K. Let F be a subfield of K and let $S := F \cap R$. Prove that S is local with maximal ideal $\mathbf{m} \cap S$, and thus conclude that R dominates S. Give an example where R is not Noetherian, but S is Noetherian.

Remark. It can happen that R is Noetherian while S is not Noetherian; see Chapter 17.

(4) Assume the notation of Theorem 4.2. Thus y is an indeterminate over the DVR C and $D = C[[y]] \cap L$, where L is a subfield of the field of fractions of $C[[y]]$ with $C[y] \subset L$. Let x be a generator of the maximal ideal of C and let $R := C[y]_{(x,y)C[y]}$. Observe that R is a two-dimensional RLR with maximal ideal $(x,y)R$ and that $C[[y]]$ is a two-dimensional RLR with maximal ideal $(x,y)C[[y]]$ that dominates R. Let $\mathbf{m} := (x,y)C[[y]] \cap D$.

(i) Using Exercise 2, prove that

$$C \;\cong\; \frac{R}{yR} \;\hookrightarrow\; \frac{D}{yD} \;\hookrightarrow\; \frac{C[[y]]}{yC[[y]]} \;\cong\; C.$$

(ii) Deduce that $C \cong \frac{D}{yD}$, and that $\mathbf{m} = (x,y)D$.

(iii) Let $k := \frac{C}{xC}$ denote the residue field of C. Prove that $\frac{D}{xD}$ is a DVR and that

$$k[y] \;\hookrightarrow\; \frac{R}{xR} \;\hookrightarrow\; \frac{D}{xD} \;\hookrightarrow\; \frac{C[[y]]}{xC[[y]]} \;\cong\; k[[y]].$$

(iv) For each positive integer n, prove that

$$\frac{R}{(x,y)^n R} \;\cong\; \frac{D}{(x,y)^n D} \;\cong\; \frac{C[[y]]}{(x,y)^n C[[y]]}.$$

Deduce that $\widehat{R} = \widehat{D} = \widehat{C}[[y]]$, where $\widehat{C}$ is the completion of C.

(v) Let P be a prime ideal of D such that $x \notin P$. Prove that there exists $b \in P$ such that $b(D/xD) = y^r(D/xD)$ for some positive integer r, and deduce that $P \subset (b,x)D$.

(vi) For $a \in P$, observe that $a = c_1 b + a_1 x$, where c_1 and a_1 are in D. Since $x \notin P$, deduce that $a_1 \in P$ and hence $a_1 = c_2 b + a_2 x$, where c_2 and a_2 are in D. Conclude that $P \subset (b, x^2)D$. Continuing this process, deduce that

$$bD \;\subseteq P \;\subseteq\; \bigcap_{n=1}^{\infty} (b, x^n)D.$$

(vii) Extending the ideals to $C[[y]]$, observe that

$$bC[[y]] \;\subseteq\; PC[[y]] \;\subseteq\; \bigcap_{n=1}^{\infty} (b, x^n)C[[y]] \;=\; bC[[y]],$$

where the last equality is because the ideal $bC[[y]]$ is closed in the topology defined by the ideals generated by the powers of x on the Noetherian local ring $C[[y]]$. Deduce that $P = bD$.

(viii) Conclude by Theorem 2.8 that D is Noetherian and hence a two-dimensional regular local domain with completion $\widehat{D} = \widehat{C}[[y]]$.

(5) Let k be a field and let $f \in k[[x, y]]$ be a formal power series of order $r \geq 2$. Let $f = \sum_{n=r}^{\infty} f_n$, where $f_n \in k[x, y]$ is a homogeneous form of degree n. If the leading form f_r factors in $k[x, y]$ as $f_r = \alpha \cdot \beta$, where α and β are coprime homogeneous polynomials in $k[x, y]$ of positive degree, prove that f factors in $k[[x, y]]$ as $f = g \cdot h$, where g has leading form α and h has leading form β.

Suggestion. Let $G = \bigoplus_{n \geq 0} G_n$ represent the polynomial ring $k[x, y]$ as a graded ring obtained by defining $\deg x = \deg y = 1$. Notice that G_n has dimension $n + 1$ as a vector space over k. Let $\deg \alpha = a$ and $\deg \beta = b$. Then $a + b = r$ and for each integer $n \geq r + 1$, we have $\dim(\alpha \cdot G_{n-a}) = n - a + 1$ and $\dim(\beta \cdot G_{n-b}) = n - b + 1$. Since α and β are coprime, we have

$$(\alpha \cdot G_{n-a}) \ \cap \ (\beta \cdot G_{n-b}) \ = \ f_r \cdot G_{n-r}.$$

Conclude that $\alpha \cdot G_{n-a} \ + \ \beta \cdot G_{n-b}$ is a subspace of G_n of dimension $n + 1$ and hence that $G_n \ = \ \alpha \cdot G_{n-a} \ + \ \beta \cdot G_{n-b}$. Let $g_a := \alpha$ and $h_b := \beta$. Since $f_{r+1} \in G_{r+1} \ = \ \alpha \cdot G_{r+1-a} \ + \ \beta \cdot G_{r+1-b} \ = \ g_a \cdot G_{b+1} \ + \ h_b \cdot G_{a+1}$, there exist forms $h_{b+1} \in G_{b+1}$ and $g_{a+1} \in G_{a+1}$ such that $f_{r+1} \ = \ g_a \cdot h_{b+1} + h_b \cdot g_{a+1}$. Since $G_{r+2} \ = \ g_a \cdot G_{b+2} + h_b \cdot G_{a+2}$, there exist forms $h_{b+2} \in G_{b+2}$ and $g_{a+2} \in G_{a+2}$ such that $f_{r+2} - g_{a+1} \cdot h_{b+1} \ = \ g_a \cdot h_{b+2} + h_b \cdot g_{a+2}$. Proceeding by induction, assume for a positive integer s that there exist forms $g_a, g_{a+1}, \ldots, g_{a+s}$ and $h_b, h_{b+1}, \ldots, h_{b+s}$ such that the power series $f - (g_a + \cdots + g_{a+s})(h_b + \cdots + h_{b+s})$ has order greater than or equal to $r + s + 1$. Using that

$$G_{r+s+1} \ = \ g_a \cdot G_{b+s+1} \ + \ h_b \cdot G_{a+s+1},$$

deduce the existence of forms $g_{a+s+1} \in G_{a+s+1}$ and $h_{b+s+1} \in G_{b+s+1}$ such that the power series $f - (g_a + \cdots + g_{a+s+1})(h_b + \cdots + h_{b+s+1})$ has order greater than or equal to $r + s + 2$.

(6) Let k be a field of characteristic zero. Prove that both

$$xy + z^3 \quad \text{and} \quad xyz + x^4 + y^4 + z^4$$

are irreducible in the formal power series ring $k[[x, y, z]]$. Thus there does not appear to be any natural generalization to the case of three variables of the result in the previous exercise.

CHAPTER 5

Describing the Basic Construction

We discuss a universal technique that yields the examples of the previous section and also leads to more examples. For this we apply the Basic Construction Equation 1.3 from Chapter 1. There are two versions of the construction that we use. At first they appear to be quite different:

- The *Inclusion Construction* 5.3 defines an intersection $A = A_{\text{inc}} := R^* \cap L$ included in R^*, where R^* is an ideal-adic completion of a Noetherian integral domain R and L is the subfield of the total quotient ring of R^* that is generated over $\mathcal{Q}(R)$ by elements $\tau_1, \ldots, \tau_s$ of R^* that are algebraically independent over R.
- The *Homomorphic Image Construction* 5.4 is an intersection $A = A_{\text{hom}}$ of a homomorphic image of an ideal-adic completion R^* of a Noetherian integral domain R with the field of fractions of R.

The two versions are explained in more detail below. Construction 5.4 includes, up to isomorphism, Construction 5.3 as a special case.

We begin with the setting for both constructions. This setting will also be used in later chapters.

Setting 5.1. Let R be a Noetherian integral domain with field of fractions $K := \mathcal{Q}(R)$. Assume there exists $z \in R$ that is a nonzero nonunit, and let R^* denote the (z)-adic completion of R.

Remark 5.2. Section 3.1 contains material on the (z)-adic completion R^* of a Noetherian integral domain R with respect to a nonzero nonunit z of R. Remark 3.3 of Chapter 2 implies that R^* has the form

$$R^* \ = \ \frac{R[[y]]}{(y-z)R[[y]]},$$

where y is an indeterminate over R. It is natural to ask for conditions in order that R^* is an integral domain, or equivalently, that $(y-z)R[[y]]$ is a prime ideal. The element $y - z$ obviously generates a prime ideal of the polynomial ring $R[y]$. Our assumption that z is a nonunit of R implies that $(y-z)R[[y]]$ is a proper ideal. We consider in Exercise 3 of this chapter examples where $(y-z)R[[y]]$ is prime and examples where it is not prime.

5.1. Two construction methods and a picture

With R, z and R^* as in Setting 5.1 we describe two methods for the construction associated with R^*. The Inclusion Construction A_{inc} results in a domain transcendental over R and contained in a power series extension of R, whereas the Homomorphic Image Construction A_{hom} results in a domain that is birational over R and is contained in a homomorphic image of a power series extension of R.

CONSTRUCTION 5.3. *Inclusion Construction*: Let $\tau_1, \dots, \tau_s \in zR^*$ be algebraically independent elements over R such that $K(\tau_1, \dots, \tau_s) \subseteq \mathcal{Q}(R^*)$.[1] We define A to be the intersection domain $A = A_{\text{inc}} := K(\tau_1, \dots, \tau_s) \cap R^*$. Thus A_{inc} is a subring of R^* and is a transcendental extension of R.

CONSTRUCTION 5.4. *Homomorphic Image Construction*: Let I be an ideal of R^* having the property that $P \cap R = (0)$ for each $P \in \operatorname{Spec} R^*$ that is associated to I. Define $A = A_{\text{hom}} := K \cap (R^*/I)$. In connection with this intersection, see Exercises 1 and 4 of this chapter. The ring A_{hom} is contained in a homomorphic image of R^* and is a birational extension of R.

NOTE 5.5. The condition in (5.4), that $P \cap R = (0)$ for every prime ideal P of R^* that is associated to I, implies that the field of fractions K of R embeds in the total quotient ring $\mathcal{Q}(R^*/I)$ of R^*/I. To see this, observe that the canonical map $R \to R^*/I$ is an injection and that regular elements of R remain regular as elements of R^*/I. In this connection see Exercise 1.

PICTURE 5.6. The diagram below shows the relationships among these rings.

$\mathcal{Q}(R^*)$ $\qquad\qquad\qquad\qquad$ R^* $\qquad$ $\mathcal{Q}(R^*/I)$

R^* $\qquad$ $L = K(\{\tau_i\})$ $\qquad\qquad\qquad\qquad$ R^*/I $\qquad$ $K = \mathcal{Q}(R)$

$\boxed{A_{\text{inc}} = R^* \cap L}$ $\qquad$ $K = \mathcal{Q}(R)$ $\qquad\qquad\qquad\qquad$ $\boxed{A_{\text{hom}} = (R^*/I) \cap K}$

transcendental $\qquad\qquad\qquad\qquad$ birational

R $\qquad\qquad\qquad\qquad$ R

(5.3) $A := L \cap R^*$ $\qquad\qquad\qquad\qquad$ (5.4) $A := K \cap (R^*/I)$

REMARKS 5.7. (1) Chapters 4, 17 and 21 feature the Inclusion Construction 5.3 that realizes the intersection domain $A = A_{\text{inc}} := K(\tau_1, \dots, \tau_s) \cap R^*$, while Chapter 16 features the Homomorphic Image Construction 5.4. Construction 5.3 yields the examples of Sections 4.1, 4.2 and 4.3; e.g., Nagata's Example 4.8 fits the format of Construction 5.3, since

$$A := \mathbb{Q}(x, y, \rho^2) \cap \mathbb{Q}[[x, y]] = \mathbb{Q}(x, y, \rho^2) \cap \mathbb{Q}[x][[y]].$$

(2) Homomorphic Image Construction 5.4 includes Inclusion Construction 5.3 as a special case, up to isomorphism. To see this, let R, z, R^* and $\tau_1, \dots, \tau_s \in zR^*$ be as in Construction 5.3. Let $t_1, \dots, t_s$ be indeterminates over R, define $S := R[t_1, \dots, t_s]$, let S^* be the z-adic completion of S and let I denote the ideal $(t_1 - \tau_1, \dots, t_s - \tau_s)S^*$. Consider the following diagram, where D_{hom} is the intersection domain result of Construction 5.4, when R and R^* there are replaced by S and S^*, and where λ is the R-algebra isomorphism of S into R^* that maps $t_i \to \tau_i$ for

[1] Thus every nonzero element of $R[\tau_1, \dots, \tau_s]$ is a regular element of R^*. See Note 5.5.

$i = 1, \dots, s$. The map λ naturally extends to a homomorphism of S^* onto R^* with the kernel of this extension being the ideal I. Thus there is an induced isomorphism of S^*/I onto R^* that we also label as λ.

(5.7.2.1)
$$\begin{array}{ccccccc}
 & & S := R[t_1,\dots,t_s] & \longrightarrow & D_{\text{hom}} := K(t_1,\dots,t_s) \cap (S^*/I) & \longrightarrow & S^*/I \\
 & & \lambda\big\downarrow & & \lambda\big\downarrow & & \lambda\big\downarrow \\
R & \longrightarrow & R[\tau_1,\dots,\tau_s] & \longrightarrow & A_{\text{inc}} := K(\tau_1,\dots,\tau_s) \cap R^* & \longrightarrow & R^*.
\end{array}$$

Since λ maps D_{hom} isomorphically onto A_{inc}, we see that Construction 5.4 includes as a special case Construction 5.3.

(3) Many of the classical examples of interesting Noetherian local domains can be obtained with Construction 5.3 where R is a regular local domain. Construction 5.3, however, is not sufficient to obtain certain other types of rings such as Ogoma's celebrated example [**110**] of a normal non-catenary Noetherian local domain. Ogoma's example can be realized, however, as an intersection with the homomorphic image of a completion as described in Construction 5.4. In Example 9.11 of Chapter 9, we construct a Noetherian local domain with geometrically regular formal fibers that is not universally catenary; this requires the Homomorphic Image Construction.

5.2. Universality

In this section we describe how the methods of Section 5.1 can be regarded as universal for the construction of certain Noetherian local domains. Consider the following general question.

QUESTION 5.8. Let k be a field and let L/k be a finitely generated field extension. What are the Noetherian local domains $(A, \mathbf{n})$ such that

(1) L is the field of fractions A, and
(2) k is a coefficient field for A?

Recall from Section 2.1, that k is a coefficient field of $(A, \mathbf{n})$ if the composite map $k \hookrightarrow A \to A/\mathbf{n}$ defines an isomorphism of k onto $A/\mathbf{n}$.

In relation to Question 5.8, we observe in Theorem 5.9 the following general facts.

THEOREM 5.9. *Let $(A, \mathbf{n})$ be a Noetherian local domain having a coefficient field k. Then there exists a Noetherian local subring $(R, \mathbf{m})$ of A such that:*

(1) *The local ring R is essentially finitely generated over k.*
(2) *If $\mathcal{Q}(A) = L$ is finitely generated over k, then R has field of fractions L.*
(3) *The field k is a coefficient field for R.*
(4) *The local ring A dominates R and $\mathbf{m}A = \mathbf{n}$.*
(5) *The inclusion map $\varphi : R \hookrightarrow A$ extends to a surjective homomorphism $\widehat{\varphi} : \widehat{R} \to \widehat{A}$ of the $\mathbf{m}$-adic completion $\widehat{R}$ of R onto the $\mathbf{n}$-adic completion $\widehat{A}$ of A.*
(6) *For the ideal $I := \ker(\widehat{\varphi})$, of the completion $\widehat{R}$ of R from item 5, we have:*
   (a) *$\widehat{R}/I \cong \widehat{A}$, so $\widehat{R}/I$ dominates A, and*

(b) $P \cap A = (0)$ *for every* $P \in \operatorname{Ass}(\widehat{R}/I)$, *and so the field of fractions* $\mathcal{Q}(A)$ *of* A *embeds in the total ring of quotients* $\mathcal{Q}(\widehat{R}/I)$ *of* $\widehat{R}/I$, *and*

(c) $A = \mathcal{Q}(A) \cap (\widehat{R}/I)$.

PROOF. Since A is Noetherian, there exist elements $t_1, \ldots, t_n \in \mathbf{n}$ such that $(t_1, \ldots, t_n)A = \mathbf{n}$. For item 2, we may assume that $L = k(t_1, \ldots, t_n)$, since every element of $\mathcal{Q}(A)$ has the form a/b, where $a, b \in \mathbf{n}$. To see the existence of the integral domain $(R, \mathbf{m})$ and to establish item 1, we set $T := k[t_1, \ldots, t_n]$ and $\mathbf{p} := \mathbf{n} \cap T$. Define $R := T_{\mathbf{p}}$ and $\mathbf{m} := \mathbf{n} \cap R$. Then $k \subseteq R \subseteq A$, $\mathbf{m}A = \mathbf{n}$, R is essentially finitely generated over k and k is a coefficient field for R. Thus we have established items 1- 4. Even without the assumption that $\mathcal{Q}(A)$ is finitely generated over k, there is a relationship between R and A that is realized by passing to completions. Let φ be the inclusion map $R \hookrightarrow A$. The map φ extends to a map $\widehat{\varphi} : \widehat{R} \to \widehat{A}$, and by Corollary 3.11.2, the map $\widehat{\varphi}$ is surjective; thus item 5 holds. Let $I := \ker \widehat{\varphi}$. Then $\widehat{R}/I = \widehat{A}$, for the first part of item 6. The remaining assertions in item 6 follow from the fact that A is a Noetherian local domain and $\widehat{A} \cong \widehat{R}/I$. Applying Remarks 3.2, we have $\widehat{R}/I$ is faithfully flat over A, and by Remark 2.21.6 the nonzero elements of A are regular on $\widehat{R}/I$.

The following commutative diagram, where the vertical maps are injections, displays the relationships among these rings:

$$
\begin{array}{ccccccc}
 & & \widehat{R} & \xrightarrow{\widehat{\varphi}} & \widehat{A} \cong \widehat{R}/I & \longrightarrow & \mathcal{Q}(\widehat{R}/I) \\
(5.1) & & \uparrow & & \uparrow & & \uparrow \\
k & \xrightarrow{\subseteq} & R & \xrightarrow{\varphi} & A := \mathcal{Q}(A) \cap (\widehat{R}/I) & \longrightarrow & \mathcal{Q}(A) \quad .
\end{array}
$$

This completes the proof of Theorem 5.9. □

In relation to Question 5.8, we obtain Corollary 5.10 by summarizing the discussion above.

COROLLARY 5.10. *Every Noetherian local domain* $(A, \mathbf{n})$ *having a coefficient field* k, *and having the property that the field of fractions* L *of* A *is finitely generated over* k *is realizable as an intersection* $L \cap \widehat{R}/I$, *where* R *is a Noetherian local domain essentially finitely generated over* k *with* $\mathcal{Q}(R) = L$, *and* I *is an ideal in the completion* $\widehat{R}$ *of* R *such that* $P \cap R = (0)$ *for each associated prime* P *of* I.

In connection with Corollary 5.10, a result proved in [**48**, Corollary 2] implies that a d-dimensional Noetherian local domain $(R, \mathbf{m})$ that is essentially finitely generated over a field k has the following property: every d-dimensional Noetherian local domain that is either normal or quasi-unmixed, and that birationally dominates R is essentially finitely generated over k, and thus essentially finitely generated over R. A Noetherian local domain $(R, \mathbf{m})$ is said to be *quasi-unmixed* if its $\mathbf{m}$-adic completion $\widehat{R}$ is equidimensional in the sense of Definition 3.16.5.

A modification of the question raised by Judy Sally from Chapter 1 is:

QUESTION 5.11. Let R be a Noetherian integral domain. What Noetherian overrings of R exist inside the field of fractions of R?

In connection with Question 5.11, the Krull-Akizuki theorem (see Theorem 2.7) implies that every birational overring of a one-dimensional Noetherian integral domain is Noetherian and of dimension at most one. On the other hand, every Noetherian domain of dimension greater than one admits birational overrings that are not Noetherian. Indeed, if R is an integral domain with $\dim R > 1$, then by [**104**, (11.9)] there exists a valuation ring V that is birational over R with $\dim V > 1$. Since a Noetherian valuation ring has dimension at most one, if $\dim R > 1$, then there exist birational overrings of R that are not Noetherian.

Remark 5.12. Corollary 5.10 is a first start towards a classification of the Noetherian local domains A having a given coefficient field k, and having the property that the field of fractions of A is finitely generated over k. A drawback with Corollary 5.10 is that it is not true for every triple R, L, I as in Corollary 5.10 that $L \cap (\widehat{R}/I)$ is Noetherian (see Examples 13.8 below). In order to have a more satisfying classification an important goal is to identify necessary and sufficient conditions that $L \cap (\widehat{R}/I)$ is Noetherian for R, L, I as in Corollary 5.10.

If a Noetherian local domain R is essentially finitely generated over a field k, then there often exist ideals I in the completion $\widehat{R}$ of R such that the intersection domain $\mathcal{Q}(R) \cap (\widehat{R}/I)$ is an interesting Noetherian local domain that birationally dominates R. This technique may be used to describe the example given by Nagata [**103**] or [**104**, Example 7, page 209], and other examples given in [**15**], [**16**], [**74**], [**110**], [**111**], [**118**], [**119**], and [**140**].

In order to give a more precise criterion for the intersection domain A to be Noetherian, we restrict Constructions 5.3 and 5.4 to a completion R^* of R with respect to a nonzero nonunit z of R. This sometimes permits an explicit description of the intersection via the approximation methods of Chapter 6. In Chapter 8 we give necessary and sufficient conditions for A to be computable as a nested union of subrings of a specific form. We also prove that the Noetherian property for the associated nested union is equivalent to flatness of a certain map.

Exercises

(1) Let A be an integral domain and let $A \hookrightarrow B$ be an injective map to an extension ring B. For an ideal I of B, prove that the following are equivalent:

  (i) The induced map $A \to B/I$ is injective, and each nonzero element of A is regular on B/I.

  (ii) The field of fractions $\mathcal{Q}(A)$ of A naturally embeds in the total quotient ring $\mathcal{Q}(B/I)$ of B/I.

  If A and B are Noetherian, prove that conditions (i) and (ii) are also equivalent to the following condition:

  (iii) For each prime ideal P of B that is associated to I we have $P \cap A = (0)$.

(2) Let $A := k[x] \hookrightarrow k[x, y] =: B$ be a map of the polynomial ring $k[x]$ into the polynomial ring $k[x, y]$ and let $I = xyB$. Prove that the induced map $A \to B/I$ is injective, but the field of fractions $\mathcal{Q}(A)$ of A does not embed into the total quotient ring $\mathcal{Q}(B/I)$ of B/I.

(3) Let z be a nonzero nonunit of a Noetherian integral domain R, let y be an indeterminate, and let $R^* = \frac{R[[y]]}{(z-y)R[[y]]}$ be the (z)-adic completion of R.

(i) If $z = ab$, where $a, b \in R$ are nonunits such that $aR + bR = R$, prove that there exists a factorization

$$z - y \;=\; (a + a_1 y + \cdots) \cdot (b + b_1 y + \cdots) \;=\; (\sum_{i=0}^{\infty} a_i y^i) \cdot (\sum_{i=0}^{\infty} b_i y^i),$$

where the $a_i, b_i \in R$, $a_0 = a$ and $b_0 = b$.

(ii) If R is a principal ideal domain (PID), prove that R^* is an integral domain if and only if zR has prime radical.

(4) Let A be an integral domain and let $A \hookrightarrow B$ be an injective map to an extension ring B. Let I be an ideal of B having the property that $I \cap A = (0)$ and every nonzero element of A is a regular element on B/I. Let $C := \mathcal{Q}(A) \cap (B/I)$.

(i) Prove that $C = \{a/b \mid a, b \in A,\; b \neq 0 \;\text{ and } a \in I + bB\;\}$.

(ii) Assume that $J \subseteq I$ is an ideal of B having the property that every nonzero element of A is a regular element on B/J. Let $D := \mathcal{Q}(A) \cap (B/J)$. Prove that $D \;\subseteq\; C$.

Suggestion: Item ii is immediate from item i. To see item i, observe that $bC = b(B/I) \cap \mathcal{Q}(A)$, and $a \in bC \iff a \in b(B/I) \iff a \in I + bB$.

CHAPTER 6

Two approximations and their connection

Our goal in this chapter is to describe in detail the Inclusion Construction 5.3 and the Homomorphic Image Construction 5.4. The first difficulty we face is identifying precisely what we have constructed—because, while the form of the example as an intersection as given in Constructions 5.3 and 5.4 of Chapter 5 is wonderfully concise, sometimes it is difficult to fathom.

We use the notation of Setting 5.1. The approximation methods in this chapter describe a subring B inside the constructed intersection domain A of Constructions 5.3 and 5.4. This subring is useful for describing A. We discuss two approximation methods corresponding to the two construction methods of Chapter 5. In Chapter 7 we use the approximation from Section 6.1 to continue the proof of Theorem 4.12 concerning the iterative examples of Section 4.3.

Associated to each of the Constructions 5.3 and 5.4, the subrings B contained in the intersection domains A approximate A in each of the constructions. In the case of Construction 5.3 we obtain an "approximation" domain B that is a nested union of localizations of polynomial rings in s variables over R. In the case of Construction 5.4, the "approximation" domain B is a nested union of birational extensions of R that are essentially finitely generated R-algebras. In both constructions B is a subring of the corresponding intersection domain A.

6.1. Approximations for the Inclusion Construction

In this section we give an explicit description of the approximation domain B of the previous paragraph for Construction 5.3. We use the last parts, the *endpieces*, of the power series $\tau_1, \dots, \tau_s$. First we describe the endpieces for a general element γ of R^*.

Endpiece Notation 6.1. Let R be a Noetherian integral domain and let $z \in R$ be a nonzero nonunit. Let R^* be the z-adic completion of R; see Remark 3.3. Each $\gamma \in zR^*$ has an expansion as a power series in z over R,

$$\gamma := \sum_{i=1}^{\infty} c_i z^i, \quad \text{where} \quad c_i \in R.[1]$$

For each nonnegative integer n we define the n^{th} *endpiece* γ_n of γ with respect to this expansion:

$$\gamma_n := \sum_{i=n+1}^{\infty} c_i z^{i-n}. \tag{6.1.1}$$

[1]We are interested in the ring $R[\gamma]$, where γ is algebraically independent over R and nonzero elements of $R[\gamma]$ are regular in R^*. By modifying γ by an element in R, we may assume $\gamma \in zR^*$.

It follows that, for each n and $r \in \mathbb{N}$, we have the relations

$$
\begin{aligned}
\gamma_n &= c_{n+1}z + z\gamma_{n+1}\ ; \quad \gamma_{n+1} = c_{n+2}z + z\gamma_{n+2}\ ; \\
\gamma_n &= c_{n+2}z + c_{n+1}z^2 + z^2\gamma_{n+2}\ ;\ \cdots . \\
\gamma_n &= c_{n+r}z \ ++ \cdots + c_{n+1}z^r + z^r\gamma_{n+r} \implies \\
\gamma_n &= az + z^r\gamma_{n+r} \quad \text{and} \quad \gamma_{n+1} = bz + z^{r-1}\gamma_{n+r}\ ,
\end{aligned} \tag{6.1.2}
$$

for some $a \in (c_{n+1}, \dots, c_{n+r})R$ and $b \in (c_{n+2}, \dots, c_{n+r})R$.

In Construction 5.3, the elements $\tau_1, \dots \tau_s \in zR^*$ are algebraically independent over the field of fractions $\mathcal{Q}(R)$ of R and have the property that every nonzero element of the polynomial ring $R[\tau_1, \dots, \tau_s]$ is a regular element of R^*. Thus $\mathcal{Q}(R[\tau_1, \dots, \tau_s])$ is contained in the total quotient ring $\mathcal{Q}(R^*)$. Hence it makes sense to define the *intersection domain* $A := R^* \cap \mathcal{Q}(R[\tau_1, \dots, \tau_s])$ inside $\mathcal{Q}(R^*)$. We set

$$U_0 := R[\tau_1, \dots, \tau_s] \subseteq R^* \cap \mathcal{Q}(R[\tau_1, \dots, \tau_s]) =: A,$$

and U_0 is a polynomial ring in s variables over R. Each $\tau_i \in zR^*$ has a representation $\tau_i := \sum_{j=1}^{\infty} r_{ij}z^j$, where the $r_{ij} \in R$. For each positive integer n, we associate with this representation of τ_i the n^{th} endpiece,

$$\tau_{in} := \sum_{j=n+1}^{\infty} r_{ij}z^{j-n}. \tag{6.1.3}$$

We define

$$U_n := R[\tau_{1n}, \dots, \tau_{sn}] \quad \text{and} \quad B_n := (1 + zU_n)^{-1}U_n \tag{6.1.4}$$

Then U_n is a polynomial ring in s variables over R, and z is in every maximal ideal of B_n, so $z \in \mathcal{J}(B_n)$, the Jacobson radical of B_n; see Section 2.1. Using (6.1.2), we have for each positive integer n a birational inclusion of polynomial rings $U_n \subseteq U_{n+1}$. We also have $U_{n+1} \subseteq U_n[1/z]$. By Remark 3.2.1, the element z is in $\mathcal{J}(R^*)$. Hence the localization B_n of U_n is also a subring of A and $B_n \subseteq B_{n+1}$. We define rings U and B associated to the construction:

$$U := \bigcup_{n=1}^{\infty} U_n = \bigcup_{n=1}^{\infty} R[\tau_{1n}, \dots, \tau_{sn}] \quad \text{and} \quad B := \bigcup_{n=1}^{\infty} B_n. \tag{6.1.5}$$

Thus the ring U is a nested union of polynomial rings over R, and the ring B, the *Approximation domain* for the construction, is a localization of U; we have

$$B = (1 + zU)^{-1}U \quad \text{and} \quad B \subseteq A := R^* \cap \mathcal{Q}(R[\tau_1, \dots, \tau_s]). \tag{6.1.6}$$

Remark 6.2. With the notation and setting of (6.1), the representation

$$\tau_i = \sum_{j=1}^{\infty} r_{ij}z^j$$

of τ_i as a power series in z with coefficients in R is not unique. Indeed, since $z \in R$, it is always possible to modify finitely many of the coefficients r_{ij} in this representation. It follows that the endpiece τ_{in} is also not unique. However, as we observe in Proposition 6.3 the rings U and U_n are uniquely determined by the τ_i.

PROPOSITION 6.3. *Assume the notation and setting of (6.1). Then the ring U and the rings U_n are independent of the representation of the τ_i as power series in z with coefficients in R. Hence also the ring B and the rings B_n are independent of the representation of the τ_i as power series in z with coefficients in R.*

PROOF. For $1 \le i \le s$, assume that τ_i and $\omega_i = \tau_i$ have representations

$$\tau_i := \sum_{j=1}^{\infty} a_{ij} z^j \qquad \text{and} \qquad \omega_i := \sum_{j=1}^{\infty} b_{ij} z^j,$$

where each $a_{ij}, b_{ij} \in R$. We define the n^{th}-endpieces τ_{in} and ω_{in} as in (6.1):

$$\tau_{in} = \sum_{j=n+1}^{\infty} a_{ij} z^{j-n} \qquad \text{and} \qquad \omega_{in} = \sum_{j=n+1}^{\infty} b_{ij} z^{j-n}.$$

Then we have

$$\tau_i = \sum_{j=1}^{\infty} a_{ij} z^j = \sum_{j=1}^{n} a_{ij} z^j + z^n \tau_{in} = \sum_{j=1}^{\infty} b_{ij} z^j = \sum_{j=1}^{n} b_{ij} z^j + z^n \omega_{in} = \omega_i.$$

Therefore, for $1 \le i \le s$ and each positive integer n,

$$z^n \tau_{in} - z^n \omega_{in} = \sum_{j=1}^{n} b_{ij} z^j - \sum_{j=1}^{n} a_{ij} z^j, \quad \text{and so} \quad \tau_{in} - \omega_{in} = \frac{\sum_{j=1}^{n} (b_{ij} - a_{ij}) z^j}{z^n}.$$

Thus $\sum_{j=1}^{n} (b_{ij} - a_{ij}) z^j \in R$ is divisible by z^n in R^* . Since $z^n R$ is closed in the (z)-adic topology on R, we have $z^n R = R \cap z^n R^*$. It follows that z^n divides the sum $\sum_{j=1}^{n} (b_{ij} - a_{ij}) z^j$ in R. Therefore $\tau_{in} - \omega_{in} \in R$. Thus the rings U_n and $U = \bigcup_{n=1}^{\infty} U_n$ are independent of the representation of the τ_i. Since $B_n = (1 + zU_n)^{-1} U_n$ and $B = \bigcup_{n=1}^{\infty} B_n$, the rings B_n and the ring B are also independent of the representation of the τ_i. $\square$

REMARK 6.4. Let R be a Noetherian integral domain, let $z \in R$ be a nonzero nonunit, let R^* denote the (z)-adic completion of R, and let $\tau_1, \ldots, \tau_s \in zR^*$ be algebraically independent elements over R, as in the setting of Construction 5.3.

(1) Assume in addition that R is local with maximal ideal $\mathbf{m}$. We observe in this case that the ring B defined in Equation 6.1.5 is the directed union of the localized polynomial rings $C_n := (U_n)_{P_n}$, where $P_n := (\mathbf{m}, \tau_{1n}, \ldots, \tau_{sn}) U_n$ and U_n is as defined in Equation 6.1.4. It is clear that $C_n \subseteq C_{n+1}$, and $P_n \cap (1 + zU_n) = \emptyset$ implies that $B_n \subseteq C_n$. We show that $C_n \subseteq B_{n+1}$: Let $\frac{a}{d} \in C_n$, where $a \in U_n$ and $d \in U_n \setminus P_n$. Then $d = d_0 + \sum_{i=1}^{s} \tau_{in} b_i$, where $d_0 \in R$ and each $b_i \in U_n$. Notice that $d_0 \notin \mathbf{m}$ since $d \notin P_n$, and so $d_0^{-1} \in R$. Thus

$$d d_0^{-1} = 1 + \sum_{i=1}^{s} \tau_{in} b_i d_0^{-1} \in (1 + zU_{n+1}),$$

since each $\tau_{in} \in zU_{n+1}$ by (6.1.2). Hence $\frac{a}{d} = \frac{ad_0}{dd_0} \in B_{n+1}$, and so

$$B = \bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} C_n.$$

(2) In the special case where $R = k[x]_{(x)}$, then $R^* = k[[x]]$ is the (x)-adic completion of R and $U_n = k[x, \tau_{1n}, \ldots, \tau_{sn}]$, as defined in Equation 6.1.4, using

Endpiece Notation 6.1. We have that $B = \bigcup k[x, \tau_{1n}, \dots, \tau_{sn}]_{P_n}$ where $P_n := (x, \tau_{1n}, \dots, \tau_{sn})k[x, \tau_{1n}, \dots, \tau_{sn}]$ by a proof similar to that for item 1.

(3) Let R be the localized polynomial ring $k[x, y_1, \dots, y_r]_{(x,y_1,\dots,y_r)}$ over a field k with variables $x, y_1, \dots, y_r$ and let $z = x$. Then $R^* = k[y_1, \dots, y_r]_{(y_1,\dots,y_r)}[[x]]$ is the (x)-adic completion of R. Define $U'_n = k[x, y_1, \dots, y_r, \tau_{1n}, \dots, \tau_{sn}]$ and $U' = \bigcup U'_n$, using Endpiece Notation 6.1, and use U'_n in place of U_n from Equation 6.1.4. Then the ring B of Equation 6.1.5 satisfies $B = \bigcup C_n$, where each $C_n = (U'_n)_{P_n}$ and each $P_n := (x, y_1, \dots, y_r, \tau_{1n}, \dots, \tau_{sn})U'_n$.

It is important to identify when the approximation domain B equals the intersection domain A of Construction 5.3. In Definition 6.5, we introduce the term "limit-intersecting" for this situation. We use the same term for the Homomorphic Images Construction 5.4 in Definition 6.12.

Definition 6.5. Using the notation in (6.1.3) and (6.1.5), we say Inclusion Construction 5.3 is *limit-intersecting* over R with respect to the τ_i if $B = A$. In this case, we refer to the sequence of elements $\tau_1, \dots \tau_s \in zR^*$ as *limit-intersecting* over R, or briefly, as *limit-intersecting* for A.

Example 6.6 illustrates the limit-intersecting property. Notice that Example 4.1 is a special case of Example 6.6 .

Example 6.6. Let $R = k[x]$ where k is a field and x is an indeterminate. Let

$$\tau_1 := \sum_{j=1}^{\infty} r_{1j}x^j, \qquad \dots \qquad , \tau_s := \sum_{j=1}^{\infty} r_{sj}x^j \in xk[[x]]$$

be algebraically independent over $k(x)$. By Remark 2.1, $A := k(x, \tau_1, \dots, \tau_s) \cap k[[x]]$ is a DVR with (x)-adic completion $\widehat{A} = A^* = k[[x]]$. The elements $\tau_1, \dots, \tau_s$ are limit-intersecting over R, that is, $A = B$, where B is as defined in (6.1.5). This follows from Theorem 9.2. We give the following direct proof.

Proof. Let $C_n = k[x, \tau_{1n}, \dots, \tau_{sn}]_{(x,\tau_{1n},\dots,\tau_{sn})}$. By Remark 6.4, it suffices to prove that $A = \bigcup_{n=1}^{\infty} C_n$. Notice that C_n is a RLR of dimension $s+1$, and $C_n \subseteq C_{n+1} \subseteq A$ with A birationally dominating C_{n+1} and C_{n+1} birationally dominating C_n. Since C_n is a regular local ring, the function that associates to each nonzero $f \in C_n$ the nonnegative integer $e := \operatorname{ord}_{C_n}(f)$, where f is in the e-th power of the maximal ideal of C_n, but not in the $(e+1)$-th power, defines a valuation; see Section 2.1.

Let $a \in A$. Using that A is birational over C_n and that C_n is a UFD, we can write $a = f_n/g_n$, where $f_n, g_n \in C_n$ are relatively prime. Notice that $a \in C_n$ if and only if $\operatorname{ord}_{C_n}(g_n) = 0$. Let ord_A denote the order function or *order valuation* associated to the one-dimensional regular local ring A by setting $\operatorname{ord}_A(a) = \operatorname{ord}_{A,\mathbf{m}_A}(a)$ for every $a \in A$, where $\mathbf{m}_A$ is the maximal ideal of A. Since $a \in A$ and since ord_A is a valuation, we have $\operatorname{ord}_A(a) = \operatorname{ord}_A(f_n) - \operatorname{ord}_A(g_n) \geq 0$; see Remark 2.2. Since A dominates C_n, we have $\operatorname{ord}_A(g_n) \geq 0$ and $\operatorname{ord}_A(g_n) = 0$ if and only if $\operatorname{ord}_{C_n}(g_n) = 0$. These statements hold for each of the rings C_n.

To show that $a \in \bigcup_{n=1}^{\infty} C_n$ and hence that $\bigcup_{n=1}^{\infty} C_n = A$, it suffices to show that $\operatorname{ord}_{C_m}(g_m) = 0$ for some positive integer m. If $\operatorname{ord}_A(g_n) > 0$, it suffices to show that with the representation $a = \frac{f_{n+1}}{g_{n+1}}$, where $f_{n+1}, g_{n+1} \in C_{n+1}$ are relatively prime, then $\operatorname{ord}_A(g_{n+1}) < \operatorname{ord}_A(g_n)$. By (6.1.2) we see that the maximal ideal $(x,\tau_{1n},\dots,\tau_{sn})C_n$ of C_n is contained in xC_{n+1}. Thus there exist elements f'_{n+1}, g'_{n+1} in C_{n+1} such that

$$f_n = xf'_{n+1} \text{ and } g_n = xg'_{n+1}. \quad \text{Hence} \quad a = \frac{f_{n+1}}{g_{n+1}} = \frac{f_n}{g_n} = \frac{xf'_{n+1}}{xg'_{n+1}} = \frac{f'_{n+1}}{g'_{n+1}}.$$

Since C_{n+1} is a UFD and f_{n+1} and g_{n+1} are relatively prime in C_{n+1}, we see that $g'_{n+1} = bg_{n+1}$ for some $b \in C_{n+1}$. Since A dominates C_{n+1} we have

$$\operatorname{ord}_A g_n = \operatorname{ord}_A x + \operatorname{ord}_A g'_{n+1} = 1 + \operatorname{ord}_A g'_{n+1} \geq 1 + \operatorname{ord}_A g_{n+1}.$$

This completes the proof that $\tau_1,\dots,\tau_s$ are limit-intersecting over R. $\square$

Example 6.7 extends and generalizes Example 6.6 to a situation where R is a polynomial ring in several variables over $k[x]$.

EXAMPLE 6.7. Let m be a positive integer and let $R := k[x,y_1,\dots,y_m]$ be a polynomial ring in independent variables $x,y_1,\dots,y_m$ over a field k. Let

$$\tau_1 := \sum_{j=1}^{\infty} r_{1j}x^j, \qquad \dots \qquad ,\tau_s := \sum_{j=1}^{\infty} r_{sj}x^j \in xk[[x]]$$

be algebraically independent over $k(x)$. Then $C := k(x,\tau_1,\dots,\tau_s) \cap k[[x]]$ is a DVR with (x)-adic completion $\widehat{C} = C^* = k[[x]]$ by Remark 2.1. Let R^* denote the (x)-adic completion of R, and let

$$A := k(x,\tau_1,\dots,\tau_s,y_1,\dots,y_m) \cap R^*.$$

Notice that $C[y_1,\dots,y_n] \subsetneq A$. The inclusion is proper since elements such as $1-xy_1$ are units of A, but are not units of the polynomial ring $C[y_1,\dots,y_m]$. Let $U_n := k[x,y_1,\dots,y_m,\tau_{1n},\dots,\tau_{sn}]$, where the τ_{in} are the n^{th} endpieces of the τ_i. Theorem 9.2 implies that

$$A = B := \bigcup_{n=1}^{\infty} B_n, \quad \text{where} \quad B_n := (1+xU_n)^{-1}U_n.$$

Thus the elements $\tau_1,\dots,\tau_s$ are limit-intersecting over R.

REMARK 6.8. The limit-intersecting property depends on the choice of the elements $\tau_1,\dots,\tau_s \in zR^*$. For example, if R is the polynomial ring $\mathbb{Q}[z,y]$, then $R^* = \mathbb{Q}[y][[z]]$. Let $s=1$, and let $\tau_1 = \tau := e^z - 1 \in zR^*$. Then τ is algebraically independent over $\mathbb{Q}(z,y)$. Let $U_0 = R[\tau]$. Example 6.7 shows that τ is limit-intersecting. On the other hand, the element $y\tau$ is not limit-intersecting. For if $U'_0 := R[y\tau,]$, then $\mathcal{Q}(U_0) = \mathcal{Q}(U'_0)$ and the intersection domain

$$A = \mathcal{Q}(U_0) \cap R^* = \mathcal{Q}(U'_0) \cap R^*$$

is the same for τ and $y\tau$. However the approximation domain B' associated to U'_0 does not contain τ. Indeed, $\tau \notin R[y\tau][1/z]$. Hence B' is properly contained in the approximation domain B associated to U_0. We have $B' \subsetneq B = A$ and the limit-intersecting property fails for the element $y\tau$.

6.2. Approximations for the Homomorphic Image Construction

Applying the notation from Setting 5.1 to the Homomorphic Image Construction 5.4, we again approach A using a sequence of "approximation rings" over R, but we use the *fronts* of the power series involved, rather than the ends. The approximation rings that are so obtained are not localizations of polynomial rings over R; instead they are localizations of finitely generated birational extensions within the field of fractions of R.

A goal of the computations here is to develop machinery for a proof that flatness of a certain extension implies that the integral domain A of the Homomorphic Image Construction 5.4 is Noetherian and is a localization of a subring of $R[1/z]$. This goal is realized in Theorem 8.1 of Chapter 8. For Construction 5.4 one no longer has an approximation of A by a nested union of polynomial rings over R. Indeed, in Construction 5.4 the extension $R \subseteq A$ is birational. However, there is an analogous approximation.

Frontpiece Notation 6.9. Let R be a Noetherian integral domain with field of fractions $K := \mathcal{Q}(R)$, let $z \in R$ be a nonzero nonunit, and let R^* denote the (z)-adic completion of R. Let I be an ideal of R^* having the property that $P \cap R = (0)$ for each $P \in \operatorname{Spec} R^*$ that is associated to I. As in Construction 5.4, define $A = A_{\text{hom}} := K \cap (R^*/I)$.

Since $I \subset R^*$, each $\gamma \in I$ has an expansion as a power series in z over R,

$$\gamma \ := \ \sum_{i=0}^{\infty} a_i z^i, \quad \text{where} \quad a_i \in R.$$

For each positive integer n we define the n^{th} *frontpiece* γ_n of γ with respect to this expansion:

$$\gamma_n \ := \ \sum_{j=0}^{n} \frac{a_j z^j}{z^n}.$$

Thus with $I := (\sigma_1, \dots, \sigma_t)R^*$, then for each σ_i we have

$$\sigma_i \ := \ \sum_{j=0}^{\infty} a_{ij} z^j, \qquad \text{where the} \qquad a_{ij} \ \in \ R,$$

and the n^{th} frontpiece σ_{in} of σ_i is

$$\sigma_{in} \ := \ \sum_{j=0}^{n} \frac{a_{ij} z^j}{z^n}. \tag{6.9.1}$$

For the Homomorphic Image Construction 5.4, we obtain approximating rings as follows: We define

$$U_n := R[\sigma_{1n}, \dots, \sigma_{tn}], \qquad \text{and } B_n := (1 + zU_n)^{-1} U_n. \tag{6.9.2}$$

The rings U_n and B_n are subrings of K. We observe in Proposition 6.11 that they may also be considered to be subrings of R^*/I. First we show in Proposition 6.10 that the approximating rings U_n and B_n are nested.

Proposition 6.10. *With the setting of Frontpiece Notation 6.9, for each integer $n \geq 0$ and for each integer i with $1 \leq i \leq t$, we have*

(1) $\sigma_{in} = -z a_{i,n+1} + z \sigma_{i,n+1}$.
(2) $(z, \sigma_i)R^* = (z, a_{i0})R^*$ *and hence* $(z, I)R^* = (z, a_{10}, \dots, a_{t0})R^*$.

(3) $(z, \sigma_i)R^* = (z, z^n\sigma_{in})R^*$ *and hence* $(z, I)R^* = (z, z^n\sigma_{1n}, \dots, z^n\sigma_{tn})R^*$. *Thus* $R \subseteq U_0$ *and we have* $U_n \subseteq U_{n+1}$ *and* $B_n \subseteq B_{n+1}$ *for each positive integer* n.

PROOF. For item 1, by Definition 6.9.1, we have $\sigma_{i,n+1} := \sum_{j=0}^{n+1} \frac{a_{ij}z^j}{z^{n+1}}$. Thus

$$z\sigma_{i,n+1} = \sum_{j=0}^{n+1} \frac{a_{ij}z^{j+1}}{z^{n+1}} = \sum_{j=0}^{n} \frac{a_{ij}z^j}{z^n} + za_{i,n+1} = \sigma_{in} + za_{i,n+1}.$$

For item 2, by definition

$$\sigma_i := \sum_{j=0}^{\infty} a_{ij}z^j = a_{i0} + z\big(\sum_{j=1}^{\infty} a_{ij}z^{j-1}\big).$$

For item 3, observe that

$$\sigma_i = \sum_{j=0}^{\infty} a_{ij}z^j = z^n\sigma_{in} + z^{n+1}\big(\sum_{j=n+1}^{\infty} a_{ij}z^{j-n-1}\big).$$

The asserted inclusions follow from this. $\square$

PROPOSITION 6.11. *Assume the setting of Frontpiece Notation 6.9 and let* n *be a positive integer. As an element of the total quotient ring of* R^*/I *the* n^{th} *frontpiece* σ_{in} *is the negative of the* n^{th} *endpiece of* σ_i *defined in Endpiece Notation 6.1, that is, for* $\sigma_i := \sum_{j=0}^{\infty} a_{ij}z^j$, *where each* $a_{ij} \in R$,

$$\sigma_{in} = -\sum_{j=n+1}^{\infty} \frac{a_{ij}z^j}{z^n} = -\sum_{j=n+1}^{\infty} a_{ij}z^{j-n} \qquad (\text{mod } I).$$

It follows that $\sigma_{in} \in K \cap (R^*/I)$, *and so* U_n *and* B_n *are subrings of* R^*/I.

PROOF. Let π denote the natural homomorphism from R^* onto R^*/I. Using that the restriction of π to R is the identity map on R, we have

$$\begin{aligned}
\sigma_i = z^n\sigma_{in} + \sum_{j=n+1}^{\infty} a_{ij}z^j \implies & z^n\sigma_{in} = \sigma_i - \sum_{j=n+1}^{\infty} a_{ij}z^j \\
\implies & \pi(z^n\sigma_{in}) = \pi(\sigma_i) - \pi\big(\sum_{j=n+1}^{\infty} a_{ij}z^j\big) \\
\implies & z^n\sigma_{in} = -z^n\pi\big(\sum_{j=n+1}^{\infty} a_{ij}z^{j-n}\big).
\end{aligned}$$

Therefore $z^n\sigma_{in} \in z^n\pi(R^*) = z^n(R^*/I)$. Since z is a regular element of R^*/I, we have $\sigma_{in} = -\pi(\sum_{j=n+1}^{\infty} a_{ij}z^{j-n})$ is an element of R^*/I. $\square$

DEFINITION 6.12. Assume the setting of Frontpiece Notation 6.9. We define the nested union U, the *Approximation domain* B and the *Intersection domain* A:

$$(6.12.1)\quad U := \bigcup_{n=1}^{\infty} U_n, \qquad B := \bigcup_{n=1}^{\infty} B_n = (1 + zU_n)^{-1}U_n, \qquad A := K \cap (R^*/I).$$

By Remark 3.2.1, the element z is in the Jacobson radical of R^*. Hence $B \subseteq A$. Construction 5.4 is said to be *limit-intersecting* if $B = A$.

6.3. The Inclusion Construct is a Homomorphic Image Construct

Even though they appear different, Proposition 6.11 shows for each of the power series in the ideal I that the approximation in Frontpiece Notation 6.9 is, modulo the ideal I, the negative of the approximation in Endpiece Notation 6.1.

In view of Remark 5.7.2 of Chapter 5, Construction 5.4 includes Construction 5.3 as a special case. When an Inclusion Construction 5.3 is translated to a Homomorphic Image Construction 5.4 under the correspondence described in (5.7.2), Proposition 6.14 shows that the approximating rings using frontpieces as in (6.9) correspond to the approximating rings using endpieces as in (6.1).

We revise the notation slightly so that we can view the Inclusion Construction as a Homomorphic Image Construction as in Remark 5.7.2:

Revised Notation 6.13. Let R, z, R^* and $\tau_1, \dots, \tau_s$ be as in Construction 5.3. Thus

$$\tau_i := \sum_{j=1}^{\infty} r_{ij} z^j \quad \text{where} \quad r_{ij} \in R.$$

Let $t_1, \dots, t_s$ be indeterminates over R, define $S := R[t_1, \dots, t_s]$, let S^* be the (z)-adic completion of S and let I denote the ideal $(t_1 - \tau_1, \dots, t_s - \tau_s)S^*$. Notice that $S^*/I \cong R^*$ implies that $P \cap S = (0)$ for each prime ideal $P \in \operatorname{Ass}(S^*/I)$. Thus we are in the setting of the Homomorphic Image Construction and can define $D := \mathcal{Q}(R)(t_1, \dots, t_s) \cap (S^*/I)$. Let $\sigma_i := t_i - \tau_i$, for each i with $1 \le i \le s$.

As in Frontpiece Notation 6.9, define σ_{in} to be the n^{th} frontpiece for σ_i over S. We consider Diagram 5.7.2.1, the same diagram as before, where λ is the R-algebra isomorphism that maps $t_i \to \tau_i$ for $i = 1, \dots, s$:

$$\begin{array}{ccccccc}
 & & S := R[t_1, \dots, t_s] & \longrightarrow & D := \mathcal{Q}(R)(t_1, \dots, t_s) \cap (S^*/I) & \longrightarrow & S^*/I \\
 & & {\scriptstyle\lambda}\big\downarrow & & {\scriptstyle\lambda}\big\downarrow & & {\scriptstyle\lambda}\big\downarrow \\
R & \longrightarrow & S' := R[\tau_1, \dots, \tau_s] & \longrightarrow & A := \mathcal{Q}(R)(\tau_1, \dots, \tau_s) \cap R^* & \longrightarrow & R^*.
\end{array}$$

As before, λ maps D isomorphically onto A via $\lambda(t_i) = \tau_i$ for every i with $1 \le i \le s$. Define V_n, C_n, V, C in analogy to U_n, B_n, U, B in Frontpiece Notation 6.9 and Equation 6.12.1 over S, that is, with S and S^* in place of R and R^*. Define U_n, B_n, U, B using Endpiece Notation 6.1 and Equations 6.1.4 and 6.1.5 over R. Then

$$\begin{aligned}
&V_n := S[\sigma_{1n}, \dots, \sigma_{sn}] = R[t_1, \dots, t_s][\sigma_{1n}, \dots, \sigma_{sn}], \\
&C_n := (1 + zV_n)^{-1} V_n, \\
(6.1) \qquad &V := \bigcup_{n=1}^{\infty} V_n, \qquad C := \bigcup_{n=1}^{\infty} C_n = (1 + zV)^{-1} V \\
&U := \bigcup_{n=1}^{\infty} U_n = \bigcup_{n=1}^{\infty} R[\tau_{1n}, \dots, \tau_{sn}] \quad \text{and} \quad B := \bigcup_{n=1}^{\infty} B_n = (1 + zU)^{-1} U.
\end{aligned}$$

Proposition 6.14. *With the setting of Revised Notation 6.13, the R-algebra isomorphism λ has the following properties:*

$$\lambda(D) = A, \quad \lambda(\sigma_{in}) = \tau_{in}, \quad \lambda(V_n) = U_n, \quad \lambda(C_n) = B_n, \quad \lambda(V) = U, \quad \lambda(C) = B,$$

for all i with $1 \le i \le s$ and all $n \in \mathbb{N}$.

PROOF. We have elements $r_{ij} \in R$ so that

$$\begin{aligned} \tau_i &:= \sum_{j=1}^{\infty} r_{ij} z^j, \qquad \tau_{in} \ := \sum_{j=n+1}^{\infty} r_{ij} z^{j-n} \\ \sigma_i &:= t_i - \tau_i = t_i - \sum_{j=1}^{\infty} r_{ij} z^j, \qquad \sigma_{in} \ := \ \frac{t_i - \sum_{j=1}^{n} r_{ij} z^j}{z^n} \\ \implies \lambda(\sigma_{in}) &= \frac{\tau_i - \sum_{j=1}^{n} r_{ij} z^j}{z^n} \\ &= \tau_{in}. \end{aligned}$$

The remaining statements of Proposition 6.14 now follow. □

REMARK 6.15. With the setting of Revised Notation 6.13, since U_n is a polynomial ring over R in the variables $\tau_{1n}, \dots \tau_{sn}$, Proposition 6.14 implies that V_n is a polynomial ring over R in the variables $\sigma_{1n}, \dots, \sigma_{sn}$. Thus

$$V_n := S[\sigma_{1n}, \dots, \sigma_{sn}] = R[t_1, \dots, t_s][\sigma_{1n}, \dots, \sigma_{sn}] = R[\sigma_{1n}, \dots, \sigma_{sn}].$$

6.4. Basic properties of the approximation domains

The following general lemma is useful in Theorem 6.17 to establish relationships among rings that arise in the Homomorphic Image Construction 5.7 and the approximations in Section 6.2.

LEMMA 6.16. *Let S be a subring of a ring T and let $x \in S$ be a regular element of T such that $\bigcap_{n=1}^{\infty} x^n T = (0)$. The following conditions are equivalent.*

(1) (i) $xS = xT \cap S$ *and* (ii) $S/xS = T/xT$.
(2) *For each positive integer n we have:*
$x^n S = x^n T \cap S, \quad S/x^n S = T/x^n T \quad$ *and* $\quad T = S + x^n T$.
(3) *The rings S and T have the same (x)-adic completion.*
(4) (i) $S = S[1/x] \cap T$, *and* (ii) $T[1/x] = S[1/x] + T$.

PROOF. To see that item 1 implies item 2, observe that

$$x^n T \cap S = x^n T \cap xS = x(x^{n-1} T \cap S),$$

so the equality $x^{n-1} S = x^{n-1} T \cap S$ implies the equality $x^n S = x^n T \cap S$. Moreover $S/xS = T/xT$ implies $T = S + xT = S + x(S + xT) = \cdots = S + x^n T$, so $S/x^n S = T/x^n T$ for every $n \in \mathbb{N}$. Therefore (1) implies (2).

It is clear that item 2 is equivalent to item 3.

To see that item 2 implies (4i), let $s/x^n \in S[1/x] \cap T$ with $s \in S$ and $n \geq 0$. Item 2 implies that $s \in x^n T \cap S = x^n S$ and therefore $s/x^n \in S$. To see (4ii), let $\frac{t}{x^n} \in T[1/x]$ with $t \in T$ and $n \geq 0$. Item 2 implies that $t = s + x^n t_1$ for some $s \in S$ and $t_1 \in T$. Therefore $\frac{t}{x^n} = \frac{s}{x^n} + t_1$. Thus (2) implies (4).

It remains to show that item 4 implies item 1. To see that (4) implies (1i), let $t \in T$ and $s \in S$ be such that $xt = s$. Then $t = s/x \in S[1/x] \cap T = S$, by (4i). Thus $xt \in xS$. To see that (4) implies (1ii), let $t \in T$. Then $\frac{t}{x} = \frac{s}{x^n} + t'$, for some $n \in \mathbb{N}$, $s \in S$ and $t' \in T$ by (4ii). Thus $t = \frac{s}{x^{n-1}} + t'x$. Hence $t - t'x = \frac{s}{x^{n-1}} \in S[1/x] \cap T = S$, by (4ii). □

Theorem 6.17 relates to the analysis of the Homomorphic Image Construction. This analysis is useful for the development of examples in later chapters.

Construction Properties Theorem 6.17. (Homomorphic Image Version) *Let R be a Noetherian integral domain with field of fractions $K := \mathcal{Q}(R)$, let $z \in R$ be a nonzero nonunit, and let R^* denote the (z)-adic completion of R. Let I be an ideal of R^* having the property that $P \cap R = (0)$ for each $P \in \operatorname{Spec} R^*$ that is associated to I. With the notation of Frontpiece Notation 6.9 and Definition 6.12, we have for each positive integer n:*

(1) *The ideals of R containing z^n are in one-to-one inclusion preserving correspondence with the ideals of R^* containing z^n. In particular, we have $(I,z)R^* = (a_{10},\dots,a_{t0},z)R^*$ and*
$$(I,z)R^* \cap R = (a_{10},\dots,a_{t0},z)R^* \cap R = (a_{10},\dots,a_{t0},z)R.$$
(2) *The ideal $(a_{10},\dots,a_{t0},z)R$ equals $z(R^*/I) \cap R$ under the identification of R as a subring of R^*/I, and the element z is in the Jacobson radical of both R^*/I and B.*
(3) $z^n(R^*/I) \cap A = z^nA, \qquad z^n(R^*/I) \cap U = z^nU, \qquad z^n(R^*/I) \cap B = z^nB.$
(4) $U/z^nU = B/z^nB = A/z^nA = R^*/(z^nR^* + I)$. *The rings A, U and B all have (z)-adic completion R^*/I, that is, $A^* = U^* = B^* = R^*/I$.*
(5) $R[1/z] = U[1/z]$, $U = R[1/z] \cap B = R[1/z] \cap A = R[1/z] \cap (R^*/I)$ *and the integral domains R, U, B and A all have the same field of fractions K.*

Proof. The first assertion of item 1 follows because R/z^nR is canonically isomorphic to R^*/z^nR^*. The next assertion of (1) follows from part 2 of Proposition 6.10. If $\gamma = \sum_{i=1}^t \sigma_i\beta_i + z\tau \in (I,z)R^* \cap R$, where $\tau, \beta_i \in R^*$, then write each $\beta_i = b_i + z\beta_i'$, where $b_i \in R$, $\beta_i' \in R^*$. Thus $\gamma - \sum_{i=1}^t a_{i0}b_i \in zR^* \cap R = zR$, and so $\gamma \in (a_{10},\dots,a_{t0},z)R$.

Since $z(R^*/I) = (z,I)R^*/I$, we have $(a_{10},\dots,a_{t0},z)R \subseteq z(R^*/I) \cap R$. The reverse inclusion in item 2 follows from $(I,z)R^* = (a_{10},\dots,a_{t0},z)R^*$. For the last part of item 2, we have that $z \in \mathcal{J}(R^*)$ and so $1 + az$ is outside every maximal ideal of R^* for every $a \in R^*$. Thus $z \in \mathcal{J}(R^*/I)$. By the definition of B in Equation 6.12.1, $z \in \mathcal{J}(B)$.

The first assertion of item 3 follows from the definition of A as $(R^*/I) \cap K$. To see that $z(R^*/I) \cap U \subseteq zU$, let $g \in z(R^*/I) \cap U$. Then $g \in U_n$, for some n, implies $g = r_0 + g_0$, where $r_0 \in R$, $g_0 \in (\sigma_{1n},\dots,\sigma_{tn})U_n$. Also $\sigma_{in} = -za_{i,n+1} + z\sigma_{i,n+1}$, and so $g_0 \in zU_{n+1} \subseteq z(R^*/I)$. Now $r_0 \in (z,\sigma_1,\dots,\sigma_t)R^* = (I,z)R^*$. Thus by item 1, $r_0 \in (a_{10},\dots,a_{t0},z)R$. Also each $a_{i0} = \sigma_i - z\sum_{j=1}^\infty a_{ij}z^{j-1} \in zU$ because $a_{i0} = z\sigma_{i1} - za_{i1}$.

Thus $r_0 \in zU$, as desired. This proves that $z^n(R^*/I) \cap U = z^nU$. Since $B = (1+zU)^{-1}U$, we also have $z^n(R^*/I) \cap B = z^nB$.

Item 4 follows from item 3 and Lemma 6.16.

For (5): if $g \in U$, then $g \in U_n$, for some n. Clearly each $\sigma_{in} \in R[1/z]$, and so $g \in R[1/z]$. To see that $U = R[1/z] \cap B$, apply Lemma 6.16 with $S = U$, $B = T$. Similarly we see that $U = R[1/z] \cap A$, since
$$R[1/z] \cap A = R[1/z] \cap (\mathcal{Q}(R) \cap (R^*/I)) = R[1/z] \cap (R^*/I).$$
It is clear that the integral domains R, U, B and A all have the same field of fractions K. $\square$

Remark 6.18. We note the following implications from Theorem 6.17 .

(1) Item 5 of Theorem 6.17 implies that the definitions in (6.12.1) of B and U are independent of

   (a) the choice of generators for I, and
   (b) the representation of the generators of I as power series in z.

(2) Item 5 of Theorem 6.17 implies that the rings $U = R[1/z] \cap (R^*/I)$ and $B = (1 + zU)^{-1}U$ are uniquely determined by z and the ideal I of R^*.

(3) Since z is in the Jacobson radical of B, item 4 of Theorem 6.17 implies that if $b \in B$ is a unit of A, then b is already a unit of B.

(4) We have the following diagram displaying the relationships among the rings.

$$
\begin{tikzcd}
\mathcal{Q}(R) \arrow[r, equal] & \mathcal{Q}(U) \arrow[r, equal] & \mathcal{Q}(B) \arrow[r, equal] & \mathcal{Q}(A) \arrow[r, "\subseteq"] & \mathcal{Q}(R^*/I) \\
R[1/z] \arrow[u] \arrow[r, equal] & U[1/z] \arrow[u] \arrow[r, "\subseteq"] & B[1/z] \arrow[u] \arrow[r, "\subseteq"] & A[1/z] \arrow[u] \arrow[r, "\subseteq"] & (R^*/I)[1/z] \arrow[u] \\
R \arrow[u] \arrow[r, "\subseteq"] & U = \cup\, U_n \arrow[u] \arrow[r, "\subseteq"] & B \arrow[u] \arrow[r, "\subseteq"] & A \arrow[u] \arrow[r, "\subseteq"] & R^*/I. \arrow[u]
\end{tikzcd}
$$

We record in Construction Properties Theorem 6.19 (Inclusion Version) some basic properties of the integral domains associated with Inclusion Construction 5.3 that follow from Construction Properties Theorem 6.17 (Homomorphic Image Version), Remark 6.18, Proposition 6.3 and Proposition 6.14.

Construction Properties Theorem 6.19. (Inclusion Version) *Let R be a Noetherian integral domain with field of fractions $K := \mathcal{Q}(R)$, let $z \in R$ be a nonzero nonunit, and let R^* denote the (z)-adic completion of R. Let $\underline{\tau} = \{\tau_1, \dots, \tau_s\}$ be a set of elements of R^* that are algebraically independent over K, so that $R[\underline{\tau}]$ is a polynomial ring in s variables over R. Define $A = A_{inc} := K(\underline{\tau}) \cap R^*$, as in Inclusion Construction 5.3. Let U_n, B_n, B and U be defined as in Equations 6.1.4 and 6.1.5. Then:*

(1) *$z^nR^* \cap A = z^nA$, $\quad z^nR^* \cap B = z^nB \quad$ and $\quad z^nR^* \cap U = z^nU$, for each $n \in N$.*

(2) *$R[\underline{\tau}][1/z] = U[1/z]$, $\quad U = R[\underline{\tau}][1/z] \cap B = R[\underline{\tau}]1/z] \cap A$, and $B[1/z]$ is a localization of $R[\underline{\tau}]$. The integral domains $R[\underline{\tau}], U, B$ and A all have the same field of fractions, namely $K(\underline{\tau})$.*

(3) *$R/z^nR = U/z^nU = B/z^nB = A/z^nA = R^*/z^nR^*$, for each $n \in N$.*

(4) *The (z)-adic completions of U, B and A are all equal to R^*, that is, $R^* = U^* = B^* = A^*$.*

(5) *The definitions in Equation 6.1.5 of B and U are independent of the representations given in Notation 6.1 for the τ_i as power series in R^*.*

(6) *If $(R, \mathbf{m})$ is local, then B is local and, in view of Remark 6.4.1, we have*

$$B \;=\; (1 + zU)^{-1}U \;=\; \bigcup_{n=1}^{\infty} (U_n)_{(\mathbf{m}, \tau_{1n}, \dots, \tau_{sn})U_n}.$$

Proposition 6.20 concerns the extension to R^* of a prime ideal of either A or B that does not contain z, and provides information about the maps from $\operatorname{Spec} R^*$ to $\operatorname{Spec} A$ and to $\operatorname{Spec} B$. We use Proposition 6.20 in Chapters 7 and 17.

Proposition 6.20. *With the notation of Construction Properties Theorem 6.19 we have:*

(1) *z is in the Jacobson radical of B, in the Jacobson radical of A, and in the Jacobson radical of R^*.*
(2) *Let q be a prime ideal of R. Then*
 (a) *qU is a prime ideal in U.*
 (b) *Either $qB = B$ or qB is a prime ideal of B.*
 (c) *If $qB \neq B$, then $qB \cap U = qU$ and $U_{qU} = B_{qB}$.*
 (d) *If $z \notin q$, then $qU \cap U_n = qU_n$ and $U_{qU} = (U_n)_{qU_n}$.*
 (e) *If $z \notin q$ and $qB \neq B$, then $qB \cap B_n = qB_n$ and*
$$(U_n)_{qU_n} \ = \ U_{qU} \ = \ B_{qB} \ = \ (B_n)_{qB_n} \ .$$
(3) *Let I be an ideal of B or of A and let $t \in \mathbb{N}$. Then $z^t \in IR^* \iff z^t \in I$.*
(4) *Let $P \in \operatorname{Spec} B$ or $P \in \operatorname{Spec} A$ with $z \notin P$. Then z is a nonzerodivisor on R^*/PR^*. Thus $z \notin Q$ for each associated prime of the ideal PR^*. Since z is in the Jacobson radical of R^*, it follows that PR^* is contained in a nonmaximal prime ideal of R^*.*
(5) *If R is local, then R^*, A and B are local. Let $\mathbf{m}_R$, $\mathbf{m}_{R^*}, \mathbf{m}_A$ and $\mathbf{m}_B$ denote the maximal ideals of R, R^*, A and B, respectively. In this case*
 - *$\mathbf{m}_B = \mathbf{m}_R B$, $\mathbf{m}_A = \mathbf{m}_R A$ and each prime ideal P of B such that $\operatorname{ht}(\mathbf{m}_B/P) = 1$ is contracted from R^*.*
 - *Let I be an ideal of B. Then IR^* is primary for $\mathbf{m}_{R^*} \iff I$ is primary for $\mathbf{m}_B$. In this case, $IR^* \cap B = I$ and $B/I \cong R^*/IR^*$.*

Proof. For item 1, since $B_n = (1 + zU_N)^{-1}U_n$, it follows that $1 + zb$ is a unit of B_n for each $b \in B_n$. Therefore z is in the Jacobson radical of B_n for each n and thus z is in the Jacobson radical of B. The statement for R^* follows from Remark 3.2.1.

For item 2, since each U_n is a polynomial ring over R, the ideal qU_n is a prime ideal of U_n and thus $qU = \bigcup_{n=0}^{\infty} qU_n$ is a prime ideal of U. Since B is a localization of U, either $qB = B$, or qB is a prime ideal of B such that $qB \cap U = qU$ and $U_{qU} = B_{qB}$.

For part d of item 2, since $U_n[1/z] = U[1/z]$ and the ideals qU_n and qU are prime ideals in U_n and U that do not contain z, the localizations $(U_n)_{qU_n}$ and U_{qU} are both further localizations of $U[1/z]$. Moreover, they both equal $U[1/z]_{qU[1/z]}$. Thus we have $U_{qU} = (U_n)_{qU_n}$. Since $U_n \subset U$, we also have $qU \cap U_n = qU_n$. Since B is a localization of U, the assertions in part e follow as in the proof of part d.

To see item 3, let I be an ideal of B. The proof for A is identical. We observe that there exist elements $b_1, \ldots, b_s \in I$ such that $IR^* = (b_1, \ldots, b_s)R^*$. If $z^t \in IR^*$, there exist $\alpha_i \in R^*$ such that
$$z^t \ = \ \alpha_1 b_1 + \cdots + \alpha_s b_s.$$
We have $\alpha_i = a_i + z^{t+1}\lambda_i$ for each i, where $a_i \in B$ and $\lambda_i \in R^*$. Thus
$$z^t[1 - z(b_1\lambda_1 + \cdots + b_s\lambda_s)] \ = \ a_1b_1 + \cdots + a_sb_s \in B \cap z^tB^* = z^tB.$$
Therefore $\gamma := 1 - z(b_1\lambda_1 + \cdots + b_s\lambda_s) \in B$. Thus $z(b_1\lambda_1 + \cdots + b_s\lambda_s) \in B \cap zR^* = zB$, and so $b_1\lambda_1 + \cdots + b_s\lambda_s \in B$. By item 1, the element z is in the Jacobson radical of B. Hence γ is invertible in B. Since $\gamma z^t \in (b_1, \cdots, b_s)B$, it follows that $z^t \in I$. If $z^t \in I$, then $z^t \in IR^*$. This proves item 3.

For item 4, assume that $P \in \operatorname{Spec} B$. The proof for $P \in \operatorname{Spec} A$ is identical. We have that

$$P \cap zB = zP \quad \text{ and so } \quad \frac{P}{zP} = \frac{P}{P \cap zB} \cong \frac{P + zB}{zB}$$

By Construction Properties Theorem 6.19.3, B/zB is Noetherian. Hence the B-module P/zP is finitely generated. Let $g_1, \dots, g_t \in P$ be such that $P = (g_1, \dots, g_t)B + zP$. Then also $PR^* = (g_1, \dots, g_t)R^* + zR^* = (g_1, \dots, g_t)R^*$, the last equality by Nakayama's Lemma.

Let $\widehat{f} \in R^*$ be such that $z\widehat{f} \in PR^*$. We show that $\widehat{f} \in PR^*$.

Since $\widehat{f} \in R^*$, we have $\widehat{f} := \sum_{i=0}^{\infty} c_i z^i$, where each $c_i \in R$. For each $m > 1$, let $f_m := \sum_{i=0}^{m} c_i z^i$, the first $m+1$ terms of this expansion of $\widehat{f}$. Then $f_m \in R \subseteq B$ and there exists an element $\widehat{h}_1 \in R^*$ so that.

$$\widehat{f} = f_m + z^{m+1}\widehat{h_1}.$$

Since $z\widehat{f} \in PR^*$, we have

$$z\widehat{f} = \widehat{a_1}g_1 + \cdots + \widehat{a_t}g_t,$$

where $\widehat{a_i} \in R^*$. The $\widehat{a_i}$ have power series expansions in z over R, and thus there exist elements $a_{im} \in R$ such that $\widehat{a_i} - a_{im} \in z^{m+1}R^*$. Thus

$$z\widehat{f} = a_{1m}g_1 + \cdots + a_{tm}g_t + z^{m+1}\widehat{h_2},$$

where $\widehat{h}_2 \in R^*$, and

$$zf_m = a_{1m}g_1 + \cdots + a_{tm}g_t + z^{m+1}\widehat{h_3},$$

where $\widehat{h_3} = \widehat{h_2} - z\widehat{h_1} \in R^*$. Since the g_i are in B, we have $z^{m+1}\widehat{h_3} \in z^{m+1}R^* \cap B = z^{m+1}B$, the last equality by Construction Properties Theorem 6.19.1. Therefore $\widehat{h_3} \in B$. Rearranging the last displayed equation above, we obtain

$$z(f_m - z^m\widehat{h_3}) = a_{1m}g_1 + \cdots + a_{tm}g_t \in P.$$

Since $z \notin P$, we have $f_m - z^m\widehat{h_3} \in P$. It follows that $\widehat{f} \in P + z^mR^* \subseteq PR^* + z^mR^*$, for each $m > 1$. Hence we have that $\widehat{f} \in PR^*$, as desired.

For item 5, if R is local, then B is local, A is local, $\mathbf{m}_B = \mathbf{m}_R B$ and $\mathbf{m}_A = \mathbf{m}_R A$ since $R/zR = B/zB = A/zA = R^*/zR^*$ and z is in the Jacobson radical of B and of A. If $z \notin P$, then item 4 implies that no power of z is in PR^*. Hence PR^* is contained in a prime ideal Q of R^* that does not meet the multiplicatively closed set $\{z^n\}_{n=1}^{\infty}$. Thus $P \subseteq Q \cap B \subsetneq \mathbf{m}_B$. Since $\operatorname{ht}(\mathbf{m}_B/P) = 1$, we have $P = Q \cap B$, so P is contracted from R^*. If $z \in P$, then $B/zB = R^*/zR^*$ implies that PR^* is a prime ideal of R^* and $P = PR^* \cap B$.

For the second part of item 5, let I be an ideal of B. By item 3, for each $t \in \mathbb{N}$, we have $z^t \in IR^* \iff z^t \in I$. If either IR^* is $\mathbf{m}_{R^*}$-primary or I is $\mathbf{m}_B$-primary, then $z^t \in I$ for some $t \in \mathbb{N}$. By Theorem 6.19.3, $B/z^tB = R^*/z^tR^*$. Hence the $\mathbf{m}_B$-primary ideals containing z^t are in one-to-one inclusion preserving correspondence with the $\mathbf{m}_{R^*}$-primary ideals that contain z^t. This completes the proof of item 5. $\square$

In many of the examples constructed in this book, the ring R is a polynomial ring (or a localized polynomial ring) in finitely many variables over a field; such rings are UFDs. We observe in Theorem 6.21 that the constructed ring B is a UFD if R is a UFD and z is a prime element.

THEOREM 6.21. *With the notation of Construction Properties Theorem 6.19:*

(1) *If R is a UFD and z is a prime element of R, then zB is a prime ideal, $B[1/z]$ is a Noetherian UFD and B is a UFD.*
(2) *If in addition R is regular, then $B[1/z]$ is a regular Noetherian UFD.*

PROOF. By Proposition 6.20.2, zB is a prime ideal. Since R is a Noetherian UFD and $R[\underline{\tau}]$ is a polynomial ring in finitely many variables over R, it follows that $R[\underline{\tau}]$ is a Noetherian UFD. By Theorem 6.19.2, the ring $B[1/z]$ is a localization of $R[\underline{\tau}]$ and thus is a Noetherian UFD; moreover $B[1/z]$ is regular if R is. By Theorem 6.19.4, The (z)-adic completion of B is R^* . By Proposition 6.20.1, z is in the Jacobson radical of R^*. Since R^* is Noetherian, it follows that $\bigcap_{n=1}^{\infty} z^n R^* = (0)$. Thus $\bigcap_{n=1}^{\infty} z^n B = (0)$. It follows that B_{zB} is a DVR [**104**, (31.5)].

By Fact 2.11, we have $B = B[1/z] \cap B_{zB}$. Therefore B is a Krull domain. Since $B[1/z]$ is a UFD and B is a Krull domain, Theorem 2.10 implies that B is a UFD. $\square$

Exercises

(1) Prove item 2 of Remark 6.4, that is, with R a polynomial ring $k[x]$ over a field k and $R^* = k[[x]]$ the (x)-adic completion of R, show that with the notation of Construction 5.3 we have $B = \bigcup C_n$, where $C_n = (U_n)_{P_n}$ and $P_n := (x, \tau_{1n}, \dots, \tau_{sn})U_n$. Thus B is a DVR that is the directed union of a birational family of localized polynomial rings in $n+1$ indeterminates.
(2) Assume the setting of Frontpiece Notation 6.9 and Definition 6.12. If J is a proper ideal of B, prove that JB^* is a proper ideal of B^*, where B^* is the (z)-adic completion of B.
(3) Assume the setting of Frontpiece Notation 6.9, and let W denote the set of elements of R^* that are regular on R^*/I. Prove that the natural homomorphism $\pi : R^* \to R^*/I$ extends to a homomorphism $\overline{\pi} : W^{-1}R^* \to W^{-1}(R^*/I)$.

CHAPTER 7

Revisiting the iterative examples

In this chapter we continue the proof of Theorem 4.12 stated in Chapter 4. We give specific examples to illustrate possible properties of the Inclusion and Homomorphic Constructions 5.6 and 5.7 of Chapter 5. We focus on the iterative examples of Section 4.3.

7.1. Properties of the iterative examples

We begin by fixing notation. We include several remarks concerning the integral domains that are used in the proof of Theorem 4.12.

Notation and Remarks 7.1. Let k be a field, let x and y be indeterminates over k, and let

$$\sigma := \sum_{i=1}^{\infty} a_i x^i \in xk[[x]] \qquad \text{and} \qquad \tau := \sum_{i=1}^{\infty} b_i y^i \in yk[[y]]$$

be formal power series that are algebraically independent over the fields $k(x)$ and $k(y)$, respectively. Let $R := k[x,y]_{(x,y)}$, and let σ_n, τ_n be the n^{th} endpieces of σ, τ defined as in (6.1). Define

$$\begin{aligned}
C_n &:= k[x,\sigma_n]_{(x,\sigma_n)}, \quad C \ := \ k(x,\sigma) \cap k[[x]] \ = \ \cup_{n=1}^{\infty} C_n \ ; \\
D_n &:= k[y,\tau_n]_{(y,\tau_n)}, \quad D \ := \ k(y,\tau) \cap k[[y]] \ = \ \cup_{n=1}^{\infty} D_n; \\
U_n &:= k[x,y,\sigma_n,\tau_n], \qquad U : = \ \cup_{n=1}^{\infty} U_n; \\
B_n &:= k[x,y,\sigma_n,\tau_n]_{(x,y,\sigma_n,\tau_n)} \quad B : = \ \cup_{n=1}^{\infty} B_n; \\
A \ &:= k(x,y,\sigma,\tau) \ \cap \ k[[x,y]].
\end{aligned} \tag{7.1.0}$$

(i) Since $k[[x,y]]$ is the (x,y)-adic completion of the Noetherian ring R, Remark 3.2.4 implies that $k[[x,y]]$ is faithfully flat over R. By Remark 2.21.7

$$(x,y)^n k[[x,y]] \ \cap \ R \ = \ (x,y)^n R$$

for each $n \in \mathbb{N}$. Equation 6.1.2 implies for each positive integer n the inclusions

$$C_n \ \subset \ C_{n+1}, \quad D_n \ \subset \ D_{n+1}, \quad \text{and} \quad B_n \ \subset \ B_{n+1}.$$

Moreover, for each of these inclusions we have birational domination of the larger local ring over the smaller, and the local rings C_n, D_n, B_n are all dominated by $k[[x,y]] = \widehat{R}$.

(ii) We observe that $(x,y)B \cap B_n = (x,y,\sigma_n,\tau_n)B_n$, for each $n \in \mathbb{N}$: To see this, let $h \in (x,y,\sigma_n,\tau_n)U_n$. Then Equation 6.1.2 implies that

$$\sigma_n \ = \ -xa_{n+1} + x\sigma_{n+1} \quad \text{and} \quad \tau_n = -yb_{n+1} + y\tau_{n+1}.$$

Hence $h \in (x,y)U_{n+1} \cap U_n \subseteq (x,y)U \cap U_n$. Since $(x,y,\sigma_n,\tau_n)U_n$ is a maximal ideal of U_n contained in $(x,y)U$, a proper ideal of U, it follows that $(x,y)U \cap U_n = (x,y,\sigma_n,\tau_n)U_n$. Thus also $(x,y)B \cap B_n = (x,y,\sigma_n,\tau_n)B$ for each $n \in \mathbb{N}$.

(iii) We also observe that $U_n[\frac{1}{xy}] = U[\frac{1}{xy}]$ for each $n \in \mathbb{N}$: By Equation 6.1.2 we have $\sigma_{n+1} \in U_n[\frac{1}{x}] \subseteq U_n[\frac{1}{xy}]$ and $\tau_{n+1} \in U_n[\frac{1}{y}] \subseteq U_n[\frac{1}{xy}]$. Hence $U_{n+1} \subseteq U_n[\frac{1}{xy}]$ for each $n \in \mathbb{N}$. Hence $U \subseteq U_n[\frac{1}{xy}]$, and $U_n[\frac{1}{xy}] = U[\frac{1}{xy}]$.

(iv) By Remark 2.1, the rings C and D are rank-one discrete valuation domains; as in Example 6.6, they are the asserted directed unions. The rings B_n are four-dimensional regular local domains that are localized polynomial rings over the field k. Thus B is the directed union of a chain of four-dimensional regular local rings, with each ring birational over the previous ring.

We prove in Theorem 7.2 other parts of Theorem 4.12.

Theorem 7.2. *Assume the setting of Notation 7.1. Then the ring A is a two-dimensional regular local domain that birationally dominates the ring B; A has maximal ideal $(x,y)A$ and completion $\widehat{A} = k[[x,y]]$. Moreover we have:*

(1) *The rings U and B are UFDs,*
(2) *B is a local Krull domain with maximal ideal $\mathbf{n} = (x,y)B$,*
(3) *The dimension of B is either 2 or 3, depending on the choice of σ and τ,*
(4) *B is Hausdorff in the topology defined by the powers of $\mathbf{n}$,*
(5) *The $\mathbf{n}$-adic completion $\widehat{B}$ of B is canonically isomorphic to $k[[x,y]]$, and*
(6) *The following statements are equivalent:*
 (a) *$B = A$.*
 (b) *B is a two-dimensional regular local domain.*
 (c) *B is Noetherian.*
 (d) *Every finitely generated ideal of B is closed in the $\mathbf{n}$-adic topology on B.*
 (e) *Every principal ideal of B is closed in the $\mathbf{n}$-adic topology on B.*

Proof. The assertions about A follow from Proposition 4.13, a consequence of Valabrega's Theorem 4.12. Since U_0 has field of fractions $k(x,y,\sigma,\tau) = \mathcal{Q}(A)$ and $U_0 \subseteq B \subseteq A$, the extension $B \hookrightarrow A$ is birational. By Equation 7.1.0 and Remarks 7.1.iv and 7.1.ii, B is the directed union of the four-dimensional regular local domains B_n and $(x,y)B \cap B_n = (x,y,\sigma_n,\tau_n)B$ for each $n \in \mathbb{N}$. Thus B is local with maximal ideal $\mathbf{n} = (x,y)B$. Since B and A are both dominated by $k[[x,y]]$, it follows that A dominates B.

We now prove that U and B are UFDs. By Equation 7.1.0 and Remark 7.1.iii, U_n is a polynomial ring over a field and $U_n[\frac{1}{xy}] = U[\frac{1}{xy}]$. Thus the ring $U[\frac{1}{xy}]$ is a UFD. For each $n \in \mathbb{N}$, the principal ideals xU_n and yU_n are prime ideals in the polynomial ring U_n. Therefore xU and yU are principal prime ideals of U. Moreover, $U_{xU} = B_{xB}$ and $U_{yU} = B_{yB}$ are DVRs since each is the contraction to the field $k(x,y,\sigma,\tau)$ of the (x)-adic or the (y)-adic valuations of $k[[x,y]]$; see Remark 2.2.

By applying Fact 2.11 with $S = U$ and $c = x$, and then with $S = U[\frac{1}{x}]$ and $c = y$, we obtain $U = U_{xU} \cap U_{yU} \cap U[\frac{1}{xy}]$. Since $U[\frac{1}{xy}] = U_n[\frac{1}{xy}]$, we have $U[\frac{1}{xy}]$ is a Krull domain, and so also U is a Krull domain; see Definition 2.3.2 and the comments there. Hence, by Nagata's Theorem 2.10, U is a UFD. Since B is a localization of U, the ring B is a UFD. This completes the proof of items 1 and 2.

Since B is dominated by $k[[x,y]]$, the intersection $\bigcap_{n=1}^{\infty} \mathbf{n}^n = (0)$. Thus B is Hausdorff in the topology defined by the powers of $\mathbf{n}$ [**108**, Proposition 4, page 381], as in Definitions 3.1. We have local injective maps $R \hookrightarrow B \hookrightarrow \widehat{R}$, and we have $\mathbf{m}^n B = \mathbf{n}^n$, $\mathbf{m}^n \widehat{R} = \widehat{\mathbf{m}}^n$ and $\widehat{\mathbf{m}}^n \cap R = \mathbf{m}^n$, for each positive integer n. Since the natural map $R/\mathbf{m}^n \to \widehat{R}/\mathbf{m}^n\widehat{R} = \widehat{R}/\widehat{\mathbf{m}}^n$ is an isomorphism, the map $R/\mathbf{m}^n \to B/\mathbf{m}^n B = B/\mathbf{n}^n$ is injective and the map $B/\mathbf{n}^n \to \widehat{R}/\mathbf{n}^n\widehat{R} = \widehat{R}/\widehat{\mathbf{m}}^n$ is surjective. Since $B/\mathbf{n}^n$ has finite length as an R-module, it follows that $R/\mathbf{m}^n \cong B/\mathbf{n}^n \cong \widehat{R}/\widehat{\mathbf{m}}^n$ for each $n \in \mathbb{N}$. Hence $\widehat{B} = \widehat{R} = k[[x,y]]$. Notice that B is a birational extension of the three-dimensional Noetherian domain $C[y,\tau]$. The dimension of B is at most 3 by Theorem 2.9, a theorem of Cohen, also see [**96**, Theorem 15.5].

This completes the proof of items 3, 4 and 5.

For item 6, by Proposition 4.13, we have A is a two-dimensional RLR. Thus $(a) \implies (b)$. Clearly $(b) \implies (c)$. Since B is local by item 2, and since the completion of a Noetherian local ring is a faithfully flat extension by Remark 3.2.4, we have $(c) \implies (d)$ by Remark 2.21.7. It is clear that $(d) \implies (e)$. To complete the proof of Theorem 7.2, it suffices to show that $(e) \implies (a)$. Since A birationally dominates B, we have $B = A$ if and only if $bA \cap B = bB$ for every element $b \in \mathbf{n}$; see Exercise 2.ii of Chapter 4. The principal ideal bB is closed in the $\mathbf{n}$-adic topology on B if and only if $bB = b\widehat{B} \cap B$. Also $\widehat{B} = \widehat{A}$ and $bA = b\widehat{A} \cap A$, for every $b \in B$. Thus (e) implies, for every $b \in B$,

$$bB = b\widehat{B} \cap B = b\widehat{A} \cap B = b\widehat{A} \cap A \cap B = bA \cap B,$$

and so $B = A$. This completes the proof of Theorem 7.2. $\square$

Depending on the choice of σ and τ, the ring B of Equation 7.1.0 may fail to be Noetherian. Example 7.3 shows that in the setting of Theorem 7.2 the ring B can be non-Noetherian and strictly smaller than $A := k(x,y,\sigma,\tau) \cap k[[x,y]]$.

EXAMPLE 7.3. Using the setting of Notation 7.1, let $\tau \in k[[y]]$ be defined to be $\sigma(y)$, that is, set $b_i := a_i$ for every $i \in \mathbb{N}$. We then have that $\theta := \frac{\sigma-\tau}{x-y} \in A$. Indeed,

$$\sigma - \tau = a_1(x-y) + a_2(x^2-y^2) + \cdots + a_n(x^n - y^n) + \cdots ,$$

and so $\theta = \frac{\sigma-\tau}{x-y} \in k[[x,y]] \cap k(x,y,\sigma,\tau) = A$. As a specific example, one may take $k := \mathbb{Q}$ and set $\sigma := e^x - 1$ and $\tau := e^y - 1$. The ring B is a localization of the ring $U := \bigcup_{n\in\mathbb{N}} k[x,y,\sigma_n,\tau_n]$.

CLAIM 7.4. *The element θ is not in B.*

PROOF. If θ is an element of B, then

$$\sigma - \tau \ \in \ (x-y)B \ \cap \ U \ = \ (x-y)U.$$

Let $S := k[x,y,\sigma,\tau]$ and let $U_n := k[x,y,\sigma_n,\tau_n]$ for each positive integer n. We have

$$U \ = \ \bigcup_{n\in\mathbb{N}} U_n \ \subseteq \ S[\frac{1}{xy}] \ \subset \ S_{(x-y)S},$$

where the last inclusion is because $xy \notin (x-y)S$. Thus $\theta \in B$ implies that

$$\sigma - \tau \in (x-y)S_{(x-y)S} \ \cap \ S \ = \ (x-y)S,$$

but this contradicts the fact that x,y,σ,τ are algebraically independent over k, and thus S is a polynomial ring over k in x,y,σ,τ. $\square$

To return to Example 7.3 and justify the assertions there, we see that $\frac{\sigma-\tau}{x-y} \notin B$, and so $B \subsetneq A$ and $(x-y)B \subsetneq (x-y)A \cap B$. Since an ideal of B is closed in the $\mathbf{n}$-adic topology if and only if the ideal is contracted from $\widehat{B}$ and since $\widehat{B} = \widehat{A}$, the principal ideal $(x-y)B$ is not closed in the $\mathbf{n}$-adic topology on B. Using Theorem 7.2, we conclude that B is a non-Noetherian three-dimensional local Krull domain having a two-generated maximal ideal such that B birationally dominates a four-dimensional regular local domain. We discuss this example in more detail in Example 8.11 of Chapter 8.

In contrast to Example 7.3, a Krull domain that birationally dominates a two-dimensional Noetherian local domain is Noetherian; see Exercise 13 in Chapter 2.

As stated in Theorem 7.2, the ring B may be Noetherian for certain choices of σ and τ. To complete the proof of Theorem 4.12, we establish in Example 7.15 below with $k = \mathbb{Q}$ that the elements $\sigma := e^x - 1$ and $\tau := e^{(e^y-1)} - 1$ give an example where $B = A$. As we show in Proposition 7.8, the critical property of τ used to prove B is Noetherian and $A = B$ is that, for $T = C[y]_{(x,y)}$, the image of τ in $\widehat{R}/Q$ is algebraically independent over $T/(Q \cap T)$, for each height-one prime ideal Q of $\widehat{R} = \mathbb{Q}[[x,y]]$ such that $Q \cap T \neq (0)$ and $xy \notin Q$. In the proof of Theorem 7.12 we show that this property holds for $\tau = e^{(e^y-1)} - 1$ by using results of Ax that yield generalizations of Schanuel's conjectures regarding algbraic relations satisfied by exponential functions [**11**, Corollary 1, p. 253].

Remark 7.5. In the setting of Notation 7.1, It seems natural to consider the ring compositum of $\widehat{C} = k[[x]]$ and $\widehat{D} = k[[y]]$. We outline in Exercise 3 of this chapter a proof due to Kunz that the subring $\widehat{C}[\widehat{D}]$ of $k[[x,y]]$ is not Noetherian.

7.2. Residual algebraic independence

Recall that an extension of Krull domains $R \hookrightarrow S$ satisfies the condition **PDE** ("pas d'éclatement", or in English "no blowing up") provided that $\operatorname{ht}(Q \cap R) \leq 1$ for each prime ideal of height one Q in S; see Definition 2.3. The iterative example leads us to consider in this section a related property as in the following definition.

Definition 7.6. Let $R \hookrightarrow S$ denote an extension of Krull domains. An element $\nu \in S$ is *residually algebraically independent with respect to* S *over* R if ν is algebraically independent over R and for every height-one prime ideal Q of S such that $Q \cap R \neq 0$, the image of ν in S/Q is algebraically independent over $R/(Q \cap R)$.

Remark 7.7. If $(R, \mathbf{m})$ is a regular local domain, or more generally an analytically normal Noetherian local domain, it is natural to consider the extension of Krull domains $R \hookrightarrow \widehat{R}$, where $\widehat{R}$ is the $\mathbf{m}$-adic completion of R, and ask about the existence of an element $\nu \in \widehat{R}$ that is residually algebraically independent with respect to $\widehat{R}$ over R. If the dimension of R is at least two and R has countable cardinality, for example, if $R = \mathbb{Q}[x,y]_{(x,y)}$, then a cardinality argument implies the existence of an element $\nu \in \widehat{R}$ that is residually algebraically independent with respect to $\widehat{R}$ over R; see Theorems 19.20 and 19.27.

We show in Proposition 19.15 and Theorem 19.27 of Chapter 19 that, if $\nu \in \widehat{\mathbf{m}}$ is residually algebraically independent with respect to $\widehat{R}$ over R, then the intersection domain $A = \widehat{R} \cap \mathcal{Q}(R[\nu])$ is the localized polynomial ring $R[\nu]_{(\mathbf{m},\nu)}$. Therefore A is Noetherian and the completion $\widehat{A}$ of A is a formal power series ring in one variable over $\widehat{R}$. As in Exercise 7 of Chapter 3, the local inclusion maps $R \hookrightarrow A \hookrightarrow \widehat{R}$ determine a surjective map of $\widehat{A}$ onto $\widehat{R}$. Since $\dim \widehat{A} > \dim \widehat{R}$, this surjective map has a nonzero kernel. Hence A is not a a *subspace* of $\widehat{R}$, that is, the topology on A determined by the powers of the maximal ideal of A is not the same as the *subspace topology* on A defined by intersecting the powers of the maximal ideal of $\widehat{R}$ with A.

The existence of an element that is almost residually algebraically independent is important in completing the proof of the iterative examples of Section 4.3, as we demonstrate in Proposition 7.8 and Theorem 7.12. In the proof of these results we use Noetherian Flatness Theorem 8.8 of Chapter 8. In the proof of Proposition 7.8 we show that our setting here fits Inclusion Construction 5.3 and the approximation procedure of Section 6.1, and so Theorem 8.8 implies that the intersection domain equals the approximation domain and is Noetherian provided a certain extension is flat.

PROPOSITION 7.8. *Let the notation be as in (7.1), and let $T = C[y]_{(x,y)C[y]}$. Thus T is a two-dimensional regular local domain with completion $\widehat{T} = k[[x, y]] = \widehat{R}$. If the image of τ in $C[[y]]/Q$ is algebraically independent over $T/(Q \cap T)$ for each height-one prime ideal Q of $C[[y]]$ such that $Q \cap T \neq (0)$ and $xy \notin Q$, then B is Noetherian and $B = A$.*

PROOF. Since $B_n \subset T[\tau_n]_{(x,y,\tau_n)} \subset B$ for each positive integer n, we have $B = \bigcup_{n=1}^{\infty} T[\tau_n]_{(x,y,\tau_n)}$. In order to show that B is Noetherian and $B = A$, it suffices by the Noetherian Flatness Theorem 8.8 to show that the map $\phi_y : T[\tau] \longrightarrow C[[y]][1/y]$ is flat; see Definition 2.20. By Remark 2.21.1, flatness is a local property. Thus it suffices to show for each prime ideal Q of $C[[y]]$ with $y \notin Q$ that the induced map $\phi_Q : T[\tau]_{Q\cap T[\tau]} \longrightarrow C[[y]]_Q$ is flat. If $\mathrm{ht}(Q \cap T[\tau]) \leq 1$, then $T[\tau]_{Q\cap T[\tau]}$ is either a field or a DVR, and ϕ_Q is flat by (2.23.3) since $C[[y]]_Q$ is torsionfree over $T[\tau]_{Q\cap T[\tau]}$. Therefore it suffices to show that $\mathrm{ht}(Q \cap T[\tau]) \leq 1$. This is clear for $Q = xC[[y]]$. On the other hand, if $xy \notin Q$, then the image of τ in $C[[y]]/Q$ is algebraically independent over $T/(Q \cap T)$, and this implies that $\mathrm{ht}(Q \cap T[\tau]) \leq 1$. We conclude that B is Noetherian and that $B = A$. $\square$

REMARK 7.9. To establish the existence of examples to which Proposition 7.8 applies, we take k to be the field $\mathbb{Q}$ of rational numbers. Thus $R := \mathbb{Q}[x, y]_{(x,y)}$ is the localized polynomial ring, and the completion of R with respect to its maximal ideal $\mathbf{m} := (x, y)R$ is $\widehat{R} := \mathbb{Q}[[x, y]]$, the formal power series ring in the variables x and y. Let $\sigma := e^x - 1 \in \mathbb{Q}[[x]]$, and $C := \mathbb{Q}[[x]] \cap \mathbb{Q}(x, \sigma)$. Thus C is an excellent DVR[1] with maximal ideal xC, and $T := C[y]_{(x,y)C[y]}$ is an excellent countable two-dimensional regular local ring with maximal ideal $(x, y)T$ and with (y)−adic completion $C[[y]]$. The UFD $C[[y]]$ has maximal ideal $\mathbf{n} = (x, y)$. Using that T is countable, an elementary proof is given below in Theorem 7.10 that there exists $\tau \in C[[y]]$ such that for each height-one prime Q of $C[[y]]$ with $Q \cap T \neq (0)$ and $y \notin Q$, the image of τ in $C[[y]]/Q$ is transcendental over $T/(Q \cap T)$. If the element τ can be found in $\mathbb{Q}[[y]]$, then by Proposition 7.8, for this choice of $\sigma \in \mathbb{Q}[[x]]$ and $\tau \in \mathbb{Q}[[y]]$ in Theorem 4.12, we have B is Noetherian and $B = A$.

THEOREM 7.10. *Let C be an excellent countable rank-one DVR with maximal ideal xC and let y be an indeterminate over C. Let $T = C[y]_{(x,y)C[y]}$. Then there*

[1]Every Dedekind domain of characteristic zero is excellent [**94**, (34.B)]. See also Remark 3.32.

exists an element $\tau \in C[[y]]$ for which the image of τ in $C[[y]]/Q$ is transcendental over $T/(Q \cap T)$, for every height-one prime ideal Q of $C[[y]]$ such that $Q \cap T \neq (0)$ and $y \notin Q$. Moreover τ is transcendental over T.

PROOF. Since C is a DVR, C is a UFD, and so are $T = C[y]_{(x,y)C[y]}$ and $C[[y]]$. Hence every height-one prime ideal Q_i of T is principal and is generated by an irreducible polynomial of $C[y]$, say $f_i(y)$. There are countably many of these prime ideals.

Let $\mathcal{U}$ be the countable set of all height-one prime ideals of $C[[y]]$ that are generated by some irreducible factor in $C[[y]]$ of some irreducible polynomial $f(y)$ of $C[y]$ *other* than y; that is, $yC[[y]]$ is *not* included in $\mathcal{U}$. Let $\{P_i\}_{i=1}^{\infty}$ be an enumeration of the prime ideals of $\mathcal{U}$. Let $\mathbf{n} := (x, y)C[[y]]$ denote the maximal ideal of $C[[y]]$.

CLAIM 7.11. *For each $i \in \mathbb{N}$, there are uncountably many distinct cosets in $((P_1 \cap \cdots \cap P_{i-1} \cap y^{i+1}C[[y]]) + P_i)/P_i$.*

PROOF. Since $y \notin P_i$, the image of y in the one-dimensional local domain $C[[y]]/P_i$ generates an ideal primary for the maximal ideal. Also $C[[y]]/P_i$ is a finite $C[[y]]$-module. Since $C[[y]]$ is (y)-adically complete it follows that $C[[y]]/P_i$ is a (y)-adically complete local domain [**96**, Theorem 8.7]. Hence, if we let $\mathcal{H} \subseteq C[[y]]$ denote a complete set of distinct coset representatives of $C[[y]]/P_i$, then $\mathcal{H}$ is uncountable.

Let a_i be an element of $P_1 \cap \cdots \cap P_{i-1} \cap y^{i+1}C[[y]]$ that is not in P_i. Then the set $a_i\mathcal{H} := \{a_i\beta \mid \beta \in \mathcal{H}\}$ represents an uncountable set of distinct coset representatives of $C[[y]]/P_i$, since, if $a_i\beta$ and $a_i\gamma$ are in the same coset of P_i and $\beta, \gamma \in \mathcal{H}$, then

$$a_i\beta - a_i\gamma \in P_i \implies \beta - \gamma \in P_i \implies \beta = \gamma,$$

Thus there are uncountably many distinct cosets of $C[[y]]/P_i$ of the form $a_i\beta + P_i$, where β ranges over $\mathcal{H}$, as desired for Claim 7.11. $\square$

To return to the proof of Theorem 7.10, we use that

$$((P_1 \cap \cdots \cap P_{i-1} \cap y^{i+1}C[[y]]) + P_i)/P_i$$

is uncountable for each i as follows: Choose $f_1 \in y^2C[[y]]$ so that the image of $y - f_1$ in $C[[y]]/P_1$ is not algebraic over $T/(P_1 \cap T)$; this is possible since the set of cosets is uncountable and so some cosets are transcendental over the countable set $T/(P_1 \cap T)$. Then the element $y - f_1 \notin P_1$, and $f_1 \notin P_1$, since the image of y is not transcendental over $T/(T \cap P_1)$. Choose $f_2 \in P_1 \cap y^3C[[y]]$ so that the image of $y - f_1 - f_2$ in $C[[y]]/P_2$ is not algebraic over $T/(P_2 \cap T)$. Note that $f_2 \in P_1$ implies the image of $y - f_1 - f_2$ is the same as the image of $y - f_1$ in $C[[y]]/P_1$ and so it is not algebraic over P_1. Successively by induction, for each positive integer n, we choose f_n as in the display

$$f_n \in P_1 \cap P_2 \cap \cdots \cap P_{n-1} \cap y^{n+1}C[[y]]$$

so that the image of $y - f_1 - ... - f_n$ in $C[[y]]/P_n$ is transcendental over $T/(T \cap P_i)$ for each i with $1 \leq i \leq n$. Then we have a Cauchy sequence $\{f_1 + \cdots + f_n\}_{n=1}^{\infty}$ in $C[[y]]$ with respect to the $(yC[[y]])$-adic topology, and so it converges to an element $a \in y^2C[[y]]$. Now

$$y - a = (y - f_1 - \ \dots \ - f_n) + (f_{n+1} + \dots),$$

where the image of $(y - f_1 - \ldots - f_n)$ in $C[[y]]/P_n$ is transcendental over $T/(P_i \cap T)$ and $f_i \in yC[[y]]$ for all $1 \le i \le n$ and $(f_{n+1} + \ldots) \in \cap_{i=1}^n P_i \cap yC[[y]]$. Therefore the image of $y - a$ in $C[[y]]/P_n$ is transcendental over $T/(P_n \cap T)$, for every $n \in \mathbb{N}$, and we have $y - a \in yC[[y]]$, as desired.

For the "Moreover" statement, suppose that τ is a root of a polynomial $f(z)$ with coefficients in T. For each prime ideal Q such that the image of τ is transcendental over $T/(T \cap Q)$, the coefficients of $f(z)$ must all be in $T \cap Q$. Since this is true for infinitely many height-one primes $T \cap Q$, and the intersection of infinitely many height-one primes in a Noetherian domain is zero, $f(z)$ is the 0 polynomial, and so τ is transcendental over T. □

Theorem 7.12 yields explicit examples for which B is Noetherian and $B = A$ in Theorem 4.12.

THEOREM 7.12. *Let x and y be indeterminates over $\mathbb{Q}$, the field of rational numbers. Then:*

(1) *There exist elements $\sigma \in x\mathbb{Q}[[x]]$ and $\tau \in y\mathbb{Q}[[y]]$ such that the following two conditions are satisfied:*
   (i) *σ is algebraically independent over $\mathbb{Q}(x)$ and τ is algebraically independent over $\mathbb{Q}(y)$.*
   (ii) $\operatorname{trdeg}_{\mathbb{Q}} \mathbb{Q}(y, \tau, \{\frac{\partial^n \tau}{\partial y^n}\}_{n\in\mathbb{N}}) > r := \operatorname{trdeg}_{\mathbb{Q}} \mathbb{Q}(x, \sigma, \{\frac{\partial^n \sigma}{\partial x^n}\}_{n\in\mathbb{N}})$*, where $\{\frac{\partial^n \tau}{\partial y^n}\}_{n\in\mathbb{N}}$ is the set of partial derivatives of τ with respect to y and $\{\frac{\partial^n \sigma}{\partial x^n}\}_{n\in\mathbb{N}}$ is the set of partial derivatives of σ with respect to x.*

(2) *If $\sigma \in x\mathbb{Q}[[x]]$ and $\tau \in y\mathbb{Q}[[y]]$ satisfy conditions i and ii, if $C = \mathbb{Q}(x, \sigma) \cap \mathbb{Q}[[x]]$, as in Notation 7.1, and if $T = C[y]_{(x,y)}$, then the image of τ in $C[[y]]/Q$ is algebraically independent over $T/(Q \cap T)$, for every height-one prime ideal Q of $C[[y]]$ such that $Q \cap T \neq (0)$ and $xy \notin Q$.*

(3) *If $\sigma \in x\mathbb{Q}[[x]]$ and $\tau \in y\mathbb{Q}[[y]]$ satisfy conditions i and ii, then the ring B of Theorem 4.12 defined for this choice of σ and τ is Noetherian and $B = A$.*

PROOF. For item 1, to establish the existence of elements σ and τ satisfying properties (i) and (ii) of Theorem 7.12, let $\sigma = e^x - 1 \in \mathbb{Q}[[x]]$ and choose for τ a hypertranscendental element in $\mathbb{Q}[[y]]$. A power series $\tau = \sum_{i=0}^{\infty} b_i y^i \in \mathbb{Q}[[y]]$ is called *hypertranscendental* over $\mathbb{Q}[y]$ if the set of partial derivatives $\{\frac{\partial^n \tau}{\partial y^n}\}_{n\in\mathbb{N}}$ is infinite and algebraically independent over $\mathbb{Q}(y)$. Two examples of hypertranscendental elements are the Gamma function and the Riemann Zeta function. (The exponential function is, of course, far from being hypertranscendental.) Thus there exist elements σ, τ that satisfy conditions (i) and (ii) of Theorem 7.12.

Another way to obtain such elements is to set $\sigma = e^x - 1$ and $\tau = e^{(e^y - 1)} - 1$. In this case, conditions i and ii of Theorem 7.12 follow from [**11**, Conjecture Σ, p. 252], a generalization of Schanuel's conjectures, which is established in Ax's paper [**11**, Corollary 1, p. 253]. To see that conditions i and ii hold, it is convenient to restate Conjecture Σ of [**11**] with different letters for the power series; let y be a variable, and use only one or two power series $s, t \in \mathbb{C}[[y]]$. Thus Conjecture Σ states that, if s and t are elements of $\mathbb{C}[[y]]$ that are $\mathbb{Q}$-linearly independent, then

$$(7.12.0)\qquad \begin{aligned} \dim_{\mathbb{Q}}(\mathbb{Q}(s,e^s)) &\geq 1+\operatorname{rank}\begin{bmatrix}\frac{\partial s}{\partial y}\end{bmatrix}.\\ \dim_{\mathbb{Q}}(\mathbb{Q}(s,t,e^s,e^t)) &\geq 2+\operatorname{rank}\begin{bmatrix}\frac{\partial s}{\partial y} & \frac{\partial t}{\partial y}\end{bmatrix}.\end{aligned}$$

Since the rank of the matrix $\begin{bmatrix}\frac{\partial e^y}{\partial y}\end{bmatrix}$ is 1, $\dim_{\mathbb{Q}}(\mathbb{Q}(y,e^y))\geq 2$, by Equation 7.12.0. By switching the variable to x, $\dim_{\mathbb{Q}}(\mathbb{Q}(x,e^x))\geq 2$. Thus $\sigma = e^x-1$ satisfies condition i.

Since just two transcendental elements generate the field $\mathbb{Q}(x,e^x)$ over $\mathbb{Q}$, we have $\dim_{\mathbb{Q}}(\mathbb{Q}(x,e^x)) = 2$. Furthermore $\partial^n\sigma/\partial x^n = e^x$ for every $n\in\mathbb{N}$, and so

$$\operatorname{trdeg}_{\mathbb{Q}}(\mathbb{Q}(x,\sigma,\{\frac{\partial^n\sigma)}{\partial x^n}\}_{n\in\mathbb{N}})) = 2;$$

that is, for r as in condition ii with this σ, we have $r=2$.

Since the rank of the matrix $\begin{bmatrix}\frac{\partial y}{\partial y} & \frac{\partial(e^y-1)}{\partial y}\end{bmatrix}$ is 1, we have

$$\dim_{\mathbb{Q}}(\mathbb{Q}(y,e^y,e^{(e^y-1)})) = \dim_{\mathbb{Q}}(\mathbb{Q}(y,e^y-1,e^y,e^{(e^y-1)}))\geq 3,$$

by Equation 7.12.0 with $s=y$ and $t=e^y-1$.

For τ we have, $\partial\tau/\partial y = \partial(e^{(e^y-1)}-1)/\partial y = e^{(e^y-1)}\cdot e^y$. Thus

$$\operatorname{trdeg}_{\mathbb{Q}}(\mathbb{Q}(y,\tau,\{\frac{\partial^n\tau}{\partial y^n}\}_{n\in\mathbb{N}}))\geq \operatorname{trdeg}_{\mathbb{Q}}(\mathbb{Q}(y,e^y,e^{(e^y-1)})) > 2,$$

by the computation above, and so conditions i and ii both hold for τ. Thus item 1 is proved.

Item 3 follows from item 2 by Proposition 7.8.

For item 2, we observe that the ring $T = C[y]_{(x,y)}$ is an overring of $R = \mathbb{Q}[x,y]_{(x,y)}$ and a subring of $\widehat{R}$ and T has completion $\widehat{T}=\widehat{R}$:

$$R=\mathbb{Q}[x,y]_{(x,y)}\longrightarrow T = C[y]_{(x,y)}\longrightarrow \widehat{R}=\widehat{T}=\mathbb{Q}[[x,y]].$$

We display the relationships among these rings.

$$\widehat{R}=\mathbb{Q}[[x,y]]$$

$$T := C[y]_{(x,y)}$$

$$C := \mathbb{Q}(x,\sigma)\cap\mathbb{Q}[[x]] = \cup\ \mathbb{Q}[x,\sigma_n]_{(x,\sigma_n)} \qquad\qquad R:=\mathbb{Q}[x,y]_{(x,y)}$$

The rings of the example

Let $\widehat{P}$ be a height-one prime ideal of $\widehat{R}$, let bars (for example, $\bar{x}$), denote images in $\overline{\widehat{R}} = \widehat{R}/\widehat{P}$ and set $P:=\widehat{P}\cap R$ and $P_1 := \widehat{P}\cap T$. Assume that $P_1\neq 0$ and that $xy\notin\widehat{P}$.

In the following commutative diagram, we identify $\mathbb{Q}[[x]]$ with $\mathbb{Q}[[\bar{x}]]$ and $\mathbb{Q}[[y]]$ with $\mathbb{Q}[[\bar{y}]]$, etc.

$$
\begin{array}{ccccc}
\mathbb{Q}[[y]] & \psi_y & \overline{\widehat{R}} = \widehat{R}/\widehat{P} & \psi_x & \mathbb{Q}[[x]] \\
 & & & & \\
 & & \overline{T} = T/P_1 & & \\
 & & & & \\
\mathbb{Q}[y]_{(y)} & \phi_y & \overline{R} = R/P & \phi_x & \mathbb{Q}[x]_{(x)}
\end{array}
$$

All maps in the diagram are injective and $\overline{\widehat{R}}$ is finite over both of the rings $\mathbb{Q}[[x]]$ and $\mathbb{Q}[[y]]$. Also $\overline{R}$ is algebraic over $\mathbb{Q}[\bar{x}]_{(\bar{x})}$, since $\operatorname{trdeg}_{\mathbb{Q}} \mathcal{Q}(\overline{R}) = 1$.

Let L denote the field of fractions of $\overline{\widehat{R}}$. We may consider $\mathbb{Q}(y, \tau, \{\frac{\partial^n \tau}{\partial y^n}\}_{n\in\mathbb{N}})$ and $\mathbb{Q}(x, \sigma, \{\frac{\partial^n \sigma}{\partial x^n}\}_{n\in\mathbb{N}})$ as subfields of L, where

$$
\operatorname{trdeg}_{\mathbb{Q}}\left(\mathbb{Q}(y, \tau, \left\{\frac{\partial^n \tau}{\partial y^n}\right\}_{n\in\mathbb{N}}\right) > \operatorname{trdeg}_{\mathbb{Q}}\left(\mathbb{Q}(x, \sigma, \left\{\frac{\partial^n \sigma}{\partial x^n}\right\}_{n\in\mathbb{N}}\right).
$$

Let d denote the partial derivative map $\frac{\partial}{\partial x}$ on $\mathbb{Q}((x))$. Since the extension L of $\mathbb{Q}((x))$ is finite and separable, d extends uniquely to a derivation $\widehat{d} : L \to L$, [**148**, Corollary 2, p. 124]. Let H denote the algebraic closure (shown in Picture 7.13.1 by a small upper a) in L of the field $\mathbb{Q}(x, \sigma, \{\frac{\partial^n \sigma}{\partial x^n}\}_{n\in\mathbb{N}})$. Let $\widehat{p}(x, y) \in \mathbb{Q}[[x, y]]$ be a prime element generating $\widehat{P}$. Claim 7.13 asserts that the images of H and $\overline{\widehat{R}}$ under $\widehat{d}$ are inside H and $(1/p')\overline{\widehat{R}}$, respectively, as shown in Picture 7.13.1.

$$
\begin{array}{cccc}
 & L := \mathcal{Q}(\overline{\widehat{R}}) & \widehat{d} & L \\
 & & & \\
 & \overline{\widehat{R}} := \overline{\mathbb{Q}[[x,y]]} & \widehat{d} & \boxed{\dfrac{1}{p'(\bar{y})}\overline{\widehat{R}}} \\
 & & & \\
 & \mathbb{Q}[[\bar{x}]] \cong \mathbb{Q}[[x]] & d := \frac{\partial}{\partial x} & \mathbb{Q}[[\bar{x}]] \\
 & & & \\
H & & \widehat{d} & H := \left(\mathbb{Q}(x, \sigma, \{\frac{\partial^n \sigma}{\partial x^n}\}_{n=1}^{\infty})\right)^a \cap L \\
 & & & \\
 & \mathbb{Q}[x, \sigma, \{\frac{\partial^n \sigma}{\partial x^n}\}_{n=1}^{\infty}] & d & \mathbb{Q}[x, \sigma, \{\frac{\partial^n \sigma}{\partial x^n}\}_{n=1}^{\infty}] \\
 & & & \\
 & \mathbb{Q} & 1_{\mathbb{Q}} & \mathbb{Q}
\end{array}
$$

Picture 7.13.1 The image of subrings of L via the extension $\widehat{d}$ of $d := \frac{\partial}{\partial x}$.

CLAIM 7.13. *With the notation above:*

(i) $\widehat{d}(H) \subseteq H$.

(ii) *There exists a polynomial* $p(x, y) \in \mathbb{Q}[[x]][y]$ *with* $p\mathbb{Q}[[x, y]] = \widehat{P}$ *and* $p(\bar{y}) = 0$.

(iii) $\widehat{d}(\bar{y}) \neq 0$ *and* $p'(\bar{y})\widehat{d}(\bar{y}) \in \overline{\overline{R}}$*, where* $p'(y) := \frac{\partial p(x,y)}{\partial y}$.

(iv) *For every element* $\lambda \in \overline{\overline{R}}$*, we have* $p'(\bar{y})\widehat{d}(\lambda) \in \overline{\overline{R}}$*, and so* $\widehat{d}(\overline{\overline{R}}) \subseteq (1/p')\overline{\overline{R}}$.

PROOF. For item i, since d maps $\mathbb{Q}(x, \sigma, \{\frac{\partial^n \sigma}{\partial x^n}\}_{n\in\mathbb{N}})$ into itself, $\widehat{d}(H) \subseteq H$.

For item ii, we have that x and y are not contained in $\widehat{P}$, and that the element $\widehat{p}(x,y) \in \mathbb{Q}[[x,y]]$ generates $\widehat{P}$ and is *regular in* y as a power series in $\mathbb{Q}[[x,y]]$ (in the sense of Zariski-Samuel [**149**, p.145]); that is, $\widehat{p}(0,y) \neq 0$. Thus by [**149**, Corollary 1, p.145] the element $\widehat{p}(x,y)$ can be written as:

$$\widehat{p}(x,y) = \epsilon(x,y)(y^n + b_{n-1}(x)y^{n-1} + \ldots + b_0(x)),$$

where $\epsilon(x,y)$ is a unit of $\mathbb{Q}[[x,y]]$ and each $b_i(x) \in \mathbb{Q}[[x]]$. Hence $\widehat{P}$ is also generated by

$$p(x,y) = p(y) := \epsilon^{-1}\widehat{p} \ = \ y^n + c_{n-1}y^{n-1} + \cdots + c_0,$$

where the $c_i \in \mathbb{Q}[[x]]$. Since $p(y)$ is the minimal polynomial of $\bar{y}$ over the field $\mathbb{Q}((x))$, we have $0 = p(\bar{y}) := \bar{y}^n + c_{n-1}\bar{y}^{n-1} + \cdots + c_1\bar{y} + c_0$.

For item iii, observe that

$$p'(y) = ny^{n-1} + c_{n-1}(n-1)y^{n-2} + \cdots + c_1,$$

and $p'(\bar{y}) \neq 0$ by minimality. Now

$$\begin{aligned} 0 = \widehat{d}(p(\bar{y})) &= \widehat{d}(\bar{y}^n + c_{n-1}\bar{y}^{n-1} + \cdots + c_1\bar{y} + c_0) \\ &= n\bar{y}^{n-1}\widehat{d}(\bar{y}) + c_{n-1}(n-1)\bar{y}^{n-2}\widehat{d}(\bar{y}) + d(c_{n-1})\bar{y}^{n-1} + \cdots \\ &\qquad\qquad + c_1\widehat{d}(\bar{y}) + d(c_1)\bar{y} + d(c_0) \\ &= \widehat{d}(\bar{y})(n\bar{y}^{n-1} + c_{n-1}(n-1)\bar{y}^{n-2} + \cdots + c_1) + \\ &\qquad\qquad + d(c_{n-1})\bar{y}^{n-1} + \cdots + d(c_1)\bar{y} + d(c_0) \\ &= \widehat{d}(\bar{y})(p'(\bar{y})) + \sum_{i=0}^{n-1} d(c_i)\bar{y}^i \\ \implies \widehat{d}(\bar{y})(p'(\bar{y})) &= -\big(\sum_{i=0}^{n-1} d(c_i)\bar{y}^i\big) \quad \text{ and } \quad \widehat{d}(\bar{y}) = \frac{-1}{p'(\bar{y})}\sum_{i=0}^{n-1} d(c_i)\bar{y}^i. \end{aligned}$$

In particular, $p'(\bar{y})\widehat{d}(\bar{y}) \in \overline{\overline{R}}$. If $d(c_i) = 0$ for every i, then $c_i \in \mathbb{Q}$ for every i; this would imply that $p(x,y) \in \mathbb{Q}[[y]]$ and either $c_0 = 0$ or c_0 is a unit of $\mathbb{Q}$. If $c_0 = 0$, $p(x,y)$ could not be a minimal polynomial for $\bar{y}$, a contradiction. If c_0 is a unit, then $p(y)$ is a unit of $\mathbb{Q}[[y]]$, and so $\widehat{P}$ contains a unit, another contradiction. Thus $\widehat{d}(\bar{y}) \neq 0$, as desired for item iii.

For item iv, observe that every element $\lambda \in \overline{\overline{R}}$ has the form:

$$\lambda = e_{n-1}(x)\bar{y}^{n-1} + \cdots + e_1(x)\bar{y} + e_0(x), \text{ where } e_i \in \mathbb{Q}[[x]].$$

Therefore:

$$\widehat{d}(\lambda) = \widehat{d}(\bar{y})\big[(n-1)e_{n-1}(x)\bar{y}^{n-2} + \cdots + e_1(x)\big] + \sum_{i=0}^{n-1}\widehat{d}(e_i(x))\bar{y}^i.$$

The sum expression on the right is in $\overline{\overline{R}}$ and, as established above, $p'(\bar{y})\widehat{d}(\bar{y}) \in \overline{\overline{R}}$, and so $p'(\bar{y})\widehat{d}(\lambda) \in \overline{\overline{R}}$. $\square$

The next claim asserts an expression for $\widehat{d}(\bar{\tau})$ in terms of the partial derivative $\frac{\partial\bar{\tau}}{\partial\bar{y}}$ of $\bar{\tau}$ with respect to $\bar{y}$.

CLAIM 7.14. $\widehat{d}(\tau) = \widehat{d}(\bar{y})\frac{\partial\bar{\tau}}{\partial\bar{y}}$.

PROOF. For every $m \in \mathbb{N}$, we have $\tau = \sum_{i=0}^{m} b_i y^i + y^{m+1}\tau_m$, where the $b_i \in \mathbb{Q}$ and $\tau_m \in \mathbb{Q}[[y]]$ is defined as in (6.1.1). Therefore

$$\widehat{d}(\bar{\tau}) = \widehat{d}(\bar{y}) \cdot \big(\sum_{i=1}^{m} ib_i\bar{y}^{i-1}\big) + \widehat{d}(\bar{y})(m+1)\bar{y}^m\bar{\tau}_m + \bar{y}^{m+1}\widehat{d}(\bar{\tau}_m).$$

Thus

$$p'(\bar{y})\widehat{d}(\bar{\tau}) = p'(\bar{y})\widehat{d}(\bar{y}) \cdot \sum_{i=1}^{m} ib_i\bar{y}^{i-1} + \bar{y}^m(p'(\bar{y})\widehat{d}(\bar{y})(m+1)\bar{\tau}_m + \bar{y}p'(\bar{y})\widehat{d}(\bar{\tau}_m)).$$

Since $\bar{\tau} = \sum_{i=0}^{\infty} b_i\bar{y}^i$ with $b_i \in \mathbb{Q}$, we have

$$\frac{\partial\bar{\tau}}{\partial\bar{y}} = \sum_{i=1}^{m} ib_i\bar{y}^{i-1} + \bar{y}^m \sum_{i=m+1}^{\infty} ib_i\bar{y}^{i-m-1}.$$

Thus, if we multiply the last equation by $p'(\bar{y})\widehat{d}(\bar{y})$, we obtain

$$p'(\bar{y})\widehat{d}(\bar{y})\frac{\partial\bar{\tau}}{\partial\bar{y}} = p'(\bar{y})\widehat{d}(\bar{y})\sum_{i=1}^{m} ib_i\bar{y}^{i-1} + p'(\bar{y})\widehat{d}(\bar{y})\bar{y}^m \sum_{i=m+1}^{\infty} ib_i\bar{y}^{i-m-1}.$$

Hence, by subtracting this last equation from the earlier expression for $p'(\bar{y})\widehat{d}(\bar{\tau})$, we obtain

$$p'(\bar{y})\widehat{d}(\bar{\tau}) - p'(\bar{y})\widehat{d}(\bar{y})\frac{\partial\bar{\tau}}{\partial\bar{y}} \in \bar{y}^m(\overline{\widehat{R}}),$$

for every $m \in \mathbb{N}$. Therefore $p'(\bar{y})\widehat{d}(\bar{\tau}) - p'(\bar{y})\widehat{d}(\bar{y})\frac{\partial\bar{\tau}}{\partial\bar{y}} \in \cap y^m(\overline{\widehat{R}}) = 0$, by Krull's Intersection Theorem 2.5. Thus $\widehat{d}(\tau) = \widehat{d}(\bar{y})\frac{\partial\bar{\tau}}{\partial\bar{y}}$, since $p'(\bar{y}) \neq 0$ and $\overline{\widehat{R}}$ is an integral domain. That is, Claim 7.14 holds. $\square$

Completion of proof of Theorem 7.12. Recall that $\overline{R}$ is algebraic over $\mathbb{Q}[x]_{(x)}$, and so $\bar{y}$ is algebraic over $\mathbb{Q}[x]_{(x)}$. Also $T := C[y]_{(x,y)}$, where $C := \mathbb{Q}(x, \sigma) \cap Q[[x]]$. Thus $\overline{T} \subseteq H$, and so H is the algebraic closure of the field $\mathcal{Q}(\overline{T}, \{\frac{\partial^n\bar{\sigma}}{\partial\bar{x}^n}\}_{n\in\mathbb{N}})$ in L. Therefore $\bar{\tau} \notin H$ if and only if $\bar{\tau}$ is transcendental over H. By hypothesis, the transcendence degree of $H/\mathbb{Q}$ is r. Since $\widehat{d}(H) \subseteq H$, if $\bar{\tau}$ were in H, then $\frac{\partial^n\bar{\tau}}{\partial\bar{y}^n} \in H$ for all $n \in \mathbb{N}$. This implies that the field $\mathbb{Q}(y, \tau, \{\frac{\partial^n\tau}{\partial y^n}\}_{n\in\mathbb{N}})$ is contained in H. This contradicts our hypothesis that $\operatorname{trdeg}_{\mathbb{Q}} \mathbb{Q}(y, \tau, \{\frac{\partial^n\tau}{\partial y^n}\}_{n\in\mathbb{N}}) > r$. Therefore the image of τ in $\widehat{R}/Q$ is algebraically independent over $T/(Q \cap T)$ for each height-one prime ideal Q of $\widehat{R}$ such that $Q \cap T \neq (0)$ and $xy \notin Q$. This completes the proof of Theorem 7.12. $\square$

Since the elements $\sigma = e^x - 1 \in \mathbb{Q}[[x]]$ and $\tau = e^{(e^y-1)} - 1 \in \mathbb{Q}[[y]]$ satisfy the conditions of Theorem 7.12 by Ax's results in [**11**] and the proof given of Theorem 7.12, we have:

EXAMPLE 7.15. For $\sigma = e^x - 1 \in \mathbb{Q}[[x]]$ and $\tau = e^{(e^y-1)} - 1 \in \mathbb{Q}[[y]]$ in Theorem 4.12, the ring B is Noetherian and $B = A$.

Exercises

(1) Let x and y be indeterminates over a field k and let R be the two-dimensional RLR obtained by localizing the mixed power series-polynomial ring $k[[x]][y]$ at the maximal ideal $(x,y)k[[x]][y]$.

(i) For each height-one prime ideal P of R different from xR, prove that R/P is a one-dimensional complete local domain.

(ii) For each nonzero prime ideal Q of $\widehat{R} = k[[x,y]]$ prove that $Q \cap R \neq (0)$. Conclude that the generic formal fiber of R is zero-dimensional.

Suggestion. For part (ii), use Theorem 3.10. For more information about the dimension of the formal fibers, see the articles of Matsumura and Rotthaus [**95**] and [**121**].

(2) Let x and y be indeterminates over a field k and let $R = k[x,y]_{(x,y)}$. As in Remark 7.7, assume that $\nu \in \widehat{\mathbf{m}}$ is residually algebraically independent with respect to $\widehat{R} = k[[x,y]]$ over R. Thus $A = \widehat{R} \cap \mathcal{Q}(R[\mathbf{n}])$ is the localized polynomial ring $R[\nu]_{(\mathbf{m},\nu)}$. Let $\mathbf{n} = (\mathbf{m},\nu)A$ denote the maximal ideal of A. Give a direct proof that A is not a subspace of $\widehat{R}$.

Suggestion. Since $\nu \in \widehat{\mathbf{m}}$ is a power series in $\widehat{R} = k[[x,y]]$, for each positive integer n, there exists a polynomial $f_n \in k[x,y]$ such that $\nu - f_n \in \widehat{\mathbf{m}}^n$. Since A is a 3-dimensional regular local ring with $\mathbf{n} = (x,y,\nu)A$, the element $\nu - f_n \notin \mathbf{n}^2$. Hence for each positive integer n, the ideal $\widehat{\mathbf{m}}^n \cap A$ is not contained in $\mathbf{n}^2$.

(3) (Kunz) Let L/k be a field extension with L having infinite transcendence degree over k. Prove that the ring $L \otimes_k L$ is not Noetherian. Deduce that the ring $k[[x]] \otimes_k k[[x]]$, which has $k((x)) \otimes_k k((x))$ as a localization, is not Noetherian.

Suggestion. Let $\{x_\lambda\}_{\lambda \in \Lambda}$ be a transcendence basis for L/k and consider the subfield $F = k(\{x_\lambda\})$ of L. The ring $L \otimes_k L$ is faithfully flat over its subring $F \otimes_k F$, and if $F \otimes_k F$ is not Noetherian, then $L \otimes_k L$ is not Noetherian. Hence it suffices to show that $F \otimes_k F$ is not Noetherian. The module of differentials $\Omega^1_{F/k}$ is known to be infinite dimensional as a vector space over F [**82**, 5.4], and $\Omega^1_{F/k} \cong I/I^2$, where I is the kernel of the map $F \otimes_k F \to F$, defined by sending $a \otimes b \mapsto ab$. Thus the ideal I of $F \otimes_k F$ is not finitely generated.

CHAPTER 8

Building Noetherian domains

In this chapter we establish that flatness of a certain map implies that the integral domain obtained via Constructions 5.3 or 5.4 is Noetherian. We use this result as formulated in Noetherian Flatness Theorem 8.8 (Inclusion Version) to complete the proofs of Theorems 4.12 and 7.2 concerning the iterative examples. More generally, we prove the following theorem.

THEOREM 8.1. *Let R be a Noetherian integral domain with field of fractions K. Let z be a nonzero nonunit of R and let R^* denote the (z)-adic completion of R. Let I be an ideal of R^* having the property that $\mathbf{p} \cap R = (0)$ for each $\mathbf{p} \in \operatorname{Ass}(R^*/I)$. The inclusion map $R \hookrightarrow (R^*/I)[1/z]$ is flat if and only if $A := K \cap (R^*/I)$ is both Noetherian and realizable as a localization of a subring of $R[1/z]$.*

In the statement of Theorem 8.1, the ring $(R^*/I)[1/z]$ is the localization of R^*/I at the multiplicative system generated by z, and the intersection $K \cap (R^*/I)$ is taken inside the total quotient ring $\mathcal{Q}(R^*/I)$ of R^*/I. As discussed in Note 5.5, the hypotheses of Theorem 8.1 imply that the field K embeds in $\mathcal{Q}(R^*/I)$.

Theorem 8.1 is is stated in terms of Homomorphic Image Construction 5.4. There are two Noetherian Flatness Theorems given in this chapter, one for each of the Constructions 5.3 and 5.4. [1] In Section 8.2, we state in Noetherian FlatnessTheorem 8.8 the corresponding version of this result for Inclusion Construction 5.3. Thus the intersection domain A constructed using Inclusion Construction 5.3 is both Noetherian and computable as a localization of an infinite directed union $U = \bigcup_{n=0}^{\infty} U_n$ of polynomial extension rings of R if and only if the map $U_0 \hookrightarrow R^*[1/z]$ is flat. The polynomial rings U_n are defined in Section 6.1 of Chapter 6.

Theorem 8.1 is implied by Noetherian Flatness Theorem 8.3 (Homomorphic Image Version) that is proved in Section 8.1. We make use of the approximation ring B of A defined in Section 6.2. The ring B is an infinite nested union of "computable rings" — while not localized polynomial rings over R, they are localizations of finitely generated birational extensions of R. In Section 8.1 we also prove a crucial lemma relating flatness and the Noetherian property.

8.1. Flatness and the Noetherian property

We use Lemma 8.2 in order to prove Noetherian Flatness Theorem 8.3 (Homomorphic Image Version). This lemma is crucial for our proof of Theorem 8.1. We

[1]When this duplication seems confusing we distinguish between the versions by labeling the one for Construction 5.3 as the "Inclusion Version" or just inserting the word "inclusion", and by labeling the theorem for Construction 5.4 as the "Homomorphic Image Version" or just inserting the word "image".

thank Roger Wiegand for observing Lemma 8.2 and its proof. For an introduction to flatness see Section 2.3 of Chapter 2.

LEMMA 8.2. *Let S be a subring of a ring T and let $z \in S$ be a regular element of T. Assume that the conditions of Lemma 6.16 are satisfied, that is $zS = zT \cap S$ and $S/zS = T/zT$. Then*

(1) *$T[1/z]$ is flat over $S \iff T$ is flat over S.*
(2) *If T is flat over S, then $D := (1+zS)^{-1}T$ is faithfully flat over $C := (1+zS)^{-1}S$.*
(3) *If T is Noetherian and T is flat over S, then $C = (1+zS)^{-1}S$ is Noetherian.*
(4) *If T and $S[1/z]$ are both Noetherian and T is flat over S, then S is Noetherian.*

PROOF. For item 1, if T flat over S, then by transitivity of flatness, Remark 2.21.13, the ring $T[1/z]$ is flat over S. For the converse, consider the exact sequence (using (6.16.3))

$$0 \to S = S[1/z] \cap T \xrightarrow{\ \alpha\ } S[1/z] \oplus T \xrightarrow{\ \beta\ } T[1/z] = S[1/z] + T \to 0,$$

where $\alpha(b) = (b, -b)$ for all $b \in S$ and $\beta(c, d) = c + d$ for all $c \in S[1/z],\ d \in T$. Since the two end terms are flat S-modules, the middle term $S[1/z] \oplus T$ is also S-flat. Therefore the direct summand T is S-flat.

For item 2, since $S \to T$ is flat, the embedding

$$C = (1+zS)^{-1}S \hookrightarrow (1+zS)^{-1}T = D$$

is flat. Since zC is in the Jacobson radical of C and $C/zC = S/zS = T/zT = D/zD$, each maximal ideal of C is contained in a maximal ideal of D, and so D is faithfully flat over C. This establishes item 2. If also T is Noetherian, then D is Noetherian, and, since D is faithfully flat over C, it follows that C is Noetherian by Remark 2.21.8, and thus item 3 holds.

For item 4, let J be an ideal of S. Since C is Noetherian by item 3 and $S[1/z]$ is Noetherian by hypothesis, there exists a finitely generated ideal $J_0 \subseteq J$ such that $J_0S[1/z] = JS[1/z]$ and $J_0C = JC$. To show $J_0 = J$, it suffices to show for each maximal ideal $\mathbf{m}$ of S that $J_0S_{\mathbf{m}} = JS_{\mathbf{m}}$. If $z \notin \mathbf{m}$, then $S_{\mathbf{m}}$ is a localization of $S[1/z]$, and so $J_0S_{\mathbf{m}} = JS_{\mathbf{m}}$, while if $z \in \mathbf{m}$, then $S_{\mathbf{m}}$ is a localization of C, and so $JCS_{\mathbf{m}} = J_0S_{\mathbf{m}}$. Therefore $J = J_0$ is finitely generated. It follows that S is Noetherian. $\square$

Noetherian Flatness Theorem 8.3 (Homomorphic Image Version) gives precise conditions for the approximation ring B of Homomorphic Image Construction 5.4 to be Noetherian.

NOETHERIAN FLATNESS THEOREM 8.3. (Homomorphic Image Version) *Let R be a Noetherian integral domain with field of fractions K. Let z be a nonzero nonunit of R and let R^* denote the (z)-adic completion of R. Let I be an ideal of R^* having the property that $\mathbf{p} \cap R = (0)$ for each $\mathbf{p} \in \operatorname{Ass}(R^*/I)$. As in (6.9.2), let*

$$U := \bigcup_{n=1}^{\infty} U_n, \quad B := \bigcup_{n=1}^{\infty} B_n = (1+zU)^{-1}U, \quad \textit{and} \quad A := K \cap (R^*/I).$$

The following statements are equivalent:

(1) *The extension* $R \hookrightarrow (R^*/I)[1/z]$ *is flat.*
(2) *The ring* B *is Noetherian.*
(3) *The extension* $B \hookrightarrow R^*/I$ *is faithfully flat.*
(4) *The ring* $A := K \cap (R^*/I)$ *is Noetherian and* $A = B$.
(5) *The ring* U *is Noetherian*
(6) *The ring* A *is both Noetherian and a localization of a subring of* $R[1/z]$.

PROOF. For (1) $\implies$ (2), if $R \hookrightarrow (R^*/I)[1/z]$ is flat, by factoring through $U[1/z] = R[1/z] \hookrightarrow (R^*/I)[1/z]$, we see that $U \hookrightarrow (R^*/I)[1/z]$ and $B \hookrightarrow (R^*/I)[1/z]$ are flat. By Lemma 8.2.2, where we let $S = U$ and $T = R^*/I$, the ring B is Noetherian.

For (2) $\implies$ (3), $B^* = R^*/I$ is flat over B, by Theorem 6.17.4 and Remark 3.2.3. By Proposition 6.20.1, $z \in \mathcal{J}(B)$, and so, using Remark 3.2.4, we have $B^* = R^*/I$ is faithfully flat over B.

For (3) $\implies$ (4), again Theorem 6.17.4 yields $B^* = R^*/I$, and so B^* is faithfully flat over B. Then

$$B = \mathcal{Q}(B) \cap R^* = \mathcal{Q}(A) \cap R^* = K(\underline{\tau}) \cap R^* = A$$

by Remark 2.21.9 and Theorem 6.19.2. By Remark 2.21.8, A is Noetherian.

For (4) $\implies$ (5), the composite embedding

$$U \hookrightarrow B = A \hookrightarrow B^* = A^* = R^*/I$$

is flat because B is a localization of U and A is Noetherian; see Remark 3.2.3. By Remark 3.2.4 again, A^* is faithfully flat over A. Thus by Lemma 8.2, parts 1 and 3, where again we let $S = U$ and $T = R^*/I$, we have $S[1/z] = U[1/z] = R[1/z]$ is Noetherian, and hence U is Noetherian by Lemma 8.2.4.

If U is Noetherian, then the localization B of U is Noetherian, and as above $B = A$. Hence A is a localization of U, a subring of $R[1/z]$. Thus (5) $\implies$ (6).

For (6) $\implies$ (1): since A is a localization of a subring D of $R[1/z]$, we have $A := \Gamma^{-1}D$, where Γ is a multiplicatively closed subset of D. Now

$$R \subseteq A = \Gamma^{-1}D \subseteq \Gamma^{-1}R[1/z] = \Gamma^{-1}A[1/z] = A[1/z],$$

and so $A[1/z]$ is a localization of R. That is, to obtain $A[1/z]$ we localize R by the elements of Γ and then localize by the powers of $1/z$. Since A is Noetherian, $A \hookrightarrow A^* = R^*/I$ is flat by Remark 3.2.2. Thus $A[1/z] \hookrightarrow (R^*/I)[1/z]$ is flat. It follows that $R \hookrightarrow (R^*/I)[1/z]$ is flat. $\square$

COROLLARY 8.4. *Let* R*,* I *and* z *be as in Noetherian Flatness Theorem 8.3 (Homomorphic Image version). If* $\dim(R^*/I) = 1$*, then* $\varphi : R \hookrightarrow W := (R^*/I)[1/z]$ *is flat and therefore the equivalent conditions of Theorem 8.3 all hold.*

PROOF. We have z is in the Jacobson radical of R^*/I by Construction Properties Theorem 6.17.2. Thus $\dim(R^*/I) = 1$ implies that $\dim W = 0$. The hypothesis on the ideal I implies that every prime ideal P of W contracts to (0) in R. Hence

$$\varphi_P : R_{P \cap R} = R_{(0)} = K \hookrightarrow W_P.$$

Thus W_P is a K-module and so a vector space over K. By Remark 2.21.2, φ_P is flat. Since flatness is a local property by Remark 2.211, the map φ is flat. $\square$

REMARK 8.5. With R, I and z as in Noetherian Flatness Theorem 8.3:

(1) There are examples where A is Noetherian and yet $B \neq A$. Thus B in such examples is non-Noetherian (see Theorem 7.2 and Example 8.11 below).
(2) There are examples where $A = B$ is non-Noetherian (see Theorem 13.6 and Examples 13.8, as well as Remark 8.10 below).
(3) A necessary and sufficient condition that $A = B$ is that A is a localization of $R[1/z] \cap A$. Indeed, Theorem 6.17.5 implies that $R[1/z] \cap A = U$ and, by Definition 6.12.1, $B = (1 + zU)^{-1}U$. Therefore the condition is sufficient. On the other hand, if $A = \Gamma^{-1}U$, where Γ is a multiplicatively closed subset of U, then by Remark 6.18.3, each $y \in \Gamma$ is a unit of B, and so $\Gamma^{-1}U \subseteq B$ and $A = B$.
(4) We discuss in Chapter 9 a family of examples where $R \hookrightarrow (R^*/I)[1/z]$ is flat.

Theorem 8.6 extends the range of applications of Noetherian Flatness Theorem 8.3.

THEOREM 8.6. *Let R be a Noetherian integral domain with field of fractions K. Let z be a nonzero nonunit of R and let R^* denote the (z)-adic completion of R. Let I be an ideal of R^* having the property that $\mathbf{p} \cap R = (0)$ for each $\mathbf{p} \in \operatorname{Ass}(R^*/I)$. Assume that I is generated by a regular sequence of R^*. If $R \hookrightarrow (R^*/I)[1/z]$ is flat, then for each $n \in \mathbb{N}$ we have*
(1) $\operatorname{Ass}(R^*/I^n) = \operatorname{Ass}(R^*/I)$,
(2) *R canonically embeds in R^*/I^n, and*
(3) *$R \hookrightarrow (R^*/I^n)[1/z]$ is flat.*

PROOF. Let $I = (\sigma_1, \ldots, \sigma_r)R^*$, where $\sigma_1, \ldots, \sigma_r$ is a regular sequence in R^*. Then the sequence $\sigma_1, \ldots, \sigma_r$ is quasi-regular in the sense of [**96**, Theorem 16.2, page 125]; that is, the associated graded ring of R^* with respect to I, which is the direct sum $R^*/I \oplus I/I^2 \oplus \ldots$, is a polynomial ring in r variables over R^*/I. The component I^n/I^{n+1} is a free (R^*/I)-module generated by the monomials of total degree n in these variables. It follows that I^n/I^{n+1} is a free (R^*/I)-module for each positive integer n. Thus $\operatorname{Ass}(I^n/I^{n+1}) = \operatorname{Ass}(R^*/I)$; that is, a prime ideal P of R^* annihilates a nonzero element of R^*/I if and only if P annihilates a nonzero element of I^n/I^{n+1}.

For item 1 we proceed by induction: assume $\operatorname{Ass}(R^*/I^n) = \operatorname{Ass}(R^*/I)$ and $n \in \mathbb{N}$. Consider the exact sequence

$$0 \to I^n/I^{n+1} \hookrightarrow R^*/I^{n+1} \to R^*/I^n \to 0. \tag{8.6.0}$$

Then $\operatorname{Ass}(R^*/I) = \operatorname{Ass}(I^n/I^{n+1}) \subseteq \operatorname{Ass}(R^*/I^{n+1})$. Also

$$\operatorname{Ass}(R^*/I^{n+1}) \subseteq \operatorname{Ass}(I^n/I^{n+1}) \cup \operatorname{Ass}(R^*/I^n) = \operatorname{Ass}(R^*/I)$$

by [**96**, Theorem 6.3, p. 38], and so it follows that $\operatorname{Ass}(R^*/I^{n+1}) = \operatorname{Ass}(R^*/I)$. Thus R canonically embeds in R^*/I^n for each $n \in \mathbb{N}$.

That $R \hookrightarrow (R^*/I^n)[1/z]$ is flat for every $n \in \mathbb{N}$ now follows by induction on n and by considering the exact sequence obtained by tensoring over R the short exact sequence (8.6.0) with $R[1/z]$. $\square$

EXAMPLE 8.7. Let $R = k[x, y]$ be the polynomial ring in the variables x, y over a field k and let $R^* = k[y][[x]]$ be the (x)-adic completion of R. Fix an element $\tau \in xk[[x]]$ such that x and τ are algebraically independent over k, and define the $k[[x]]$-algebra homomorphism $\phi : k[y][[x]] \to k[[x]]$, by setting $\phi(y) = \tau$. Then

$\ker(\phi) = (y - \tau)R^* =: I$. Notice that $\phi(R) = k[x, \tau] \cong R$ because x and τ are algebraically independent over k. Hence $I \cap R = (0)$. Also I is a prime ideal generated by a regular element of R^*, and $(I, x)R^* = (y, x)R^*$ is a maximal ideal of R^*. Corollary 8.4 and Theorem 8.6 imply that for each positive integer n, the intersection ring $A_n := (R^*/I^n) \cap k(x, y)$ is a one-dimensional Noetherian local domain having (x)-adic completion R^*/I^n. Since x generates an ideal primary for the unique maximal ideal of R^*/I^n, the ring R^*/I^n is also the completion of A_n with respect to the powers of the unique maximal ideal $\mathbf{n}_n$ of A_n. The ring A_1 is a DVR since R^*/I is a DVR by Remark 2.1. For $n > 1$, the completion of A_n has nonzero nilpotent elements and hence the integral closure of A_n is not a finitely generated A_n-module, Remarks 3.13. The inclusion $I^{n+1} \subsetneq I^n$ and the fact that A_n has completion R^*/I^n imply that $A_{n+1} \subsetneq A_n$ for each $n \in \mathbb{N}$. Hence the rings A_n form a strictly descending chain

$$A_1 \supset A_2 \supset \cdots \supset A_n \supset \cdots$$

of one-dimensional local birational extensions of $R = k[x, y]$.

8.2. The Inclusion Version

As remarked in (5.7.2), Inclusion Construction 5.4 is a special case of Homomorphic Images Construction 5.4 . Thus we obtain the following result analogous to Noetherian Flatness Theorem 8.3 (Homomorphic Image Version) for Inclusion Constructions.

Noetherian Flatness Theorem 8.8. (Inclusion Version) *Let R be a Noetherian integral domain with field of fractions K. Let z be a nonzero nonunit of R and let R^* denote the (z)-adic completion of R. Let $\tau_1, \dots, \tau_s \in zR^*$ be algebraically independent elements over K such that the field $K(\tau_1, \dots, \tau_s)$ is a subring of the total quotient ring of R^*.*[2] *As in Equations 6.1.4, 6.1.5 and 6.1.6 of Notation 6.1, we define* $U_n := R[\tau_{1n}, \dots, \tau_{sn}]$, $B_n = (1 + zU_n)^{-1}U_n$,

$$A := K(\tau_1, \dots, \tau_s) \cap R^*, \quad U := \bigcup_{n=1}^{\infty} U_n, \quad \text{and} \quad B := \bigcup_{n=1}^{\infty} B_n = (1+zU)^{-1}U.$$

The following statements are equivalent:

1. *The extension $U_0 := R[\tau_1, \dots \tau_s] \hookrightarrow R^*[1/z]$ is flat.*
2. *The ring B is Noetherian.*
3. *The extension $B \hookrightarrow R^*$ is faithfully flat.*
4. *The ring A is Noetherian and $A = B$.*
5. *The ring U is Noetherian*
6. *The ring A is Noetherian and is a localization of a subring of $U_0[1/z]$.*
7. *The ring A is Noetherian and is a localization of a subring of $U[1/z]$.*

Proof. For the proof of Theorem 8.8, use the identifications in (6.13) and (6.14); then apply Noetherian Flatness Theorem 8.3 (Homomorphic Images Theorem). □

Remarks 8.9. (1) Noetherian Flatness Theorem 8.8 completes the proofs of the iterative examples Theorems 4.12 and 7.2; see Example 7.15.

[2] This condition implies that the τ_i are regular elements of R^*. See Note 5.5

(2) The original proof given for Noetherian Flatness Theorem 8.8 (Inclusion Version) in [**59**] is an adaptation of a proof given by Heitmann in [**74**, page 126]. Heitmann considers the case where there is one transcendental element τ and defines the corresponding extension U to be a *simple PS-extension of R for z*. Heitmann proves in this case that a certain monomorphism condition on a sequence of maps is equivalent to U being Noetherian [**74**, Theorem 1.4].

In Chapters 9, 10, 13, 17 and 18 we study the condition in Noetherian Flatness Theorem 8.8 (inclusion Version) that the embedding $U_0 \to R^*[1/z]$ is flat. Consider an extension of polynomial subrings of $R^*[1/z]$

$$U_0 = S := R[s_1, \dots, s_m] \overset{\varphi}{\hookrightarrow} T := R[t_1, \dots, t_n] \overset{\psi}{\hookrightarrow} R^*[1/z],$$

where $m, n \in \mathbb{N}$, each of the sets $\{s_i\}$ and $\{t_j\}$ is algebraically independent over R and ψ is flat. Then flatness of φ implies flatness of $U_0 \hookrightarrow R^*[1/z]$. This idea is the basis for Insider Construction 13.1. Moreover, non-flatness of the extension φ sometimes implies non-flatness of the extension $U_0 \hookrightarrow R^*[1/z]$; see Theorem 13.3.

REMARK 8.10. Examples where $A = B$ and A is not Noetherian show that it is possible for A to be a localization of U and yet for A, and therefore also U, to fail to be Noetherian; see Example 18.1 and Theorem 18.5. Thus the equivalent conditions of Noetherian Flatness Theorem 8.8 are not implied by the property that A is a localization of U.

EXAMPLE 8.11. Recall that in Example 7.3 we may take $k = \mathbb{Q}$ and $\sigma := e^x - 1$ and $\tau := e^y - 1$. We have $\theta := \frac{\sigma - \tau}{x - y}$ is in $A := \mathbb{Q}[[x, y]] \cap \mathbb{Q}(x, y, \sigma, \tau)$ and not in $B := \bigcup_{n \in \mathbb{N}} \mathbb{Q}[x, y, \sigma_n, \tau_n]_{(x,y,\sigma_n,\tau_n)}$, where σ_n and τ_n are the endpieces defined in (6.1.1). Then A is Noetherian and $B \subsetneq A$. Set $C := \bigcup_{n \in \mathbb{N}} \mathbb{Q}[x, y, \sigma_n]_{(x,y,\sigma_n)}$. Then $C = \mathbb{Q}[y]_{(y)}[[x]] \cap \mathbb{Q}(x, y, \sigma)$ is an excellent two-dimensional regular local domain; see Theorem 9.2 and Corollary 9.7 of Chapter 9. Inside the (y)-adic completion of C, we define $U := \bigcup_{n \in \mathbb{N}} C[\tau_n]$ as in (6.1.5). The ring B is the localization of U at the multiplicative system $1 + yU$, and the rings B and U are not Noetherian. It follows that A is not a localization of U by Theorem 8.8.

PROOF. By Claim 7.4, $\theta \notin B$. If U were Noetherian, then B would be Noetherian. But the maximal ideal of B is $(x, y)B$, so if B were Noetherian, then it would be a regular local domain with completion $\mathbb{Q}[[x, y]]$ by Construction Properties Theorem 6.19. Since the completion of a local Noetherian ring is a faithfully flat extension by Remark 3.2.3, and since the field of fractions of B is $\mathbb{Q}(x, y, \sigma, \tau)$, then B would equal A by Remark 2.21.9.

That A is Noetherian follows from Valabrega's Theorem 4.2. If A were a localization of U, then A would be a localization of B. But each of A and B has a unique maximal ideal and the maximal ideal of A contains the maximal ideal of B. Therefore $B \subsetneq A$ implies that A is not a localization of B. $\square$

The following diagram displays the situation concerning possible implications among certain statements for Inclusion Construction 5.3 and the approximations in Section 6.1:

$\boxed{R^*[1/a] \text{ is flat over } U_0 = R[\underline{\tau}]}$ $\qquad$ $\boxed{U \text{ Noetherian}}$

$\boxed{A \text{ is a localization of } U}$ $\qquad$ $\boxed{A \text{ Noetherian}}$

REMARK 8.12. In connection with the flatness property, if $U_0 := R[\underline{\tau}] \hookrightarrow R^*[1/z]$ is flat, then for each $P \in \operatorname{Spec} R^*[1/z]$ one has that $\operatorname{ht} P \geq \operatorname{ht}(P \cap U_0)$. We show in Chapter 11 that conversely this height inequality in certain contexts implies flatness.

Inclusion Construction 5.3 might seem simpler to use than its more general form, Homomorphic Image Construction 5.4. The reader may wonder why we bother with the latter construction at all. In fact Homomorphic Image Construction 5.4 has more flexibility and may result in more unusual examples than does Inclusion Construction 5.3. As Proposition 8.13 shows, the universally catenary property extends from a Noetherian local base domain to Noetherian rings constructed using Inclusion Construction 5.3. On the other hand, Remark 8.14 and Example 9.11 show that examples constructed with Homomorphic Image Construction 5.4 may result in a non-universally catenary Noetherian local domain when the base domain is universally catenary, Noetherian and local.

PROPOSITION 8.13. *Assume the notation of Noetherian Flatness Theorem 8.8 (Inclusion Version) with the additional assumption that* $(R, \mathbf{m})$ *is a universally catenary Noetherian local domain. That is, the rings* A *and* B *constructed using Inclusion Construction 5.3 with base ring* R, *are given by Equation 6.1.6, for some choice of* $z \in \mathbf{m}$, $n \in \mathbb{N}$ *and* $\tau_1, \ldots, \tau_n \in R^*$, *the* (z)*-adic completion of* R. *Then:*

(1) *If* A *is Noetherian, then* A *is a universally catenary Noetherian local domain.*
(2) *If* B *is Noetherian, then* $B = A$ *and* B *is a universally catenary local domains.*

PROOF. By Construction Properties Theorem 6.19.4, $R^* = B^* = A^*$. By Proposition 6.20, A and B are local and their maximal ideals are $\mathbf{m}A$ and $\mathbf{m}B$, respectively. The $\mathbf{m}$-, $\mathbf{m}A$- and $\mathbf{m}B$-adic completions of R, A and B, respectively, all equal the $\mathbf{m}R^*$-adic completion of R^*, and so $\widehat{R} = \widehat{A} = \widehat{B}$. By Ratliff's Theorem 3.17, a Noetherian local domain C is universally catenary if and only if its completion $\widehat{C}$ is equidimensional. By assumption R is universally catenary and so $\widehat{R}$ is equidimensional. Thus, if A is Noetherian, then A is also universally catenary. If B is Noetherian, then $B = A$, by Noetherian Flatness Theorem 8.8, and so B is universally catenary. $\square$

REMARK 8.14. In contrast to Proposition 8.13, suppose $(R, \mathbf{m})$ is a universally catenary Noetherian local domain and we apply Homomorphic Image Construction 5.4, for some appropriate nonzero element $z \in \mathbf{m}$ and ideal I of R^*, the (z)-adic completion of R. Then the constructed domains A and B are again local and they satisfy

$$A^* = B^* = R^*/I, \quad \text{and so} \quad \widehat{A} = \widehat{B} = \widehat{R}/I\widehat{R},$$

by Construction Properties Theorem 6.17.4. It is not necessarily true that $\widehat{R}/I\widehat{R}$ is equidimensional. In Example 9.11 we construct a Noetherian local domain A that is not universally catenary by using Homomorphic Image Construction 5.4 applied to the universally catenary ring R that is the localized polynomial ring in 3 variables over a field.

Exercise

(1) For the strictly descending chain of one-dimensional local domains

$$A_1 \supset A_2 \supset \cdots \supset A_n \supset \cdots$$

that are birational extensions of $R = k[x,y]$ given in Example 8.7, describe the integral domain $D := \bigcap_{n=1}^{\infty} A_n$.

Suggestion: Since $\mathbf{n}_n \cap R = (x,y)R$, we have $R_{(x,y)R} \subset A_n$ for each $n \in \mathbb{N}$. By Exercise 4 of Chapter 5, the ring A_n may be described as

$$A_n = \{\ a/b \mid a, b \in R,\ b \neq 0 \ \text{ and } a \in I^n + bR^*\ \}.$$

Show that $a \in I^n + bR^*$ for all $n \in \mathbb{N}$ if and only if $a/b \in R_{(x,y)R}$.

CHAPTER 9

Examples where flatness holds

We continue the notation of the preceding chapters: R is a Noetherian integral domain with field of fractions K, z is a nonzero nonunit of R, and R^* is the (z)-adic completion of R. We consider R^* as a power series ring in z over R in the sense of Remark 3.3. Assume that I is an ideal of R^* such that $\mathbf{p} \cap R = (0)$ for each $\mathbf{p} \in \operatorname{Ass}(R^*/I)$.

In Sections 9.1 and 9.2 we present several examples of Homomorphic Image Construction 5.4 where the flatness condition of Noetherian Flatness Theorem 8.3 holds; that is, the map $R \hookrightarrow (R^*/I)[1/z]$ is flat. We also describe several of these examples using Inclusion Construction 5.3. Inclusion Construction 5.3 is in certain ways more transparent.

In Section 9.3 we investigate special properties of the intersection domain $A := K \cap (R^*/I)$ from Homomorphic Image Construction 5.4, such as having geometrically regular formal fibers. We present in Example 9.11 an example of a Noetherian local domain that is not universally catenary, but has geometrically regular formal fibers. Homomorphic Image Construction 5.4 permits the construction of such examples, whereas sometimes Inclusion Construction 5.3 does not; see Proposition 8.13 and Remark 8.14.

9.1. Polynomial rings over special DVRs

In view of Noetherian Flatness Theorem 8.3, it is natural to ask about the existence of ideals I of the (z)-adic completion R^* such that $R \hookrightarrow (R^*/I)[1/z]$ is flat. In the case where R is a polynomial ring over a field and R^* is the completion with respect to one of the variables, Polynomial Example Theorem 9.5 presents explicit ideals I of R^* such that $R \hookrightarrow (R^*/I)[1/z]$ is flat. The intersection domains A obtained with Homomorphic Image Construction 5.4 using these ideals are polynomial rings over special DVRs, and they are equal to their approximation domains B.

First we prove Polynomial Example Theorem 9.2 for Inclusion Construction 5.3; this version is useful for many of our examples and it is vital to the Insider Construction in Chapters 10 and13. The proof is easier for this form. The isomorphisms obtained from Notation 6.13 and Proposition 6.14 yield Polynomial Example Theorem 9.5 for Homomorphic Image Construction 5.4. We give localized versions of both Polynomial Example Theorems 9.2 and 9.5 in Localized Polynomial Example Theorem 9.7.

We present the setting used for the versions of the first two Polynomial Example Theorems together. For convenience we also include the definitions of the intersection and approximation domains corresponding to the two constructions from Sections 5.1, 6.1 and 6.2.

Setting and Notation 9.1. Let x be an indeterminate over a field k. Let r be a nonnegative integer and s a positive integer. Assume that $\tau_1, \dots, \tau_s \in xk[[x]]$ are algebraically independent over $k(x)$ and let $y_1, \dots, y_r$ and $t_1, \dots, t_s$ be additional indeterminates. We define the following rings:

$$\text{(9.1.a)} \quad R := k[x, y_1, \dots, y_r], \;\; R^* = k[y_1, \dots, y_r][[x]], \;\; V = k(x, \tau_1, \dots, \tau_s) \cap k[[x]].$$

Notice that R^* is the (x)-adic completion of R and V is a DVR by Remark 2.1.

We use the base ring R to define as in Construction 5.3 and Section 6.1

$$(9.1_{\text{incl}}) \qquad A_{\text{incl}} := k(x, y_1, \dots, y_r, \tau_1, \dots \tau_s) \cap R^*, \;\; B_{\text{incl}} := (1 + xU_{\text{incl}})^{-1} U_{\text{incl}},$$

where $U_{\text{incl}} := \bigcup_{n\in\mathbb{N}} R[\tau_{1n}, \dots \tau_{sn}]$, each τ_{in} is the n^{th} endpiece of τ_i and each $\tau_{in} \in R^*$, for $1 \le i \le s$. By Construction Properties Theorem 6.19.4, the ring R^* is the (x)-adic completion of each of the rings $A_{\text{incl}}, B_{\text{incl}}$ and U_{incl}.

Set $S := k[x, y_1, \dots, y_r, t_1, \dots, t_s]$, let S^* be the (x)-adic completion of S and let $\sigma_i := t_i - \tau_i$, for each i. We define $I := (t_1 - \tau_1, \dots, t_s - \tau_s)S^* = (\sigma_1, \dots, \sigma_s)S^*$.

With S as the base ring, we define as in Construction 5.4 and Section 6.2

$$(9.1_{\text{hom}})$$
$$A_{\text{hom}} := k(x, y_1, \dots, y_r, t_1, \dots, t_s) \cap (S^*/I), \;\; B_{\text{hom}} := (1 + xU_{\text{hom}})^{-1} U_{\text{hom}},$$

where $U_{\text{hom}} := \bigcup_{n\in\mathbb{N}} S[\sigma_{1n}, \dots, \sigma_{sn}]$, each σ_{in} is the n^{th} frontpiece of σ_i and each $\sigma_{in} \in \mathcal{Q}(S) \cap (S^*/I)$, for $1 \le i \le s$, by Proposition 6.11. By Construction Properties Theorem 6.17.4, the ring S^*/I is the (x)-adic completion of each of the rings $A_{\text{hom}}, B_{\text{hom}}$ and U_{hom}.

Proposition 6.14 and Notation 6.13 imply the following isomorphisms

$$\text{(9.1.b)} \quad U_{\text{incl}} \cong U_{\text{hom}}, \qquad A_{\text{incl}} \cong A_{\text{hom}}, \qquad B_{\text{incl}} \cong B_{\text{hom}} \qquad R^* \cong S^*/I.$$

In Polynomial Example Theorems 9.2, 9.5 and 9.7, the rings A and B are often excellent; see Definition 3.31. This is not always true; see Remark 9.4 below.

Polynomial Example Theorem 9.2. (Inclusion Version) *Assume Setting and Notation 9.1 with R, R^*, V, A_{inc} and B_{inc} as defined in Equations 9.5.a and 9.1_{incl}. Then:*

1. *The canonical map $\alpha : R[\tau_1, \dots \tau_s] \hookrightarrow R^*[1/x]$ is flat.*
2. *$B_{incl} = A_{incl}$ is Noetherian of dimension $r + 1$ and is the localization $(1 + xV[y_1, \dots, y_r])^{-1} V[y_1, \dots, y_r]$ of the polynomial ring $V[y_1, \dots, y_r]$ over the DVR V. Thus A_{incl} is a regular integral domain.*
3. *B_{incl} is a directed union of localizations of polynomial rings in $r + s + 1$ variables over k.*
4. *If k has characteristic zero, then the ring $B_{incl} = A_{incl}$ is excellent.*

Proof. The map

$$k[x, \tau_1, \dots \tau_s] \hookrightarrow k[[x]][1/x]$$

is flat by Remark 2.21.4 since $k[[x]][1/x]$ is a field. By Fact 2.22

$$k[x, \tau_1, \dots \tau_s] \otimes_k k[y_1, \dots, y_r] \hookrightarrow k[[x]][1/x] \otimes_k k[y_1, \dots, y_r]$$

is flat. We also have $k[[x]][1/x] \otimes_k k[y_1, \dots, y_r] \cong k[[x]][y_1, \dots, y_r][1/x]$ and

$$k[x, \tau_1, \dots \tau_s] \otimes_k k[y_1, \dots, y_r] \cong k[x, y_1, \dots, y_r][\tau_1, \dots \tau_s].$$

Hence the natural inclusion map

$$k[x, y_1, \dots, y_r][\tau_1, \dots \tau_s] \overset{\beta}{\hookrightarrow} k[[x]][y_1, \dots, y_r][1/x]$$

is flat. Also $k[[x]][y_1,\dots,y_r] \hookrightarrow k[y_1,\dots,y_r][[x]]$ is flat since it is the map taking a Noetherian ring to an ideal-adic completion; see Remark 3.2.2. Therefore

$$k[[x]][y_1,\dots,y_r][1/x] \overset{\delta}{\hookrightarrow} k[x,y_1,\dots,y_r][[x]][1/x]$$

is flat. It follows that the map

$$k[x,y_1,\dots,y_r][\tau_1,\dots\tau_s] \overset{\delta\circ\beta}{\hookrightarrow} R^*[1/x] = k[y_1,\dots,y_r][[x]][1/x]$$

is flat. Thus the Noetherian Flatness Theorem 8.8 implies items 1 - 3.

If k has characteristic zero, then V is excellent by Remark 3.32; hence item 4 follows from item 2 since excellence is preserved under localization of a finitely generated algebra by Remark 3.32. For more details see [**94**, (34.B),(33.G) and (34.A)], [**44**, Chap. IV]. $\square$

We observe in Proposition 9.3 that over a perfect field k of characteristic $p > 0$ (so that $k = k^{1/p}$) a one-dimensional form of the construction in Polynomial Example Theorem 9.2 yields a DVR that is not a Nagata ring, defined in Definition 2.3.1, and thus is not excellent; see Remark 3.32, [**96**, p. 264], [**94**, Theorem 78, Definition 34.A].

PROPOSITION 9.3. *Let k be a perfect field of characteristic $p > 0$, let the element τ of $xk[[x]]$ be such that x and τ are algebraically independent over k and set $V := k(x,\tau)\cap k[[x]]$. Then V is a DVR for which the integral closure $\overline{V}$ of V in the purely inseparable field extension $k(x^{1/p},\tau^{1/p})$ is not a finitely generated V-module. Hence V is not a Nagata ring and so is not excellent.*

PROOF. It is clear that V is a DVR with maximal ideal xV. Since x and τ are algebraically independent over k, $[k(x^{1/p},\tau^{1/p}) : k(x,\tau)] = p^2$. Let W denote the integral closure of V in the field extension $k(x^{1/p},\tau)$ of degree p over $k(x,\tau)$. Notice that

$$W = k(x^{1/p},\tau) \cap k[[x^{1/p}]] \quad \text{and} \quad \overline{V} = k(x^{1/p},\tau^{1/p}) \cap k[[x^{1/p}]]$$

are both DVRs having residue field k and maximal ideal generated by $x^{1/p}$. Thus $\overline{V} = W + x^{1/p}\overline{V}$. If $\overline{V}$ were a finitely generated W-module, then by Nakayama's Lemma it would follow that $W = \overline{V}$. This is impossible because $\overline{V}$ is not birational over W. It follows that $\overline{V}$ is not a finitely generated V-module, and hence V is not a Nagata ring. $\square$

REMARK 9.4. Let $V = k(x,\tau)\cap k[[x]]$ and A_{incl} be as in Setting and Notation 9.1 with $s = r = 1$, and suppose that k is a perfect field with characteristic $p > 0$. By Proposition 9.3, the ring V is not excellent. By Polynomial Example Theorem 9.5.2, $A_{incl} = (1 + xV[y])^{-1}V[y]$, and so the ring V is a homomorphic image of A_{incl}. Since excellence is preserved by taking homomorphic images, the two-dimensional regular ring A_{incl} is not excellent in this situation; see Remark 3.32. In general, over a perfect field of characteristic $p > 0$, the Noetherian regular ring $A = B$ obtained in Polynomial Example Theorem 9.2 fails to be excellent.

We now state and prove the homomorphic image version of the theorem.

POLYNOMIAL EXAMPLE THEOREM 9.5. (Homomorphic Image Version) *Assume Setting and Notation 9.1 with A_{hom} and B_{hom} as in Equation 9.1$_{hom}$. Then*

(1) *The canonical map $\alpha : S \hookrightarrow (S^*/I)[1/x]$ is flat.*

(2) *$B_{hom} = A_{hom}$ is Noetherian of dimension $r+1$ and is a localization of the polynomial ring $V[y_1, \ldots, y_r]$ over the DVR V. Thus A_{hom} is a regular integral domain.*
(3) *B_{hom} is a directed union of localizations of polynomial rings in $r+s+1$ variables over k.*
(4) *If k has characteristic zero, then $B_{hom} = A_{hom}$ is excellent.*

PROOF. The ideal $I := (t_1 - \tau_1, \ldots, t_s - \tau_s)S^*$ is a prime ideal of S^* and $S^*/I \cong k[y_1, \ldots, y_r][[x]]$. The fact that $\tau_1, \ldots, \tau_s$ are algebraically independent over $k(x)$ implies that $I \cap S = (0)$.

We identify Homomorphic Image Construction 5.4 for the ring S with Inclusion Construction 5.3 for the ring R as in the following diagram slightly modified from Chapter 5, where λ is the R-algebra isomomorphism that maps $t_i \to \tau_i$ for $i = 1, \ldots, s$, and $K := \mathcal{Q}(R)$.

(9.5.1)
$$\begin{array}{ccccccc}
 & & S := R[t_1, \ldots, t_s] & \longrightarrow & A_{\text{hom}} := K(t_1, \ldots, t_s) \cap (S^*/I) & \longrightarrow & S^*/I \\
 & & \lambda\big\downarrow & & \lambda\big\downarrow & & \lambda\big\downarrow \\
R & \longrightarrow & R[\tau_1, \ldots, \tau_s] & \longrightarrow & A_{\text{incl}} := K(\tau_1, \ldots, \tau_s) \cap R^* & \longrightarrow & R^*.
\end{array}$$

In view of the identifications displayed in this diagram, the items of the homomorphic image version of Polynomial Example Theorem 9.5 follow from the corresponding items of the inclusion version, Polynomial Example Theorem 9.2. $\square$

REMARK 9.6. As we indicate in the proof of Theorem 9.5, the examples constructed using the inclusion notation of Setting and Notation 9.1 exactly correspond with the part using the homomorphic image notation. They are essentially the same examples. This is also shown by the identifications indicated in the proof of Theorem 9.5. More details on this identification are in Chapters 5 and 6. Thus there exists a one-dimensional ring A over a perfect field k of characteristic $p > 0$ constructed using Homomorphic Image Construction 5.4 and Setting 9.1 that is Noetherian but not excellent. This ring corresponds with the identifications to the ring described in Remark 9.4 with Inclusion Construction 5.3.

We give below a localized form of Polynomial Example Theorems 9.5 and 9.2. Then the rings R, A, and B are local. The ring B is a localization of $U = \bigcup_{n=1}^{\infty} U_n$, where each $U_n = R[\tau_{in}]$, and B is also a localization of $U' = \bigcup_{n=1}^{\infty} U'_n$, where each $U'_n = k[x, y_j, \tau_{in}]$. This simpler second form U' of U is used in Chapters 17 and 18.

LOCALIZED POLYNOMIAL EXAMPLE THEOREM 9.7. *If we adjust Setting and Notation 9.1 so that the base rings are the regular local rings*

$$R := k[x, y_1, \ldots, y_r]_{(x, y_1, \ldots, y_r)} \quad \textit{and} \quad S := k[x, y_1, \ldots, y_r, t_1, \ldots, t_s]_{(x, y_1, \ldots, y_r, t_1, \ldots, t_s)},$$

then the conclusions of Polynomial Example Theorems 9.2 and 9.5 are still valid. In particular:

(1) *For Inclusion Construction 5.3, with the notation of Equation 9.1$_{incl}$,*

$$A_{incl} = B_{incl} = V[y_1, \ldots, y_r]_{(x, y_1, \ldots, y_r)},$$

is a Noetherian regular local ring, and the extension $R[t_1,\dots,t_s] \to R^*[1/x]$ *is flat. In addition,*

$$B_{incl} = \bigcup_{n=1}^{\infty} (U_n)_{\mathbf{m}_n} = U_{\mathbf{m}_U} = \bigcup_{n=1}^{\infty} (U'_n)_{\mathbf{m}'_n} = U'_{\mathbf{m}_{U'}}, \text{ where } U = \bigcup_{n=1}^{\infty} U_n,\ U' = \bigcup_{n=1}^{\infty} U'_n,$$

$$U_n = k[x,y_1,\dots,y_r]_{(x,y_1,\dots,y_r)}[\tau_{1n},\dots,\tau_{sn}], \quad \mathbf{m}_n = (x,y_1,\dots,y_r,\tau_{1n},\dots,\tau_{sn})U_n,$$

$$U'_n = k[x,y_1,\dots,y_r,\tau_{1n},\dots,\tau_{sn}], \quad \mathbf{m}'_n = (x,y_1,\dots,y_r,\tau_{1n},\dots,\tau_{sn})U'_n,$$

$$\mathbf{m}_U = (x,y_1,\dots,y_r)U \quad \text{and} \quad \mathbf{m}'_U = (x,y_1,\dots,y_r)U'.$$

(2) *For Homomorphic Image Construction 5.6, with the notation of Equation 9.1*$_{hom}$, $A_{hom} = V[y_1,\dots,y_r]_{(x,y_1,\dots,y_r)} \cong B_{hom}$.
(3) $A_{incl} \cong A_{hom}$.
(4) *The canonical map* $\alpha : S \hookrightarrow (S^*/I)[1/x]$ *is flat, for the "adjusted"* S.

Proof. The proofs of Theorems 9.2 and 9.5 apply to the localized polynomial rings. The statements about the rings U and U' follow from Remark 6.4. Item 3 follows from Diagram 9.5.1. □

Example 9.8. Let S be as in Localized Polynomial Example Theorem 9.7. Then $t_1 - \tau_1, \dots, t_s - \tau_s$ is a regular sequence in S^*. Let $I = (t_1 - \tau_1, \dots, t_s - \tau_s)S^*$ as in Localized Polynomial Example Theorem 9.7. Then Theorem 8.6 implies that $S \hookrightarrow (S^*/I^n)[1/x]$ is flat for each positive integer n. Using I^n in place of I, Theorem 9.7.2 implies the existence for every r and n in $\mathbb{N}$ of a Noetherian local domain A having dimension $r+1$ such that the (x)-adic completion A^* of A has nilradical $\mathbf{n}$ with $\mathbf{n}^{n-1} \neq (0)$.

Here are some more specific examples to which Polynomial Example Theorems 9.5, 9.2 and 9.7 apply. Example 9.9 shows that the dimension of U can be greater than the dimension of $B_{\rm hom}$.

Examples 9.9. Assume the setting and notation of (9.1).

(1) Let $S := k[x,t_1,\dots,t_s]$, that is, there are no y variables and let S^* denote the (x)-adic completion of S. Then $I = (t_1 - \tau_i, \dots, t_s - \tau_s)$ and, by Theorem 9.5,

$$(S^*/I) \cap \mathcal{Q}(S) = A_{\rm hom} = B_{\rm hom}$$

is the DVR obtained by localizing U at the prime ideal xU. In this example $S[1/x] = U[1/x]$ has dimension $s+1$ and so $\dim U = s+1$, while $\dim(S^*/I) = \dim A_{\rm hom} = \dim B_{\rm hom} = 1$.

(2) Essentially the same example as in item 1 can be obtained by using Theorem 9.2 as follows. Let $R = k[x]$, then $R^* = k[[x]]$ and

$$A_{\rm incl} = k(x,\tau_1,\dots,\tau_s) \cap k[[x]] \quad \text{and} \quad A_{\rm incl} = B_{\rm incl},$$

by Theorem 9.2. In this case $U_{\rm incl}$ is a directed union of polynomial rings over k,

$$U_{\rm incl} = \bigcup_{n=1}^{\infty} k[x][\tau_{1n},\dots,\tau_{sn}],$$

where the τ_{in} are the $n^{\rm th}$ endpieces of the τ_i as in Section 6.1. By Proposition 6.11, the endpieces are related to the frontpieces of the homomorphic image construction.

(3) Applying Localized Polynomial Example Theorem 9.7, one can modify Example 9.9.1 by taking S to be the $(s+1)$-dimensional regular local domain

$k[t_1,\dots,t_s,x]_{(t_1,\dots,t_s,x)}$. In this case $S[1/x] = U[1/x]$ has dimension s, while we still have $S^*/I \cong k[[x]]$. Thus $\dim U = s+1$ and $\dim(S^*/I) = 1 = \dim A_{\text{hom}} = \dim B_{\text{hom}}$.

One can also obtain a local version of Example 9.9.2 using the inclusion construction with $R = k[x]_{(x)}$ and applying Theorem 9.2. We again have $R^* = k[[x]]$.

With S as in either (9.9.1) or (9.9.3), the domains B_n constructed from S as in Section 6.2 of Chapter 6 are $(s+1)$-dimensional regular local domains dominated by $k[[x]]$ and having k as a coefficient field. In either case, since $(S^*/I)[1/x]$ is a field, the extension $S \hookrightarrow (S^*/I)[1/x]$ is flat. Thus by Theorem 8.3 the family $\{B_n\}_{n\in\mathbb{N}}$ is a directed union of $(s+1)$-dimensional regular local domains whose union B is Noetherian, and is, in fact a DVR.

(4) In the notation of (9.1) with the adjustment of Localized Polynomial Example Theorem 9.7, let $r = 1$ and $y_1 = y$. Thus $S = k[x,y,t_1,\dots,t_s]_{(x,y,t_1,\dots,t_s)}$. Then $S^*/I \cong k[y]_{(y)}[[x]]$. By Theorem 9.7.2, the extension $S \hookrightarrow (S^*/I)[1/x]$ is flat. Let $V = k[[x]] \cap k(x,\tau_1,\dots,\tau_s)$. Then V is a DVR and $(S^*/I) \cap \mathcal{Q}(S) \cong V[y]_{(x,y)}$ is a 2-dimensional regular local domain that is the directed union of $(s+2)$-dimensional regular local domains.

9.2. Transfer to the intersection of two ideals

We use Homomorphic Image Construction 5.4 and assume the notation of Noetherian Flatness Theorem 8.3. In Theorem 9.10 we show that in certain circumstances the flatness, Noetherian and computability properties associated with ideals I_1 and I_2 in place of the ideal I of R^* as described in Theorem 8.3 transfer to their intersection $I_1 \cap I_2$.[1] We use Theorem 9.10 in Section 9.3 to show in Theorem 9.13 that the property of regularity of the generic formal fibers also transfers in certain cases to the intersection domain associated with the ideal $I_1 \cap I_2$.

Theorem 9.10. *Let R be a Noetherian integral domain with field of fractions K, let z be a nonzero nonunit of R, and let R^* denote the (z)-adic completion of R. Let I_1 and I_2 be ideals of R^* such that*

(i) $z^n \in I_1 + I_2$ *for some positive integer n,*
(ii) $\mathbf{p} \cap R = (0)$, *for each $\mathbf{p} \in \operatorname{Ass}(R^*/I_1)$, and each $\mathbf{p} \in \operatorname{Ass}(R^*/I_2)$, and*
(iii) $R \hookrightarrow (R^*/I_i)[1/z]$ *is flat for each i.*

We set $A_i := K \cap (R^*/I_i)$ and set $A := K \cap (R^*/(I_1 \cap I_2))$. Then:

(1) *The ideal $I := I_1 \cap I_2$ satisfies the conditions of Noetherian Flatness Theorem 8.3 and the map $R \hookrightarrow (R^*/I)[1/z]$ is flat. The (z)-adic completion A^* of A is R^*/I, and the (z)-adic completion A_i^* of A_i is R^*/I_i, for $i = 1,2$.*
(2) *The ring $A^*[1/z] \cong (A_1^*)[1/z] \oplus (A_2^*)[1/z]$. If $Q \in \operatorname{Spec}(A^*)$ and $z \notin Q$, then $(A^*)_Q$ is a localization of either A_1^* or A_2^*.*
(3) *We have $A \subseteq A_1 \cap A_2$, and $(A_1)[1/z] \cap (A_2)[1/z] \subseteq A_P$ for each $P \in \operatorname{Spec} A$ with $z \notin P$. Thus we have $A[1/z] = (A_1)[1/z] \cap (A_2)[1/z]$.*

Proof. By Theorem 8.3, the (z)-adic completion A_i^* of A_i is R^*/I_i. Since $\operatorname{Ass}(R^*/(I_1 \cap I_2)) \subseteq \operatorname{Ass}(R^*/I_1) \cup \operatorname{Ass}(R^*/I_2)$, the condition on associated primes of Theorem 8.3 holds for the ideal $I_1 \cap I_2$. The natural R-algebra homomorphism $\pi : R^* \to (R^*/I_1) \oplus (R^*/I_2)$ has kernel $I_1 \cap I_2$. Further, the localization of π at z is onto because $(I_1 + I_2)R^*[1/z] = R^*[1/z]$. Thus $(R^*/(I_1 \cap I_2))[1/z] \cong$

[1] A generalization of Theorem 9.10 is given in Theorem 16.9.

$(R^*/I_1)[1/z] \oplus (R^*/I_2)[1/z] = (A_1^*)[1/z] \oplus (A_2^*)[1/z]$ is flat over R. Therefore A is Noetherian and $A^* = R^*/I$ is the (z)-adic completion of A.

If $Q \in \operatorname{Spec}(A^*)$ and $z \notin Q$, then A^*_Q is a localization of

$$A^*[1/z] \cong (A_1^*)[1/z] \oplus (A_2^*)[1/z].$$

Every prime ideal of $(A_1^*)[1/z] \oplus (A_2^*)[1/z]$ has the form either $(Q_1)A_1^*[1/z] \oplus (A_2^*)[1/z]$ or $(A_1^*)[1/z] \oplus Q_2A_2^*[1/z]$, where $Q_i \in \operatorname{Spec}(A_i^*)$. It follows that A^*_Q is a localization of either A_1^* or A_2^*.

Since R^*/I_i is a homomorphic image of $R^*/(I_1 \cap I_2)$, the intersection ring $A \subseteq A_1 \cap A_2$. We show that $A_1[1/z] \cap A_2[1/z] \subseteq A[1/z]$. For this, let $P \in \operatorname{Spec} A$ with $z \notin P$. Since $A^* = R^*/I$ is faithfully flat over A, there exists $P^* \in \operatorname{Spec}(A^*)$ with $P^* \cap A = P$. Then $z \notin P^*$ implies $A^*_{P^*}$ is either $(A_1^*)_{P_1^*}$ or $(A_2^*)_{P_2^*}$, where $P_i^* \in \operatorname{Spec}(A_i^*)$. By symmetry, we may assume $A^*_{P^*} = (A_1^*)_{P_1^*}$. Let $P_1 = P_1^* \cap A_1$. Since $A_P \hookrightarrow A^*_{P^*}$ and $(A_1)_{P_1} \hookrightarrow (A_1^*)_{P_1^*}$ are faithfully flat, we have

$$A_P = A^*_{P^*} \cap K = (A_1^*)_{P_1^*} \cap K = (A_1)_{P_1} \supseteq (A_1)[1/z].$$

It follows that $(A_1)[1/z] \cap (A_2)[1/z] \subseteq A_P$. Thus we have

$$(A_1)[1/z] \ \cap \ (A_2)[1/z] \ \subseteq \ \bigcap\{A_P \mid P \in \operatorname{Spec} A \text{ and } z \notin P\} \ = \ A[1/z].$$

Since $A[1/z] \subseteq (A_i)[1/z]$, for $i = 1, 2$, we have $A[1/z] = (A_1)[1/z] \cap (A_2)[1/z]$. $\square$

The ring B of Example 9.11 is a two-dimensional Noetherian local domain that birationally dominates a three-dimensional regular local domain and is such that B is not universally catenary. The completion of B has two minimal primes one of dimension one and one of dimension two and this implies B is not universally catenary by Theorem 3.17. It follows that B is not a homomorphic image of a regular local ring because every homomorphic image of a regular local ring, or even of a Cohen-Macaulay local ring, is universally catenary by Remark 3.19. We present in Chapter 16 other examples of Noetherian local domains of various dimensions that are not universally catenary and that have properties similar to those of the Noetherian local domain B of Example 9.11.

EXAMPLE 9.11. Let k be a field of characteristic zero[2] and let x, y, z be indeterminates over k. Let $R = k[x, y, z]_{(x,y,z)}$, let K denote the field of fractions of R, and let $\tau_1, \tau_2, \tau_3 \in xk[[x]]$ be algebraically independent over $k(x, y, z)$. Let R^* denote the (x)-adic completion of R. We take the ideal I to be the intersection of the two prime ideals $Q := (z - \tau_1, y - \tau_2)R^*$, which has height 2, and $P := (z - \tau_3)R^*$, which has height 1. Then R^*/P and R^*/Q are examples of the form considered in Examples 9.9. Thus $(R^*/P)[1/x]$ and $(R^*/Q)[1/x]$ are both flat over R. Here $R^*/P \cong k[y]_{(y)}[[x]]$; the ring $V := k[[x]] \cap k(x, \tau_3)$ is a DVR, and $A_1 := (R^*/P) \cap K \cong V[y]_{(x,y)}$ is a two-dimensional regular local domain that is a directed union of three-dimensional RLRs, while $A_2 := (R^*/Q) \cap K$ is a DVR.

Since $\tau_1, \tau_3 \in xk[[x]]$, the ideal $(z - \tau_1, z - \tau_3)R^*$ has radical $(x, z)R^*$. Hence the ideal $P + Q$ is primary for the maximal ideal $(x, y, z)R^*$, so, in particular, P is not contained in Q. Therefore the representation $I = P \cap Q$ is irredundant and $\operatorname{Ass}(R^*/I) = \{P, Q\}$. Since $P \cap R = Q \cap R = (0)$, the ring R injects into R^*/I. Let $A := K \cap (R^*/I)$.

[2]The characteristic zero assumption implies that the intersection rings A_1 and A_2 as constructed below are excellent; see Theorem 9.5.4.

By (9.10.1), the inclusion $R \hookrightarrow (R^*/I)[1/x]$ is flat. By Theorem 8.3, the ring A is Noetherian and is a localization of a subring of $R[1/x]$. The map $A \hookrightarrow \widehat{A}$ of A into its completion factors through the map $A \hookrightarrow A^* = R^*/I$. Since R^*/I has minimal primes P/I and Q/I with $\dim R^*/P = 2$ and $\dim R^*/Q = 1$, and since $\widehat{A}$ is faithfully flat over $A^* = R^*/I$, the ring $\widehat{A}$ is not equidimensional. It follows that A is not universally catenary by Theorem 3.17.

9.3. Regular maps and geometrically regular formal fibers

We show in Corollary 9.14 that the ring $A = B$ of Example 9.11 has geometrically regular formal fibers.[3] Another example of a Noetherian local domain that is not universally catenary but has geometrically regular formal fibers is given in [**44**, (18.7.7), page 144] using a gluing construction; also see [**43**, (1.1)].

Proposition 9.12. *Let $R, z,\ R^*,\ A,\ B$ and I be as in Theorem 8.3. Assume that the map $\psi_P : R_{P\cap R} \longrightarrow (R^*/I)_P$ is regular, for each $P \in \operatorname{Spec}(R^*/I)$ with $z \notin P$. Then $A = B$ and moreover:*

(1) *A is Noetherian and the map $A \longrightarrow A^* = R^*/I$ is regular.*
(2) *If R is semilocal with geometrically regular formal fibers and z is in the Jacobson radical of R, then A has geometrically regular formal fibers.*

Proof. Since flatness is a local property by (2.21.1), and regularity of a map includes flatness, the map $\psi_z : R \longrightarrow (R^*/I)[1/z]$ is flat. By Theorem 8.3, the intersection ring A is Noetherian with (z)-adic completion $A^* = R^*/I$. Hence $A \longrightarrow A^*$ is flat.

Let $Q \in \operatorname{Spec}(A)$, let $\mathbf{q} = Q \cap R$, let $k(Q)$ denote the field of fractions of A/Q, and let $A^*_{QA^*} = (A \setminus Q)^{-1}A^*$.

Case 1: $z \in Q$. Then $R/\mathbf{q} = A/Q = A^*/QA^*$. By Equation 3.21.0, we have

$$A^* \otimes_A k(Q) \;=\; \frac{A^*_{QA^*}}{QA^*_{QA^*}} \;=\; \frac{A_Q}{QA_Q} \;=\; k(Q).$$

Thus regularity holds in this case.

Case 2: $z \notin Q$. Let L be a finite algebraic field extension of $k(Q)$. We show the ring $A^* \otimes_A L$ is regular. There is a natural embedding $A^* \otimes_A k(Q) \hookrightarrow A^* \otimes_A L$. Let $W \in \operatorname{Spec}(A^* \otimes_A L)$ and let $W' = W \cap (A^* \otimes_A k(Q))$. We have maps

$$\operatorname{Spec}(A^* \otimes_A k(Q)) \overset{\theta,\cong}{\to} \operatorname{Spec}\Big(\frac{A^*_{QA^*}}{QA^*_{QA^*}}\Big) \quad \text{and} \quad \operatorname{Spec}\Big(\frac{A^*_{QA^*}}{QA^*_{QA^*}}\Big) \overset{\psi}{\to} \operatorname{Spec} A^*,$$

since $A^*_{QA^*}/QA^*_{QA^*} = A^* \otimes_A k(Q)$ by Equation 3.21.0, and $A^* \to A^*_{QA^*}/QA^*_{QA^*}$. Let P be the prime ideal $P := \psi(\theta(W')) \in \operatorname{Spec}(A^*)$; then $P \cap A = Q$.

By assumption the map

$$R_{\mathbf{q}} \longrightarrow (R^*/I)_P = A^*_P$$

is regular. Since $z \notin Q$, it follows that $R_{\mathbf{q}} = U_{Q\cap U} = A_Q$ and that $k(\mathbf{q}) = k(Q)$. Thus the ring $A^*_P \otimes_{A_Q} L$ is regular. Therefore $(A^* \otimes_A L)_W$, which is a localization of this ring, is regular.

For item 2, we use a theorem of Rotthaus [**120**, (3.2), p. 179]: If R is a Noetherian semilocal ring with geometrically regular formal fibers and I is an ideal

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[3] A generalization of Corollary 9.14 is given in Corollary 16.19

of R contained in the Jacobson radical of R, then the I-adic completion of R also has geometrically regular formal fibers; see also [**96**, Remark 2, p. 260]. Thus R^* has geometrically regular formal fibers. Since the formal fibers of R^*/I are a subset of the formal fibers of R^*, the map $A^* = R^*/I \longrightarrow \widehat{A} = \widehat{(R^*/I)}$ is regular. By item 1, the map $A \to A^*$ is regular. The composition of two regular maps is regular [**96**, Thm. 32.1 (i)]. Therefore A has geometrically regular formal fibers, that is, the map $A \longrightarrow \widehat{A}$ is regular. $\square$

Theorem 9.13. *Let R be a Noetherian integral domain with field of fractions K. Let z be a nonzero nonunit of R and let R^* denote the (z)-adic completion of R. Assume that I_1 and I_2 are ideals of R^* such that every prime ideal $P \in \operatorname{Ass}(R^*/I_i)$ satisfies $P \cap R = (0)$, for $i = 1, 2$. Also assume*

(1) *R is semilocal with geometrically regular formal fibers and z is in the Jacobson radical of R.*
(2) *$(R^*/I_1)[1/z]$ and $(R^*/I_2)[1/z]$ are flat R-modules and the ideal $I_1 + I_2$ of R^* contains some power of z.*
(3) *For $i = 1, 2$, $A_i := K \cap (R^*/I_i)$ has geometrically regular formal fibers.*

Then $A := K \cap (R^*/(I_1 \cap I_2)) = B$ and A has geometrically regular formal fibers.

Proof. Let $I = I_1 \cap I_2$. Since R has geometrically regular formal fibers, by Proposition 9.12.2, it suffices to show for $W \in \operatorname{Spec}(R^*/I)$ with $z \notin W$ that $R_{W_0} \longrightarrow (R^*/I)_W$ is regular, where $W_0 := W \cap R$. As in Theorem 9.10, we have $(R^*/I)[1/z] = (R^*/I_1)[1/z] \oplus (R^*/I_2)[1/z]$ and $(R^*/I)_W$ is a localization of either R^*/I_1 or R^*/I_2. Also $A_i^* = R^*/I_i$, for $i = 1, 2$. Suppose $(R^*/I)_W = (R^*/I_1)_{W_1}$, where $W_1 \in \operatorname{Spec}(R^*/I_1)$. Then $R_{W_0} = (A_1)_{W_1 \cap A_1}$ and $(A_1)_{W_1 \cap A_1} \longrightarrow (R^*/I_1)_{W_1}$ is regular. A similar argument holds if $(R^*/I)_W$ is a localization of R^*/I_2. Thus, in either case, $R_{W_0} \longrightarrow (R^*/I)_W$ is regular. $\square$

Corollary 9.14. *The ring $A = B$ of Example 9.11 has geometrically regular formal fibers, that is, the map $\phi : A \hookrightarrow \widehat{A}$ is regular.*

Proof. Since the field k of this example has characteristic zero, it is enough to observe the fibers are regular to conclude that they are geometrically regular; see Remark 3.24. By the definition of R and the observations given in (9.11), the hypotheses of (9.13) are satisfied. $\square$

Exercises

(1) Describe Example 9.9.4 in terms of Inclusion Construction 5.3. In particular, determine the appropriate base ring R for this construction.
(2) For the rings A and A^* of Example 9.11, prove that A^* is universally catenary.

CHAPTER 10

Introduction to the insider construction

We describe in this chapter and in Chapter 13 a version of Inclusion Construction 5.3 that we call the "Insider Construction". We present in this chapter several examples using this Insider Construction including two classical examples. Insider Construction 13.1 simplifies the verification of properties of examples constructed using Inclusion Construction 5.3 such as whether the constructed domains are or are not Noetherian. This chapter is an introduction dealing with special cases of Insider Construction 13.1. In Chapter 13 we present this "Insider Construction" in more generality and with more detail.

For the examples considered in this chapter, we begin with a base ring that is a localized polynomial ring in two or three variables over a field. We construct two "insider" integral domains inside an "outside" ring A^{out}, where A^{out} is constructed from Localized Polynomial Example Theorem 9.7 with the Inclusion Construction 5.3. By Localized Polynomial Example Theorem 9.7, the intersection domain A^{out} is equal to an approximation integral domain B^{out} that is the nested union of localized polynomial rings from Section 6.1. The two insider integral domains contained in A^{out} are: A^{ins}, an intersection of a field with a power series ring as in Construction 5.3, and B^{ins}, a nested union of localized polynomial rings that "approximates" A^{ins} as in Section 6.1.

As we describe in Proposition 10.6, the condition that the insider approximation domain B^{ins} is Noetherian is related to flatness of a map of polynomial rings corresponding to the extension $B^{\text{ins}} \hookrightarrow B^{\text{out}}$. Flatness of this map implies that the insider approximation domain B^{ins} is Noetherian, that the insider intersection domain A^{ins} equals B^{ins} and that A^{ins} is Noetherian. We apply this observation in Proposition 10.2 and Example 10.7 to conclude that certain constructed rings are Noetherian. We use Theorem 13.6 from Chapter 13 to show that other constructed rings are not Noetherian.

We present more details concerning flatness of polynomial extensions in Chapter 11.

10.1. The Nagata example

In Proposition 10.2 we adapt the example of Nagata given as Example 4.8 in Chapter 4, to fit the Insider Construction.

SETTING 10.1. Let k be a field, let x and y be indeterminates over k, and set

$$R := k[x,y]_{(x,y)} \qquad \text{and} \qquad R^* := k[y]_{(y)}[[x]].$$

The power series ring R^* is the xR-adic completion of R. Let $\tau \in xk[[x]]$ be a transcendental element over $k(x)$. Since R^* is an integral domain, every nonzero element of the polynomial ring $R[\tau]$ is a regular element of R^*. Thus the field $k(x, y, \tau)$ is a subfield of $\mathcal{Q}(R^*)$. We define the *intersection domain* A_τ corresponding to τ by $A_\tau := k(x,y,\tau) \cap R^*$.

Let f be a polynomial in $R[\tau]$ that is algebraically independent over $\mathcal{Q}(R)$, for example, $f = (y+\tau)^2$. We set $A_f := \mathcal{Q}(R[f]) \cap R^*$ to be the *intersection domain* corresponding to f. Since $R[f] \subseteq R[\tau]$, we have $k(x,y,f) = \mathcal{Q}(R[f]) \subseteq \mathcal{Q}(R^*)$. In the language of the introduction to this chapter, A_τ is the "outsider" intersection domain A^{out} and A_f, which is contained in A_τ, is the "insider" intersection domain A^{ins}.

Recall from Section 6.1 and Remark 6.4 that there are natural "approximating" outsider and insider domains associated to A_τ and A_f, namely,

$$B_\tau := \bigcup_{\mathbf{n}\in\mathbb{N}} k[x,y,\tau_n]_{(x,y,\tau_n)} \quad \text{and} \quad B_f := \bigcup_{\mathbf{n}\in\mathbb{N}} k[x,y,f_n]_{(x,y,\tau_n)}, \tag{10.1.0}$$

where the τ_n are the n^{th} endpieces of τ and the f_n are the n^{th} endpieces of f. The rings B_τ and B_f are nested unions of localized polynomial rings over k in 3 variables. (These approximating domains are the C_n of Remark 6.4.)

By Localized Polynomial Example Theorem 9.7.1, the extension $T := R[\tau] \xhookrightarrow{\psi} R^*[1/x]$ is flat, where ψ is the inclusion map, and the ring A_τ is Noetherian and is equal to the "computable" nested union B_τ. Let $S := R[f] \subseteq R[\tau]$ and let φ be the embedding $\varphi : S := R[f] \xhookrightarrow{\varphi} T = R[\tau]$. Put $\alpha := \psi \circ \varphi : S \to R^*[1/x]$. Then we have the following commutative diagram:

$$\begin{tikzcd}
 & & R^*[1/x] \\
R \arrow[r, phantom, "\subseteq"] & S := R[f] \arrow[ur, "\alpha:=\psi\varphi"] \arrow[r, "\varphi"] & T := R[\sigma,\tau] \arrow[u, "\psi"]
\end{tikzcd} \tag{10.1.1}$$

In Proposition 10.2, we present a different proof of the result of Nagata given in [**104**, Example 7, pp.209-211]. This example is described in Example 4.8 in the case where the ground field $k = \mathbb{Q}$.

Proposition 10.2. *With the notation of Setting 10.1, let $f := (y+\tau)^2$. In the Nagata Example 4.8, the ring $B_f = A_f$ and B_f is Noetherian with completion $k[[x,y]]$. Therefore B_f is a two-dimensional regular local domain.*

Proof. Since $T = R[\tau]$ is a free S-module with free basis $\langle 1, y+\tau\rangle$, the map $S \xhookrightarrow{\varphi} T$ is flat, by Remark 2.21.2. As mentioned above, the map $T \xhookrightarrow{\psi} R^*[1/x]$ is flat. Therefore as displayed in Diagram 10.1.1, the map $S \xhookrightarrow{\alpha} R^*[1/x]$ is flat; see Remark 2.2113. By Noetherian Flatness Theorem 8.8 and Remark 6.4, B_f is Noetherian and $B_f = A_f$. $\square$

Remark 10.3. In Nagata's original example [**104**, Example 7,pp. 209-211], the field k has characteristic different from 2. This assumption is not necessary for showing that the domain B_f of Proposition 10.2 is a two-dimensional regular local domain.

10.2. Other examples inside the polynomial example

To describe other examples, we modify Setting 10.1 as follows.

SETTING 10.4. Let k be a field, let x, y, z be indeterminates over k, and set

$$R := k[x,y,z]_{(x,y,z)} \qquad \text{and} \qquad R^* := k[y,z]_{(y,z)}[[x]].$$

The power series ring R^* is the xR-adic completion of R. Let σ and τ in $xk[[x]]$ be algebraically independent over $k(x)$. In this section describe some examples using just τ and other examples using both σ and τ. We define the *intersection domains* A_τ and $A_{\sigma,\tau}$ corresponding to τ and σ, τ as follows:

$$A_\tau := k(x,y,z,\tau) \cap R^* \qquad \text{and} \qquad A_{\sigma,\tau} := k(x,y,z,\sigma,\tau) \cap R^*$$

In the following examples we define f to be an element of either $R[\tau]$ or $R[\sigma,\tau]$ that is algebraically independent over R. The *intersection domain* corresponding to f is $A_f = \mathcal{Q}(R[f]) \cap R^*$. In the language of the beginning of this chapter, A_τ or $A_{\sigma,\tau}$ is the "outsider" intersection domain A^{out}, and A_f is the "insider" intersection domain A^{ins}. The domain A_f is contained in A_τ or $A_{\sigma,\tau}$.

As in Setting 10.1 for the Nagata Example, there are natural "approximating" domains associated to A_τ, $A_{\sigma,\tau}$ and A_f, respectively, namely, B_τ, $B_{\sigma,\tau}$ and B_f, respectively. The rings B_τ, $B_{\sigma,\tau}$ and B_f are nested unions of localized polynomial rings over k in 4 or 5 variables.

REMARK 10.5. With Setting 10.4, let $T := R[\sigma,\tau]$ and let $S := R[f]$, where f is a polynomial in $R[\sigma,\tau]$ that is algebraically independent over $\mathcal{Q}(R)$. Notice that the inclusion map $\psi : T \hookrightarrow R^*[1/x]$ is flat. Let $\varphi : S \hookrightarrow T$ denote the inclusion map from S to T. Then we have the following commutative diagram with additional maps shown:

(10.5.a)
$$\begin{tikzcd}
 & & & R^*[1/x] & \\
 & & & & T[1/x] \arrow[ul, "\psi_x"'] \\
R & \subseteq & S:=R[f] \arrow[r, "\varphi"'] \arrow[uur, "\alpha:=\psi\circ\varphi"] \arrow[urr, "\varphi_x:=i_x\circ\varphi"'] & T:=R[\tau] \arrow[uu, "\psi"'] \arrow[ur, "i_x"'] &
\end{tikzcd}$$

We note the following:

(1) Since σ and τ are algebraically independent over R, the element $f \in R[\sigma,\tau]$ has a unique expression

$$f = c_{00} + c_{10}\sigma + c_{01}\tau + \cdots + c_{ij}\sigma^i\tau^j + \cdots + c_{mn}\sigma^m\tau^n,$$

where the $c_{ij} \in R$. The c_{ij} with at least one of i or j nonzero are the *nonconstant coefficients* of f. The ideal $L := (c_{10}, c_{01}, \ldots, c_{mn})R$ is the *ideal generated by the nonconstant coefficients* of f. Lemma 11.20 implies

(10.5.b) $$\varphi \text{ is flat} \iff LR = R.$$

(2) If φ is flat then α is the composition of two flat homomorphisms and so α is also flat, by Remark 2.2113.

(3) To conclude that α is flat, it suffices to show that $\varphi_x : S \hookrightarrow T[1/x]$ is flat, since the inclusion map $T[1/x] \hookrightarrow R^*[1/x]$ is flat.

(4) Theorem 13.6.1 implies that $\varphi_x : S \hookrightarrow T[1/x]$ is flat if and only if the nonconstant coefficients of f generate the unit ideal of $R[1/x]$.

Proposition 10.6 is a preliminary result regarding flatness; there is a more extensive discussion in Chapter 13.

PROPOSITION 10.6. *With the notation of Remark 10.5, let $\varphi_x : S \hookrightarrow T[1/x]$ denote the composition of $\varphi : S \hookrightarrow T$ with the localization map from T to $T[1/x]$. If $\varphi_x : S \hookrightarrow T[1/x]$ is flat, then the approximating domain B_f from Section 6.1 is Noetherian and is equal to the intersection domain A_f.*

PROOF. In Diagram 10.5.a, the map ψ is flat. Thus $\psi' : T[1/x] \hookrightarrow R^*[1/x]$ is flat. If $\varphi_x : S \hookrightarrow T[1/x]$ is flat, then $\alpha = \psi' \circ \varphi_x$ in Diagram 10.5.a is the composition of two flat maps. By Remark 2.21.13, α is flat, and hence by Noetherian Flatness Theorem 8.8, the approximating domain B_f from Section 6.1 is Noetherian and is equal to the intersection domain A_f. $\square$

We use Proposition 10.6 to show the Noetherian property for the following example of Rotthaus [**118**], Example 4.10 of Chapters 4.

EXAMPLE 10.7. (Rotthaus) With the Setting 10.4, let $f := (y + \sigma)(z + \tau)$ and consider the insider domain A_f, contained in the outsider domain $A_{\sigma,\tau}$. The nonconstant coefficients of $f = yz + \sigma z + \tau y + \sigma\tau$ as a polynomial in $R[\sigma, \tau]$ are $\{1, z, y\}$. They do generate the unit ideal of $R[1/x]$, and so, since we assume Remark 10.5.4 for now, we have φ_x is flat. Thus, by Proposition 10.6, the associated nested union domain B_f is Noetherian and is equal to A_f.

EXAMPLES 10.8. (1) With the Setting 10.4, let $f := y\sigma + z\tau$. We show in Examples 13.8 that the map $R[f] \hookrightarrow R[\sigma, \tau][1/x]$ is not flat and that $A_f = B_f$, i.e., A is "limit-intersecting" as in Definition 6.5, but is not Noetherian. Thus we have a situation where the intersection domain equals the approximation domain, but is not Noetherian.

(2) The following is a related simpler example: again with the notation of Setting 10.4, let $f := y\tau + z\tau^2 \in R[\tau] \subseteq A_\tau$. Then the constructed approximation domain B_f is not Noetherian by Theorem 13.6. Moreover, B_f is equal to the intersection domain $A_f := R^* \cap k(x, y, z, f)$ by Corollary 13.4.

In dimension two (the two variable case), an immediate consequence of Valabrega's Theorem 4.2 is the following.

THEOREM 10.9. (Valabrega) *Let x and y be indeterminates over a field k and let $R = k[x, y]_{(x,y)}$. Then $\widehat{R} = k[[x, y]]$ is the completion of R. If L is a field between the fraction field of R and the fraction field of $k[y]\,[[x]]$, then $A = L \cap \widehat{R}$ is a two-dimensional regular local domain with completion $\widehat{R}$.*

Example 10.8 shows that the dimension three analog to Valabrega's result fails. With $R = k[x, y, z]_{(x,y,z)}$ the field $L = k(x, y, z, f)$ is between $k(x, y, z)$ and the fraction field of $k[y, z]\,[[x]]$, but $L \cap \widehat{R} = L \cap R^*$ is not Noetherian.

EXAMPLE 10.10. The following example is given in Section 22.4. With the notation of Setting 10.4, let $f = (y + \sigma)^2$ and $g = (y + \sigma)(z + \tau)$. It is shown in Chapter 22 that the intersection domain $A := R^* \cap k(x, y, z, f, g)$ properly contains its associated approximation domain B and that both A and B are non-Noetherian.

CHAPTER 11

The flat locus of a polynomial ring extension

Let R be a Noetherian ring, let n be positive integer and let $z_1, \dots, z_n$ be indeterminates over R. In this chapter we examine the flat locus of an extension φ of polynomial rings of the form

$$S \ := \ R[f_1, \dots, f_m] \overset{\varphi}{\hookrightarrow} R[z_1, \dots, z_n] \ =: \ T, \tag{11.01}$$

where the f_j are polynomials in $R[z_1, \dots, z_n]$ that are algebraically independent over R.[1] We are motivated to examine the flat locus of the extension φ by the flatness condition of Proposition 10.6 in the Insider Construction of Chapter 10.

We discuss in Section 11.1 a general result on flatness. Then in Section 11.2 we consider the Jacobian ideal of the map $\varphi : S \hookrightarrow T$ of (11.01) and describe the nonsmooth and nonflat loci of this map. In Section 11.3 we discuss applications to polynomial extensions. Related results are given in the papers of Picavet [**113**] and Wang [**139**].

11.1. Flatness criteria

Recall that a Noetherian local ring $(R, \mathbf{m})$ of dimension d is *Cohen-Macaulay* if there exist elements $x_1, \dots, x_d$ in $\mathbf{m}$ that form a regular sequence as defined in Chapter 2; see [**96**, pages 134, 136].

The following definition is useful in connection with what is called the "local flatness criterion" [**96**, page 173].

Definition 11.1. Let I be an ideal of a ring A.

(1) An A-module N is *separated* for the I-adic topology if $\bigcap_{n=1}^{\infty} I^n N = (0)$.
(2) An A-module M is said to be *I-adically ideal-separated* if $\mathfrak{a} \otimes M$ is separated for the I-adic topology for every finitely generated ideal $\mathfrak{a}$ of A.

Remark 11.2. In Theorem 11.3, we use the following result on flatness. Let I be an ideal of a Noetherian ring A and let M be an I-adically ideal-separated A-module. By [**96**, part (1) $\iff$ (3) of Theorem 22.3], we have M is A-flat $\iff$ the following two conditions hold: (a) $I \otimes_A M \cong IM$, and (b) M/IM is (A/I)-flat.

Theorem 11.3 is a general result on flatness involving the Cohen-Macaulay property and a trio of Noetherian local rings.

Theorem 11.3. *Let $(R, \mathbf{m}), (S, \mathbf{n})$ and (T, ℓ) be Noetherian local rings, and assume there exist local maps:*

$$R \longrightarrow S \longrightarrow T,$$

[1] In general for a commutative ring T and a subring R, we say that elements $f_1, \dots, f_m \in T$ are *algebraically independent* over R if, for indeterminates $t_1, \dots, t_m$ over R, the only polynomial $G(t_1, \dots, t_m) \in R[t_1, \dots, t_m]$ with $G(f_1, \dots, f_m) = 0$ is the zero polynomial.

such that

(i) $R \to T$ *is flat and* $T/\mathbf{m}T$ *is Cohen-Macaulay, and*
(ii) $R \to S$ *is flat and* $S/\mathbf{m}S$ *is a regular local ring.*

Then the following statements are equivalent:

(1) $S \to T$ *is flat.*
(2) *For each prime ideal* $\mathbf{w}$ *of* T*, we have* $ht(\mathbf{w}) \geq ht(\mathbf{w} \cap S)$*.*
(3) *For each prime ideal* $\mathbf{w}$ *of* T *such that* $\mathbf{w}$ *is minimal over* $\mathbf{n}T$*, we have* $\mathrm{ht}(\mathbf{w}) \geq \mathrm{ht}(\mathbf{n})$*.*

Proof. The implication (2) $\Longrightarrow$ (3) is obvious and the implication (1) $\Longrightarrow$ (2) is clear by Remark 2.21.10. To prove (3) $\Longrightarrow$ (1), we apply Remark 11.2 with $A = S$, $I = \mathbf{m}S$ and $M = T$. Thus it suffices to show:

(a) $\mathbf{m}S \otimes_S T \cong \mathbf{m}T$.
(b) The map $S/\mathbf{m}S \longrightarrow T/\mathbf{m}T$ is faithfully flat.

Proof of (a): Since $R \hookrightarrow S$ is flat, we have $\mathbf{m}S \cong \mathbf{m}R \otimes_R S$. Therefore

$$\mathbf{m}S \otimes_S T \cong (\mathbf{m} \otimes_R S) \otimes_S T \cong \mathbf{m} \otimes_R T \cong \mathbf{m}T,$$

where the last isomorphism follows because the map $R \to T$ is flat.

Proof of (b): By assumption, $T/\mathbf{m}T$ is Cohen-Macaulay and $S/\mathbf{m}S$ is a regular local ring. We also have $T/\mathbf{n}T = (T/\mathbf{m}T) \otimes_{S/\mathbf{m}S} (S/\mathbf{n})$. By [**96**, Theorem 23.1], if

$$\dim(T/\mathbf{m}T) \;=\; \dim(S/\mathbf{m}S) + \dim(T/\mathbf{n}T), \tag{11.3.c}$$

then $S/\mathbf{m}S \to T/\mathbf{m}T$ is flat.

Let $\mathbf{w} \in \operatorname{Spec} T$ be such that $\mathbf{n}T \subseteq \mathbf{w}$. Since the map $S \to T$ is a local homomorphism, we have $\mathbf{w} \cap S = \mathbf{n}$. By [**96**, Theorem 15.1i] we have

$$\mathrm{ht}(\mathbf{w}) \;\leq\; \mathrm{ht}(\mathbf{n}) \;+\; \dim(T_{\mathbf{w}}/\mathbf{n}T_{\mathbf{w}}).$$

If $\mathbf{w}$ is minimal over $\mathbf{n}T$, then $\dim(T_{\mathbf{w}}/\mathbf{n}T_{\mathbf{w}}) = 0$, and hence $\mathrm{ht}\,\mathbf{w} \leq \mathrm{ht}\,\mathbf{n}$. By assumption $\mathrm{ht}(\mathbf{w}) \geq \mathrm{ht}(\mathbf{n})$ and therefore $\mathrm{ht}(\mathbf{w}) = \mathrm{ht}(\mathbf{n})$ for every minimal prime divisor $\mathbf{w}$ of $\mathbf{n}T$. Thus $\mathrm{ht}(\mathbf{n}) = \mathrm{ht}(\mathbf{n}T)$.

Since $\mathbf{n}$ is generated up to radical by $\mathrm{ht}(\mathbf{n})$ elements, we have

$$\begin{aligned}\dim(T/\mathbf{n}T) &= \dim(T) - \mathrm{ht}(\mathbf{n}T)\\ &= \dim(T) - \mathrm{ht}(\mathbf{w})\\ &= \dim(T) - \mathrm{ht}(\mathbf{n})\end{aligned}$$

Let $P \in \operatorname{Spec} T$ be a minimal prime of $\mathbf{m}T$. Since $R \to T$ is flat,

$$\mathrm{ht}\,P \;=\; \mathrm{ht}\,\mathbf{m} \;+\; \dim(T_P/\mathbf{m}T_P) \;=\; \mathrm{ht}\,\mathbf{m},$$

the first equality by [**96**, Theorem 15.1ii] and the second equality because P is minimal over $\mathbf{m}T$. Thus $\mathrm{ht}(\mathbf{m}T) = \mathrm{ht}(\mathbf{m})$. Since $R \to S$ is flat, a similar argument gives $\mathrm{ht}(\mathbf{m}S) = \mathrm{ht}(\mathbf{m})$. Therefore

$$\begin{aligned}\dim T/\mathbf{n}T &= \dim(T) - \mathrm{ht}(\mathbf{n})\\ &= \dim(T) - \mathrm{ht}(\mathbf{m}T) - (\mathrm{ht}(\mathbf{n}) - \mathrm{ht}(\mathbf{m}S))\\ &= \dim(T/\mathbf{m}T) - \dim(S/\mathbf{m}S)).\end{aligned}$$

Therefore Equation 11.3.c holds, and so $S/\mathbf{m}S \longrightarrow T/\mathbf{m}T$ is flat, by [**96**, Theorem 23.1]. Since $S \to T$ is a local homomorphism, $S/\mathbf{m}S \longrightarrow T/\mathbf{m}T$ is faithfully flat. This completes the proof of Theorem 11.3. $\square$

In Theorem 11.4 we present a result closely related to Theorem 11.3 with a Cohen-Macaulay hypothesis on all the fibers of $R \to T$ and a regularity hypothesis on all the fibers of $R \to S$. A ring homomorphism $f : A \to B$ has *Cohen-Macaulay fibers* with respect to f if, for every $P \in \operatorname{Spec} A$, the ring $B \otimes_A k(P)$ is Cohen-Macaulay, where $k(P)$ is the field of fractions of A/P. For more information about the fibers of a map, see Discussion 3.21 and Definition 3.22.

THEOREM 11.4. *Let $(R, \mathbf{m}), (S, \mathbf{n})$ and (T, ℓ) be Noetherian local rings, and assume there exist local maps:*

$$R \longrightarrow S \longrightarrow T,$$

such that

(i) *$R \to T$ is flat with Cohen-Macaulay fibers, and*
(ii) *$R \to S$ is flat with regular fibers.*

Then the following statements are equivalent:

(1) *$S \to T$ is flat with Cohen-Macaulay fibers.*
(2) *$S \to T$ is flat.*
(3) *For each prime ideal $\mathbf{w}$ of T, we have $ht(\mathbf{w}) \geq ht(\mathbf{w} \cap S)$.*
(4) *For each prime ideal $\mathbf{w}$ of T such that $\mathbf{w}$ is minimal over $\mathbf{n}T$, we have $ht(\mathbf{w}) \geq ht(\mathbf{n})$.*

PROOF. The implications $(1) \implies (2)$ and $(3) \implies (4)$ are obvious and the implication $(2) \implies (3)$ is clear by Remark 2.21.10. By Theorem 11.3, item 4 implies that $S \to T$ is flat.

To show Cohen-Macaulay fibers for $S \to T$, it suffices to show, for each prime ideal Q of T, if $P := Q \cap S$ then T_Q/PT_Q is Cohen-Macaulay. Let $Q \cap R = q$. By passing to $R/q \subseteq S/qS \subseteq T/qT$, we may assume $Q \cap R = (0)$. Let $\operatorname{ht} P = n$. Since $R \to S_P$ has regular fibers and $P \cap R = (0)$, the ideal PS_P is generated by n elements. Moreover, faithful flatness of the map $S_P \to T_Q$ implies that the ideal PT_Q has height n by Remark 2.21.10. Since T_Q is Cohen-Macaulay, a set of n generators of PS_P forms a regular sequence in T_Q. Hence T_Q/PT_Q is Cohen-Macaulay [**96**, Theorems 17.4 and 17.3]. $\square$

Since flatness is a local property by Remark 2.21.4, the following two corollaries are immediate from Theorem 11.4; see also [**113**, Théorème 3.15].

COROLLARY 11.5. *Let T be a Noetherian ring and let $R \subseteq S$ be Noetherian subrings of T. Assume that $R \to T$ is flat with Cohen-Macaulay fibers and that $R \to S$ is flat with regular fibers. Then $S \to T$ is flat if and only if, for each prime ideal P of T, we have* $\operatorname{ht}(P) \geq \operatorname{ht}(P \cap S)$.

As a special case of Corollary 11.5, we have:

COROLLARY 11.6. *Let R be a Noetherian ring and let $z_1, \ldots, z_n$ be indeterminates over R. Assume that $f_1, \ldots, f_m \in R[z_1, \ldots, z_n]$ are algebraically independent over R. Then*

(1) *$\varphi : S := R[f_1, \ldots, f_m] \hookrightarrow T := R[z_1, \ldots, z_n]$ is flat if and only if, for each prime ideal P of T, we have* $\operatorname{ht}(P) \geq \operatorname{ht}(P \cap S)$.
(2) *For $Q \in \operatorname{Spec} T$, $\varphi_Q : S \to T_Q$ is flat if and only if for each prime ideal $P \subseteq Q$ of T, we have* $\operatorname{ht}(P) \geq \operatorname{ht}(P \cap S)$.

PROOF. Since S and T are polynomial rings over R, the maps $R \to S$ and $R \to T$ are flat with regular fibers. Hence both assertions follow from Corollary 11.5. $\square$

11.2. The Jacobian ideal and the smooth and flat loci

We use the following definitions as in Swan [**134**].

DEFINITION 11.7. Let R be a ring. An R-algebra A is said to be *quasi-smooth* over R if for every R-algebra B and ideal N of B with $N^2 = 0$, every R-algebra homomorphism $g : A \to B/N$ lifts to an R-algebra homomorphims $f : A \to B$. In the commutative diagram below, let the maps $\theta : R \to A$ and $\psi : R \to B$ be the canonical ring homomorphisms that define A and B as R-algebras and let the map $\pi : B \to B/N$ be the canonical quotient ring map

$$\begin{array}{ccc} R & \xrightarrow{\theta} & A \\ {\scriptstyle\psi}\Big\downarrow & & {\scriptstyle g}\Big\downarrow \\ B & \xrightarrow{\pi} & B/N \ . \end{array}$$

If A is quasi-smooth over R, then there exists an R-algebra homomorphism $f : A \to B$ that preserves commutativity of the diagram. If A is finitely presented and quasi-smooth over R, then A is said to be *smooth* over R. If A is essentially finitely presented and quasi-smooth over R, then A is said to be *essentially smooth* over R; see Chapter 2 for the definitions of finitely presented and essentially finitely presented.

We return to the extension φ of polynomial rings from Equation 11.01

$$S \ := \ R[f_1, \dots, f_m] \ \overset{\varphi}{\hookrightarrow} \ R[z_1, \dots, z_n] \ =: \ T,$$

where the f_j are polynomials in $R[z_1, \dots, z_n]$ that are algebraically independent over R.

DEFINITIONS AND REMARKS 11.8. (1) The *Jacobian ideal* J of the extension $S \hookrightarrow T$ is the ideal of T generated by the $m \times m$ minors of the $m \times n$ matrix $\mathcal{J}$ defined as follows:

$$\mathcal{J} := \left(\frac{\partial f_i}{\partial z_j} \right)_{1 \le i \le m, \ 1 \le j \le n} .$$

(2) For the extension $\varphi : S \hookrightarrow T$, the *nonflat locus* of φ is the set $\mathcal{F}$, where

$$\mathcal{F} := \{Q \in \operatorname{Spec}(T) \mid \text{the map} \quad \varphi_Q : S \to T_Q \text{ is not flat } \}.$$

We also define the set $\mathcal{F}_{\min}$ and the ideal F of T as follows:

$$\mathcal{F}_{\min} := \{ \text{ minimal elements of } \mathcal{F} \} \ \text{ and } \ F := \bigcap \{Q \mid Q \in \mathcal{F}\}.$$

By [**96**, Theorem 24.3], the set $\mathcal{F}$ is closed in the Zariski topology on $\operatorname{Spec} T$. Hence

$$\mathcal{F} \ = \ \mathcal{V}(F) \ := \ \{P \in \operatorname{Spec} T \mid F \subseteq P \ \}.$$

Thus the set $\mathcal{F}_{\min}$ is a finite set and is equal to the set $\operatorname{Min}(F)$ of minimal primes of the ideal F of T.

Since a flat homomorphism satisfies the going-down theorem by Remark 3.2.9, Corollary 11.6 implies that

(i) $\mathcal{F}_{\min} \subseteq \{Q \in \operatorname{Spec} T \mid \operatorname{ht} Q < \operatorname{ht}(Q \cap S)\}$, and

(ii) If $Q \in \mathcal{F}_{\min}$, then every prime ideal $P \subsetneq Q$ satisfies $\operatorname{ht} P \geq \operatorname{ht}(P \cap S)$.

Example and Remarks 11.9. (1) Let k be a field, let x and y be indeterminates over k and set $f = x$, $g = (x-1)y$. Then $k[f,g] \overset{\varphi}{\hookrightarrow} k[x,y]$ is not flat.

Proof. For the prime ideal $P := (x-1) \in \operatorname{Spec}(k[x,y])$, we see that $\operatorname{ht}(P) = 1$, but $\operatorname{ht}(P \cap k[f,g]) = 2$; thus the extension is not flat by Corollary 11.6. $\square$

(2) The Jacobian ideal J of f and g in (1) is given by:

$$J \;=\; (\det \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\delta g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix})k[x,y] \;=\; (\det \begin{pmatrix} 1 & 0 \\ y & x-1 \end{pmatrix})k[x,y] \;=\; (x-1)k[x,y].$$

(3) In the example of item 1, the nonflat locus is equal to the set of prime ideals Q of $k[x,y]$ that contain the Jacobian ideal $(x-1)k[x,y]$, thus $J = F$.

(4) One can also describe the example of item 1 by taking the base ring R to be the polynomial ring $k[x]$ rather than the field k. Then both $T = R[y]$ and $S = R[g]$ are polynomial rings in one variable over R with $g = (x-1)y$. The Jacobian ideal J is the ideal of T generated by $\frac{\partial g}{\partial y} = x - 1$, so is the same as in item 1.

Remark 11.10. A homomorphism $f : R \to \Lambda$ of Noetherian rings is said to be regular, see Definition 3.25, if f is flat and has geometrically regular fibers. In the case where Λ is a finitely generated R-algebra, the map f is regular if and only if it is smooth as can be seen by taking $\Lambda = A$ in [**134**, Corollary 1.2].

We record in Theorem 11.11 observations about smoothness and flatness that follow from well-known properties of the Jacobian.

Theorem 11.11. *Let R be a Noetherian ring, let $z_1, \ldots, z_n$ be indeterminates over R, and let $f_1, \ldots, f_m \in R[z_1, \ldots, z_n]$ be algebraically independent over R. Consider the embedding $\varphi : S := R[f_1, \ldots, f_m] \hookrightarrow T := R[z_1, \ldots, z_n]$. Let J denote the Jacobian ideal of φ, and let F and $\mathcal{F}_{min}$ be as in* (11.8)*. Then*

1. *$Q \in \operatorname{Spec} T$ does not contain $J \iff \varphi_Q : S \to T_Q$ is essentially smooth. Thus J defines the nonsmooth locus of φ.*
2. *If $Q \in \operatorname{Spec} T$ does not contain J, then $\varphi_Q : S \to T_Q$ is flat. Thus $J \subseteq F$.*
3. *$\mathcal{F}_{min} \subseteq \{Q \in \operatorname{Spec} T \mid J \subseteq Q$ and $\operatorname{ht}(Q \cap S) > \operatorname{ht} Q\}$.*
4. *$\mathcal{F}_{min} \subseteq \{Q \in \operatorname{Spec} T \mid J \subseteq Q,\ \operatorname{ht} Q < \dim S$ and $\operatorname{ht}(Q \cap S) > \operatorname{ht} Q\}$.*
5. *φ is flat $\iff$ for every $Q \in \operatorname{Spec}(T)$ such that $J \subseteq Q$ and $\operatorname{ht}(Q) < \dim S$, we have $\operatorname{ht}(Q \cap S) \leq \operatorname{ht}(Q)$.*
6. *If $\operatorname{ht} J \geq \dim S$, then φ is flat.*

Proof. For item 1, we show that our definition of the Jacobian ideal J given in (11.8) agrees with the description of the smooth locus of an extension given in Elkik [**32**] and Swan [**134**, Section 4]. To see this, we review the description of the smooth locus as given in Elkik and Swan:

Let $u_1, \ldots, u_m$ be indeterminates over $R[z_1, \ldots, z_n]$ and identify

$$R[z_1, \ldots, z_n] \quad \text{with} \quad \frac{R[u_1, \ldots, u_m][z_1, \ldots, z_n]}{(\{u_i - f_i\}_{i=1,\ldots,m})}.$$

Since $u_1, \ldots, u_m$ are algebraically independent, the ideal J generated by the minors of $\mathcal{J}$ is the Jacobian ideal of the extension φ by means of this identification. We make this more explicit as follows.

Let $B := R[u_1, \dots, u_m, z_1, \dots, z_n]$ and $I = (\{f_i - u_i\}_{i=1,\dots,m})B$. Consider the following commutative diagram

$$\begin{array}{ccc} S := R[f_1, \dots, f_m] & \longrightarrow & T := R[z_1, \dots, z_n] \\ \cong\big\downarrow & & \cong\big\downarrow \\ S_1 := R[u_1, \dots, u_m] & \longrightarrow & T_1 := B/I \end{array}$$

Let $\Delta := \Delta(g_1, \dots, g_s)$ be the ideal of $T \cong T_1$ generated by the $s \times s$-minors of $\left(\frac{\partial g_i}{\partial z_j}\right)$, if $1 \le s \le m$; and $\Delta = T$, if $s = 0$. Define as in Elkik[**32**] and Swan [**134**, Section 4]:

$$\text{(11.11.a)} \quad H \ = \ H_{T_1/S_1}| : = \ \text{the radical of} \sum_{g_1, \cdots, g_s} \Delta(g_1, \dots, g_s)[(g_1, \dots, g_s) :_B I],$$

where the sum is taken over all s with $0 \le s \le m$, for all choices of s polynomials $g, \dots, g_s$ from $I = (\{f_1 - u_1, \dots, f_m - u_m\})B$.

To establish item 1 of Theorem 11.11, we show that $H = \operatorname{rad}(J)$. Since u_i is a constant with respect to z_j, we have $\left(\frac{\partial (f_i - u_i)}{\partial z_j}\right) = \left(\frac{\partial f_i}{\partial z_j}\right)$. Thus $J \subseteq H$.

For $g_1, \dots, g_s \in I$, the $s \times s$-minors of $\left(\frac{\partial g_i}{\partial z_j}\right)$ are contained in the $s \times s$-minors of $\left(\frac{\partial f_i}{\partial z_j}\right)$. Thus it suffices to consider s polynomials

$$g_1, \dots, g_s \ \in \ \{f_1 - u_1, \dots, f_m - u_m\}.$$

Since $f_1 - u_1, \dots, f_m - u_m$ is a regular sequence in B, for $s < m$, we have $[(g_1, \dots, g_s) :_B I] = (g_1, \dots, g_s)B$. Thus the $m \times m$-minors of $\left(\frac{\partial f_i}{\partial z_j}\right)$ generate H up to radical, and so $H = \operatorname{rad}(J)$.

Hence by [**32**] or [**134**, Theorem 4.1], T_Q is essentially smooth over S if and only if Q does not contain J.

Item 2 follows from item 1 because essentially smooth maps are flat. In view of Corollary 11.6 and (11.8.2), item 3 follows from item 2.

If $\operatorname{ht} Q \ge \dim S$, then $\operatorname{ht}(Q \cap S) \le \dim S \le \operatorname{ht} Q$. Hence the set

$$\begin{aligned} & \{Q \in \operatorname{Spec} T \mid J \subseteq Q \text{ and } \operatorname{ht}(Q \cap S) > \operatorname{ht} Q\} \\ = \ & \{Q \in \operatorname{Spec} T \mid J \subseteq Q, \ \operatorname{ht} Q < \dim S \text{ and } \operatorname{ht}(Q \cap S) > \operatorname{ht} Q\}. \end{aligned}$$

Thus item 3 is equivalent to item 4.

The ($\implies$) direction of item 5 is clear [**96**, Theorem 9.5]. For the ($\impliedby$) direction of item 5 and for item 6, it suffices to show $\mathcal{F}_{\min}$ is empty, and this holds by item 4. $\square$

REMARKS 11.12. (1) For φ as in Theorem 11.11, it would be interesting to identify the set $\mathcal{F}_{\min} = \operatorname{Min}(F)$. In particular we are interested in conditions for $J = F$ and/or conditions for $J \subsetneq F$. Example 11.9 is an example where $J = F$, whereas Examples 11.15 contains several examples where $J \subsetneq F$.
(2) If R is a Noetherian integral domain, then the zero ideal is not in $\mathcal{F}_{\min}$ and so $F \ne \{0\}$.
(3) In view of Theorem 11.11.3, we can describe $\mathcal{F}_{\min}$ precisely as

$$\mathcal{F}_{\min} = \{Q \in \operatorname{Spec} T \mid J \subseteq Q, \operatorname{ht}(Q \cap S) > \operatorname{ht} Q \text{ and } \forall P \subsetneq Q, \operatorname{ht}(P \cap S) \le \operatorname{ht}(P)\}.$$

(4) Item 3 of Theorem 11.11 implies that for each prime ideal Q of $\mathcal{F}_{\min}$ there exist prime ideals P_1 and P_2 of S with $P_1 \subsetneq P_2$ such that Q is minimal over both P_1T and P_2T.

Corollary 11.13 is immediate from Theorem 11.11.

COROLLARY 11.13. *Let k be a field, let $z_1, \ldots, z_n$ be indeterminates over k and let $f, g \in k[z_1, \ldots, z_n]$ be algebraically independent over k. Consider the embedding $\varphi : S := k[f, g] \hookrightarrow T := k[z_1, \ldots, z_n]$. Assume that the associated Jacobian ideal J is nonzero.*[2] *Then*

(1) $\mathcal{F}_{min} \subseteq \{$*minimal primes* Q *of* J *with* $\operatorname{ht}(Q \cap S) > \operatorname{ht} Q = 1$ $\}$.
(2) φ *is flat* $\iff$ *for every height-one prime ideal* $Q \in \operatorname{Spec} T$ *such that* $J \subseteq Q$ *we have* $\operatorname{ht}(Q \cap S) \leq 1$.
(3) *If* $\operatorname{ht} J \geq 2$, *then* φ *is flat.*

REMARK 11.14. In the case where k is algebraically closed, another argument can be used for Corollary 11.13.3: Each height-one prime ideal $Q \in \operatorname{Spec} T$ has the form $Q = hT$ for some polynomial $h \in T$. If φ is not flat, then there exists a prime ideal Q of T of height one, such that $\operatorname{ht}(Q \cap S) = 2$. Then $Q \cap S$ has the form $(f - a, g - b)S$, where $a, b \in k$. Thus $f - a = f_1 h$ and $g - b = g_1 h$ for some polynomials $f_1, g_1 \in T$. Now the Jacobian ideal J of f, g is the same as the Jacobian ideal of $f - a, g - b$, and an easy computation shows that $J \subseteq hT$. Therefore $\operatorname{ht} J \leq 1$.

EXAMPLES 11.15. Let k be a field of characteristic different from 2 and let x, y, z be indeterminates over k.

(1) With $f = x$ and $g = xy^2 - y$, consider $S := k[f, g] \overset{\varphi}{\hookrightarrow} T := k[x, y]$. Then $J = (2xy - 1)T$. Since $\operatorname{ht}((2xy-1)T \cap S) = 1$, φ is flat by Corollary 11.13.2. But φ is not smooth, since J defines the nonsmooth locus and $J \neq T$; see Theorem 11.11.1. Here we have $J \subsetneq F = T$.

(2) With $f = x$ and $g = yz$, consider $S := k[f, g] \overset{\varphi}{\hookrightarrow} T := k[x, y, z]$. Then $J = (y, z)T$. Since $\operatorname{ht} J \geq 2$, φ is flat by Corollary 11.13.3. Again φ is not smooth since $J \neq T$.

(3) The examples given in items 1 and 2 may also be described by taking $R = k[x]$. In item 1, we then have $S := R[xy^2 - y] \hookrightarrow R[y] =: T$. The Jacobian $J = (2xy - 1)T$ is the same but is computed now as just a derivative. In item 2, we have $S := R[yz] \hookrightarrow R[y, z] =: T$. The Jacobian $J = (y, z)T$ is now computed by taking the partial derivatives $\frac{\partial(yz)}{\partial y}$ and $\frac{\partial(yz)}{\partial z}$.

(4) Let $R = k[x]$ and $S = R[xyz] \hookrightarrow R[y, z] =: T$. Then $J = (xz, xy)T$. Thus J has two minimal primes xT and $(y, z)T$. Notice that $xT \cap S = (x, xyz)S$ is a prime ideal of S of height two, while $(y, z)T \cap S$ has height one. Therefore $J \subsetneq F = xT$.

(5) Let $R = k[x]$ and $S = R[xy + xz] \hookrightarrow R[y, z] =: T$. Then $J = xT$.

(6) Let $R = k[x]$ and $S = R[xy + z^2] \hookrightarrow R[y, z] =: T$. Then $J = (y, z)T$. Hence $S \hookrightarrow T$ is flat but not regular.

(7) Let $R = k[x]$ and $S = R[xy + z] \hookrightarrow R[y, z] =: T$. Then $J = T$. Hence $S \hookrightarrow T$ is a regular map.

COROLLARY 11.16. *With the notation of Theorem 11.11, we have*

[2] This is automatic if the field k has characteristic zero.

(1) *If* $Q \in \mathcal{F}_{min}$, *then* Q *is a nonmaximal prime of* T.
(2) $\mathcal{F}_{min} \subseteq \{Q \in \operatorname{Spec} T : J \subseteq Q,\ \dim(T/Q) \geq 1$ *and* $\operatorname{ht}(Q \cap S) > \operatorname{ht} Q\}$.
(3) φ *is flat* $\iff$ $\operatorname{ht}(Q \cap S) \leq \operatorname{ht}(Q)$ *for every nonmaximal* $Q \in \operatorname{Spec}(T)$ *with* $J \subseteq Q$.
(4) *If* $\dim R = d$ *and* $\operatorname{ht} J \geq d + m$, *then* φ *is flat.*

Proof. For item 1, suppose $Q \in \mathcal{F}_{\min}$ is a maximal ideal of T. Then $\operatorname{ht} Q < \operatorname{ht}(Q \cap S)$ by Theorem 11.11.3. By localizing at $R \setminus (R \cap Q)$, we may assume that R is local with maximal ideal $Q \cap R := \mathbf{m}$. Since Q is maximal, T/Q is a field finitely generated over $R/\mathbf{m}$. By the Hilbert Nullstellensatz [**96**, Theorem 5.3], T/Q is algebraic over $R/\mathbf{m}$ and $\operatorname{ht} Q = \operatorname{ht}(\mathbf{m}) + n$. It follows that $Q \cap S = P$ is maximal in S and $\operatorname{ht} P = \operatorname{ht}(\mathbf{m}) + m$. The algebraic independence hypothesis for the f_i implies that $m \leq n$, and therefore that $\operatorname{ht} P \leq \operatorname{ht} Q$. This contradiction proves item 1. Item 2 follows from Theorem 11.11.3 and item 1.

Item 3 follows from Theorem 11.11.5 and item 1, and item 4 follows from Theorem 11.11.6. □

As an immediate corollary to Theorem 11.11 and Corollary 11.16, we have:

Corollary 11.17. *Let* R *be a Noetherian ring, let* $z_1, \ldots, z_n$ *be indeterminates over* R *and let* $f_1, \ldots, f_m \in R[z_1, \ldots, z_n]$ *be algebraically independent over* R. *Consider the embedding* $\varphi : S := R[f_1, \ldots, f_m] \hookrightarrow T := R[z_1, \ldots, z_n]$, *let* J *be the Jacobian ideal of* φ *and let* F *be the radical ideal that describes the nonflat locus of* φ *as in Definition 11.8.2. Then* $J \subseteq F$ *and either* $F = T$, *that is,* φ *is flat, or* $\dim(T/Q) \geq 1$, *for each* $Q \in \operatorname{Spec}(T)$ *that is minimal over* F.

11.3. Applications to polynomial extensions

Proposition 11.18 considers behavior of the extension $\varphi : S \hookrightarrow T$ with respect to prime ideals of R.

Proposition 11.18. *Let* R *be a commutative ring, let* $z_1, \ldots, z_n$ *be indeterminates over* R, *and let* $f_1, \ldots, f_m \in R[z_1, \ldots, z_n]$ *be algebraically independent over* R. *Consider the embedding* $\varphi : S := R[f_1, \ldots, f_m] \hookrightarrow T := R[z_1, \ldots, z_n]$.

(1) *If* $\mathbf{p} \in \operatorname{Spec} R$ *and* $\varphi_{\mathbf{p}T} : S \to T_{\mathbf{p}T}$ *is flat, then* $\mathbf{p}S = \mathbf{p}T \cap S$ *and the images* $\overline{f_i}$ *of the* f_i *in* $T/\mathbf{p}T \cong (R/\mathbf{p})[z_1, \ldots, z_n]$ *are algebraically independent over* $R/\mathbf{p}$.
(2) *If* φ *is flat, then for each* $\mathbf{p} \in \operatorname{Spec}(R)$ *we have* $\mathbf{p}S = \mathbf{p}T \cap S$ *and the images* $\overline{f_i}$ *of the* f_i *in* $T/\mathbf{p}T \cong (R/\mathbf{p})[z_1, \ldots, z_n]$ *are algebraically independent over* $R/\mathbf{p}$.

Proof. Item 2 follows from item 1, so it suffices to prove item 1. Assume that $T_{\mathbf{p}T}$ is flat over S. Then $\mathbf{p}T \neq T$ and it follows from [**96**, Theorem 9.5] that $\mathbf{p}T \cap S = \mathbf{p}S$. If the $\overline{f_i}$ were algebraically dependent over $R/\mathbf{p}$, then there exist indeterminates $t_1, \ldots, t_m$ and a polynomial $G \in R[t_1, \ldots, t_m] \setminus \mathbf{p}R[t_1, \ldots, t_m]$ such that $G(f_1, \ldots, f_m) \in \mathbf{p}T$. This implies $G(f_1, \ldots, f_m) \in \mathbf{p}T \cap S$. But $f_1, \ldots, f_m$ are algebraically independent over R and $G(t_1, \ldots, t_m) \notin \mathbf{p}R[t_1, \ldots, t_m]$ implies $G(f_1, \ldots, f_m) \notin \mathbf{p}S = \mathbf{p}T \cap S$, a contradiction. □

Proposition 11.19. *Let* R *be a Noetherian integral domain containing a field of characteristic zero. Let* $z_1, \ldots, z_n$ *be indeterminates over* R *and let* $f_1, \ldots, f_m \in R[z_1, \ldots, z_n]$ *be algebraically independent over* R. *Consider the embedding* $\varphi : S :=$

$R[f_1, \dots, f_m] \hookrightarrow T := R[z_1, \dots, z_n]$. Let J be the associated Jacobian ideal and let F be the reduced ideal of T defining the nonflat locus of φ . Then

(1) *If $\mathbf{p} \in \operatorname{Spec} R$ and $J \subseteq \mathbf{p}T$, then $\varphi_{\mathbf{p}T} : S \to T_{\mathbf{p}T}$ is not flat. Thus we also have $F \subseteq \mathbf{p}T$.*
(2) *If the embedding $\varphi : S \hookrightarrow T$ is flat, then for every $\mathbf{p} \in \operatorname{Spec}(R)$ we have $J \nsubseteq \mathbf{p}T$.*

Proof. Item 2 follows from item 1, so it suffices to prove item 1. Let $\mathbf{p} \in \operatorname{Spec} R$ with $J \subseteq \mathbf{p}T$, and suppose $\varphi_{\mathbf{p}T}$ is flat. Let $\overline{f_i}$ denote the image of f_i in $T/\mathbf{p}T$. Consider

$$\overline{\varphi} : \overline{S} := (R/\mathbf{p})[\overline{f_1}, \dots, \overline{f_m}] \to \overline{T} := (R/\mathbf{p})[z_1, \dots, z_n].$$

By Proposition 11.18, $\overline{f_1}, \dots, \overline{f_m}$ are algebraically independent over $\overline{R} := R/\mathbf{p}$. Since the Jacobian ideal commutes with homomorphic images, the Jacobian ideal of $\overline{\varphi}$ is zero. Thus for each $Q \in \operatorname{Spec} \overline{T}$ the map $\overline{\varphi}_Q : \overline{S} \to \overline{T}_Q$ is not smooth. But taking $Q = (0)$ gives $\overline{T}_Q$ is a field separable over the field of fractions of $\overline{S}$ and hence $\overline{\varphi}_Q$ is a smooth map. This contradiction completes the proof. $\square$

Theorem 11.20 follows from [**113**, Proposition 2.1] in the case of one indeterminate z, so in the case where $T = R[z]$.

Theorem 11.20. *Let R be a Noetherian integral domain, let $z_1, \dots, z_n$ be indeterminates over R, and let $T = R[z_1, \dots z_n]$. Suppose $f \in T \setminus R$. Then the following are equivalent:*

(1) *$R[f] \to T$ is flat.*
(2) *For each prime ideal q of R, we have $qT \cap R[f] = qR[f]$.*
(3) *For each maximal ideal q of R, we have $qT \cap R[f] = qR[f]$.*
(4) *The nonconstant coefficients of f generate the unit ideal of R.*
(5) *$R[f] \to T$ is faithfully flat.*

Proof. Since $f \in T \setminus R$ and R is an integral domain, the ring $R[f]$ is a polynomial ring in the indeterminate f over R. Thus the map $R \to R[f]$ is flat with regular fibers. Hence Corollary 11.6 implies that $R[f] \hookrightarrow T$ is flat $\iff$ for each $Q \in \operatorname{Spec} T$ we have $\operatorname{ht} Q \geq \operatorname{ht}(Q \cap R[f])$. Let $q := Q \cap R$. We have $\operatorname{ht} q = \operatorname{ht} qR[f] = \operatorname{ht} qT$. Thus $R[f] \hookrightarrow T$ is flat implies for each $q \in \operatorname{Spec} R$ that $qT \cap R[f] = qR[f]$. Moreover, if $P := Q \cap R[f]$ properly contains $qR[f]$, then $\operatorname{ht} P = 1 + \operatorname{ht} q$, while if Q properly contains qT, then $\operatorname{ht} Q \geq 1 + \operatorname{ht} q$. Therefore (1) $\iff$ (2) follows from Corollary 11.6. It is obvious that (2) $\implies$ (3).

(3) $\implies$ (4): Let $a \in R$ be the constant term of f. If the nonconstant coefficients of f are contained in a maximal ideal q of R, then $f - a \in qT \cap R[f]$. Since R is an integral domain, the element $f - a$ is transcendental over R and $f - a \notin qR[f]$ since $R[f]/qR[f]$ is isomorphic to the polynomial ring $(R/q)[x]$. Therefore $qT \cap R[f] \neq qR[f]$ if the nonconstant coefficients of f are in q.

(4) $\implies$ (2): Let $q \in \operatorname{Spec} R$ and consider the map

(11.1)
$$R[f] \otimes_R R/q = R[f]/qR[f] \xrightarrow{\ \varphi\ } T \otimes_R R/q = T/qT \cong (R/q)[z_1, \dots, z_n].$$

Since the nonconstant coefficients of f generate the unit ideal of R, the image of f in $(R/q)[z_1, \dots, z_n]$ has positive degree. This implies that φ is injective and $qT \cap R[f] = qR[f]$.

This completes a proof that items (1), (2), (3) and (4) are equivalent. To show that these equivalent statements imply (5), it suffices to show for $P \in \operatorname{Spec}(R[f])$ that $PT \neq T$. Let $q = P \cap R$, and let $\kappa(q)$ denote the field of fractions of R/q. Let $\overline{f}$ denote the image of f in $R[f]/qR[f]$. Then $R[f]/qR[f] \cong (R/q)[\overline{f}]$, a polynomial ring in one variable over R/q. Tensoring the map φ of equation 11.1 with $\kappa(q)$ gives an embedding of the polynomial ring $\kappa(q)[\overline{f}]$ into $\kappa(q)[z_1, \dots, z_n]$. The image of P in $\kappa(q)[\overline{f}]$ is either zero or a maximal ideal of $\kappa(q)[\overline{f}]$. In either case, its extension to $\kappa(q)[z_1, \dots, z_n]$ is a proper ideal. Therefore $PT \neq T$. It is obvious that (5) $\implies$ (1), so this completes the proof of Theorem 11.20. □

Corollary 11.21. *Let R be a Noetherian integral domain, let $z_1, \dots, z_n$ be indeterminates over R, and let $T = R[z_1, \dots z_n]$. Suppose $f \in T \setminus R$. Let L denote the ideal of R generated by the nonconstant coefficients of f. Then LT defines the nonflat locus of the map $R[f] \hookrightarrow T$.*

Proof. Let $Q \in \operatorname{Spec} T$ and let $q = Q \cap R$. Tensoring the map $R[f] \to T$ with R_q, we see that $R[f] \hookrightarrow T_Q$ is flat if and only if $R_q[f] \hookrightarrow T_Q$ is flat. Consider the extensions:

$$R_q[f] \overset{\theta}{\hookrightarrow} R_q[z_1, \dots, z_n] \overset{\psi}{\hookrightarrow} T_Q.$$

Since ψ is a localization the composite $\psi \circ \theta$ is flat if θ is flat. By (4) $\implies$ (1) of Theorem 11.20, $LR_q = R_q$ implies $R[f] \hookrightarrow T_Q$ is flat. Thus $L \nsubseteq Q$ implies the map $R[f] \hookrightarrow T_Q$ is flat.

Assume $L \subseteq Q$. Then $L \subseteq q$, and we have $f \in qT_Q \cap R_q[f]$, whereas $f \notin qR_q[f]$. Thus $R_q[f] \hookrightarrow T_Q$ and hence $R[f] \hookrightarrow T_Q$ is not flat. Therefore L defines the nonflat locus of the map $R[f] \hookrightarrow T$. □

Remark 11.22. A different proof that (4) $\implies$ (1) in Theorem 11.20 is as follows: Let v be another indeterminate and consider the commutative diagram

$$\begin{tikzcd} R[v] \arrow[r] \arrow[d, "\pi"'] & T[v] = R[z_1, \dots, z_n, v] \arrow[d, "\pi'"'] \\ R[f] \arrow[r, "\varphi"] & \frac{R[z_1, \dots, z_n, v]}{(v - f(z_1, \dots, z_n))}. \end{tikzcd}$$

where π maps $v \to f$ and π' is the canonical quotient homomorphism. By [**94**, Corollary 2, p. 152] or [**96**, Theorem 22.6 and its Corollary, p. 177], φ is flat if the coefficients of $f - v$ generate the unit ideal of $R[v]$. Moreover, the coefficients of $f - v$ as a polynomial in $z_1, \dots, z_n$ with coefficients in $R[v]$ generate the unit ideal of $R[v]$ if and only if the nonconstant coefficients of f generate the unit ideal of R. For if $a \in R$ is the constant term of f and $a_1, \dots, a_r$ are the nonconstant coefficients of f, then $(a_1, \dots, a_r)R = R$ clearly implies that $(a - v, a_1, \dots, a_r)R[v] = R[v]$. On the other hand, if $(a - v, a_1, \dots, a_r)R[v] = R[v]$, then setting $v = a$ implies that $(a_1, \dots, a_r)R = R$.

We observe in Proposition 11.23 that item 1 implies item 4 of Theorem 11.20 also holds for more than one polynomial f; see also [**113**, Theorem 3.8] for a related result concerning flatness.

Proposition 11.23. *Let $z_1, \dots, z_n$ be indeterminates over an integral domain R. Let $f_1, \dots, f_m$ be polynomials in $R[z_1, \dots, z_n] := T$ that are algebraically independent over $\mathcal{Q}(R)$. If the inclusion map $\varphi : S := R[f_1, \dots f_m] \to T$ is flat, then the nonconstant coefficients of each of the f_i generate the unit ideal of R.*

PROOF. The algebraic independence of the f_i implies that the inclusion map $R[f_i] \hookrightarrow R[f_1, \dots, f_m]$ is flat, for each i with $1 \le i \le m$. If $S \longrightarrow T$ is flat, then so is the composition $R[f_i] \longrightarrow S \longrightarrow T$, and the statement follows from Theorem 11.20. $\square$

Exercises

(1) Let k be an algebraically closed field of characteristic zero and let T denote the polynomial ring $k[x]$. Let $f \in T$ be a polynomial of degree $d \ge 2$ and let $S := k[f]$.

(i) Prove that the map $S \hookrightarrow T$ is free and hence flat.

(ii) Prove that the prime ideals $Q \in \operatorname{Spec} T$ for which $S \to T_Q$ is not a regular map are precisely the primes Q such that the derivative $\frac{df}{dx} \in Q$.

(iii) Deduce that $S \hookrightarrow T$ is not smooth.

(2) With $S = k[x, xy^2 - y] \hookrightarrow T = k[x, y]$ and $J = (2xy - 1)T$ as in Examples 11.15.1, prove that $\operatorname{ht}(J \cap S) = 1$.

Suggestion. Show that $J \cap S \cap k[x] = (0)$ and use that, for A an integral domain, prime ideals of the polynomial ring $A[y]$ that intersect A in (0) are in one-to-one correspondence with prime ideals of $K[y]$, where $K = \mathcal{Q}(A)$ is the field of fractions of A.

(3) Let $z_1, \dots, z_n$ be indeterminates over a ring R, and let $T = R[z_1, \dots z_n]$. Fix an element $f \in T \setminus R$. Modify the proof of (3) implies (4) of Theorem 11.20 to prove that $qT \cap R[f] = qR[f]$ for each maximal ideal q of R implies that the nonconstant coefficients of f generate the unit ideal of R without the assumption that the ring R is an integral domain.

Suggestion. Assume that the nonconstant coefficients of f are contained in a maximal ideal q of R. Observe that one may assume that f as a polynomial in $R[z_1, \dots, z_n]$ has zero as its constant term and that the ring R is local with maximal ideal q. Let M be a monomial in the support of f of minimal total degree and let $b \in R$ denote the coefficient of M for f. Then b is nonzero, but $f \in qR[f]$ implies that $b \in qb$ and this implies, by Nakayama's lemma, that $b = 0$.

(4) Let k be a field and let $T = k[[u, v, w, z]]$ be the formal power series ring over k in the variables u, v, w, z. Define a k-algebra homomorphism φ of T into the formal power series ring $k[[x, y]]$ by defining

$$\varphi(u) = x^4, \quad \varphi(v) = x^3 y, \quad \varphi(w) = xy^3, \quad \varphi(z) = y^4.$$

Let $P = \ker(\varphi)$ and let $I = (v^3 - u^2 w,\ w^3 - z^2 v)T$. Notice that $I \subset P$, and that the ring $\varphi(T) = k[[x^4, x^3 y, xy^3, y^4]]$ is not Cohen-Macaulay. Let $S = T/I$, and let $R = k[[u, z]] \subset T$.

(a) Prove that $P \cap R = (0)$.

(b) Prove that the ring S is Cohen-Macaulay and a finite free R-module.

(c) Prove that PS is a minimal prime of S and S/PS is not flat over R.

Suggestion. To see that S is module finite over R, observe that

$$\frac{S}{(u, z)S} = \frac{T}{(u, z, v^3 - u^2 w, w^3 - z^2 v)T},$$

and the ideal $(u, z, v^3 - u^2 w, w^3 - z^2 v)T$ is primary for the maximal ideal of T. Hence by Theorem 3.10, S is a finite R-module.

(5) Let k be a field and let $A = k[x, xy] \subset k[x, y] = B$, where x and y are indeterminates. Let $R = k[x] + (1 - xy)B$.
   (a) Prove that R is a proper subring of B that contains A.
   (b) Prove that B is a flat R-module.
   (c) Prove that B is contained in a finitely generated R-module.
   (d) Prove that R is not a Noetherian ring.
   (e) Prove that $P = (1-xy)B$ is a prime ideal of both R and B with $R/P \cong k[x]$ and $B/P \cong R[x, 1/x]$.
   (f) Prove that the map $\operatorname{Spec} B \to \operatorname{Spec} R$ is one-to-one but not onto.

   Question. What prime ideals of R are not finitely generated?

CHAPTER 12

Height-one primes and limit-intersecting elements

Let z be a nonzero nonunit of a normal Noetherian integral domain R and let R^* denote the (z)-adic completion of R. As in Construction 5.3, we consider in this chapter the structure of a subring A of R^* of the form $A := \mathcal{Q}(R)(\tau_1, \tau_2, \dots \tau_s) \cap \widehat{R}$, where $\tau_1, \tau_2, \dots, \tau_s \in zR^*$ are algebraically independent elements over R and every nonzero element of $R[\tau_1, \tau_2, \dots, \tau_s]$ is regular on R^*.

If the intersection ring A can be expressed as a directed union B of localized polynomial extension rings of R as in Section 6.1, then the computation of A is easier. Recall that $\tau_1, \tau_2, \dots, \tau_s$ are called *limit-intersecting* for A if the ring A is such a directed union; see Definitions 6.5 and 6.12. The main result of Section 12.1 is Weak Flatness Theorem 12.7 (Inclusion Version). In this theorem we give criteria for $\tau_1, \tau_2, \dots, \tau_s$ to be limit-intersecting for A. We present a version of this result for Homomorphic Image Construction 5.4 in Weak Flatness Theorem 12.8 (Homomorphic Image Version).

In Section 12.2 with the setting of extensions of Krull domains, we continue to analyse the properties of height-one primes considered in Section 12.1. We obtain results that we use in Chapters 19 - 22.

Weak Flatness Theorem 12.7 is used in Examples 13.8 to obtain a family of examples where the approximating ring B is equal to the intersection ring A and is not Noetherian. In Chapter 21 we consider stronger forms of the limit-intersecting condition that are useful for constructing examples.

12.1. The limit-intersecting condition

In this section we prove two versions of the Weak Flatness Theorem that describe conditions in order that the intersection domain A be equal to the approximation domain B, that is the construction is limit-intersecting. For this purpose, we consider the following properties of an extension of commutative rings:

Definitions 12.1. Let $S \hookrightarrow T$ be an extension of commutative rings.

(1) We say that the extension $S \hookrightarrow T$ is *weakly flat*, or that T is *weakly flat* over S, if every height-one prime ideal P of S with $PT \neq T$ satisfies $PT \cap S = P$.
(2) We say that the extension $S \hookrightarrow T$ is *height-one preserving*, or that T is a *height-one preserving* extension of S, if for every height-one prime ideal P of S with $PT \neq T$ there exists a height-one prime ideal Q of T with $PT \subseteq Q$.
(3) For $d \in \mathbb{N}$, we say that $\varphi : S \hookrightarrow T$ *satisfies* LF_d if, for each $P \in \operatorname{Spec} T$ with $\operatorname{ht} P \leq d$, the composite map $S \to T \to T_P$ is flat.

REMARK 12.2. Let $\varphi : S \hookrightarrow T$ be an extension of commutative rings, and let $P \in \operatorname{Spec} T$. With $Q := P \cap S$, the composite map $S \to T \to T_P$ factors through S_Q, and the map $S \to T_P$ is flat if and only if the map $S_Q \to T_P$ is faithfully flat.

PROPOSITION 12.3. *Let $S \hookrightarrow T$ be an extension of commutative rings where S is a Krull domain.*

(1) *If every nonzero element of S is regular on T and each height-one prime ideal of S is contracted from T, then $S = T \cap \mathcal{Q}(S)$.*
(2) *If $S \hookrightarrow T$ is a birational extension and each height-one prime of S is contracted from T, then $S = T$.*
(3) *If T is a Krull domain and $T \cap \mathcal{Q}(S) = S$, then each height-one prime of S is the contraction of a height-one prime of T, and the extension $S \hookrightarrow T$ is height-one preserving and weakly flat.*

PROOF. Item 1 follows from item 2. For item 2, recall from Definition 2.3.2 that $S = \cap\{S_{\mathbf{p}} \mid \mathbf{p}$ is a height-one prime ideal of $S\}$. We show that $T \subseteq S_{\mathbf{p}}$, for each height-one prime ideal of S. Since $\mathbf{p}$ is contracted from T, there exists a prime ideal $\mathbf{q}$ of T such that $\mathbf{q} \cap S = \mathbf{p}$; see Exercise 9 of Chapter 2. Then $S_{\mathbf{p}} \subseteq T_{\mathbf{q}}$ and $T_{\mathbf{q}}$ birationally dominates $S_{\mathbf{p}}$. Since $S_{\mathbf{p}}$ is a DVR, we have $S_{\mathbf{p}} = T_{\mathbf{q}}$. Therefore $T \subseteq S_{\mathbf{p}}$, for each $\mathbf{p}$. It follows that $T = S$.

For item 3, since T is a Krull domain, Definition 2.3.2 implies that

$$T = \bigcap\{T_{\mathbf{q}} \mid \mathbf{q} \text{ is a height-one prime ideal of } T\}.$$

Hence

$$S = T \cap \mathcal{Q}(S) = \bigcap\{T_{\mathbf{q}} \cap \mathcal{Q}(S) \mid \mathbf{q} \text{ is a height-one prime ideal of } T\}.$$

Since each $T_{\mathbf{q}}$ is a DVR, Remark 2.1 implies that $T_{\mathbf{q}} \cap \mathcal{Q}(S)$ is either the field $\mathcal{Q}(S)$ or a DVR birational over S. By the discussion in Definition 2.3.2, for each height-one prime $\mathbf{p}$ of S, the localization $S_{\mathbf{p}}$ is a DVR of the form $T_{\mathbf{q}} \cap \mathcal{Q}(S)$. It follows that each height-one prime ideal $\mathbf{p}$ of S is contracted from a height-one prime ideal $\mathbf{q}$ of T, and that T is height-one preserving and weakly flat over S. □

Corollary 12.4 demonstrates the relevance of the weakly flat property for an extension of a Krull domain.

COROLLARY 12.4. *Let $S \hookrightarrow T$ be an extension of commutative rings where S is a Krull domain such that every nonzero element of S is regular on T and $PT \neq T$ for every height-one prime ideal P of S.*

(i) *If $S \hookrightarrow T$ is weakly flat, then $S = \mathcal{Q}(S) \cap T$.*
(ii) *If T is Krull, then T is weakly flat over $S \iff S = \mathcal{Q}(S) \cap T$. Moreover, in this setting, these equivalent conditions imply that $S \hookrightarrow T$ is height-one preserving.*

PROOF. For item i, each height-one prime ideal of S is contracted from T. Thus by Proposition 12.3.1, $S = \mathcal{Q}(S) \cap T$.

For item ii, we apply Proposition 12.3.3. □

REMARKS 12.5. Let $S \hookrightarrow T$ be an extension of commutative rings.

(a) If $S \hookrightarrow T$ is flat, then $S \hookrightarrow T$ is weakly flat; see [**96**, Theorem 9.5].

(b) Let G be a multiplicative system in S consisting of units of T. Then $S \hookrightarrow G^{-1}S$ is flat and every height-one prime ideal of $G^{-1}S$ is the extension of a height-one prime ideal of S. Thus $S \hookrightarrow T$ is weakly flat $\iff$ $G^{-1}S \hookrightarrow T$ is weakly flat.

REMARKS 12.6. Let $S \hookrightarrow T$ be an extension of Krull domains.

(a) If $S \hookrightarrow T$ is flat, then $S \hookrightarrow T$ is height-one preserving and satisfies PDE. See Definition 2.3.3 and [**12**, Chapitre 7, Proposition 15, page 19].
(b) Let G be a multiplicative system in S consisting of units of T. It follows as in Remarks 12.5.b that:
   (i) $S \hookrightarrow T$ is height-one preserving $\iff$ $G^{-1}S \hookrightarrow T$ is height-one preserving.
   (ii) $S \hookrightarrow T$ satisfies PDE $\iff$ $G^{-1}S \hookrightarrow T$ satisfies PDE.
(c) If each height-one prime of S is the radical of a principal ideal, in particular, if S is a UFD, then the extension $S \hookrightarrow T$ is height-one preserving. To see this, let P be a height-one prime of S and suppose that P is the radical of the principal ideal xS. Then $PT \neq T$ if and only if xT is a proper principal ideal of T. Every proper principal ideal of a Krull domain is contained in a height-one prime. Hence if $PT \neq T$, then PT is contained in a height-one prime of T.

With these results and remarks in hand, we return to the investigation of the structure of the intersection domain A mentioned in the introduction to this chapter: When does A equal the approximation domain B? We first consider the intersection domain A of Inclusion Construction 5.3 and the approximation ring B of Section 6.1. We show in Weak Flatness Theorem 12.7 that, if the base ring R of the construction is a normal Noetherian domain and the extension

$$R[\tau_1, \ldots, \tau_s] \hookrightarrow R^*[1/z]$$

is weakly flat, then the intersection domain A is equal to the approximation domain B; that is, $\tau_1, \ldots, \tau_s$ are limit-intersecting in the sense of Definition 6.5.

WEAK FLATNESS THEOREM 12.7. (Inclusion Version) *Let R be a normal Noetherian integral domain and let $z \in R$ be a nonzero nonunit. Let R^* denote the (z)-adic completion of R and let $\tau_1, \ldots, \tau_s \in R^*$ be algebraically independent over R. Assume that every nonzero element of the polynomial ring $R[\tau_1, \ldots, \tau_s]$ is regular on R^*. Let $A = \mathcal{Q}(R)(\tau_1, \ldots, \tau_s) \cap R^*$ and let B be the approximation domain defined in Section 6.1. Consider the following statements:*

(1) *$A = B$; that is, $\tau_1, \ldots, \tau_s$ are limit-intersecting in the sense of Definition 6.5.*
(2) *The extension $R[\tau_1, \ldots, \tau_s] \hookrightarrow R^*[1/z]$ is weakly flat.*
(3) *The extension $B \hookrightarrow R^*[1/z]$ is weakly flat.*
(4) *The extension $B \hookrightarrow R^*$ is weakly flat.*

Then

(a) *Items 2, 3, and 4 are equivalent.*
(b) *Item 2 $\implies$ item 1.*
(c) *If R^* is normal, then the four items are equivalent.*

PROOF. For (a), we show item 4 $\implies$ item 3 $\implies$ item 2 $\implies$ item 4. To see that item 4 $\implies$ item 3, we have

$$B \overset{\text{w.f.}}{\hookrightarrow} R^* \overset{\text{flat}}{\hookrightarrow} R^*[1/z].$$

Thus, for a height-one prime ideal P of B with $PR^*[1/z] \neq R^*[1/z]$, we have $PR^* \neq R^*$ and $z \notin P$, and so $PR^*[1/z] \cap B = PR^* \cap B = P$, where the last equality uses $B \overset{\text{w.f.}}{\hookrightarrow} R^*$. Thus item 3 holds.

Item 3 $\implies$ item 2: We have $B \overset{\text{w.f.}}{\hookrightarrow} R^*[1/z]$ implies $B[1/z] \overset{\text{w.f.}}{\hookrightarrow} R^*[1/z]$, by Remarks 12.5.b. By Construction Properties Theorem 6.19.2, $B[1/z]$ is a localization of $R[\tau_1, \ldots, \tau_s]$. Thus, by Remark 12.5.b, we have $R[\tau_1, \ldots, \tau_s] \overset{\text{w.f.}}{\hookrightarrow} R^*[1/z]$.

To see that item 2 $\implies$ item 4, let $P \in \operatorname{Spec} B$ have height one and suppose $PR^* \neq R^*$. If $z \in P$, then, by Construction Properties Theorem 6.19.3, we have $P/zB = PR^*/zR^*$, and so $PR^* \cap B = P$ in this case. Thus we assume $z \notin P$; then $PB[1/z] \cap B = P$.

By assumption, $R[\tau_1, \ldots, \tau_s] \hookrightarrow R^*[1/z]$ is weakly flat. Since $B[1/z]$ is a localization of $R[\tau_1, \ldots, \tau_s]$ and of B, Remark 12.5.b implies that $B[1/z] \hookrightarrow R^*[1/z]$ is weakly flat. Since $PR^*[1/z] \neq R^*[1/z]$, we have $PR^*[1/z] \cap B[1/z] = PB[1/z]$. Thus $PR^* \cap B = P$ and so $B \hookrightarrow R$ is weakly flat, as desired.

We show item 2 $\implies$ item 1: Since B is a Krull domain and the extension $B \hookrightarrow A$ is birational, by Proposition 12.3.2, it suffices to show that every height-one prime ideal $\mathbf{p}$ of B is contracted from A. As in the proof of item 2 $\implies$ item 4, Construction Properties Theorem 6.19.3 implies that each height-one prime of B containing zB is contracted from A.

Let $\mathbf{p}$ be a height-one prime of B that does not contain zB. Consider the prime ideal $\mathbf{q} = R[\tau_1, \ldots, \tau_s] \cap \mathbf{p}$. Since $B[1/z]$ is a localization of the ring $R[\tau_1, \ldots, \tau_s]$, we see that $B_{\mathbf{p}} = R[\tau_1, \ldots, \tau_s]_{\mathbf{q}}$ and so $\mathbf{q}$ has height one in $R[\tau_1, \ldots, \tau_s]$. The weakly flat hypothesis implies $\mathbf{q}R^* \cap R[\tau_1, \ldots, \tau_s] = \mathbf{q}$. Hence there exists a prime ideal $\mathbf{w}$ of R^* with $\mathbf{w} \cap R[\tau_1, \ldots, \tau_s] = \mathbf{q}$. This implies that $\mathbf{w} \cap B = \mathbf{p}$ and thus also $(\mathbf{w} \cap A) \cap B = \mathbf{p}$. Hence every height-one prime ideal of B is the contraction of a prime ideal of A. Thus $A = B$ as desired.

To prove (c), we assume R^* is a normal Noetherian domain. Thus R^* is a Krull domain; see Definition 2.3.2. We prove item 1 $\implies$ item 4: Since $B = A = \mathcal{Q}(B) \cap R^*$, Proposition 12.3 implies the extension $B \hookrightarrow R^*$ is weakly flat. $\square$

In Theorem 12.8, we present a version of Weak Flatness Theorem 12.7 that applies to Homomorphic Image Construction 5.4.

WEAK FLATNESS THEOREM 12.8. (Homomorphic Image Version) *Let R be a normal Noetherian integral domain and let $z \in R$ be a nonzero nonunit. Let R^* denote the (z)-adic completion of R and let I be an ideal of R^* having the property that $P \cap R = (0)$ for each associated prime ideal P of I. Let the rings A and B be as defined in Section 6.12. Assume that B is a Krull domain. Then*

(1) *If the extension $R \hookrightarrow (R^*/I)[1/z]$ is weakly flat, then $A = B$, that is, the construction is limit-intersecting as in Definition 6.12.*
(2) *If R^*/I is a normal integral domain, then the following statements are equivalent:*
    (a) *$A = B$.*
    (b) *$R \hookrightarrow (R^*/I)[1/z]$ is weakly flat.*

(c) *The extension* $B \hookrightarrow (R^*/I)[1/z]$ *is weakly flat.*
(d) *The extension* $B \hookrightarrow R^*/I$ *is weakly flat.*

PROOF. Theorem 6.17.3 implies that each height-one prime of B containing zB is contracted from R^*/I. Using Frontpiece Notation 6.9, Definition 6.12 and Theorem 6.17, we have $B[1/z]$ is a localization of $R[1/z] = U[1/z]$. Since $R \hookrightarrow (R^*/I)[1/z]$ is weakly flat, it follows that $B \hookrightarrow (R^*/I)[1/z]$ is weakly flat by Remark 12.5.b. Therefore $B \hookrightarrow R^*/I$ is weakly flat. By Proposition 12.3.1, we have $B = (R^*/I) \cap \mathcal{Q}(B) = A$. This proves item 1.

For item 2, since R^*/I is a normal integral domain, $A = (R^*/I) \cap \mathcal{Q}(R)$ is a Krull domain. As noted in the proof of item 1, Theorem 6.17 implies that each height-one prime of B containing zB is contracted from R^*/I and $B[1/z]$ is a localization of $R[1/z] = U[1/z]$. It follows that (b), (c) and (d) are equivalent. By Proposition 12.3.3, (a) $\implies$ (d), and by Proposition 12.3.1, (d) $\implies$ (a). $\square$

12.2. Height-one primes in extensions of Krull domains

We observe in Proposition 12.9 that a weakly flat extension of Krull domains is height-one preserving.

PROPOSITION 12.9. *If* $\phi : S \hookrightarrow T$ *is a weakly flat extension of Krull domains, then* ϕ *is height-one preserving. Moreover, for every height-one prime ideal* P *of* S *with* $PT \neq T$ *there is a height-one prime ideal* Q *of* T *with* $Q \cap S = P$.

PROOF. Let $P \in \operatorname{Spec} S$ with $\operatorname{ht} P = 1$ be such that $PT \neq T$. Since T is weakly flat over S, we have $PT \cap S = P$. Then $S \setminus P$ is a multiplicatively closed subset of T and $PT \cap (S \setminus P) = \emptyset$. Let Q' be an ideal of T that contains PT and is maximal with respect to $Q' \cap (S \setminus P) = \emptyset$. Then Q' is a prime ideal of T and $Q' \cap S = P$. Let a be a nonzero element of P and let $Q \subseteq Q'$ be a minimal prime divisor of aT. Since T is a Krull domain, Q has height one. We have $a \in Q \cap S$. Hence $(0) \neq Q \cap S \subseteq P$. Since $\operatorname{ht} P = 1$, we have $Q \cap S = P$. $\square$

The height-one preserving condition does *not* imply weakly flat as we demonstrate in Example 12.10.

EXAMPLE 12.10. Let x and y be variables over a field k, let $R = k[[x]][y]_{(x,y)}$ and let $C = k[[x, y]]$. There exists an element $\tau \in \mathbf{n} = (x, y)C$ that is algebraically independent over $\mathcal{Q}(R)$. For any such element τ, let $S = R[\tau]_{(\mathbf{m},\tau)}$. Since R is a UFD, the ring S is also a UFD and the local inclusion map $\varphi : S \hookrightarrow C$ is height-one preserving. There exists a height-one prime ideal P of S such that $P \cap R = 0$. Since the map $S \hookrightarrow C$ is a local map, we have $PC \neq C$. Because φ is height-one preserving, there exists a height-one prime ideal Q of C such that $PC \subseteq Q$. Since C is the $\mathbf{m}$-adic completion $\widehat{R}$ of R and the generic formal fiber of R is zero-dimensional, $\dim(C \otimes_R \mathcal{Q}(R)) = 0$. Hence $Q \cap R \neq 0$. We have $P \subseteq Q \cap S$ and $P \cap R = (0)$. It follows that P is strictly smaller than $Q \cap S$, so $Q \cap S$ has height greater than one. Therefore the extension $\varphi : S \hookrightarrow C$ is not weakly flat.

Proposition 12.11 describes weakly flat and PDE extensions. We define PDE in Definition 2.3.3. For extensions of Krull domains we show in Proposition 12.12 below that PDE is equivalent to LF_1, which is defined in Definition 12.1.3.

PROPOSITION 12.11. *Let* $\varphi : S \hookrightarrow T$ *be an extension of Krull domains.*

(1) *φ is weakly flat $\iff$ for every height-one prime ideal $P \in \operatorname{Spec} S$ such that $PT \neq T$ there is a height-one prime ideal $Q \in \operatorname{Spec} T$ with $P \subseteq Q \cap S$ such that the induced map on the localizations*

$$\varphi_Q : S_{Q \cap S} \longrightarrow T_Q$$

is faithfully flat.

(2) *φ satisfies PDE $\iff$ for every height-one prime ideal $Q \in \operatorname{Spec} T$, the induced map on the localizations*

$$\varphi_Q : S_{Q \cap S} \longrightarrow T_Q$$

is faithfully flat.

PROOF. We use in both (1) and (2) that for each height-one prime $P \in \operatorname{Spec} S$ the induced map $\varphi_P : S_P \longrightarrow (S \setminus P)^{-1}T$ is flat since a domain extension of a DVR is always flat by Remark 2.23.3; and φ_P is faithfully flat $\iff$ P does not extend to the whole ring in $(S \setminus P)^{-1}T$, a property that is equivalent to the existence of a prime in T lying over P in S.

For the proof of (1), to see ($\Longleftarrow$), we use that φ_Q a faithfully flat map implies φ_Q satisfies the going-down property; see Remark 2.21.10. Hence $Q \cap S$ is of height one, so $P = Q \cap S$, and thus $PT \cap S = P$. For ($\Longrightarrow$), suppose $P \in \operatorname{Spec} S$ has height one and φ is weakly flat. Then Proposition 12.9 implies the existence of $Q \in \operatorname{Spec} T$ of height one such that $Q \cap S = P$. Since T_Q is a localization of $(S \setminus P)^{-1}T$, we see that φ_Q is faithfully flat.

For the proof of (2), ($\Longrightarrow$) is clear by the remark in the first sentence of the proof, and ($\Longleftarrow$) follows from the fact that a faithfully flat map satisfies the going-down property. $\square$

Proposition 12.12 is an immediate consequence of Proposition 12.11:

PROPOSITION 12.12. *Let $\varphi : S \hookrightarrow T$ be an extension of Krull domains. Then φ satisfies PDE if and only if φ satisfies LF_1.*

We show in Proposition 12.13 that an extension of Krull domains satisfying both the LF_1 condition and the height-one preserving condition is weakly flat. Example 12.14 shows that LF_1 alone does not imply weakly flat.

PROPOSITION 12.13. *Let $S \hookrightarrow T$ be an extension of Krull domains that is height-one preserving and satisfies PDE. Then T is weakly flat over S. That is, if $S \hookrightarrow T$ is height-one preserving and satisfies LF_1, then T is weakly flat over S.*

PROOF. Let $P \in \operatorname{Spec} S$ be such that $\operatorname{ht}(P) = 1$ and $PT \neq T$. Since $S \hookrightarrow T$ is height-one preserving, PT is contained in a prime ideal Q of T of height one. The PDE hypothesis on $S \hookrightarrow T$ implies that $Q \cap S$ has height one. It follows that $Q \cap S = P$, and so $PT \cap T = P$; that is, the extension is weakly flat. The last statement holds by Proposition 12.12. $\square$

Without the assumption that the extension $S \hookrightarrow T$ is height-one preserving, it can happen that the extension satisfies PDE and yet is not weakly flat as we demonstrate in Example 12.14. Since PDE and height-one preserving imply weakly flat, this example also shows that PDE does not imply height-one preserving.

EXAMPLE 12.14. Let X, Y, Z, W be indeterminates over a field k and define

$$S \ := \ k[x,y,z,w] \ = \ \frac{k[X,Y,Z,W]}{(XY-ZW)} \quad \text{and} \quad T \ := \ S[\frac{x}{z}].$$

Since $w = \frac{yx}{z}$, the ring $T = k[y, z, \frac{x}{z}]$. Since $\mathcal{Q}(T)$ has transcendence degree 3 over k, the elements $y, z, \frac{x}{z}$ are algebraically independent over k and $T = k[y, z, \frac{x}{z}]$ is a polynomial ring in three variables over k. Let $A = k[X, Y, Z, W]$ and let $F = XY - ZW$. Then $S = A/FA$ and and the partials of F generate a maximal ideal of A. It follows that $S_{\mathbf{p}}$ is regular for each nonmaximal prime ideal $\mathbf{p}$ of S; see for example [**96**, Theorem 30.3]. Since S is Cohen-Macaulay, it follows from Serre's normality theorem [**96**, Theorem 23.8] that S is a normal Noetherian domain. Hence S is a Krull domain. The ideal $P := (y, z)S$ is a height-one prime ideal of S, because it corresponds to the height-one prime ideal $(Y, Z)A/FA$ of A/FA. Since $PT = (y, z)T$ and $(y, z)T \cap S = (y, z, x, w)S$, a maximal ideal of S, the extension $S \hookrightarrow T$ is not weakly flat.

Another way to realize this example is to let r, s, t be indeterminates over the field k, and let $S = k[r, s, rt, st] \hookrightarrow k[r, s, t] = T$. Here we set $r = y, s = z, rt = w$ and $st = x$. Then $P = (r, s)S$. We have

$$\begin{aligned} T \ &= \ \bigcap \{ S_Q \mid Q \in \operatorname{Spec} S \ , \ \operatorname{ht} Q = 1 \ \text{ and } \ Q \neq P \ \} \\ &= \ \bigcup_{n=1}^{\infty} (S :_{\mathcal{Q}(S)} P^n) \ = \ S\,[\frac{1}{y}] \ \cap \ S\,[\frac{1}{z}]\ ; \end{aligned} \tag{6.2.13.0}$$

for the last equality, see Exercise 3 at the end of this chapter and [**13**]. It is straightforward to see that $T \subseteq \bigcup_{n=1}^{\infty}(S :_{\mathcal{Q}(S)} P^n)$. The reverse inclusion follows because $\operatorname{ht} PT > 1$. To see the other equality in Equation 12.14.0, we use the uniqueness of the family of essential valuation rings of the Krull domain S and that an intersection of localizations of S is again a Krull domain for which the family of essential valuation rings is a subset of the family of essential valuation rings for S; see Definitions 2.3.2.b. Therefore T is an intersection of localizations of S. Thus the extension $S \hookrightarrow T$ satisfies PDE.

REMARKS 12.15. (1) By Proposition 12.9, an injective map of Krull domains that is weakly flat is also height-one preserving. Thus the equivalent conditions of Theorem 12.7 imply that $B \hookrightarrow R^*$ is height-one preserving.

(2) If the ring B in Theorem 12.7 is Noetherian, then, by Noetherian Flatness Theorem 8.8, $A = B$ and the equivalent conclusions of Theorem 12.7 hold, since flatness implies weak flatness.

(3) Theorem 13.6 of Chapter 13 yields examples where the constructed rings A and B are equal, but are not Noetherian. The limit-intersecting property holds for these examples. These examples are described in Examples 13.8.

(4) As we note in Remark 19.37, Examples 19.34 and 19.36 give extensions of Krull domains that are weakly flat but do not satisfy PDE.

QUESTION 12.16. Let $(C, \mathbf{n})$ be a complete Noetherian local domain that dominates a quasilocal Krull domain $(D, \mathbf{m})$. Assume that the inclusion map $D \hookrightarrow C$ is height-one preserving, and that $\tau \in \mathbf{n}$ is algebraically independent over D. Does it follow that the local inclusion map $\varphi : \ S := D[\tau]_{(\mathbf{m},\tau)} \hookrightarrow C$ is height-one preserving?

Discussion 12.17. If D has torsion divisor class group, then S also has torsion divisor class group and by item c of Remark 12.6, the extension $S \hookrightarrow C$ is height-one preserving, and so the answer to Question 12.16 is affirmative in this case. To consider the general case, let P be a height-one prime ideal of S that is not assumed to be the radical of a principal ideal. One may then consider the following cases:

Case (i): If $\operatorname{ht}(P \cap D) = 1$, then $P = (P \cap D)S$. Since $D \hookrightarrow C$ is height-one preserving, $(P \cap D)C \subseteq Q$, for some height-one prime ideal Q of C. Then $PC = (P \cap D)SC \subseteq Q$ as desired.

Case (ii): Suppose $P \cap D = (0)$. Let U denote the multiplicative set of nonzero elements of D. Let t be an indeterminate over D and let $S_1 = D[t]_{(\mathbf{m},t)}$. Consider the following commutative diagram where the map from S_1 to S is the D-algebra isomorphism taking t to τ and λ is the extension mapping $C[[t]]$ onto C.

$$\begin{array}{ccccccc}
 & & U^{-1}S_1 & \xrightarrow{\subseteq} & U^{-1}C[t]_{(\mathbf{n},t)} & & \\
 & & \cup\Big\uparrow & & \cup\Big\uparrow & & \\
D & \xrightarrow{\subseteq} & S_1 = D[t]_{(\mathbf{m},t)} & \xrightarrow{\subseteq} & C[t]_{(\mathbf{n},t)} & \xrightarrow{\subseteq} & C[[t]] \\
=\Big\downarrow & & \cong\Big\downarrow & & & & \lambda\Big\downarrow \\
D & \xrightarrow{\subseteq} & S = D[\tau]_{(\mathbf{m},\tau)} & & & \xrightarrow{\varphi} & C.
\end{array}$$

Under the above isomorphism of S with S_1, the prime ideal P corresponds to a height-one prime ideal P_1 of S_1 such that $P_1 \cap D = (0)$. Since $U^{-1}S_1$ is a localization of a polynomial ring in one variable over a field, the extended ideal $P_1U^{-1}S_1$ is a principal prime ideal. Therefore P_1 is contained in a proper principal ideal of $U^{-1}C[t]_{(\mathbf{n},t)}$.

However, in the above diagram it can happen that the inclusion map

$$U^{-1}S_1 \ \hookrightarrow \ U^{-1}C[t]_{(\mathbf{n},t)}$$

may fail to be faithfully flat. As an example to illustrate this, let

$$D := k[x, y = e^x - 1]_{(x,y)} \ \hookrightarrow \ k[[x]] \ =: \ C$$

and let $P = (xt - y)D[t]$. Then P extends to the whole ring in $U^{-1}C[t]_{(\mathbf{n},t)}$ since $t - \frac{y}{x}$ is a unit of $U^{-1}C[t]_{(\mathbf{n},t)}$.

Exercises

(1) Let $T = k[x, y, z]$ be a polynomial ring in the 3 variables x, y, z over a field k, and consider the subring $S = k[xy, xz, yz]$ of T.
- (a) Prove that the field extension $\mathcal{Q}(T)/\mathcal{Q}(S)$ is algebraic with $[\mathcal{Q}(T) : \mathcal{Q}(S)] = 2$.
- (b) Deduce that $xy,\ xz,\ yz$ are algebraically independent over k, so S is a polynomial ring in 3 variables over k.
- (c) Prove that the extension $S \hookrightarrow T$ is height-one preserving, but is not weakly flat.
- (d) Prove that $T \cap \mathcal{Q}(S) = S[x^2, y^2, z^2]$ is a Krull domain that properly contains S.
- (e) Prove that the map $S \hookrightarrow T[\frac{1}{xyz}]$ is flat.

(f) Prove that $S[\frac{1}{xyz}] = T[\frac{1}{xyz}]$. (Notice that $S[\frac{1}{xyz}]$ is not a localization of S since xyz is not in $\mathcal{Q}(S)$.)

(2) In the case where T is also a Krull domain, give a direct proof using primary decomposition of the assertion in Corollary 12.4 that $S = \mathcal{Q}(S) \cap T$ implies T is weakly flat over S.

Suggestion. Let $\mathbf{p}$ be a height-one prime ideal of S and let $0 \neq a \in \mathbf{p}$. Since T is a Krull domain, the principal ideal aT has an irredundant primary decomposition

$$aT = Q_1 \cap \cdots \cap Q_s,$$

where each Q_i is primary for a height-one prime ideal P_i of T.

(b) Show that $aS = \mathcal{Q}(S) \cap aT$.

(c) Show that after relabeling there exists an integer $t \in \{1, \ldots, s\}$ such that the ideal $Q_1 \cap \ldots \cap Q_t \cap S$ is the $\mathbf{p}$-primary component of aS. Conclude that $P_i \cap S = \mathbf{p}$, for some i.

(3) Let A be an integral domain and let I be an ideal generated by the nonzero elements $a_1, \ldots, a_r$ of A. Let $\mathcal{F} = \{P \in \operatorname{Spec} A \mid I \nsubseteq P\}$. For each $n \in \mathbb{N}$ define $I^{-n} := (A :_{\mathcal{Q}(A)} I^n)$. Show that

$$\bigcup_{n=1}^{\infty} I^{-n} = \bigcap_{i=1}^{r} A\left[\frac{1}{a_i}\right] = \bigcap_{P \in \mathcal{F}} A_P.$$

Remark. This exercise is a result of Jim Brewer [**13**, Prop. 1.4 and Theorem 1.5]

(4) Let $(R, \mathbf{m})$ be a 3-dimensional regular local domain with $\mathbf{m} = (x, y, z)R$, let $\mathbf{p} = xR$ and let $V = R_{\mathbf{p}}$. Thus V is an essential valuation ring for the Krull domain R, and $R/\mathbf{p}$ is a 2-dimensional regular local domain. Let $w = \frac{x-y^2}{z}$ and let $T = R[w]_{(y,z,w)R[w]}$.

(a) Prove that T is a 3-dimensional regular local domain such that $T \subset V$.

(b) Prove that V is also an essential valuation ring for the Krull domain T.

(c) Find a generator for the height-one prime ideal $\mathbf{q}$ of T such that $T_{\mathbf{q}} = V$.

(d) Prove that $T/\mathbf{q}$ is a 2-dimensional local domain that is not regular.

CHAPTER 13

Insider construction details,

In this chapter we continue the development of Insider Construction 13.1 begun in Chapter 10. In Section 13.1 we expand the notation for Insider Construction 13.1 to the case where the base ring R is a Noetherian domain that is not necessarily a polynomial ring over a field. As before we first construct an "outside" Noetherian domain A^{out} using Inclusion Construction 5.3. We require that this intersection domain A^{out} is equal to its corresponding approximation integral domain B^{out} that is the nested union of localized polynomial rings from Section 6.1. Then we construct inside $A^{\text{out}} = B^{\text{out}}$ two "insider" integral domains: A^{inside}, an intersection of a field with a power series ring as in Construction 5.3, and B^{inside}, a nested union of localized polynomial rings that "approximates" A^{inside} as in Section 6.1. We show that B^{inside} is Noetherian and equal to A^{inside} if a certain map of polynomial rings over R is flat.

In Section 13.1, we describe background and notation for the construction. Theorem 13.3 in Section 13.2 gives necessary and sufficient conditions for the integral domains constructed with Insider Construction 13.1 to be Noetherian and equal. In Section 13.3, we use the analysis of flatness for polynomial extensions from Chapter 11 to obtain a general flatness criterion for the Insider Construction. This yields examples where the constructed domains A and B are equal and are not Noetherian.

In Section 13.4 we discuss the preservation of excellence for the insider construction. We give in Theorem 13.11 necessary and sufficient conditions for the Insider Construction to be excellent.

In Chapter 15, we use Insider Construction 13.1 to establish, for each integer $d \geq 3$ and each integer h with $2 \leq h \leq d-1$, the existence of a d-dimensional regular local domain $(A, \mathbf{n})$ that has a prime ideal P of height h such that the extension $P\widehat{A}$ is not integrally closed. In Chapter 17, we use the Insider Construction to obtain, for each positive integer n, an explicit example of a 3-dimensional non-Noetherian local unique factorization domain B such that the maximal ideal of B is 2-generated, B has precisely n prime ideals of height two, and each prime ideal of B of height two is not finitely generated.

13.1. Describing the construction

We use the following setting and details for the Insider Construction in this chapter. This setting includes Noetherian domains that are not necessarily local and thus generalizes Settings 10.1 and 10.4.

Insider Construction 13.1. Let R be a Noetherian integral domain. Let z be a nonzero nonunit of R and let R^* be the (z)-adic completion of R. The intersection domain of Construction 5.3 and the corresponding approximation domain of Section 6.1 are inside R^*. Assume that $\tau_1, \dots, \tau_n \in zR^*$ are algebraically independent over R and are such that nonzero elements of $R[\tau_1, \dots, \tau_n]$ are regular on R^*. Let $\underline{\tau}$ abbreviate the list $\tau_1, \dots, \tau_n$. We define the *intersection domain corresponding to* $\underline{\tau}$ to be the ring $A_{\underline{\tau}} := K(\tau_1, \dots, \tau_n) \cap R^*$. The Noetherian Example Theorem 8.8 implies that $A_{\underline{\tau}}$ is simultaneously Noetherian and a nested union $B_{\underline{\tau}}$ of certain associated localized polynomial rings over R using $\underline{\tau}$ if and only if the extension $T := R[\underline{\tau}] = R[\tau_1, \dots, \tau_n] \overset{\psi}{\hookrightarrow} R^*[1/z]$ is flat. Moreover, if this flatness condition holds, then $A_{\underline{\tau}}$ is a localization of a subring of $T[1/z]$ and $A_{\underline{\tau}}[1/z]$ is a localization of T.

We assume that $\psi : T \hookrightarrow R^*[1/z]$ is flat, so that the intersection domain $A_{\underline{\tau}}$ is Noetherian and equals its associated approximation domain. We take the outside domain to be $A^{\text{out}} := A_{\underline{\tau}} = B_{\underline{\tau}}$ so that $B^{\text{out}} = A^{\text{out}}$. Then we construct new "insider" examples inside $A^{\text{out}} = A_{\underline{\tau}}$ as follows: We choose elements $f_1, \dots, f_m$ of $T := R[\underline{\tau}]$, considered as polynomials in the τ_i with coefficients in R and abbreviated by $\underline{f}$. Assume that $f_1, \dots, f_m$ are algebraically independent over K; thus $m \le n$. As above we define $A_{\underline{f}} := K(f_1, \dots, f_m) \cap R^*$ to be the *intersection domain corresponding to* $\underline{f}$. We let $B_{\underline{f}}$ be the *approximation domain corresponding to* $\underline{f}$ that approximates $A_{\underline{f}}$, obtained using the f_i as in Section 6.1. Sometimes we refer to $B_{\underline{f}}$ as the *nested union domain corresponding to* $A_{\underline{f}}$. We define $A^{\text{inside}} := A_{\underline{f}}$, and $B^{\text{inside}} := B_{\underline{f}}$.

Recall that the nested union domains $B_{\underline{\tau}}$ and $B_{\underline{f}}$ are localizations of $R[\underline{\tau}]$ and $R[\underline{f}]$ respectively by Construction Properties Theorem 6.19.2. Clearly $B_{\underline{f}} \subseteq B_{\underline{\tau}}$ are local domains with $B_{\underline{\tau}}$ dominating $B_{\underline{f}}$.

Set $S := R[\underline{f}] = R[f_1, \dots, f_m]$, let φ be the embedding

$$\varphi : S := R[\underline{f}] \overset{\varphi}{\hookrightarrow} T := R[\underline{\tau}], \tag{13.1.1}$$

and let ψ be the inclusion map: $R[\underline{\tau}] \hookrightarrow R^*[1/z]$. Put $\alpha := \psi \circ \varphi : S \to R^*[1/z]$. Then we have

$$\begin{tikzcd} & & R^*[1/z] \\ R \arrow[r, phantom, "\subseteq"] & S := R[\underline{f}] \arrow[ur, "\alpha:=\psi\varphi"] \arrow[r, "\varphi"] & T := R[\underline{\tau}] \arrow[u, "\psi"'] \end{tikzcd} \tag{13.1.2}$$

We show in Proposition 10.6 of Chapter 10 for the special case where R is a localized polynomial ring over a field in two or three variables that, if $\varphi_z : S \hookrightarrow T[1/z]$ is flat, then $A_{\underline{f}}$ is Noetherian and is equal to the corresponding approximating domain $B_{\underline{f}}$. In Section 13.2 we make a more thorough analysis of conditions for $A_{\underline{f}}$ to be Noetherian and equal to $B_{\underline{f}}$. In Section 13.3, we present examples where $B_{\underline{f}} = A_{\underline{f}}$ is not Noetherian.

REMARK 13.2. If R is a Noetherian local domain, then R^* is local and hence the intersection domains $A_{\underline{\tau}}$ and $A_{\underline{f}}$ are also local with $A_{\underline{f}}$ possibly non-Noetherian. By Remark 6.4.1 the approximating domain $B_{\underline{f}}$ is also local.

13.2. The flat locus of the Insider Construction

We assume the notation of Insider Construction 13.1 for this discussion and refer the reader to Section 6.1 for details concerning the approximation domains $B_{\underline{\tau}}$ and $B_{\underline{f}}$ corresponding to the intersection domains $A_{\underline{\tau}}$ and $A_{\underline{f}}$, respectively, of (13.1).

Theorem 8.8 is the basis for our construction of examples.

In the notation of Diagram 13.1.2, let

$$F := \cap\{P \in \operatorname{Spec}(T) \mid \varphi_P : S \to T_P \text{ is not flat }\}. \tag{13.2.1}$$

Thus, as in (11.8.2), the ideal F defines the nonflat locus of the map $\varphi : S \to T$.

For $Q^* \in \operatorname{Spec}(R^*[1/z])$, we consider flatness of the localization $\varphi_{Q^*\cap T}$ of the map φ in Equation 13.1.1:

$$\varphi_{Q^*\cap T} : S \longrightarrow T_{Q^*\cap T} \tag{13.2.2}$$

Theorem 13.3 enables us to recover information about the flatness of α in Diagram 13.1.2 from the map $\varphi : S \to T$.

THEOREM 13.3. *Let R be Noetherian domain, let z be a nonzero nonunit of R and let R^* be the z-adic completion of R. With the notation of Diagram 13.1.2 and Equations 13.2.1 and 13.2.2, we have*

(1) *For $Q^* \in \operatorname{Spec}(R^*[1/z])$, the map $\alpha_{Q^*} : S \to (R^*[1/z])_{Q^*}$ is flat if and only if the map $\varphi_{Q^*\cap T}$ in Equation 13.2.2 is flat.*
(2) *The following are equivalent:*
   (i) *The ring A is Noetherian and $A = B$.*
   (ii) *The ring B is Noetherian.*
   (iii) *For every maximal $Q^* \in \operatorname{Spec}(R^*[1/z])$, the map $\varphi_{Q^*\cap T}$ in Equation 13.2.2 is flat.*
   (iv) $FR^*[1/z] = R^*[1/z]$.
(3) *The map $\varphi_z : S \hookrightarrow T[1/z]$ is flat if and only if $FT[1/z] = T[1/z]$. Moreover, either of these equivalent conditions implies B is Noetherian and $B = A$. It then follows that $A[1/z]$ is a localization of S.*
(4) *If z is in the Jacobson radical of R and the conditions of item 2 or item 3 hold, then* $\dim R = \dim A = \dim R^*$.

PROOF. For item 1, we have $\alpha_{Q^*} = \psi_{Q^*} \circ \varphi_{Q^*\cap T} : S \to T_{Q^*\cap T} \to (R^*[1/z])_{Q^*}$. Since the map ψ_{Q^*} is faithfully flat, the composition α_{Q^*} is flat if and only if $\varphi_{Q^*\cap T}$ is flat [**96**, (1) and (3), p. 46].

For item 2, the equivalence of (i) and (ii) is part of Theorem 8.8. The equivalence of (ii) and (iii) follows from item 1 and Theorem 8.8. For the equivalence of (iii) and (iv), we use $FR^* \neq R^* \iff F \subseteq Q^* \cap T$, for some Q^* maximal in $\operatorname{Spec}(R^*[1/z]) \iff$ the map in Equation 13.2.2 fails to be flat.

The first statement of item 3 follows from the definition of F and the fact that the nonflat locus of $\varphi : S \to T$ is closed. Theorem 8.8 implies the final statement of item 3.

Item 4 follows by Remark 3.2.4. $\square$

13.3. The nonflat locus of the Insider Construction

To examine the map $\alpha : S \to R^*[1/z]$ in more detail, we consider the following:

PROPOSITION 13.4. *Let R be a normal Noetherian domain. With the notation of Insider Construction 13.1 and Equation 13.2.1, we have*

1. $\operatorname{ht}(FR^*[1/z]) > 1 \iff \alpha : S \to R^*[1/z]$ *satisfies LF_1.*
2. *Assume that R^* is a normal domain and that each height-one prime of R is the radical of a principal ideal. Then the equivalent conditions of item 1 imply that $B = A$.*

PROOF. Item 1 follows from the definition of LF_1; see Definition 12.1.3.

For item 2, assume R^* is a normal domain and each height-one prime of R is the radical of a principal ideal. Then the extension $R \hookrightarrow R^*$ is height-one preserving by Remark 12.6.c. By Proposition 12.13 the extension is weakly flat. Theorem 12.7 implies that $B = A$. $\square$

QUESTION 13.5. Does item 2 of Proposition 13.4 hold without the condition that every height-one prime ideal is the radical of a principal ideal?

THEOREM 13.6. *Let R be a Noetherian integral domain, let z be a nonzero nonunit of R and let R^* be the z-adic completion of R. With the notation of Insider Construction 13.1, assume $m = 1$, that is, there is only one polynomial $f_1 = f$, $S := R[f]$ and $T := R[\tau_1, \dots, \tau_n]$. Assume that $f \in T \setminus R$. Let $B = B_f$ and $A = A_f$ be the approximation domain and intersection domain associated to f over R, and let L be the ideal in R generated by the nonconstant coefficients of f as a polynomial in T. Then:*

1. *The ideal $LR^*[1/z]$ defines the nonflat locus of $\alpha : S \hookrightarrow R^*[1/z]$.*
2. *The ideal $LR^*[1/z]$ defines the nonflat locus of $\beta : B \hookrightarrow R^*[1/z]$.*
3. *The following are equivalent:*
   (a) *B is Noetherian.*
   (b) *B is Noetherian and $B = A$.*
   (c) *The extension $\alpha : S \hookrightarrow R^*[1/z]$ is flat.*
   (d) *For each $Q^* \in \operatorname{Spec} R^*[1/z]$, we have $LR^*[1/z]_{Q^*} = R^*[1/z]_{Q^*}$.*
   (e) *For each $Q^* \in \operatorname{Spec} R^*[1/z]$, we have $LR_q = R_q$, where $q = Q^* \cap R$.*
4. *If $\operatorname{ht} LR^*[1/z] = d$, then the map $\alpha : S \hookrightarrow R^*[1/z]$ satisfies LF_{d-1}, but not LF_d, as defined in Definition 12.1.3.*
5. $\varphi_z : S \hookrightarrow T[1/z]$ *is flat* $\iff LT[1/z] = T[1/z] \iff LR[1/z] = R[1/z]$.
6. *The equivalent conditions in item 5 imply the insider approximation domain B is Noetherian and is equal to the insider intersection domain A.*

PROOF. For item 1, let $Q^* \in \operatorname{Spec}(R^*[1/z])$. Theorem 13.3.1 implies the map

$$\alpha_{Q^*} : S \hookrightarrow (R^*[1/z])_{Q^*} \quad \text{is flat} \quad \iff \varphi_{Q^* \cap T} : S \hookrightarrow T_{Q^* \cap T} \quad \text{is flat.}$$

By Corollary 11.21, the ideal LT defines the nonflat locus of $\varphi : S \hookrightarrow T$. Thus

$$\varphi_{Q^* \cap T} \quad \text{is flat} \quad \iff LT \nsubseteq Q^* \cap T.$$

Since L is an ideal of R, we have

$$LT \nsubseteq Q^* \cap T \iff LR^*[1/z] \nsubseteq Q^*.$$

Thus $LR^*[1/z]$ defines the nonflat locus of α.

In view of item 1, to prove item 2, it suffices to show for each $Q^* \in \operatorname{Spec} R^*[1/z]$:

$$\alpha_{Q^*} : S \hookrightarrow R^*[1/z]_{Q^*} \text{ is flat } \iff \beta_{Q^*} : B \hookrightarrow R^*[1/z]_{Q^*} \text{ is flat.}$$

By Remarks 2.21.1, we have

$$\alpha_{Q^*} \text{ is flat } \iff S_{Q^* \cap S} \hookrightarrow R^*[1/z]_{Q^*} \text{ is flat.}$$

Similarly, we have

$$\beta_{Q^*} \text{ is flat } \iff B_{Q^* \cap B} \hookrightarrow R^*[1/z]_{Q^*} \text{ is flat.}$$

Since $z \notin Q^*$ and $B[1/z]$ is a localization of S, we have $S_{Q^* \cap S} = B_{Q^* \cap B}$ because $B_{Q^* \cap B}$ dominates and is a localizaton of $S_{Q^* \cap S}$. This completes the proof of item 2.

For item 3, (a), (b) and (c) are equivalent by Noetherian Flatness Theorem 8.8. By item 1, (c) and (d) are equivalent. Since L is an ideal of R, (d) is equivalent to (e); that is $L \nsubseteq Q^* \iff L \nsubseteq Q^* \cap R = q$.

For item 4, assume that $\operatorname{ht}(LR^*[1/z]) = d$. Let $Q^* \in \operatorname{Spec}(R^*[1/z])$. The map $\alpha_{Q^*} : S \hookrightarrow (R^*[1/z])_{Q^*}$ is flat $\iff$ $L \nsubseteq Q^*$ by item 1. Thus α_{Q^*} is flat for every Q^* with $\operatorname{ht} Q^* < d$, and so α satisfies LF_{d-1}. On the other hand, there exists $Q^* \in \operatorname{Spec} R^*[1/z]$ such that $L \subseteq Q^*$ and $\operatorname{ht} Q^* = d$. By item 1, the map α_{Q^*} is not flat. Thus α does not satisfy LF_d.

For item 5, Corollary 11.21 states that LT is the nonflat locus of the map $\varphi : S \hookrightarrow T$. Thus $S \hookrightarrow T[1/z]$ is flat $\iff$ $LT[1/z] = T[1/z]$. Since L is an ideal of R, and $T[1/z]$ is a polynomial ring over $R[1/z]$, we have $LT[1/z] = T[1/z] \iff LR[1/z] = R[1/z]$.

If $S \hookrightarrow T[1/z]$ is flat, then Theorem 13.3.3 implies that B is Noetherian and $B = A$. Thus item 6 holds. □

Example 13.7 illustrates that in Theorem 13.6 the map $\varphi_z : S \hookrightarrow T[1/z]$ may fail to be flat even though the map $\alpha : S \hookrightarrow R^*[1/z]$ is flat.

EXAMPLE 13.7. Let $R = k[z]$, where z is an indeterminate over the field k. Let $\tau \in zk[[z]]$ be such that z and τ are algebraically independent over k. Let $T = R[\tau]$, let $f = (1-z)\tau$, and let $S = R[f]$. The ideal L of R generated by the nonconstant coefficients of f is $L = (1-z)R$. The map $\varphi_z : S \hookrightarrow T[1/z]$ is not flat, but the map $\alpha : S \hookrightarrow R^*[1/z]$ is flat since $R^*[1/z]$ is a field.

We return to Example 10.8 and establish a more general result.

EXAMPLES 13.8. Let $d \in \mathbb{N}$ be greater than or equal to 2, and let $x, y_1, \dots, y_d$ be indeterminates over a field k. Let R be either

(1) The polynomial ring $R := k[x, y_1, \dots, y_d]$ with (x)-adic completion $R^* = k[y_1, \dots, y_d][[x]]$, or
(2) The localized polynomial ring $R := k[x, y_1, \dots, y_d]_{(x, y_1, \dots, y_d)}$ with (x)-adic completion R^*.

Let $f := y_1\tau_1 + \cdots + y_d\tau_d$, where $\tau_1, \dots, \tau_d \in xR^*$ are algebraically independent over R. Let $S := R[f]$ and let $T := R[\tau_1, \dots, \tau_d]$. We regard f as a polynomial in $\tau_1, \dots, \tau_d$ over R. By Theorem 13.6.4, the map $\varphi_x : S \hookrightarrow T[1/x]$ satisfies LF_{d-1}, but fails to satisfy LF_d because the ideal $L = (y_1, \dots, y_d)R[1/x]$ of nonconstant coefficients of f has height d. Since $d \geq 2$, the map $\varphi_x : S \hookrightarrow T[1/x]$ satisfies LF_1. Moreover, S is a UFD and hence, by Proposition 13.4, we have $A = B$, that is, the element f is "limit-intersecting". However, since φ_x does not satisfy LF_d, the map φ_x is not flat and thus B is not Noetherian by Theorem 13.3.2.

13.4. Preserving excellence with the Insider Construction

We assume the following setting in this section.

SETTING 13.9. Let $(R, \mathbf{m})$ be an excellent normal local domain with field of fractions K. Let z be a nonzero element of $\mathbf{m}$ and let R^* denote the (z)-adic completion of R. Assume that the s elements $\tau_1, \dots, \tau_s \in zR^*$ are such that

$$R[\tau_1, \dots, \tau_s] \longrightarrow R^*[1/z]$$

is flat. Thus we have

$$B := \varinjlim R[\tau_{1n}, \dots, \tau_{sn}]_{(\mathbf{m}, \tau_{1n}, \dots, \tau_{sn})} = A := K(\tau_1, \dots, \tau_s) \cap R^*.$$

Let $t_1, \dots, t_s, v$ be variables over R, let $\underline{t}$ denote the s-tuple $(t_1, \dots, t_s)$, and let $f(\underline{t}) := f(t_1, \dots, t_s)$ be a nonzero polynomial in $(\mathbf{m}, \underline{t})R[\underline{t}]$. Also let $\underline{\tau}$ denote the s-tuple $(\tau_1, \dots, \tau_s)$, let $\rho := f(\underline{\tau})$ denote $f(\tau_1, \dots, \tau_s)$, let $S := (R[\underline{t}, v]/(f(\underline{t}) - v))_{(\mathbf{m}, \underline{t}, v)}$ and $E := (R[f(\underline{t})])_{(\mathbf{m}, f(\underline{t}))}$. We have the following commutative diagram of R-algebra injections:

(13.9.1)
$$\begin{tikzcd}
R[v]_{(\mathbf{m},v)} \arrow[r, "\phi"] \arrow[d, "\mu_1"] & S := (R[\underline{t}, v]/(f(\underline{t}) - v))_{(\mathbf{m}, \underline{t}, v)} \arrow[d, "\nu_1"] & \\
E := R[f(\underline{t})]_{(\mathbf{m}, f(\underline{t}))} \arrow[r, "\phi"] \arrow[d, "\mu_2"] & R[\underline{t}]_{(\mathbf{m}, \underline{t})} \arrow[d, "\nu_2"] & \\
R[\rho]_{(\mathbf{m},\rho)} = R[f(\underline{\tau})]_{(\mathbf{m},\rho)} \arrow[r, "\phi"] & B_0 := R[\underline{\tau}]_{(\mathbf{m}, \underline{\tau})} \arrow[r, "\psi"] & R^*_z
\end{tikzcd}$$

where μ_1 and ν_1 are the R-algebra isomorphisms defined by $\mu_1(v) = f(\underline{t})$, respectively $\nu_1(t_i) = t_i$ and $\nu_1(v) = f(\underline{t})$, and μ_2 and ν_2 take $\underline{t} \to \underline{\tau}$. We use the identifications established by the μ_i and ν_i throughout the following and we identify the three maps ϕ. With the same assumptions as above we put

$$\rho_n := (f(\underline{\tau}) - f(\sum_{j=1}^{n} a_{1j} z^j, \dots, \sum_{j=1}^{n} a_{sj} z^j))/z^n$$

and let

$$C := \varinjlim R[\rho_n]_{(\mathbf{m}, \rho_n)} = \bigcup_{n=1}^{\infty} R[\rho_n]_{(\mathbf{m}, \rho_n)} \qquad \text{and} \qquad D := K(\rho) \cap R^*. \tag{13.9.2}$$

Note that $C[1/z]$ is a localization of E.

Now define F to be the ideal of S such that

$$FS_z = \bigcap \{P \in \operatorname{Spec}(S_z) | \, (S_z)_P \text{ is not flat over the ring } R[\rho]_{(\mathbf{m},\rho)}\};$$

By Definitions and Remarks 11.8.2 or [**96**, (24.3)], F describes the nonflat locus of S_z over the ring $R[\rho]_{(\mathbf{m},\rho)}$, that is,

$$\{P \in \operatorname{Spec}(S_z) \mid P \supset F\} = \{P \in \operatorname{Spec}(S_z) \mid F \, (S_z)_P \text{ is not flat over } R[\rho]_{(\mathbf{m},\rho)}\}.$$

Since S_z is an integral domain, the ideal $F \neq 0$.

With the notation of (13.9), we examine conditions under which B is excellent implies C is excellent.

NOTATION 13.10. Assume the setting of (13.9). Also assume that B is excellent. Consider the local injective morphism

$$R[\rho]_{(\mathbf{m},\rho)} \xrightarrow{\phi} S := R[\tau_1, \dots, \tau_s]_{(\mathbf{m},\tau_1,\dots,\tau_s)}.$$

Let $\phi_z : R[\rho]_{(\mathbf{m},\rho)} \longrightarrow S_z$ denote the composition of ϕ followed by the canonical map of S to S_z. Let H be the ideal of S such that HS_z describes the nonregular locus of the map ϕ_z, that is,

$$HS_z = \bigcap\{P \in \operatorname{Spec}(S_z) | (\phi_y)_P : (R[\rho]_{(\mathbf{m},\rho)})_{\phi_z^{-1}(P)} \longrightarrow (S_z)_P \text{ is not regular }\}.$$

If $P \in \operatorname{Spec}(S_z)$ and $HS_z \subseteq P$ then $(\phi_z)_P$ is not a regular map by [**44**, IV, 6.8.7, page 153]. For an explicit description of the ideal H, see Theorem 11.11.

THEOREM 13.11. *Assume the setting and notation of (13.9) and (13.10). The following are equivalent:*

(a) *The ring C is excellent.*
(b) $HR^*_z = R^*_z$.
(c) *ϕ_z is a regular morphism.*

PROOF. That conditions (b) and (c) are equivalent is clear from the definition of a regular morphism.

(a)$\Longrightarrow$(b) We have embeddings:

$$C \xrightarrow{\Phi} B \xrightarrow{\Psi} R^* \xrightarrow{\Gamma} \widehat{R}.$$

Since C, B and R^* are all excellent with the same completion $\widehat{R}$, [**96**, Theorem 32.1] implies Φ is regular. Let $Q \in \operatorname{Spec}(R^*)$ with $z \notin Q$. After localization the map

$$C_{(\Psi\circ\Phi)^{-1}(Q)} \xrightarrow{\Phi_Q} B_{\Psi^{-1}(Q)}$$

remains regular. Since $z \notin Q$ the map Φ_Q can be identified with

$$R[v]_{(\psi\circ\phi)^{-1}(Q)} \xrightarrow{\phi_Q} R[t_1, \dots, t_s, v]/(f(\underline{t}) - v))_{\psi^{-1}(Q)} := S_{\psi^{-1}(Q)}$$

showing that ϕ_Q is regular. Hence $H \not\subseteq \psi^{-1}(Q)$. It follows that $HR^*_z = R^*_z$.

(b)$\Longrightarrow$(a) By Theorem 13.3 the ring C is Noetherian with z-adic completion R^*. Therefore the completion of C with respect to the powers of its maximal ideal is $\widehat{R}$. Therefore C is formally equidimensional. Hence by [**96**, Theorem 31.6, page 251], C is universally catenary.

To show C is excellent, it remains to show that C is a G-ring. Consider the morphisms

$$C \xrightarrow{\Phi} B \quad \text{and} \quad C \xrightarrow{\Gamma\circ\Psi\circ\Phi} \widehat{R}.$$

Since C and B are Noetherian, $\widehat{R}$ is faithfully flat over both C and B. Hence the map Φ is faithfully flat by Remark 2.21.14. A straightforward argument using Definition 3.29 of G-ring shows that C is a G-ring if the map Φ is regular in the sense of Definition 3.25; see [**96**, Theorem 32.2].

To see that Φ is regular, let $P \in \operatorname{Spec}(C)$. If $z \in P$ then $R^* \cap \mathcal{Q}(C) = C$ implies $zR^* \cap \mathcal{Q}(C) = zC$. Therefore $C/zC = R^*/zR^* = R/zR$. Since R is excellent, the ring $\widehat{R} \otimes_C k(P)$ is geometrically regular over $k(P) = (C/PC)_P$. If $z \notin P$ we show that for every finite field extension L of $k(P)$ the ring $B \otimes_C L$ is regular. Let $\overline{W}$ be a prime ideal in $B \otimes_C L$ which contracts to a prime ideal W in B. Then W lies over P in C. Let $Q \in \operatorname{Spec}(R^*)$ be a prime ideal with $\Psi^{-1}(Q) = W$. Since

$\Phi^{-1}(W) = (\Psi \circ \Phi)^{-1}(Q) = P$, we know that $z \notin Q$ and thus $H \not\subseteq Q$ implying that $H \not\subseteq W$. Therefore the map

$$R[v]_{(\psi\circ\phi)^{-1}(Q)} \xrightarrow{\phi_Q} R[t_1,\dots,t_s,v]/(f(\underline{t})-v))_{\psi^{-1}(Q)}$$

is regular. Since $z \notin Q$ we have the following identifications:

$$\begin{array}{ccc} R[v]_{(\psi\circ\phi)^{-1}(Q)} & \xrightarrow{\phi_Q} & R[t_1,\dots,t_s,v]/(f(\underline{t})-v))_{\psi^{-1}(Q)} \\ \| & & \| \\ C_P & \xrightarrow{\Phi_Q} & B_W \end{array}$$

which imply that Φ_Q is regular. Hence the ring $B_W \otimes_{C_P} L$ is regular. But $(B \otimes_C L)_{\overline{W}}$ is a localization of $B_W \otimes_{C_P} L$. Therefore $B \otimes_C L$ is regular. Thus Φ is a regular morphism and C is excellent. $\square$

REMARK 13.12. (1) For the example of Nagata described in Example 4.8 and in Proposition 10.2, the map from $R[\rho]_{(\mathbf{m},\rho)} \to S_x$ may be identified as the inclusion map

$$E := k[x,y,(t+y)^2]_{(x,y,(t+y)^2)} \xrightarrow{\phi_x} k[x,y,t]_{(x,y,t)}[1/x] = S_x.$$

Let $Q = (y,t)S_x$ and let $P = Q \cap E = (y,(t+y)^2)E$. Then PS_x is Q-primary and properly contained in Q. Therefore the map ϕ_x is not regular.

(2) For the example of Rotthaus described in Example 4.10, the map from $R[\rho]_{(\mathbf{m},\rho)} \to S_y$ may be identified as the inclusion map

$$k[x,y,z,(t_1+x)(t_2+z)]_{\mathbf{n}} \xrightarrow{\phi_y} k[x,y,z,t_1,t_2]_{x,y,z,t_1,t_2}[1/y] = S_y,$$

where $\mathbf{n} = (x,y,z,(t_1+x)(t_2+z))k[x,y,z,(t_1+x)(t_2+z)]$. Let

$$E \;:=\; k[x,y,z,(t_1+x)(t_2+z)]_{(x,y,z,(t_1+x)(t_2+z)}$$

and let P be defined as $P := (x,z,(t_1+x)(t_2+z))E$. Then $S_y/PS_y = k(y)[t_1,t_2]/(t_1t_2)$ is a nonregular ring, so ϕ_y is not a regular map.

CHAPTER 14

Excellent rings: motivation and explanation

The goal of this chapter is to motivate and explain the concept of excellence.[1] We describe the desirable attributes of an excellent ring and discuss why they are useful. In considering this, we are led to a discussion of the singular locus and the Jacobian criterion. For a Noetherian local ring $(R, \mathbf{m})$ with $\mathbf{m}$-adic completion $\widehat{R}$, the fibers of the map $R \hookrightarrow \widehat{R}$ play an important role. We also consider Henselian rings and the Henselization of a Noetherian local ring.

14.1. Basic properties and background

In the 1950s, M. Nagata constructed an example in characteristic $p > 0$ of a normal Noetherian local domain $(R, \mathbf{m})$ such that the $\mathbf{m}$-adic completion $\widehat{R}$ is not reduced [**104**, Example 6, p.208], [**102**]. He constructed another example of a normal Noetherian local domain $(R, \mathbf{m})$ that contains a field of characteristic zero and has the property that $\widehat{R}$ is not an integral domain [**104**, Example 7, p.209]; see Example 4.8, Remarks 4.9 and Section 10.1 for information about this example. The existence of these examples motivated the search for appropriate conditions to impose on a Noetherian local ring R that will ensure that R does not exhibit the bad behavior of these examples. We consider the following questions:

Questions 14.1.

(1) What properties should a "nice" Noetherian ring have?
(2) What properties on a Noetherian local ring ensure good behavior with respect to completion?
(3) What properties on a Noetherian ring ensure "nice" properties of finitely generated algebras over the given ring?

In the 1960s, A. Grothendieck systematically investigated Noetherian rings that are exceptionally well behaved. He called these rings "excellent". The intent of his definition of excellent rings is that these rings should have the same nice properties as the rings in classical algebraic geometry. The rings in classical algebraic geometry are the affine rings

$$A = k[x_1, \ldots, x_n]/I,$$

where k is a field and I is a radical ideal of the polynomial ring $S_n := k[x_1, \ldots, x_n]$, where radical ideal is as defined Chapter 2.

There are three fundamental properties of affine rings that are relevant for the definition of excellent rings. The first two are straightforward:

Property 1: Every algebra of finite type over an affine ring is again an affine ring.

[1]Much of the material in this chapter comes from the article [**123**].

Property 2: Every affine ring is universally catenary.

We recall for Property 3 the following definition.

DEFINITION 14.2. Let A be a Noetherian ring. The *singular locus* of A is:

$$\mathrm{Sing}(A) = \{P \in \mathrm{Spec}(A) \mid A_P \text{ is not a regular local ring}\}.$$

Let k be an algebraically closed field. An *affine algebraic variety* V is a subset of affine n-space k^n for some positive integer n such that V is the zero set of an ideal I of the polynomial ring $S_n = k[x_1, \dots, x_n]$:

$$V \;=\; \mathcal{Z}(I) \;=\; \{a \in k^n \mid f(a) \;=\; 0, \text{ for all } f \in I \;\}.$$

It is clear that $V = \mathcal{Z}(\sqrt{I})$. We define the *singular locus* of V to be $\mathrm{Sing}(S_n/\sqrt{I})$.

It is a basic fact of algebraic geometry that the singular locus of an affine algebraic variety is again an affine algebraic variety [**46**, Theorem 5.3] This motivates Property 3:

Property 3: For every affine ring A over an arbitrary field k, the singular locus $\mathrm{Sing}(A)$ is closed in the Zariski topology of $\mathrm{Spec}(A)$, that is, there is an ideal $J \subseteq A$ such that $\mathrm{Sing}(A) = \mathcal{V}(J)$.

Property 3 is the key to the definition of excellent rings.

JACOBIAN CRITERION 14.3. Let $A = S/I$, where $S = k[x_1, \dots, x_n]$ is a polynomial ring over a perfect field k. Let P be a prime ideal of S with $I \subseteq P$, let $p = P/I$, and let r be the height of I_P in S_P. Assume that $I = (f_1, \dots, f_s)S$. The *Jacobian criterion* asserts the equivalence of the following statements:

(1) The map $\psi : k \hookrightarrow A_p$ is a regular homomorphism.
(2) rank $(\partial f_i/\partial x_j)\mathrm{mod}(P) \;=\; r$.
(3) $P \;\not\supseteq$ the ideal generated by the $r \times r$ minors of $(\partial f_i/\partial x_j)$.

Here regular map is as in Definition 3.25; that is, ψ is flat and has geometrically regular fibers. Equivalently, ψ is flat and, for each prime ideal P of A and each finite algebraic field extension L of k, the ring $A_P \otimes_k L$ is a regular local ring. Since k is a field, A_P is a free k-module and so the extension ψ is flat by Remark 2.21.2. If k is a perfect field, every algebraic extension is separable algebraic; this implies that $A_P \otimes_k L$ is a regular local ring when A_P is. Thus, in the case that k is a perfect field, the map $k \hookrightarrow A_p$ is regular if and only if A_p is a regular local ring. For a proof of the Jacobian criterion for regularity see [**96**, Theorem 30.3 and Remark 2, p. 235].

If k is not a perfect field, items 1 and 2 of the Jacobian criterion 14.2.0 are no longer equivalent. If k is a non-perfect field, item 2 of Criterion 14.2.0 is equivalent to the statement that A_p is smooth[2] over k, a condition that is stronger than the statement that A_p is a regular local ring[**96**, Theorem 30.3].

Example 14.4 is an example of a local ring over a non-perfect field k that is a regular local ring, but is not smooth over k.

EXAMPLE 14.4. Let k be a field of characteristic $p > 0$ such that k is not perfect, that is, k^p is properly contained in k. Let $a \in k \setminus k^p$ and let $f = x^p - a$. Then $L = k[x]/(f)$ is a proper purely inseparable extension field of k. Since $\partial f/\partial x = 0$, the Jacobian criterion for smoothness shows that L fails to be smooth over k. However, L is a field and thus a regular local ring. There exists a k^p-derivation

[2]For the definition of smoothness see Definition 11.7.

$D : k[x] \to k[x]$ with $D(f) \neq 0$. This reflects the basic idea of Nagata's Jacobian criterion for regularity [**96**, Theorem 30.10].

If k is a field of characteristic $p > 0$, the main idea of Nagata's Jacobian criterion for regularity is to include the k^{p^n}-derivations of k for all $n \in \mathbb{N}$ in addition to the partial derivatives $\partial f_i/\partial x_j$.

A first approach towards enlarging the class of affine rings might be to consider those rings that admit "Jacobian criteria", but this class is rather small. The following example from [**123**, p. 319] illustrates an example of an excellent Noetherian local ring that fails to satisfy Jacobian criteria:

Example 14.5. Let $\sigma = e^{(e^x-1)} \in \mathbb{Q}[x]$. Notice that σ and $\partial\sigma/\partial x$ are algebriacally independent over $\mathbb{Q}(x)$. As in Example 4.1, consider the intersection domain

$$A \ := \ \mathbb{Q}(x,\sigma) \cap \mathbb{Q}[[x]].$$

By Remark 2.1, A is a DVR with maximal ideal xA and field of fractions $\mathbb{Q}(x,\sigma)$. We have $\mathbb{Q}[x]_{(x)} \subset A \subset \mathbb{Q}[[x]]$. If $d : A \hookrightarrow \mathbb{Q}[[x]]$ is a derivation, then $d(\sigma) = d(x)\partial\sigma/\partial x$. Since σ and $\partial\sigma/\partial x$ are algebraically independent over $\mathbb{Q}(x)$ it follows that $d(\sigma) \notin A$ whenever $d(x) \neq 0$. Hence there is only the trivial derivation from A into itself. Since every DVR containing a field of characteristic 0 is excellent [**44**, Chap IV], [**118**], the ring A is excellent.

There is, however, an important class of Noetherian local rings that admit Jacobian and regularity criteria, namely, the class of complete Noetherian local rings. These criteria were established by Nagata and Grothendieck and are similar to the above mentioned criterion. The main objective of the theory of excellent rings is to exploit Jacobian criteria on the completion $\widehat{A}$ of an excellent local ring A in order to describe certain properties of A, even though the ring A itself may fail to admit Jacobian criteria. This theory requires considerable theoretical background. For example, Grothendieck's theory of formal smoothness and regularity was developed to work out the connection between a local ring A and its completion $\widehat{A}$.

In the following discussion we sketch the main ideas in the definition of excellent rings.

Discussion 14.6. Let (A, m) be a Noetherian local ring. By Cohen's structure theorems the m-adic completion $\widehat{A}$ of A is the homomorphic image of a formal power series ring over a ring K, where K is either a field or a complete discrete valuation ring, that is, $\widehat{A} \cong K[[x_1,\dots,x_n]]/I$; see Remarks 3.13.3. The singular locus $\operatorname{Sing}\widehat{A}$ of $\widehat{A}$ is closed by the Jacobian criterion on complete Noetherian local rings [**96**, Corollary to Theorem 30.10]. One way to guarantee closure of the singular locus of A is to require that the singular loci of A and $\widehat{A}$ be generated by an ideal J of A and the extension $J\widehat{A}$ of J to $\widehat{A}$, that is, to require:

$$\text{(14.6.a)}\quad \operatorname{Sing}(A) \ = \ \mathcal{V}(J) \quad \text{and} \quad \operatorname{Sing}(\widehat{A}) \ = \ \mathcal{V}(J\widehat{A}), \quad \text{for some ideal } J \text{ of } A.$$

We show in Proposition 14.7 below that Condition 14.6.a is equivalent to Condition 14.6.b.

$$\text{(14.6.b)}\qquad \text{for all } Q \in \operatorname{Spec}(\widehat{A}), \quad A_{Q\cap A} \ \text{ is regular } \iff \widehat{A}_Q \ \text{ is regular.}$$

We first make some observations about Condition 14.6.b. The backward direction " $\Leftarrow$ " of Condition 14.6.b is always satisfied: By Remark 3.2.2, $\widehat{A}$ is flat over A

for every Noetherian local ring A. Thus the induced morphism $A_{Q\cap A} \longrightarrow \widehat{A}_Q$ is faithfully flat. Since flatness descends regularity [**96**, Theorem 23.7(i)], if $\widehat{A}_Q$ is regular, then $A_{Q\cap A}$ is regular.

The forward direction " $\Rightarrow$ " of Condition 14.6.b requires an additional assumption about the map $f : A \to \widehat{A}$. If we assume that the fiber of f over $P = Q \cap A$ is regular as in Definition 3.22, it follows that the ring $\widehat{A}_Q/P\widehat{A}_Q$ is regular. By [**96**, Theorem 23.7(ii)], the ring $\widehat{A}_Q$ is regular, if A_P and $\widehat{A}_Q/P\widehat{A}_Q$ are both regular.

Proposition 14.7. *Let $(A, \mathbf{m})$ be a Noetherian local ring and let $\widehat{A}$ be its $\mathbf{m}$-adic completion. Condition 14.6.a is equivalent to Condition 14.6.b.*

Proof. It is clear that Condition 14.6.a implies Condition 14.6.b. Assume Condition 14.6.b and let I be the radical ideal of $\widehat{A}$ such that $\operatorname{Sing}(\widehat{A}) = \mathcal{V}(I)$. Then $I = \bigcap_{i=1}^n Q_i$, where the Q_i are prime ideals of $\widehat{A}$. Let $P_i = Q_i \cap A$ for each i and let $I \cap A = J$. Then $J = \bigcap_{i=1}^n P_i$. We observe that $\operatorname{Sing}(A) = \mathcal{V}(J)$. Since $\widehat{A}_{Q_i}$ is not regular, Condition 14.6.b implies that A_{P_i} is not regular. Let $P \in \operatorname{Spec} A$. If $J \subseteq P$, then $P_i \subseteq P$ for some i, and $P_i \subseteq P$ implies that A_{P_i} is a localization of A_P. Therefore A_P is not regular.

Assume that $J \not\subseteq P$. There exists $Q \in \operatorname{Spec} \widehat{A}$ such that $Q \cap A = P$, and it is clear that $I \not\subseteq Q$. Hence $\widehat{A}_Q$ is regular, and thus by Condition 14.6.b the ring A_P is regular. Therefore $\operatorname{Sing} A = \mathcal{V}(J)$.

It remains to observe that $\sqrt{J\widehat{A}} = I$. Clearly $\sqrt{J\widehat{A}} \subseteq I$. Let $Q \in \operatorname{Spec} \widehat{A}$ with $J\widehat{A} \subseteq Q$. Then $J \subseteq Q \cap A := P$ and A_P is not regular. By Condition 14.6.b, the ring $\widehat{A}_Q$ is not regular, so $I \subseteq Q$. $\square$

As in Chapter 3, if A is a ring and $p \in \operatorname{Spec}(A)$, then $k(p)$ denotes the field of fractions $\mathcal{Q}(A/p)$ of A/p. By permutability of localization and residue class formation $k(p) = A_p/pA_p$.

Let $(A, \mathbf{m})$ be a Noetherian local ring and let $\widehat{A}$ be its $\mathbf{m}$-adic completion. Recall from Definition 3.28 that the formal fibers of A are the rings $\widehat{A} \otimes_A k(p) = (A \setminus p)^{-1}(\widehat{A}/p\widehat{A})$ where $p \in \operatorname{Spec}(A)$.

As we observe in the paragraph above Proposition 14.7, Condition 14.6.b holds for a local ring A if the formal fibers of A are regular. In this case by Proposition 14.7 the singular locus of A is closed. Thus we have:

Corollary 14.8. *Let A be a Noetherian local ring such that the formal fibers of A are regular. Then the singular locus* $\operatorname{Sing} A$ *is closed.*

In order to obtain that every algebra essentially of finite type over A also has the property that its singular locus is closed, the stronger condition that the formal fibers of A are geometrically regular as in Definition 3.23 is needed.

Remark 14.9. Let $(A, \mathbf{m})$ be a Noetherian local ring and let $f : A \to B$ be a map to a Noetherian A-algebra. For $P \in \operatorname{Spec} A$, let $C = (A \setminus P)^{-1}(B/f(P)B$ denote the fiber over P in B.

In Definition 3.23, a prime ideal P in A is said to be geometrically regular if for every finite extension field F of $k(P)$ the condition for the fiber over P to be regular, given in Definition 3.22, holds true for every finite purely inseparable field extension $k \subseteq L$. That is, suppose that $B \otimes_k L$ is regular for every finite purely inseparable field extension $k \subseteq L$. Then the ring $B \otimes L$ is regular for every finite field extension $k \subseteq L$.

14.2. Excellent rings

In the case of a Noetherian local ring, Grothendieck defined excellence as follows:

DEFINITION 14.10. Let A be a Noetherian local ring. Then A is *excellent* if

(a) The formal fibers of A are geometrically regular, that is, for all $P \in \operatorname{Spec}(A)$, the ring $\widehat{A} \otimes_A L$ is regular for all finite purely inseparable field extensions $k(P) \subseteq L$.

(b) A is universally catenary.

Notice that Condition (iii) of Definition 3.31 about closedness of the singular locus of finitely generated A-algebras is not included in Definition 14.10. For the case of the local ring A, this follows from Corollary 14.8.

If A is an excellent local ring then its completion $\widehat{A}$ inherits many properties from A. In particular, we have :

THEOREM 14.11. [**44**, Section on excellence] *Let (A, m) be an excellent local ring, $Q \in \operatorname{Spec}(A)$, and $P = Q \cap A$ its contraction to A. Then the ring A_P is regular (normal, reduced, Cohen-Macaulay, Gorenstein, respectively) if and only if the ring $\widehat{A}_Q$ is regular (normal, reduced, Cohen-Macaulay, Gorenstein, respectively).*

COROLLARY 14.12. *If (A, m) is an excellent local ring then* $\operatorname{Sing}(A)$ *is closed in* $\operatorname{Spec}(A)$.

If A is not a local ring the formal fibers of A are the formal fibers of the local rings A_m where m is a maximal ideal of A. We say that A has geometrically regular formal fibers if the local rings A_m for all maximal ideals $m \subseteq A$ have geometrically regular formal fibers. If A is a semilocal ring with geometrically regular formal fibers then $\operatorname{Sing}(A)$ is again closed in $\operatorname{Spec}(A)$. If A is a non-semilocal ring with geometrically regular formal fibers then it is possible that $\operatorname{Sing}(A)$ is no longer closed in $\operatorname{Spec}(A)$, as in an example by Nishimura, [**106**]. Therefore the definition of a general (non semilocal) excellent ring requires an additional condition that guarantees that the singular locus of A and of all algebras of finite type over A are closed in their respective spectra. A definition of execllence equivalent to that given in Definition 3.31 is the following:

DEFINITION 14.13. A Noetherian ring A is excellent, if

(1) The formal fibers of A are geometrically regular.
(2) For every finitely generated A-algebra B the singular locus $\operatorname{Sing}(B)$ of B is closed in $\operatorname{Spec}(B)$.
(3) A is univerally catenary.

In working with excellent rings it is often helpful to make use of the notion of a regular morphism. Recall that in Definition 3.25 a homomorphism $f : A \longrightarrow B$ of Noetherian rings is defined to be regular if f is flat and if for every prime ideal $P \in \operatorname{Spec}(A)$ the ring $B \otimes_A k(P)$ is geometrically regular over $k(P)$.

Following Matsumura, in Definition 3.29 we have defined a Noetherian ring A to be a G-ring if the natural map $A_P \longrightarrow \widehat{A_P}$ is regular for all $P \in \operatorname{Spec} A$. If this condition is satisfied for all maximal ideals of A, then it also holds for all prime ideals of A.

Let A be a Noetherian ring and assume that $f : A \to B$ defines B as an A-algebra of finite type. Assume that $g : B \to C$ is a map of Noetherian rings. By Matsumura [**96**, Theorem 32.1] the composition $gf : A \to C$ is regular if g is regular. Since B is an A-algebra of finite type, the following theorem applies:

THEOREM 14.14. [**134**, Corollary 1.2] *Let $f : A \longrightarrow B$ be a morphism of Noetherian rings with B of finite type over A. Then the following are equivalent:*

(1) *f is regular.*
(2) *f is smooth, i.e, B is a smooth A-algebra.*

Since B is of finite type over A, the non-smooth locus of B over A, that is, the set

$$T = \{P \in \operatorname{Spec}(B) \mid B_P \text{ is not smooth over } A\},$$

is closed in $\operatorname{Spec}(B)$. Thus there is an ideal $J \subseteq B$ with $T = V(J)$. By Elkik [**32**], the ideal J can be computed by using the Jacobian matrix of B over A. For the above situation $A \xrightarrow{f} B \xrightarrow{g} C$, one can then conclude that gf is a regular morphism if and only if $JC = C$.

14.3. Henselian rings

Let $(R, \mathbf{m})$ be a local ring and let $k = R/\mathbf{m}$ denote the residue field of R. Recall that R is *Henselian* if the following condition known as "Hensel's Lemma" holds for R:

$$\begin{aligned}
&\text{For every monic polynomial } f(x) \in A[x] \text{ such that the image}\\
&\overline{f(x)} \in \overline{A[x]} = A[x]/\mathbf{m}[x] \cong k[x] \text{ can be expressed as } \overline{f(x)} = \alpha(x)\cdot\beta(x),\\
&\text{for some relatively prime monic polynomials } \alpha(x) \text{ and } \beta(x) \in \overline{A[x]},\\
&\text{there exist monic polynomials } g(x) \text{ and } h(x) \in A[x] \text{ with}\\
&f(x) = g(x)\cdot h(x), \alpha(x) = \overline{g(x)} \text{ and } \beta(x) = \overline{h(x)}.
\end{aligned} \tag{14.3.0}$$

Hensel's Lemma holds for every complete local ring [**96**, Theorem 8.3]. In particular the ring of p-adic integers satisfies Hensel's Lemma. The Henselian property was first observed in algebraic number theory around 1910 for the ring of p-adic integers. Many popular Noetherian local rings fail to be Henselian; see for example Exercise 14.1.

Since a Noetherian local ring $(R, \mathbf{m})$ is a subring of its $\mathbf{m}$-adic completion $\widehat{R}$, every Noetherian local ring is dominated by a Noetherian Henselian ring. The Henselization R^h is the smallest Henselian subring of $\widehat{R}$ that dominates R; see Definition 2.3.4a.

In this section we describe an approach to the construction of the Henselization of R developed in [**116**] and discussed in [**123**] that is different from that used in [**104**] and discussed in Definition 2.3.4. The approach in [**116**] uses the concept of an ètale morphism as in Definitions 14.15.

DEFINITIONS 14.15. Let $(R, \mathbf{m})$ be a local ring.

(1) Let $\varphi : (R, \mathbf{m}) \to (A, \mathbf{n})$ be a local homomorphism with A essentially finite over R; that is A is a localization of an R-algebra that is a finitely generated R-module. Then A is *ètale* over R if the following condition holds: for every R-algebra B and ideal N of B with $N^2 = 0$, every R-algebra homomorphism $g : A \to B/N$ has a unique lifting to an R-algebra homomorphims $f : A \to B$. Thus A is *étale* over R if for every commutative diagram of the form below, where the maps from $R \to A$ and $R \to B$ are the canonical ring homomorphisms that define A and B as R-algebras and the map $B \to B/N$ is the canonical quotient ring map

$$\begin{array}{ccc} R & \longrightarrow & A \\ \Big\downarrow & & \Big\downarrow{\scriptstyle g} \\ B & \longrightarrow & B/N, \end{array}$$

there exists a unique R-algebra homomorphism $f : A \to B$ that preserves commutativity of the diagram.

(2) A local ring $(A, \mathbf{n})$ is an *etale neighborhood* of R if A is ètale over R and $R/\mathbf{m} \cong A/\mathbf{n}$; that is, there is no residue field extension.

REMARKS 14.16. Let $(R, \mathbf{m})$ be a Noetherian local ring.

(1) The Henselization R^h of R is usually much smaller than the $\mathbf{m}$-adic completion $\widehat{R}$ of R. The Henselization R^h is an algebraic extension of R, whereas $\widehat{R}$ is often of infinite (uncountable) transcendence degree over R.

(2) If R is an excellent normal local domain, then R^h is the algebraic closure of R in $\widehat{R}$; that is, every element in $\widehat{R} \setminus R^h$ is transcendental over R, see, for example, [**104**, (44.1)].

Exercises

(1) Let x be an indeterminate over a field k and let R denote the localized polynomial ring $k[x]_{(xk[x])}$. Show that R is *not* Henselian.
Suggestion. Consider the polynomial $f(y) = y^2 + y + x \in R[y]$.

CHAPTER 15

Integral closure under extension to the completion

This chapter is related to the general question: "What properties of ideals of a Noetherian local ring A with maximal ideal $\mathbf{n}$ are preserved under extension to the $\mathbf{n}$-adic completion $\widehat{A}$?" We consider in this chapter the integral closure property (see Definition 15.1.4).

Using Insider Construction 13.1 described in Chapter 13, we present in this chapter an example of a 3-dimensional regular local domain $(A, \mathbf{n})$ having a height-two prime ideal P with the property that the extension $P\widehat{A}$ of P to the $\mathbf{n}$-adic completion $\widehat{A}$ of A is not integrally closed. The ring A in the example is a 3-dimensional regular local domain that is a nested union of 5-dimensional regular local domains.

More generally, we use this same technique to establish, for each integer $d \geq 3$ and each integer h with $2 \leq h \leq d-1$, the existence of a d-dimensional regular local domain $(A, \mathbf{n})$ having a prime ideal P of height h with the property that the extension $P\widehat{A}$ is not integrally closed, where $\widehat{A}$ is the $\mathbf{n}$-adic completion of A. A regular local domain having a prime ideal with this property is necessarily not excellent, see item 7 of Remark 15.7.

We discuss in Section 15.1 conditions in order that integrally closed ideals of a ring R extend to integrally closed ideals of R', where R' is an R-algebra. In particular, we consider under what conditions integrally closed ideals of a Noetherian local ring A extend to integrally closed ideals of the completion $\widehat{A}$ of A.

15.1. Integral closure under ring extension

For properties of integral closure of ideals, rings and modules we refer to the book of Swanson and Huneke [**135**]. In particular, we use the following definition.

DEFINITIONS 15.1. Let I be an ideal of a ring R.

1. (1) An element $r \in R$ is *integral over I* if there exists a positive integer n and a monic polynomial $f(x) = x^n + \sum_{i=1}^{n} a_i x^{n-i}$ such that $f(r) = 0$ and such that $a_i \in I^i$ for each i with $1 \leq i \leq n$.
2. (2) If J is an ideal contained in I and $JI^{n-1} = I^n$ for some positive integer n, then J is said to be a *reduction* of I.
3. (3) The *integral closure* $\overline{I}$ of I is the set of elements of R integral over I.
4. (4) If $I = \overline{I}$, then I is said to be *integrally closed*.
5. (5) The ideal I is said to be *normal* if I^n is integrally closed for every $n \geq 1$.

REMARKS 15.2. We record the following facts about an ideal I of a ring R.

(1) An element $r \in R$ is integral over I if and only if I is a reduction of the ideal $L = (I, r)R$. To see this equivalence, observe that for a monic polynomial $f(x)$ as in Definition 15.1.1, we have

$$f(r) = 0 \iff r^n = -\sum_{i=1}^{n} a_i r^{n-i} \in IL^{n-1} \iff L^n = IL^{n-1}.$$

(2) It is well known that $\overline{I}$ is an integrally closed ideal [**135**, Corollary 1.3.1].
(3) An ideal is integrally closed if and only if it is not a reduction of a properly bigger ideal.
(4) A prime ideal is always integrally closed. More generally, a radical ideal is always integrally closed.
(5) Let a, b be elements in a Noetherian ring R and let $I := (a^2, b^2)R$. The element ab is integral over I. If a, b form a regular sequence, then $ab \notin I$ and the ideal I is not integrally closed. More generally, if $h \geq 2$ and $a_1, \ldots, a_h$ form a regular sequence in R and $I := (a_1^h, \ldots, a_h^h)R$, then I is not integrally closed.
(6) If R is a normal domain, then I is a normal ideal of R if and only if the Rees algebra $R[It]$ is a normal domain, where the Rees algebra is defined in Definition 15.3; see the book of Swanson and Huneke [**135**, Prop. 5.2.1, p.95].

DEFINITION 15.3. Let I be an ideal of a ring R, and let t be a variable over R. The *Rees algebra of* I is the subring of $R[t]$ defined as

$$R[It] \ := \ \{ \sum_{i=0}^{n} a_i t^i \mid n \in \mathbb{N} \ ; \ a_i \in I^i \ \} \ = \ \bigoplus_{n \geq 0} I^n t^n,$$

where $I^0 = R$.

Our work in this chapter is motivated by the following questions:

QUESTIONS 15.4.

(1) Craig Huneke: "Does there exist an analytically unramified Noetherian local ring $(A, \mathbf{n})$ that has an integrally closed ideal I for which the extension $I\widehat{A}$ to the $\mathbf{n}$-adic completion $\widehat{A}$ is not integrally closed?"
(2) Sam Huckaba: "If there is such an example, can the ideal of the example be chosen to be a normal ideal?" See Definition 15.1.6.

Related to Question 15.4.1, we construct in Example 15.8 a 3-dimensional regular local domain A having a height-two prime ideal $I = P = (f, g)A$ such that $I\widehat{A}$ is not integrally closed. Thus the answer to Question 15.4.1 is "yes". This example also shows that the answer to Question 15.4.2 is again "yes". Since f, g form a regular sequence and A is Cohen-Macaulay, the powers P^n of P have no embedded associated primes and therefore are P-primary [**94**, (16.F), p. 112], [**96**, Ex. 17.4, p. 139]. Since the powers of the maximal ideal of a regular local domain are integrally closed, the powers of P are integrally closed, that is, P is a normal ideal.

Thus the Rees algebra $A[Pt] = A[ft, gt]$ is a normal domain while the Rees algebra $\widehat{A}[ft, gt]$ is not integrally closed.

REMARKS 15.5. Without the assumption that A is analytically unramified, there exist examples even in dimension one where an integrally closed ideal of a Noetherian local domain A fails to extend to an integrally closed ideal in $\widehat{A}$. If A is reduced but analytically ramified, then the zero ideal of A is integrally closed, but its extension to $\widehat{A}$ is not integrally closed.

Examples of reduced analytically ramified Noetherian local rings have been known for a long time. By Remark 3.13.5, the examples of Akizuki and Schmidt mentioned in Classical Examples 1.4 of Chapter 1 are analytically ramified Noetherian local domains. Another example due to Nagata is given in [**104**, Example 3, pp. 205-207]. (See also [**104**, (32.2), p. 114], and Remarks 4.9.2.)

Let R be a commutative ring and let R' be an R-algebra. In Remark 15.7 we list cases where extensions to R' of integrally closed ideals of R are again integrally closed. In this connection we use the following definition.

DEFINITION 15.6. Let R' be an R-algebra. The algebra R' is said to be *quasi-normal* over R if R' is flat over R and the following condition $(N_{R,R'})$ holds: If C is any R-algebra and D is a C-algebra in which C is integrally closed, then also $C \otimes_R R'$ is integrally closed in $D \otimes_R R'$. In this case we also say the map $R \to R'$ is *quasi-normal*.

REMARKS 15.7. Let R be a commutative ring and let R' be an R-algebra.

(1) By a result of Lipman [**85**, Lemma 2.4], if R' satisfies $(N_{R,R'})$ and I is an integrally closed ideal of R, then IR' is integrally closed in R'.

(2) A regular map of Noetherian rings is normal by Remark 3.26, and a normal map of Noetherian rings is quasi-normal [**44**, IV,(6.14.5)]. Hence a regular map of Noetherian rings is quasi-normal.

(3) Assume that R and R' are Noetherian rings and that R' is a flat R-algebra. Let I be an integrally closed ideal of R. The flatness of R' over R implies every $P' \in \operatorname{Ass}(R'/IR')$ contracts in R to some $P \in \operatorname{Ass}(R/I)$ [**96**, Theorem 23.2]. Thus by the previous item, if the map $R \to R'_{P'}$ is regular for each $P' \in \operatorname{Ass}(R'/IR')$, then IR' is integrally closed.

(4) Principal ideals of an integrally closed domain are integrally closed.

(5) In general, integral closedness of ideals is a local condition. If R' is an R-algebra that is a normal ring in the sense that for every prime ideal P' of R', the local ring $R'_{P'}$ is an integrally closed domain, then the extension to R' of every principal ideal of R is integrally closed by item 3. In particular, if $(A, \mathbf{n})$ is an analytically normal Noetherian local domain, then every principal ideal of A extends to an integrally closed ideal of $\widehat{A}$.

(6) Let $(A, \mathbf{n})$ be a Noetherian local ring and let $\widehat{A}$ be the $\mathbf{n}$-adic completion of A. Since $A/\mathbf{q} \cong \widehat{A}/\mathbf{q}\widehat{A}$ for every $\mathbf{n}$-primary ideal $\mathbf{q}$ of A, the $\mathbf{n}$-primary ideals of A are in one-to-one inclusion preserving correspondence with the $\widehat{\mathbf{n}}$-primary ideals of $\widehat{A}$. It follows that an $\mathbf{n}$-primary ideal I of A is a reduction of a properly larger ideal of A if and only if $I\widehat{A}$ is a reduction of a properly larger ideal of $\widehat{A}$. Therefore an $\mathbf{n}$-primary ideal I of A is integrally closed if and only if $I\widehat{A}$ is integrally closed.

(7) If R is an integrally closed domain, then for every ideal I and element x of R we have $\overline{xI} = x\overline{I}$. If $(A, \mathbf{n})$ is analytically normal and also a UFD, then

every height-one prime ideal of A extends to an integrally closed ideal of $\widehat{A}$. In particular if A is a regular local domain, then $P\widehat{A}$ is integrally closed for every height-one prime ideal P of A. If $(A, \mathbf{n})$ is a 2-dimensional local UFD, then every nonprincipal integrally closed ideal of A has the form xI, where I is an $\mathbf{n}$-primary integrally closed ideal and $x \in A$. In particular, this is the case if $(A, \mathbf{n})$ is a 2-dimensional regular local domain. In view of item 5, it follows that every integrally closed ideal of A extends to an integrally closed ideal of $\widehat{A}$ in the case where A is a 2-dimensional regular local domain.

(8) If $(A, \mathbf{n})$ is an excellent local ring, then the map $A \hookrightarrow \widehat{A}$ is quasi-normal by [**44**, (7.4.6) and (6.14.5)], and in this case every integrally closed ideal of A extends to an integrally closed ideal of $\widehat{A}$.

(9) Let $(A, \mathbf{n})$ be a Noetherian local domain and let A^h denote the Henselization of A. Every integrally closed ideal of A extends to an integrally closed ideal of A^h. This follows because A^h is a filtered direct limit of étale A-algebras; ; see the paper of Lipman [**85**, (i), (iii), (vii) and (ix), pp. 800-801]. Since the map from A to its completion $\widehat{A}$ factors through A^h, every integrally closed ideal of A extends to an integrally closed ideal of $\widehat{A}$ if and only if every integrally closed ideal of A^h extends to an integrally closed ideal of $\widehat{A}$.

15.2. Extension to the completion

We use results from Chapters 8, 9 and 13 In the construction of the following example.

Construction of Example 15.8. Let k be a field of characteristic zero and let x, y and z be indeterminates over k. Let $R := k[x,y,z]_{(x,y,z)}$ and let R^* be the (x)-adic completion of R. Thus $R^* = k[y,z]_{(y,z)}[[x]]$, the formal power series ring in x over the 2-dimensional regular local ring $k[y,z]_{(y,z)}$.

Let α and β be elements of $xk[[x]]$ that are algebraically independent over $k(x)$. Set

$$f \;=\; (y-\alpha)^2, \quad g \;=\; (z-\beta)^2, \quad \text{ and } \; A \;=\; k(x,y,z,f,g) \;\cap\; R^*.$$

Let $T := R[\alpha, \beta]$ and $S := R[f,g]$. Notice that S and T are both polynomial rings in 2 variables over R and that S is a subring of T. Indeed, with $u := y - \alpha$ and $v := z - \beta$, we have $T = R[u,v]$ and $S = R[u^2, v^2]$. Let

$$A^{\text{out}} \;:=\; R^* \;\cap\; k(x,y,z,\alpha,\beta).$$

By Localized Polynomial Example Theorem 9.7, the map $T \hookrightarrow R^*[1/x]$ is flat and A^{out} is a 3-dimensional regular local domain that is a directed union of 5-dimensional regular local domains. By Construction Properties Theorem 6.19.4, the rings A^{out} and A have (x)-adic completion R^*.

By Theorem 13.3, the ring A is Noetherian and equal to its approximation domain B provided that the map $S \hookrightarrow T[1/x]$ is flat. In our situation, the ring T is a free S-module with $\{1,\ y-\alpha,\ z-\beta,\ (y-\alpha)(z-\beta)\}$ as a free module basis. Therefore the map $S \hookrightarrow T[1/x]$ is flat. The following commutative diagram where all the labeled maps are the natural inclusions displays this situation:

$$
\begin{tikzcd}
B = A = R^* \cap \mathcal{Q}(S) \arrow[r, "\gamma_1"] & A^{\mathrm{out}} = R^* \cap \mathcal{Q}(T) \arrow[r, "\gamma_2"] & R^* = A^* \\
S = R[f,g] \arrow[u, "\delta_1"] \arrow[r, "\varphi"] & T = R[\alpha,\beta] \arrow[u, "\delta_2"] \arrow[r, equal] & T \arrow[u, "\psi"]
\end{tikzcd}
\tag{15.1}
$$

In order to better understand the structure of A, we recall some of the details of the approximation domain B associated to A.

Approximation Technique 15.9. With k, x, y, z, f, g, R and R^* as in Construction 15.8, we have

$$f = y^2 + \sum_{j=1}^{\infty} b_j x^j, \qquad g = z^2 + \sum_{j=1}^{\infty} c_j x^j,$$

where $b_j \in k[y]$ and $c_j \in k[z]$. The r^{th} *endpieces* for f and g are the sequences $\{f_r\}_{r=1}^{\infty}$, $\{g_r\}_{r=1}^{\infty}$ of elements in R^* defined for each $r \geq 1$ by:

$$f_r := \sum_{j=r+1}^{\infty} \frac{b_j x^j}{x^r} \qquad \text{and} \qquad g_r := \sum_{j=r+1}^{\infty} \frac{c_j x^j}{x^r}.$$

Then $f = y^2 + xb_1 + xf_1 = y^2 + xb_1 + x^2 b_2 + x^2 f_2 = \ldots$ and similar equations hold for g. Thus we have:

$$f = y^2 + xb_1 + x^2 b_2 + \ldots x^t b_t + x^t f_t; \quad g = y^2 + xc_1 + x^2 c_2 + \ldots x^t c_t + x^t g_t, \tag{15.9.0}$$

for each $t \geq 1$.

For each integer $r \geq 1$, the ring $B_r = (U_r)_{\mathbf{m}_r}$, where U_r is the polynomial ring $k[x, y, z, f_r, g_r]$ and $\mathbf{m}_r$ is the maximal ideal $(x, y, z, f_r, g_r)U_r$. We define the approximation domain $B := \bigcup_{r=1}^{\infty} B_r$.

Theorem 15.10. *Let A be the ring constructed in (15.8) and let $P = (f,g)A$, where f and g are as in (15.8). Then*

1. *The ring $A = B$ is a 3-dimensional regular local domain that has (xA)-adic completion $A^* = R^* = k[y,z]_{(y,z)}[[x]]$. Moreover A is a nested union of five-dimensional regular local domains.*
2. *The ideal P is a height-two prime ideal of A.*
3. *The ideal PA^* is not integrally closed in A^*.*
4. *The completion $\widehat{A}$ of A is $\widehat{R} = k[[x,y,z]]$ and $P\widehat{A}$ is not integrally closed.*

Proof. As we remark in Construction 15.8, the xA-adic completion of A is R^*, A is Noetherian and $A = B$ is a nested union of five-dimensional regular local domains. Thus item 1 follows from Theorem 13.3, parts 3 and 4.

For item 2, it suffices to observe that P has height two and that, for each positive integer r, $P_r := (f,g)U_r$ is a prime ideal of U_r. We have $f = y^2 + xb_1 + xf_1$ and $g = z^2 + xc_1 + xg_1$. It is clear that $(f,g)k[x,y,z,f,g]$ is a height-two prime ideal. Since $U_1 = kx, y, z, f_1, g_1]$ is a polynomial ring over k in the variables x, y, z, f_1, g_1, we see that

$$P_1 U_1[1/x] = (xb_1 + xf_1 + y^2, xc_1 + xg_1 + z^2)U_1[1/x]$$

is a height-two prime ideal of $U_1[1/x]$. Indeed, setting $f = g = 0$ is equivalent to setting $f_1 = -b_1 - y^2/x$ and $g_1 = -c_1 - z^2/x$. Therefore the residue class ring $(U_1/P_1)[1/x]$ is isomorphic to the integral domain $k[x, y, z][1/x]$. Since U_1 is Cohen-Macaulay and f, g form a regular sequence, and since $(x, f, g)U_1 = (x, y^2, z^2)U_1$ is an ideal of height three, we see that x is in no associated prime ideal of $(f, g)U_1$ (see, for example [**96**, Theorem 17.6]). Therefore $P_1 = (f, g)U_1$ is a height-two prime ideal, and so the same holds for P_1B_1.

For $r > 1$, by Equation 15.9.0, there exist elements $u_r \in k[x, y]$ and $v_r \in k[x, z]$ such that $f = x^r f_r + u_r x + y^2$ and $g = x^r g_r + v_r x + z^2$. An argument similar to that given above shows that $P_r = (f, g)U_r$ is a height-two prime ideal of U_r. Since U is the nested union of the U_r we have that $(f, g)U$ is a height-two prime ideal of U. Since B is a localization of U we see that $(f, g)B$ is a height-two prime ideal of $B = A$.

For items 3 and 4, $R^* = B^* = A^*$ by Construction 15.8 and it follows that $\widehat{A} = k[[x, y, z]]$. To see that $PA^* = (f, g)A^*$ and $P\widehat{A} = (f, g)\widehat{A}$ are not integrally closed, observe that $\xi := (y - \alpha)(z - \beta)$ is integral over PA^* and $P\widehat{A}$ since $\xi^2 = fg \in P^2$. On the other hand, $y - \alpha = u$ and $z - \beta = v$ form a regular sequence in A^* and $\widehat{A}$. Since $P = (u^2, v^2)A$, an easy computation shows that $uv \notin P\widehat{A} = (u^2, v^2)\widehat{A}$; see Exercise 1. Since $PA^* \subseteq P\widehat{A}$, this completes the proof. $\square$

In Example 15.11, we observe that this same technique may be used to construct, for each integer $d \geq 3$ and each integer h with $2 \leq h \leq d-1$, a d-dimensional regular local domain $(A, \mathbf{n})$ having a prime ideal P of height h with the property that the extension $P\widehat{A}$ is not integrally closed.

EXAMPLE 15.11. Let k be a field of characteristic zero and for an integer $n \geq 2$ let $x, y_1, \ldots, y_n$ be indeterminates over k. Let R denote the $d := n+1$ dimensional regular local ring obtained by localizing the polynomial ring $k[x, y_1, \ldots, y_n]$ at the maximal ideal generated by $(x, y_1, \ldots, y_n)$. Let h be an integer with $2 \leq h \leq n$ and let $\tau_1, \ldots, \tau_h \in xk[[x]]$ be algebraically independent over $k(x)$. For each i with $1 \leq i \leq h$, define $f_i = (y_i - \tau_i)^h$, and set $u_i = y_i - \tau_i$. Consider the rings

$$S \ := \ R[f_1, \ldots, f_h] \quad \text{and} \quad T \ := \ R[\tau_1, \ldots, \tau_h] \ = \ R[u_1, \ldots, u_h].$$

Notice that S and T are polynomial rings in h variables over R and that T is a finite free integral extension of S. The set

$$\{u_1^{e_1} \cdot u_2^{e_2} \cdot \ldots \cdot u_h^{e_h} \mid 0 \leq e_i \leq h-1\}$$

is a free module basis for T as an S-module. Therefore the map $S \hookrightarrow T[1/x]$ is flat. Let R^* denote the (x)-adic completion of R, and define

$$A \ := \ R^* \ \cap \ \mathcal{Q}(S) \quad \text{and} \quad A^{\text{out}} \ := \ R^* \ \cap \ \mathcal{Q}(T).$$

The following commutative diagram where the labeled maps are the natural inclusions displays this situation:

$$\begin{tikzcd}
B = A = R^* \cap \mathcal{Q}(S) \arrow[r, "\gamma_1"] & A^{\text{out}} = R^* \cap \mathcal{Q}(T) \arrow[r, "\gamma_2"] & R^* = A^* \\
S = R[f_1, \ldots, f_h] \arrow[u, "\delta_1"] \arrow[r, "\varphi"] & T = R[\tau_1, \ldots, \tau_h] \arrow[u, "\delta_2"] \arrow[r, equal] & T \arrow[u, "\psi"]
\end{tikzcd}$$

Since the map $S \hookrightarrow T$ is flat, Theorem 13.3 implies that the ring A is a d-dimensional regular local ring and is equal to its approximation domain B. Let $P := (f_1, \ldots, f_h)A$, we see that an argument similar to that given in Theorem 15.10 shows that P is a prime ideal of A of height h. We have $y_i - \tau_i = u_i \in A^*$. Let $\xi = \prod_{i=1}^h u_i$. Then $\xi^h = f_1 \cdots f_h \in P^h$ implies ξ is integral over PA^* and $P\widehat{A}$. Since $u_1, \ldots, u_h$ are a regular sequence in A^* and $\widehat{A}$, it follows that $\xi \notin P\widehat{A}$; see the thesis of Taylor [**136**, Theorem 1]. Therefore the extended ideals PA^* and $P\widehat{A}$ are not integrally closed.

15.3. Comments and Questions

In connection with Theorem 15.10 it is natural to ask the following question.

Question 15.12. For P and A as in Theorem 15.10, is P the only prime ideal of A that does not extend to an integrally closed ideal of $\widehat{A}$?

Comments 15.13. In relation to Example 15.8 and to Question 15.12, consider the following commutative diagram, where the labeled maps are the natural inclusions:

$$\begin{array}{ccccc}
B = A = R^* \cap \mathcal{Q}(S) & \xrightarrow{\gamma_1} & A^{\text{out}} = R^* \cap \mathcal{Q}(T) & \xrightarrow{\gamma_2} & R^* = A^* \\
\delta_1 \Big\uparrow & & \delta_2 \Big\uparrow & & \psi \Big\uparrow \\
S = R[f,g] & \xrightarrow{\varphi} & T = R[\alpha,\beta] & = & T
\end{array}$$

Referring to the diagram above, we observe the following:

(1) Theorem 13.3 implies that $A[1/x]$ is a localization of S and $A^{\text{out}}[1/x]$ is a localization of T. By Polynomial Example Theorem 9.5 of Chapter 9, A^{out} is excellent. Notice, however, that A is not excellent since there exists a prime ideal P of A such that $P\widehat{A}$ is not integrally closed by Remark 15.7.8. The excellence of A^{out} implies the map $\gamma_2 : A^{\text{out}} \to A^*$ is regular [**44**, (7.8.3 v)]. Thus for each $Q^* \in \operatorname{Spec} A^*$ with $x \notin Q^*$ the map $\psi_{Q^*} : T \to A^*_{Q^*}$ is regular. It follows that $\psi_x : T \to A^*[1/x]$ is regular.

(2) Let $Q^* \in \operatorname{Spec} A^*$ be such that $x \notin Q^*$ and let $\mathbf{q'} = Q^* \cap T$. Assume that $\varphi_{\mathbf{q'}} : S \to T_{\mathbf{q'}}$ is regular. By item 1 and [**96**, Theorem 32.1], the map $S \to A^*_{Q^*}$ is regular. Thus $(\gamma_2 \circ \gamma_1)_{Q^*} : A \to A^*_{Q^*}$ is regular.

(3) Let I be an ideal of A. Since A^{out} and A^* are excellent and both have completion $\widehat{A}$, Remark 15.7.8 shows that the ideals IA^{out}, IA^* and $I\widehat{A}$ are either all integrally closed or all fail to be integrally closed.

(4) In this setting, the Jacobian ideal of $\varphi : S \hookrightarrow T$ gives information about the smoothness and regularity of φ by Remark 11.10 and Theorem 11.11.1. The Jacobian ideal of $\varphi : S := k[x,y,z,f,g] \hookrightarrow T := k[x,y,z,\alpha,\beta]$ is the ideal of T generated by the determinant of the matrix
$$\mathcal{J} := \begin{pmatrix} \frac{\partial f}{\partial \alpha} & \frac{\partial g}{\partial \alpha} \\ \frac{\partial f}{\partial \beta} & \frac{\partial g}{\partial \beta} \end{pmatrix}.$$
Since the characteristic of the field k is zero, this ideal is $(y-\alpha)(z-\beta)T$.

In Proposition 15.14, we relate the behavior of integrally closed ideals in the extension $\varphi : S \to T$ to the behavior of integrally closed ideals in the extension $\gamma_2 \circ \gamma_1 : A \to A^*$.

PROPOSITION 15.14. *With the setting of Theorem 15.10, let I be an integrally closed ideal of A such that $x \notin Q$ for each $Q \in \operatorname{Ass}(A/I)$. Let $J = I \cap S$. If JT is integrally closed (resp. a radical ideal) then IA^* is integrally closed (resp. a radical ideal).*

PROOF. Since the map $A \to A^*$ is flat, Remark 15.7.3 implies that x is not in any associated prime ideal of IA^*. Therefore IA^* is contracted from $A^*[1/x]$ and it suffices to show $IA^*[1/x]$ is integrally closed (resp. a radical ideal). Our hypothesis implies $I = IA[1/x] \cap A$. By Comment 15.13.1, the ring $A[1/x]$ is a localization of S. Thus every ideal of $A[1/x]$ is the extension of its contraction to S. It follows that $IA[1/x] = JA[1/x]$. Thus $IA^*[1/x] = JA^*[1/x]$.

By Comment 15.13.1, the map $T \to A^*[1/x]$ is regular. If JT is integrally closed, then Remark 15.7.3 implies that $JA^*[1/x]$ is integrally closed. If JT is a radical ideal, then the zero ideal of $\frac{T}{JT}$ is integrally closed. The regularity of the map $\frac{T}{JT} \to \frac{A^*[1/x]}{JA^*[1/x]}$ implies that the zero ideal of $\frac{A^*[1/x]}{JA^*[1/x]}$ is integrally closed. Since the integral closure of the zero ideal is the nilradical, it follows that $JA^*[1/x]$ is a radical ideal. □

PROPOSITION 15.15. *With the setting of Theorem 15.10 and Comment 15.13, let $Q \in \operatorname{Spec} A$ be such that QA^* (or equivalently $Q\widehat{A}$) is not integrally closed. Then*

(1) *Q has height two and $x \notin Q$.*
(2) *There exists a minimal prime ideal Q^* of QA^* such that with $\mathbf{q}' = Q^* \cap T$, the map $\varphi_{\mathbf{q}'} : S \to T_{\mathbf{q}'}$ is not regular.*
(3) *Q contains $f = (y - \alpha)^2$ or $g = (z - \beta)^2$.*
(4) *Q is contained in $\mathbf{n}^2$, where $\mathbf{n}$ is the maximal ideal of A.*

PROOF. We have $\dim A = 3$, the maximal ideal of A extends to the maximal ideal of A^*, and principal ideals of A^* are integrally closed by Remark 15.7.7. Thus the height of Q is two. By Construction Properties Theorem 6.19, we have $A^*/xA^* = A/xA = R/xR$. Hence $x \notin Q$. This proves item 1.

By Remark 15.7.3, there exists a minimal prime ideal Q^* of QA^* such that $(\gamma_2 \circ \gamma_1)_{Q^*} : A \to A^*_{Q^*}$ is not regular. Thus item 2 follows from Comment 15.13.2.

For item 3, let Q^* and $\mathbf{q}'$ be as in item 2. Since $(\gamma_2 \circ \gamma_1)_{Q^*}$ is not regular it is not essentially smooth [**44**, 6.8.1]. By Theorem 11.11.1, $(y - \alpha)(z - \beta) \in \mathbf{q}'$. Hence $f = (y - \alpha)^2$ or $g = (z - \beta)^2$ is in $\mathbf{q}'$ and thus in Q. This proves item 3.

Suppose $w \in Q$ is a regular parameter for A; that is $w \in \mathbf{n} \backslash \mathbf{n}^2$. Then A/wA and A^*/wA^* are two-dimensional regular local domains. By Remark 15.7.7 QA^*/wA^* is integrally closed, but this implies that QA^* is integrally closed, which contradicts our hypothesis that QA^* is not integrally closed. This proves item 4. □

QUESTION 15.16. In the setting of Theorem 15.10 and Comment 15.13, let $Q \in \operatorname{Spec} A$ with $x \notin Q$ and let $\mathbf{q} = Q \cap S$. If QA^* is integrally closed, does it follow that $\mathbf{q}T$ is integrally closed?

QUESTION 15.17. In the setting of Theorem 15.10 and Comment 15.13, if a prime ideal Q of A contains f or g, but not both, and does not contain a regular parameter of A, does it follow that QA^* is integrally closed ?

In Example 15.8, the three-dimensional regular local domain A contains height-one prime ideals P such that $\widehat{A}/P\widehat{A}$ is not reduced. This motivates us to ask:

QUESTION 15.18. Let $(A, \mathbf{n})$ be a three-dimensional regular local domain and let $\widehat{A}$ denote the $\mathbf{n}$-adic completion of A. If for each height-one prime ideal P of A, the extension $P\widehat{A}$ is a radical ideal, i.e., the ring $\widehat{A}/P\widehat{A}$ is reduced, does it follow that $P\widehat{A}$ is integrally closed for each $P \in \operatorname{Spec} A$?

REMARK 15.19. A problem analogous to that considered here in the sense that it also deals with the behavior of ideals under extension to completion is addressed by Loepp and Rotthaus in [**88**]. They construct nonexcellent Noetherian local domains to demonstrate that tight closure need not commute with completion.

Exercise

(1) Let u, v be a regular sequence in a commutative ring R. Prove that $uv \notin (u^2, v^2)R$.

Suggestion: Use that if a, b are in R and $au = bv$, then $b \in uR$.

CHAPTER 16

Catenary local rings with geometrically normal formal fibers Oct. 29, 2013

In this chapter we consider the catenary property in a Noetherian local ring $(R, \mathbf{m})$ having geometrically normal formal fibers.[1] We prove that the Henselization R^h of R is universally catenary, and relate the catenary and universally catenary properties of R to the fibers of the map $R \hookrightarrow R^h$; see definitions below. Henselization is discussed in Definition 2.3.4a. We present for each integer $n \geq 2$ an example of a catenary Noetherian local integral domain of dimension n that has geometrically regular formal fibers and is not universally catenary.

16.1. History, terminology and summary

Recall that a ring R is *catenary* if, for every pair of comparable prime ideals $P \subset Q$ of R, every saturated chain of prime ideals from P to Q has the same length, see Definitions 3.16.3. The ring R is *universally catenary* if every finitely generated R-algebra is catenary.

Krull proves in [**81**] that every integral domain that is finitely generated as an extension ring of a field is catenary. Cohen proves in [**23**] that every complete Noetherian local ring is catenary. These results motivated the question of whether every Noetherian ring (or equivalently every Noetherian local integral domain) is catenary. Using power series rings, Nagata answers this question by giving an example of a family of non-catenary Noetherian local domains in [**101**]; see also [**104**, Example 2, pages 203-205]. Each domain in this family is not integrally closed and has the property that its integral closure is catenary.

These examples of Nagata motivated the question of whether the integral closure of a Noetherian local domain is catenary. Work on this question continued for over 20 years with Ratliff being a leading researcher in this area, [**114**], [**115**]. In 1980, T. Ogoma resolved this question by establishing the existence of a 3-dimensional Henselian Nagata local domain that is integrally closed but not catenary [**110**]; the Henselian property is defined in Definition 2.3.4. Heitmann in [**77**] gives a simplified presentation of Ogoma's example.

Heitmann in [**75**] obtains the following notable characterization of the complete Noetherian local rings that are the completion of a UFD. He proves that every complete Noetherian local ring $(R, \mathbf{m})$ that has depth at least two [2] and has the property that no element in the prime subring of R is a zerodivisor on R is the

[1] The material in this chapter comes from a paper we wrote that is included in a volume dedicated to Shreeram S. Abhyankar in celebration of his seventieth birthday. In his mathematical work Ram has opened up many avenues. In this chapter we are pursuing one of these related to power series and completions.

[2] See Definition 16.16.

completion of a Noetherian local UFD. Heitmann uses this result to establish the existence of a 3-dimensional Noetherian local UFD that is not universally catenary.

We recall that a Noetherian local ring $(R, \mathbf{m})$ with $\mathbf{m}$-adic completion $\widehat{R}$ has *geometrically normal* (respectively *geometrically regular) formal fibers* if for each prime P of R and for each finite algebraic extension k' of the field $k(P) := R_P/PR_P$, the ring $\widehat{R} \otimes_R k(P) \otimes_{k(P)} k'$ is normal (respectively regular), see Definition 3.23.

In this chapter we investigate the catenary property in Noetherian local rings having geometrically normal formal fibers. In Example 16.11 we apply a technique developed in previous chapters to construct, for each integer $n \geq 2$, an example of a catenary Noetherian local integral domain of dimension n with geometrically regular formal fibers that is not universally catenary.

Let $(R, \mathbf{m})$ be a Noetherian local ring. We denote the Henselization of R by R^h. Recall that $(R, \mathbf{m})$ is *formally equidimensional*, or in other terminology *quasi-unmixed*, provided all the minimal primes of the $\mathbf{m}$-adic completion $\widehat{R}$ have the same dimension. A theorem of Ratliff relating the universal catenary property to properties of the completion, Theorem 3.18 of Chapter 3, is crucial for our work.

In Section 16.2 we present results concerning conditions for a Noetherian local ring $(R, \mathbf{m})$ to be universally catenary. The theorem of Ratliff mentioned above leads to our observation in Proposition 16.1 that a Henselian Noetherian local ring having geometrically normal formal fibers is universally catenary.

Assume that $(R, \mathbf{m})$ is a Noetherian local integral domain having geometrically normal formal fibers. Corollary 16.4 implies that the Henselization R^h of R is universally catenary; moreover, if the integral closure $\overline{R}$ of R is local, then R is universally catenary. Theorem 16.6 asserts that R is universally catenary if and only if the set Γ_R is empty, where

$$\Gamma_R \ := \ \{P \in \operatorname{Spec}(R^h) \mid \dim(R^h/P) < \dim(R/P \cap R))\}.$$

We also prove that the subset Γ_R of $\operatorname{Spec} R^h$ is stable under generalization in the sense that if $Q \in \Gamma_R$ and $P \in \operatorname{Spec} R^h$ is such that $P \subseteq Q$, then $P \in \Gamma_R$. Thus Γ_R satisfies a strong "going down" property.

In Theorem 16.7 we prove for R as above that R is catenary but is not universally catenary if and only if the set Γ_R is nonempty and each prime ideal P in Γ_R has dimension one. It follows in this case that Γ_R is a finite subset of the minimal primes of R^h. Thus, as we observe in Corollary 16.8, if R is catenary but not universally catenary, this is signaled by the existence of dimension one minimal primes of the $\mathbf{m}$-adic completion $\widehat{R}$ of R. If R is catenary, each minimal prime of $\widehat{R}$ having dimension different from the dimension of R must have dimension one.

In Section 16.3 we present examples to illustrate the results of Section 16.2. We apply a construction involving power series, homomorphic images and intersections developed in previous chapters, see Chapter 8.

The construction begins with a Noetherian integral domain that may be taken to be a polynomial ring in several indeterminates over a field. In Theorem 16.9 we extend this construction by proving that in certain circumstances it is possible to transfer the flatness, Noetherian and computability properties of integral domains associated with ideals $I_1, \dots, I_n$ of R to the integral domain associated with their intersection $I = I_1 \cap \cdots \cap I_n$.

We apply these concepts in Examples 16.10 - 16.12 to obtain Noetherian local domains that are not universally catenary. In Remark 16.13, we specifiy precisely which of these rings are catenary. These domains illustrate the results of Section 16.2, because in Section 16.6 we prove that they have geometrically regular formal fibers.

We thank M. Brodmann and R. Sharp for raising a question on catenary and universally catenary rings that motivated our work in this chapter.

16.2. Geometrically normal formal fibers

Throughout this section $(R, \mathbf{m})$ is a Noetherian local ring. We use Theorem 3.18, the result of Ratliff mentioned above, that relates the universally catenary property to properties of the completion in order to prove:

Proposition 16.1. *Let $(R, \mathbf{m})$ be a Henselian Noetherian local ring having geometrically normal formal fibers.*

(1) *For each prime ideal P of R, the extension $P\widehat{R}$ to the $\mathbf{m}$-adic completion of R is also a prime ideal.*
(2) *The ring R is universally catenary.*

Proof. Item 2 follows from item 1 and Theorem 3.18. In order to prove item 1, observe that the completion of R/P is $\widehat{R}/P\widehat{R}$, and R/P is a Noetherian Henselian local integral domain having geometrically normal formal fibers. Hence, by passing from R to R/P, to prove item 1, it suffices to prove that if R is a Henselian Noetherian local integral domain having geometrically normal formal fibers, then the completion $\widehat{R}$ of R is an integral domain.

Since R has normal formal fibers, the completion $\widehat{R}$ of R is reduced. Hence the integral closure $\overline{R}$ of R is a finitely generated R-module [**104**, (32.2)]. Moreover, since R is Henselian, $\overline{R}$ is local [**104**, (43.12)].

The completion $\widehat{\overline{R}}$ of $\overline{R}$ is $\widehat{R} \otimes_R \overline{R}$ [**104**, (17.8)]. Since the formal fibers of R are geometrically normal, the formal fibers of $\overline{R}$ are also geometrically normal. It follows that $\widehat{\overline{R}}$ is normal [**96**, Corollary, page 184], and hence an integral domain because $\overline{R}$ is local. Since $\widehat{R}$ is a flat R-module, $\widehat{R}$ is a subring of $\widehat{\overline{R}}$. Therefore $\widehat{R}$ is an integral domain. $\square$

Remark 16.2. Let $(R, \mathbf{m})$ be a Noetherian local domain. An interesting result proved by Nagata [**104**, (43.20)] establishes the existence of a one-to-one correspondence between the minimal primes of the Henselization R^h of R and the maximal ideals of the integral closure $\overline{R}$ of R. Moreover, if a maximal ideal $\overline{\mathbf{m}}$ of $\overline{R}$ corresponds to a minimal prime q of R^h, then the integral closure of the Henselian local domain R^h/q is the Henselization of $\overline{R}_{\overline{\mathbf{m}}}$ [**104**, Ex. 2, page 188], [**100**]. Therefore $\mathrm{ht}(\overline{\mathbf{m}}) = \dim(R^h/q)$.

Remark 16.3. Let $(R, \mathbf{m})$ be a Noetherian local ring, let $\widehat{R}$ denote the $\mathbf{m}$-adic completion of R, and let R^h denote the Henselization of R. It is well known that the canonical map $R \hookrightarrow R^h$ is a regular map with zero-dimensional fibers, and that $\widehat{R}$ is also the completion of R^h with respect to its unique maximal ideal $\mathbf{m}^h = \mathbf{m}R^h$. The following statements are equivalent:

(1) The map $R \hookrightarrow \widehat{R}$ has (geometrically) normal fibers.
(2) The map $R^h \hookrightarrow \widehat{R}$ has (geometrically) normal fibers.

Let P be a prime ideal of R and let $U = R \setminus P$. Then $PR^h = P_1 \cap \cdots \cap P_n$, where the P_i are the minimal prime ideals of PR^h. Then $P\widehat{R} = (\cap_{i=1}^n P_i)\widehat{R}$. Since $\widehat{R}$ is faithfully flat over R^h, finite intersections distribute over this extension, and $P\widehat{R} = \cap_{i=1}^n (P_i\widehat{R})$. Let $S = U^{-1}(\widehat{R}/P\widehat{R})$ denote the fiber over P in $\widehat{R}$ and let $q_i = P_iS$. The ideals $q_1, \ldots, q_n$ of S intersect in (0) and are pairwise comaximal because for $i \neq j$, $(P_i + P_j) \cap U \neq \emptyset$. Therefore $S \cong \prod_{i=1}^n (S/q_i)$. Since a Noetherian ring is normal if and only if it is a finite product of normal Noetherian domains, the fiber over P in $\widehat{R}$ is normal if and only if the the fiber over each of the P_i in $\widehat{R}$ is normal.

Corollary 16.4. *Let R be a Noetherian local domain having geometrically normal formal fibers. Then*

(1) *The Henselization R^h of R is universally catenary.*
(2) *If the integral closure $\overline{R}$ of R is again local, then R is universally catenary.*

In particular, if R is a normal Noetherian local domain having geometrically normal formal fibers, then R is universally catenary.

Proof. For item 1, the Henselization R^h of R is a Noetherian local ring having geometrically normal formal fibers, and so Proposition 16.1 implies that R^h is universally catenary. For item 2, if the integral closure of R is local, then, by Remark 16.2, the Henselization R^h has a unique minimal prime. Since R^h is universally catenary, the completion $\widehat{R}$ is equidimensional, and hence R is universally catenary. □

Theorem 16.5 relates the catenary property of R to the height of maximal ideals in the integral closure of R.

Theorem 16.5. *Let $(R, \mathbf{m})$ be a Noetherian local domain of dimension d and let $\overline{R}$ denote the integral closure of R. If $\overline{R}$ contains a maximal ideal $\overline{\mathbf{m}}$ with $\operatorname{ht}(\overline{\mathbf{m}}) = r \notin \{1, d\}$, then there exists a saturated chain of prime ideals in R of length $\leq r$. Hence in this case R is not catenary.*

Proof. Since $\overline{R}$ has only finitely many maximal ideals [**104**, (33.10)], there exists $b \in \overline{\mathbf{m}}$ such that b is in no other maximal ideal of $\overline{R}$. Let $R' = R[b]$ and let $\mathbf{m}' = \overline{\mathbf{m}} \cap R'$. Notice that $\overline{\mathbf{m}}$ is the unique prime ideal of $\overline{R}$ that contains $\mathbf{m}'$. By the Going-up Theorem[**96**, (9.3)], $\operatorname{ht} \mathbf{m}' = r$. Since R' is a finitely generated R-module and is birational over R, there exists a nonzero element $a \in \mathbf{m}$ such that $aR' \subseteq R$. It follows that $R[1/a] = R'[1/a]$. The maximal ideals of $R[1/a]$ have the form $PR[1/a]$, where $P \in \operatorname{Spec} R$ is maximal with respect to not containing a. For $P \in \operatorname{Spec} R$ such that $PR[1/a]$ is maximal in $R[1/a]$, there are no prime ideals strictly between P and $\mathbf{m}$ by Theorem 2.6. If $\operatorname{ht} P = h$, then there exists in R a saturated chain of prime ideals through P of length $h + 1$. Thus to show R is not catenary, it suffices to establish the existence of a maximal ideal of $R[1/a]$ having height different from $d - 1$. Since $R[1/a] = R'[1/a]$, the maximal ideals of $R[1/a]$ correspond to the prime ideals P' in R' maximal with respect to not containing a. Since $\operatorname{ht} \mathbf{m}' > 1$, there exists $c \in \mathbf{m}'$ such that c is not in any minimal prime of aR' nor in any maximal ideal of R' other than $\mathbf{m}'$. Hence there exist prime ideals of R' containing c and not containing a. Let $P' \in \operatorname{Spec}(R')$ be maximal with respect to $c \in P'$ and $a \notin P'$. Then $P' \subset \mathbf{m}'$, so $\operatorname{ht} P' \leq r - 1 < d - 1$. It follows that there exists a saturated chain of prime ideals of R of length $\leq r$, and hence R is not catenary. $\square$

THEOREM 16.6. *Let $(R, \mathbf{m})$ be a Noetherian local integral domain having geometrically normal formal fibers and let R^h denote the Henselization of R. Consider the set*

$$\Gamma_R \;:=\; \{P \in \operatorname{Spec}(R^h) \mid \dim(R^h/P) < \dim(R/(P \cap R))\}.$$

The following statements then hold.

(1) *For $\mathbf{p} \in \operatorname{Spec}(R)$, the ring $R/\mathbf{p}$ is not universally catenary if and only if there exists $P \in \Gamma_R$ such that $\mathbf{p} = P \cap R$.*
(2) *The set Γ_R is empty if and only if R is universally catenary.*
(3) *If $\mathbf{p} \subset \mathbf{q}$ are prime ideals in R and if there exists $Q \in \Gamma_R$ with $Q \cap R = \mathbf{q}$, then there also exists $P \in \Gamma_R$ with $P \cap R = \mathbf{p}$.*
(4) *If $Q \in \Gamma_R$, then each prime ideal P of R^h such that $P \subseteq Q$ is also in Γ_R, that is, the subset Γ_R of $\operatorname{Spec} R^h$ is stable under generalization.*

PROOF. The map of $R/\mathbf{p}$ to its $\mathbf{m}$-adic completion $\widehat{R}/\mathbf{p}\widehat{R}$ factors through $R^h/\mathbf{p}R^h$. Since $R \hookrightarrow \widehat{R}$ has geometrically normal fibers, so does the map $R^h \hookrightarrow \widehat{R}$. Proposition 16.1 implies that each prime ideal P of R^h extends to a prime ideal $P\widehat{R}$. Therefore, by Theorem 3.18, the ring $R/\mathbf{p}$ is universally catenary if and only if $R^h/\mathbf{p}R^h$ is equidimensional if and only if there does not exist $P \in \Gamma_R$ with $P \cap R = \mathbf{p}$. This proves items 1 and (2). To prove item 3, observe that if $R/\mathbf{p}$ is universally catenary, then $R/\mathbf{q}$ is also universally catenary [**96**, Theorem 31.6].

It remains to prove item 4. Let $P \in \operatorname{Spec} R^h$ be such that $P \subset Q$, and let $\operatorname{ht}(Q/P) = n$. Since the fibers of the map $R \hookrightarrow R^h$ are zero-dimensional, the contraction to R of an ascending chain of primes

$$P \;=\; P_0 \;\subsetneq\; P_1 \;\subsetneq\; \cdots \;\subsetneq P_n \;=\; Q$$

of R^h is a strictly ascending chain of primes from $\mathbf{p} := P \cap R$ to $\mathbf{q} := Q \cap R$. Hence $\operatorname{ht}(\mathbf{q}/\mathbf{p}) \geq n$. Since R^h is catenary, we have

$$\dim(R^h/P) \;=\; n + \dim(R^h/Q) \;<\; n + \dim(R/\mathbf{q}) \;\leq\; \dim(R/\mathbf{p}),$$

where the strict inequality is because $Q \in \Gamma_R$. Therefore $P \in \Gamma_R$. $\square$

THEOREM 16.7. *Let $(R, \mathbf{m})$ be a Noetherian local integral domain having geometrically normal formal fibers and let Γ_R be defined as in Theorem 16.6. The ring R is catenary but not universally catenary if and only if*

(i) *the set Γ_R is nonempty, and*
(ii) *$\dim R^h/P = 1$, for each prime ideal $P \in \Gamma_R$.*

If these conditions holds, then each $P \in \Gamma_R$ is a minimal prime of R^h, and Γ_R is a finite nonempty open subset of $\operatorname{Spec} R^h$.

PROOF. Assume that R is catenary but not universally catenary. By Theorem 16.6, the set Γ_R is nonempty and there exist minimal primes P of R^h such that $\dim(R^h/P) < \dim(R^h)$. By Remark 16.2, if a maximal ideal $\overline{\mathbf{m}}$ of $\overline{R}$ corresponds to a minimal prime P of R^h, then $\operatorname{ht}(\overline{\mathbf{m}}) = \dim(R^h/P)$. Since R is catenary, Theorem 16.5 implies that the height of each maximal ideal of the integral closure $\overline{R}$ of R is either one or $\dim(R)$. Therefore $\dim(R^h/P) = 1$ for each minimal prime

P of R^h for which $\dim(R^h/P) \neq \dim(R^h)$. Item 4 of Theorem 16.6 implies each $P \in \Gamma_R$ is a minimal prime of R^h and $\dim R^h/P = 1$.

For the converse, assume that Γ_R is nonempty and each prime $P \in \Gamma_R$ has dimension one. Then R is not universally catenary by item 2 of Theorem 16.6 and by item 4 of Theorem 16.6, each prime ideal in Γ_R is a minimal prime of R^h and therefore lies over (0) in R. To show R is catenary, it suffices to show for each nonzero nonmaximal prime ideal $\mathbf{p}$ of R that $\text{ht}(\mathbf{p}) + \dim(R/\mathbf{p}) = \dim(R)$ [**96**, Theorem 31.4]. Let $P \in \text{Spec}(R^h)$ be a minimal prime of $\mathbf{p}R^h$. Since R^h is flat over R with zero-dimensional fibers, $\text{ht}(\mathbf{p}) = \text{ht}(P)$. Let Q be a minimal prime of R^h with $Q \subseteq P$. Then $Q \notin \Gamma_R$. For by assumption every prime of Γ_R has dimension one, so if Q were in Γ_R, then $Q = P$. But $P \cap R = \mathbf{p}$, which is nonzero, and $Q \cap R = (0)$. Therefore $Q \notin \Gamma_R$ and hence $\dim(R^h/Q) = \dim(R^h)$. Since R^h is catenary, it follows that $\text{ht}(P) + \dim(R^h/P) = \dim(R^h)$. Since $P \notin \Gamma_R$, we have $\dim(R/\mathbf{p}) = \dim(R^h/P)$. Therefore $\text{ht}(\mathbf{p}) + \dim(R/\mathbf{p}) = \dim(R)$ and R is catenary. $\square$

COROLLARY 16.8. *If R has geometrically normal formal fibers and is catenary but not universally catenary, then there exist in the $\mathbf{m}$-adic completion $\widehat{R}$ of R minimal prime ideals q such that $\dim(\widehat{R}/q) = 1$.*

PROOF. By Theorem 16.7, each prime ideal $Q \in \Gamma_R$ has dimension one and is a minimal prime of R^h. Moreover, $Q\widehat{R} := q$ is a minimal prime of $\widehat{R}$. Since $\dim(R^h/Q) = 1$, we have $\dim(\widehat{R}/q) = 1$. $\square$

16.3. A method for constructing examples

In Theorem 16.9 we show in certain circumstances that the flatness, Noetherian and computability properties associated with ideals $I_1, \ldots, I_n$ of the (z)-adic completion R^* of R as described in Theorem 8.3 transfer to the integral domain associated with their intersection. In Section 16.6, we show that the property of regularity of formal fibers also transfers in certain cases to the integral domain associated with their intersection. Theorem 16.9 is a generalization of Theorem 9.10.

THEOREM 16.9. *Assume that R is a Noetherian integral domain with field of fractions K, $z \in R$ is a nonzero nonunit, R^* is the (z)-adic completion of R, and $I_1, \ldots, I_n$ are ideals of R^* such that, for each $i \in \{1, \ldots, n\}$, each associated prime of R^*/I_i intersects R in (0). Also assume the map $R \hookrightarrow (R^*/I_i)[1/z]$ is flat for each i and that the localizations at z of the I_i are pairwise comaximal; that is, for all $i \neq j$, $(I_i + I_j)R^*[1/z] = R^*[1/z]$. Let $I := I_1 \cap \cdots \cap I_n$, $A := K \cap (R^*/I)$ and, for $i \in \{1, 2, \ldots n\}$, let $A_i := K \cap (R^*/I_i)$. Then*

1. *Each associated prime of R^*/I intersects R in (0).*
2. *The map $R \hookrightarrow (R^*/I))[1/z]$ is flat, so the ring A is Noetherian and is equal to its associated approximation ring B. The (z)-adic completion A^* of A is R^*/I, and the (z)-adic completion A_i^* of A_i is R^*/I_i, for $i \in \{1, \ldots, n\}$.*
3. *The ring $A^*[1/z] \cong (A_1^*)[1/z] \oplus \cdots \oplus (A_n^*)[1/z]$. If $Q \in \text{Spec}(A^*)$ and $z \notin Q$, then $(A^*)_Q$ is a localization of precisely one of the A_i^*.*
4. *We have $A \subseteq A_1 \cap \cdots \cap A_n$ and $\cap_{i=1}^n (A_i)[1/z] \subseteq A_P$ for each $P \in \text{Spec}\, A$ with $z \notin P$. Thus we have $A[1/z] = \cap_{i=1}^n (A_i)[1/z]$.*

PROOF. By Theorem 8.3, the (z)-adic completion A_i^* of A_i is R^*/I_i. Since $\mathrm{Ass}(R^*/I) \subseteq \bigcup_{i=1}^n \mathrm{Ass}(R^*/I_i)$, the condition on associated primes of Theorem 8.3 holds for the ideal I. The natural R-algebra homomorphism $\pi : R^* \to \bigoplus_{i=1}^n (R^*/I_i)$ has kernel I. Further, the localization of π at z is onto because $(I_i + I_j)R^*[1/z] = R^*[1/z]$ for all $i \neq j$. Thus $(R^*/I)[1/z] \cong \bigoplus_{i=1}^n (R^*/I_i)[1/z] = \bigoplus_{i=1}^n (A_i^*)[1/z]$ is flat over R. Therefore A is Noetherian and is equal to its associated approximation ring B, and $A^* = R^*/I$ is the (z)-adic completion of A.

If $Q \in \mathrm{Spec}(A^*)$ and $z \notin Q$, then A_Q^* is a localization of

$$A^*[1/z] \cong (A_1^*)[1/z] \oplus \ \cdots \ \oplus \ (A_n^*)[1/z].$$

Every prime ideal of $\bigoplus_{i=1}^n (A_i^*)[1/z]$ has the form $(Q_i)A_i^*[1/z] \oplus \bigoplus_{j \neq i} (A_j^*)[1/z]$, where $Q_i \in \mathrm{Spec}(A_i^*)$ for a unique $i \in \{1, \dots, n\}$. It follows that A_Q^* is a localization of A_i^* for precisely this i.

Since R^*/I_i is a homomorphic image of R^*/I, the intersection ring $A \subseteq A_i$ for each i. Let $P \in \mathrm{Spec}\, A$ with $z \notin P$. Since $A^* = R^*/I$ is faithfully flat over A, there exists $P^* \in \mathrm{Spec}(A^*)$ with $P^* \cap A = P$. Then $z \notin P^*$ implies $A_{P^*}^* = (A_i^*)_{P_i^*}$, where $P_i^* \in \mathrm{Spec}(A_i^*)$ for some $i \in \{1, \dots, n\}$. Let $P_i = P_i^* \cap A_i$. Since $A_P \hookrightarrow A_{P^*}^*$ and $(A_i)_{P_i} \hookrightarrow (A_i^*)_{P_i^*}$ are faithfully flat, we have

$$A_P = A_{P^*}^* \cap K = (A_i^*)_{P_i^*} \cap K = (A_i)_{P_i} \supseteq (A_i)[1/z].$$

It follows that $\bigcap_{i=1}^n (A_i)[1/z] \subseteq A_P$. Thus we have

$$\bigcap_{i=1}^n (A_i)[1/z] \ \subseteq \ \bigcap \{A_P \mid P \in \mathrm{Spec}\, A \text{ and } z \notin P\} \ = \ A[1/z].$$

Since $A[1/z] \subseteq (A_i)[1/z]$, for each i, we have $A[1/z] = \bigcap_{i=1}^n (A_i)[1/z]$. $\square$

16.4. Examples that are not universally catenary

Example 9.11 is an example of a two-dimensional Noetherian local domain A having geometrically regular formal fibers such that the completion $\widehat{A}$ of A has two minimal primes one of dimension one and one of dimension two. Thus A is not universally catenary. We generalize this example in the following.

We construct in Example 16.10 a two-dimensional Noetherian local domain having geometrically normal formal fibers such that the completion has any desired finite number of minimal primes of dimensions one and two.

EXAMPLE 16.10. Let r and s be positive integers and let R be the localized polynomial ring in three variables $R := k[x, y, z]_{(x,y,z)}$, where k is a field of characteristic zero and the field of fractions of R is $K := k(x, y, z)$. Then the (x)-adic completion of R is $R^* := k[y, z]_{(y,z)}[[x]]$. Let $\tau_1, \dots, \tau_r, \beta_1, \beta_2, \dots, \beta_s, \gamma \in xk[[x]]$ be algebraically independent power series over $k(x)$. Define

$$Q_i \ := \ (z - \tau_i, y - \gamma)R^*, \ \text{ for } \ i \in \{1, \dots, r\} \ \text{ and } \ P_j \ := \ (z - \beta_j)R^*, \ \text{ for } \ j \in \{1, \dots, s\}.$$

We apply Theorem 16.9 with $I_i = Q_i$ for $1 \le i \le r$, and $I_{r+j} = P_j$ for $1 \le j \le s$. Then the I_i satisfy the comaximality condition at the localization at x. Let $I := I_1 \cap \cdots \cap I_{r+s}$ and let $A := K \cap (R^*/I)$. For J an ideal of R* containing I, let $\bar{J}$ denote the image of J in R^*/I. Then, for each i with $1 \le i \le r$, $\dim((R^*/I)/\bar{Q}_i) = \dim(R^*/Q_i) = 1$ and, for each j with $1 \le j \le s$, $\dim((R^*/I)/\bar{P}_j) = 2$. Thus A^* contains r minimal primes of dimension one and s minimal primes of dimension two.

Since A^* modulo each of its minimal primes is a regular local ring, the completion $\widehat{A}$ of A also has precisely r minimal primes of dimension one and s minimal primes of dimension two.

The integral domain A birationally dominates R and is birationally dominated by each of the A_i. Corollary 16.19 implies that A has geometrically regular formal fibers. Since $\dim(A) = 2$, A is catenary.

We show in Example 16.11 that for every integer $n \geq 2$ there is a Noetherian local domain $(A, \mathbf{m})$ of dimension n that has geometrically regular formal fibers and is catenary but not universally catenary.

EXAMPLE 16.11. Let $R = k[x, y_1, \dots, y_n]_{(x,y_1,\dots,y_n)}$ be a localized polynomial ring of dimension $n+1$ where k is a field of characteristic zero. Let $\sigma, \tau_1, \dots, \tau_n \in xk[[x]]$ be $n+1$ algebraically independent elements over $k(x)$ and consider in $R^* = k[y_1, \dots, y_n]_{(y_1,\dots,y_n)}[[x]]$ the ideals

$$I_1 = (y_1 - \sigma)R^* \quad \text{and} \quad I_2 = (y_1 - \tau_1, \dots, y_n - \tau_n)R^*.$$

Then the ring

$$A = k(x, y_1, \dots, y_n) \cap (R^*/(I_1 \cap I_2))$$

is the desired example. The completion $\widehat{A}$ of A has two minimal primes $I_1\widehat{A}$ having dimension n and $I_2\widehat{A}$ having dimension one. By Corollary 16.19, A has geometrically regular formal fibers. Therefore the Henselization A^h has precisely two minimal prime ideals P, Q which may be labeled so that $P\widehat{A} = I_1\widehat{A}$ and $Q\widehat{A} = I_2\widehat{A}$. Thus $\dim(A^h/P) = n$ and $\dim(A^h/Q) = 1$. By Theorem 16.7, A is catenary but not universally catenary.

In Example 16.12 we construct for each positive integer t and specified nonnegative integers $n_1, \dots, n_t$ with $n_1 \geq 1$, a t-dimensional Noetherian local domain A that has geometrically regular formal fibers and birationally dominates a $t+1$-dimensional regular local domain such that the completion $\widehat{A}$ of A has, for each r with $1 \leq r \leq t$, exactly n_r minimal primes $\mathbf{p}_{rj}$ of dimension $t+1-r$. Moreover, each $\widehat{A}/\mathbf{p}_{rj}$ is a regular local ring of dimension $t+1-r$. If $n_i > 0$ for some $i \neq 1$, then A is not universally catenary and is not a homomorphic image of a regular local domain. It follows from Remark 16.2 that the derived normal ring $\overline{A}$ of A has exactly n_r maximal ideals of height $t+1-r$ for each r with $1 \leq r \leq t$.

EXAMPLE 16.12. Let t be a positive integer and let n_r be a nonnegative integer for each r with $1 \leq r \leq t$. Assume that $n_1 \geq 1$. We construct a t-dimensional Noetherian local domain A that has geometrically regular formal fibers such that $\widehat{A}$ has exactly n_r minimal primes of dimension $t+1-r$ for each r. Let $x, y_1 \dots, y_t$ be indeterminates over a field k of characteristic zero.

Let $R = k[x, y_1, \dots, y_t]_{(x,y_1,\dots,y_t)}$, let $R^* = k[y_1, \dots, y_t]\,[[x]]_{(x,y_1,\dots,y_t)}$ denote the (x)-adic completion of R and let K denote the field of fractions of R. For every $r, j, i \in \mathbb{N}$ such that $1 \leq r \leq t$, $1 \leq j \leq n_r$ and $1 \leq i \leq r$, choose elements $\{\tau_{rji}\}$ of $xk[[x]]$ so that the set $\bigcup\{\tau_{rji}\}$ is algebraically independent over $k(x)$.

For each r, j with $1 \leq r \leq t$ and $1 \leq j \leq n_r$, define the prime ideal $P_{rj} := (y_1 - \tau_{rj1}, \dots, y_r - \tau_{rjr})$ of height r in R^*. Notice that R^*/P_{rj} is a regular local ring of dimension $t+1-r$. Theorem 9.5 implies that the extension $R \hookrightarrow (R^*/P_{rj})[1/x]$ is flat, and that the intersection domain $A_{rj} := K \cap (R^*/P_{rj})$ is a regular local ring of dimension $t+1-r$ that has (x)-adic completion R^*/P_{rj}.

Let $I := \bigcap P_{rj}$ be the intersection of all the prime ideals P_{rj}. Since the $\tau_{rji} \in xk[[x]]$ are algebraically independent over $k(x)$, the sum of any two of these ideals P_{rj} and P_{mi}, where we may assume $r \le m$, has radical $(x, y_1, \dots, y_m)R^*$, and thus $(P_{rj} + P_{mi})R^*[1/x] = R^*[1/x]$. It follows that the representation of I as the intersection of the P_{rj} is irredundant and $\operatorname{Ass}(R^*/I) = \{P_{rj} \mid 1 \le r \le t, 1 \le j \le n_r\}$. Since each $P_{rj} \cap R = (0)$, we have $R \hookrightarrow R^*/I$, and the intersection domain $A := K \cap (R^*/I)$ is well defined.

By Theorem 16.9, the map $R \hookrightarrow (R^*/I)[1/x]$ is flat, A is Noetherian and A is a localization of a subring of $R[1/x]$. Since R^*/P_{rj} is a regular local ring of dimension $t+1-r$, the minimal primes of $\widehat{A}$ all have the form $\mathbf{p}_{rj} := P_{rj}\widehat{A}$ and each $\widehat{A}/\mathbf{p}_{rj}$ is a regular local ring of dimension $t+1-r$. The ring A birationally dominates the $(t+1)$-dimensional regular local domain R and all the stated properties hold once we prove Corollary 16.19.

Remark 16.13. By Theorem 16.7, the ring A constructed in Example 16.12 is catenary if and only if each minimal prime of $\widehat{A}$ has dimension either one or t. By taking $n_r = 0$ for $r \notin \{1, t\}$ in Example 16.12, we obtain additional examples of catenary Noetherian local domains A of dimension t having geometrically regular formal fibers for which the completion $\widehat{A}$ has precisely n_t minimal primes of dimension one and n_1 minimal primes of dimension t; thus A is not universally catenary.

Remark 16.14. Let $(A, \mathbf{n})$ be a Noetherian local domain constructed as in Example 16.12 and let A^h denote the Henselization of A. Since each minimal prime of $\widehat{A}$ is the extension of a minimal prime of A^h and also the extension of a minimal prime of A^*, the minimal primes of A^h and A^* are in a natural one-to-one correspondence. Let P be the minimal prime of A^h corresponding to a minimal prime $\mathbf{p}$ of A^*. Since the minimal primes of A^* extend to pairwise comaximal prime ideals of $A^*[1/x]$, for each prime ideal $Q \supset P$ of A^h with $x \notin Q$, the prime ideal P is the unique minimal prime of A^h contained in Q. Let $\mathbf{q} := Q \cap A$. We have $\operatorname{ht}\mathbf{q} = \operatorname{ht} Q$, and either $\dim(A/\mathbf{q}) > \dim(A^h/Q)$ or else every saturated chain of prime ideals of A containing $\mathbf{q}$ has length less than $\dim A$.

In this connection, we ask:

Question 16.15. Let $(A, \mathbf{n})$ be a Noetherian local domain constructed as in Example 16.12. If A is not catenary, what can be said about the cardinality of the set

$$\Gamma_A \ := \ \{P \in \operatorname{Spec}(A^h) \mid \dim(A^h/P) \ < \ \dim(A/(P \cap A))\}?$$

Is the set Γ_A ever infinite?

16.5. The depth of the constructed rings

We thank Lucho Avramov for suggesting we consider the depth of the rings constructed in Example 16.12, where depth is defined as follows.

Definition 16.16. Let I be an ideal in a Noetherian ring R and let M be a finitely generated R-module such that $IM \neq M$. Elements $x_1, \dots, x_d$ in I are said to form a *regular sequence on* M, or an *M-sequence*, if x_1 is not a zerodivisor on M and for i with $2 \le i \le d$, the element x_i is not a zerodivisor on $M/(x_1, \dots, x_{i-1})M$.

It is known that maximal M-sequences of elements of I exist and all maximal M-sequences of elements of I have the same length, see [**96**, Theorem 16.7] or [**78**, Theorem 121]. This integer is called the *grade of* I on M and denoted $G(I, M)$. If R is a Noetherian local ring with maximal ideal $\mathbf{m}$, and M is a nonzero finitely generated R-module, then the grade of $\mathbf{m}$ on M is also called the *depth* of M. In particular the depth of R is $G(\mathbf{m}, R)$.

REMARK 16.17. The catenary rings that arise from the construction in Example 16.12 all have depth one. However, Example 16.12 can be used to construct, for each integer $t \geq 3$ and integer d with $2 \leq d \leq t-1$, an example of a non-catenary Noetherian local domain A of dimension t and depth d having geometrically regular formal fibers. The (x)-adic completion A^* of A has precisely two minimal primes, one of dimension t and one of dimension d. To establish the existence of such an example, with notation as in Example 16.12, we set $m = t - d + 1$ and take $n_r = 0$ for $r \notin \{1, m\}$ and $n_1 = n_m = 1$. Let

$$P_1 := P_{11} = (y_1 - \tau_{111})R^* \text{ and } P_m := P_{m1} = (y_1 - \tau_{m11}, \dots, y_m - \tau_{m1m})R^*.$$

Consider $A^* = R^*/(P_1 \cap P_m)$ and the short exact sequence

$$0 \longrightarrow \frac{P_1}{P_1 \cap P_m} \longrightarrow \frac{R^*}{P_1 \cap P_m} \longrightarrow \frac{R^*}{P_1} \longrightarrow 0.$$

Since P_1 is principal and not contained in P_m, we have $P_1 \cap P_m = P_1 P_m$ and $P_1/(P_1 \cap P_m) \cong R^*/P_m$. It follows that depth $A^* = \operatorname{depth}(R^*/P_m) = d$; [**78**, page 103, ex 14] or [**18**, Prop. 1.2.9, page 11]. Since the local ring A and its (x)-adic completion have the same completion $\widehat{A}$ with respect to their maximal ideals, we have $\operatorname{depth} A = \operatorname{depth} \widehat{A} = \operatorname{depth} A^*$ [**96**, Theorem 17.5]. By Remark 16.2, the derived normal ring $\overline{A}$ of A has precisely two maximal ideals one of height t and one of height d.

16.6. Regular maps and geometrically regular formal fibers

We show in Corollary 16.19 that each of the rings A constructed in Examples 16.10, 16.11 and 16.12 has geometrically regular formal fibers.

THEOREM 16.18. *Assume that R, K, z and R^* are as in Noetherian Flatness Theorem 8.3. For a positive integer n, let $I_1, \dots, I_n$ be ideals of R^* such that each associated prime of R^*/I_i intersects R in (0), for $i = 1, \dots, n$. Let $I := I_1 \cap \cdots \cap I_n$. Assume that*

(1) *R is semilocal with geometrically regular formal fibers and z is in the Jacobson radical of R.*
(2) *Each $(R^*/I_i)[1/z]$ is a flat R-module and, for each $i \neq j$, the ideals $I_i(R^*)[1/z]$ and $I_j R^*[1/z]$ are comaximal in $(R^*)[1/z]$.*
(3) *For $i = 1, \dots, n$, $A_i := K \cap (R^*/I_i)$ has geometrically regular formal fibers.*

Then $A := K \cap (R^*/I)$ is equal to its approximation domain B, and has geometrically regular formal fibers.

PROOF. Since R has geometrically regular formal fibers, by (9.12.2), it suffices to show for $W \in \operatorname{Spec}(R^*/I)$ with $z \notin W$ that $R_{W_0} \longrightarrow (R^*/I)_W$ is regular, where

$W_0 := W \cap R$. As in Theorem 16.9, we have

$$(R^*/I)[1/z] \;=\; (R^*/I_1)[1/z] \;\oplus \cdots \oplus\; (R^*/I_n)[1/z].$$

It follows that $(R^*/I)_W$ is a localization of R^*/I_i for some $i \in \{1, \ldots, n\}$. If $(R^*/I)_W = (R^*/I_i)_{W_i}$, where $W_i \in \operatorname{Spec}(R^*/I_i)$, then $R_{W_0} = (A_i)_{W_i \cap A_i}$ and $(A_i)_{W_i \cap A_i} \longrightarrow (R^*/I_i)_{W_i}$ is regular. Thus $R_{W_0} \longrightarrow (R^*/I)_W$ is regular. $\square$

Corollary 16.19. *The rings A of Examples 16.10, 16.11 and 16.12 have geometrically regular formal fibers, that is, the map $\phi : A \to \widehat{A}$ is regular.*

Proof. By the definition of R and the observations given in (9.12), the hypotheses of (16.18) are satisfied. $\square$

Exercises

(1) Let $(R, \mathbf{m})$ be a catenary Noetherian local domain having geometrically normal formal fibers. If R is not universally catenary, prove that R has depth one.

    Suggestion: Use Theorem 16.7 and [**18**, Prop. 1.2.13].

(2) Let R be an integral domain with field of fractions K and let R' be a subring of K that contains R. If $P \in \operatorname{Spec} R$ is such that $R' \subseteq R_P$, prove that there exists a unique prime ideal $P' \in \operatorname{Spec} R'$ such that $P' \cap R = P$.

CHAPTER 17

Non-Noetherian insider examples of dimension 3,

In this chapter we use Insider Construction 13.1 of Section 13.1 to construct examples where the insider approximation domain B is local and non-Noetherian, but is very close to being Noetherian. The localizations of B at all nonmaximal prime ideals are Noetherian, and most prime ideals of B are finitely generated. Sometimes just one prime ideal is not finitely generated.

In Section 17.1 we describe, for each positive integer m, a three-dimensional local unique factorization domain B such that the maximal ideal of B is two-generated, B has precisely m prime ideals of height two, each prime ideal of B of height two is not finitely generated and all the other prime ideals of B are finitely generated. We give more details about a specific case where there is precisely one nonfinitely generated prime ideal. Section 17.2 contains the verification of the properties of the three-dimensional examples. A similar example is given by John David in [**28**]. In Chapter 18 we present a generalization to dimension four.

17.1. A family of examples in dimension 3

In this section we construct examples as described in Examples 17.1. In Discussion 17.5 we give more details for a special case of the example with exactly one nonfinitely generated prime ideal. We display the prime spectrum for this special case in Diagram 17.3.2.

EXAMPLES 17.1. For each positive integer m, we construct an example of a non-Noetherian local integral domain $(B, \mathbf{n})$ such that:

(1) $\dim B = 3$.
(2) The ring B is a UFD that is not catenary, as defined in (3.16.3).
(3) The maximal ideal $\mathbf{n}$ of B is generated by two elements.
(4) The $\mathbf{n}$-adic completion of B is a two-dimensional regular local domain.
(5) For every non-maximal prime ideal P of B, the ring B_P is Noetherian.
(6) The ring B has precisely m prime ideals of height two.
(7) Every prime ideal of B of height two is not finitely generated; all other prime ideals of B are finitely generated.

To establish the existence of the examples in Examples 17.1, we use the following notation: Let k be a field, let x and y be indeterminates over k, and set

$$R := k[x,y]_{(x,y)}, \quad K := k(x,y) \quad \text{and} \quad R^* := k[y]_{(y)}[[x]].$$

The power series ring R^* is the xR-adic completion of R. Let $\tau \in xk[[x]]$ be transcendental over $k(x)$. For each integer i with $1 \le i \le m$, let $p_i \in R \setminus xR$ be such that $p_1R^*, \ldots, p_mR^*$ are m prime ideals. For example, if each $p_i \in R \setminus (x,y)^2R$, then each p_iR^* is prime in R^*. In particular one could take $p_i = y - x^i$. Let

$p := p_1 \cdots p_m$. We set $f := p\tau$ and consider the injective R-algebra homomorphism $S := R[f] \hookrightarrow R[\tau] =: T$. In this construction the polynomial rings S and T have the same field of fractions $K(f) = K(\tau)$. Hence the intersection domain

$$A \; := \; A_f \; := \; R^* \,\cap\, K(f) \; = \; R^* \,\cap\, K(\tau) := \; A_\tau. \tag{17.1.0}$$

By Valabrega's Theorem 4.2, A is a two-dimensional regular local domain with maximal ideal $(x,y)A$ and the $(x,y)A$-adic completion of A is $k[[x,y]]$.

Let $\tau := c_1x + c_2x^2 + \cdots + c_ix^i + \cdots \in xk[[x]]$, where the $c_i \in k$ and define for each $n \in \mathbb{N}_0$ the "n^{th} endpiece" τ_n of τ by

$$\tau_n \; := \; \sum_{i=n+1}^{\infty} c_i x^{i-n} \; = \; \frac{\tau - \sum_{i=1}^{n} c_i x^i}{x^n}. \tag{17.1.a}$$

As in Equation 6.1.2 we have the following relation between the n^{th} and $(n+1)^{\text{st}}$ endpieces τ_n and τ_{n+1}:

$$\tau_n \; = \; c_{n+1}x \; + \; x\tau_{n+1}. \tag{17.1.b}$$

Define $f_n := p\tau_n$, set $U_n = k[x,y][f_n] = k[x,y,f_n]$, a three-dimensional polynomial ring over R, and set $B_n = (U_n)_{(x,y,f_n)} = k[x,y,f_n]_{(x,y,f_n)}$, a three-dimensional localized polynomial ring. Similarly set $U_{\tau n} = k[x,y,\tau_n]$, a three-dimensional polynomial ring containing U_n, and $B_{\tau n} = k[x,y,\tau_n]_{(x,y,\tau_n)}$, a localized polynomial ring containing $U_{\tau n}$ and B_n. Let U, B, U_τ and B_τ be the nested union approximation domains defined as follows:

$$U \; := \; \bigcup_{n=0}^{\infty} U_n \; \subseteq \; U_\tau \; := \; \bigcup_{n=0}^{\infty} U_{\tau n}; \qquad B \; := \; \bigcup_{n=0}^{\infty} B_n \; \subseteq \; B_\tau \; := \; \bigcup_{n=0}^{\infty} B_{\tau n} \; \subseteq \; A.$$

REMARK 17.2. By Construction Properties Theorem 6.19.2, with adjustments using Remark 6.4, parts 3 and 4, we have

(17.2.0)

$$U[1/x] = U_0[1/x] = k[x,y,f][1/x]; \quad U_\tau[1/x] = U_{\tau,0}[1/x] = k[x,y,\tau][1/x],$$

$B[1/x]$ is a localization of $S = R[f]$ and $B[1/x]$ is a localization of B_n. Similarly, $B_\tau[1/x]$ is a localization of $T = R[\tau]$.

We establish in Theorem 17.9 of Section 17.2 that the rings B of Examples 17.1 have properties 1 through 7 and also some additonal properties.

Assuming properties 1 through 7 of Examples 17.1, we describe the ring B of Examples 17.1 in the case where $m = 1$ and $p = p_1 = y$ as follows:

EXAMPLE 17.3. Let the notation be as in Examples 17.1. Thus

$$R = k[x,y]_{(x,y)}, \quad f = y\tau, \quad f_n = y\tau_n, \quad B_n = R[y\tau_n]_{(x,y,y\tau_n)}, \quad B = \bigcup_{n=0}^{\infty} B_n.$$

As we show in Section 17.2, the ideal $Q := (y, \{y\tau_n\}_{n=0}^{\infty})B$ is the unique prime ideal of B of height 2. Moreover, Q is not finitely generated and is the only prime ideal of B that is not finitely generated. We also have $Q = yA \cap B$, and $Q \cap B_n = (y, y\tau_n)B_n$ for each $n \geq 0$.

To identify the ring B up to isomorphism, we include the following details: By Equation 17.1.b, we have $\tau_n = c_{n+1}x + x\tau_{n+1}$. Thus we have

$$f_n \; = \; xf_{n+1} \; + \; yxc_{n+1}. \tag{17.3.1}$$

The family of equations (17.3.1) uniquely determines B as a nested union of the three-dimensional RLRs $B_n = k[x, y, f_n]_{(x,y,f_n)}$.

We recall the following terminology of [**149**, page 325].

DEFINITION 17.4. If a ring C is a subring of a ring D, a prime ideal P of C is *lost* in D if $PD \cap C \neq P$.

DISCUSSION 17.5. Assuming properties 1 through 7 of Examples 17.1, if q is a height-one prime of B, then B/q is Noetherian if and only if q is not contained in Q. This is clear since q is principal, Q is the unique prime of B that is not finitely generated, and, by Cohen's Theorem 2.8, a ring is Noetherian if each prime ideal of the ring is finitely generated.

The height-one primes q of B may be separated into several types as follows:

Type I. The primes $q \not\subseteq Q$ have the property that B/q is a one-dimensional Noetherian local domain. These primes are contracted from A, i.e., they are not lost in A. To see this, consider $q = gB$ where $g \notin Q$. Then gA is contained in a height one prime P of A. Hence $g \in (P \cap B) \setminus Q$, and so $P \cap B \neq Q$. Since $\mathbf{m}_B A = \mathbf{m}_A$, we have $P \cap B \neq \mathbf{m}_B$. Therefore $P \cap B$ is a height-one prime containing q, so $q = P \cap B$ and $B_q = A_P$.

There are infinitely many primes q of type I, because every element of $\mathbf{m}_B \setminus Q$ is contained in a prime q of type I. Thus $\mathbf{m}_B \subseteq Q \cup \bigcup\{q \text{ of Type I}\}$. Since $\mathbf{m}_B$ is not the union of finitely many strictly smaller prime ideals, there are infinitely many primes q of Type I.

Type I*. Among the primes of Type I, we label the prime ideal xB as Type I*. The prime ideal xB is special since it is the unique height-one prime q of B for which R^*/qR^* is not complete. If q is a height-one prime of B such that $x \notin qR^*$, then $x \notin q$ by Proposition 6.20.3. Thus R^*/qR^* is complete with respect to the powers of the nonzero principal ideal generated by the image of x mod qR^*. Notice that $R^*/xR^* \cong k[y]_{yk[y]}$.

If q is a height-one prime of B not of Type I, then $\overline{B} = B/q$ has precisely three prime ideals. These prime ideals form a chain: $(\overline{0}) \subset \overline{Q} \subset \overline{(x,y)B} = \overline{\mathbf{m}_B}$.

Type II. We define the primes of Type II to be the primes $q \subset Q$ such that q has height one and is contracted from a prime p of $A = k(x, y, f) \cap R^*$, i.e., q is not lost in A. For example, the prime $y(y+\tau)B$ is of Type II by Lemma 17.13. For q of Type II, the domain B/q is dominated by the one-dimensional Noetherian local domain A/p. Thus B/q is a non-Noetherian generalized local ring in the sense of Cohen; that is, the unique maximal ideal $\overline{\mathbf{n}}$ of B/q is finitely generated and $\cap_{i=1}^{\infty} \overline{\mathbf{n}}^{\,i} = (0)$, [**23**].

For q of Type II, the maximal ideal of B/q is not principal. This follows because a generalized local domain having a principal maximal ideal is a DVR [**104**, (31.5)].

There are infinitely many height-one primes of Type II, for example, $y(y+x^t\tau)B$ for each $t \in \mathbb{N}$ by Lemma 17.12. For q of Type II, the DVR B_q is birationally dominated by A_p. Hence $B_q = A_p$ and the ideal $\sqrt{qA} = p \cap yA$. [1]

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[1]Bruce Olberding has pointed out that the existence of prime ideals q of Type II answers a question asked by Anderson-Matijevic-Nichols in [**10**, page 17]. Their question asks whether in an integral domain every nonzero finitely generated prime ideal P that satisfies $\bigcap_{n=1}^{\infty} P^n = (0)$ and that is minimal over a principal ideal has $\operatorname{ht} P = 1$. For q of Type II, the ring $\overline{B} = B/q$ is a

That each element $y(y + x^t\tau)$ is irreducible and thus generates a height-one prime ideal, is done in greater generality in Lemma 17.12.

Type III. The primes of Type III are the primes $q \subset Q$ such that q has height one and is not contracted from A, i.e., q is lost in A. For example, the prime yB and the prime $(y + x^t y\tau)B$ for $t \in \mathbb{N}$ are of Type III by Lemma 17.13. Since the elements y and $y + x^t y\tau$ are in $\mathbf{m}_B$ and are not in $\mathbf{m}_B^2$ and since B is a UFD, these elements are necessarily prime. There are infinitely many such prime ideals by Lemma 17.12. For q of Type III, we have $\sqrt{qA} = yA$.

If $q = yB$ or $q = (y + x^t y\tau)B$, then the image $\overline{\mathbf{m}_B}$ of $\mathbf{m}_B$ in B/q is principal. It follows that the intersection of the powers of $\overline{\mathbf{m}_B}$ is Q/q and B/q is not a generalized local ring. To see that $\bigcap_{i=1}^{\infty} \overline{\mathbf{m}_B} \neq (0)$, we argue as follows: If P is a principal prime ideal of a ring and P' is a prime ideal properly contained in P, then P' is contained in the intersection of the powers of P; see [**78**, page 7, ex. 5] and Exercise 17.3

The picture of $\mathrm{Spec}(B)$ is shown below.

$\mathbf{m}_B := (x, y)B$

$Q := (y, \{f_i\})B$

$xB \in \boxed{\text{Type I}}$ $\boxed{\text{Type II}}$ $yB \in \boxed{\text{Type III}}$

(0)

Diagram 17.3.2

In Remarks 17.6 we examine the height-one primes of B from a different perspective.

Remarks 17.6. (1) Assume the notation of Example 17.3. If w is a nonzero prime element of B such that $w \notin Q$, then wA is a prime ideal in A and is the unique prime ideal of A lying over wB. To see this, observe that $w \notin yA$ since $w \notin Q = yA \cap B$. It follows that $y \notin p$, for every prime ideal $p \in \mathrm{Spec}\, A$ that is a minimal prime of wA. Thus $p \cap B \neq Q$. Since we assume the properties of Examples 17.1 hold, $p \cap B$ has height one. Therefore $p \cap B = wB$. Hence the DVR B_{wB} is birationally dominated by A_p, and thus $B_{wB} = A_p$. This implies that p is the unique prime of A lying over wB. We also have $wB_{wB} = pA_p$. Since A is a UFD and p is the unique minimal prime of wA, it follows that $wA = p$. In particular, q is not lost in A; see Definition 17.4.

If q is a height-one prime ideal of B that is contained in Q, then yA is a minimal prime of qA, and q is of Type II or III depending on whether or not qA has other minimal prime divisors.

generalized local domain with precisely 3 prime ideals. An element in the maximal ideal $\overline{\mathbf{m}_B}$ not in the other nonzero prime ideal generates an ideal primary for $\overline{\mathbf{m}_B}$. Since $\mathrm{ht}\, \overline{\mathbf{m}_B} = 2$, this yields a negative answer to the question.

To see this, observe that if yA is the only prime divisor of qA, then qA has radical yA and $yA \cap B = Q$ implies that Q is the radical of $qA \cap B$. Thus q is lost in A and q is of Type III.

On the other hand, if there is a minimal prime ideal $p \in \operatorname{Spec} A$ of qA that is different from yA, then y is not in $p \cap B$ and hence $p \cap B \neq Q$. Since Q is the only prime ideal of B of height two, it follows that $p \cap B$ is a height-one prime and thus $p \cap B = q$. Thus q is not lost in A and q is of Type II.

We observe that for every Type II prime q there are exactly two minimal primes of qA; one of these is yA and the other is a height-one prime p of A such that $p \cap B = q$. For every height-one prime ideal p of A such that $p \cap B = q$, we have B_q is a DVR that is birationally dominated by A_p and hence $B_q = A_p$. The uniqueness of B_q implies that there is precisely one such prime ideal p of A.

An example of a height-one prime ideal q of Type II is $q := (y^2 + y\tau)B$. The prime ideal $qA = (y^2 + y\tau)A$ has the two minimal primes yA and $(y + \tau)A$.

(2) The ring B/yB is a rank 2 valuation domain. This can be seen directly or else one may apply a result of Heinzer and Sally[**72**, Prop. 3.5(iv)]; see Exercise 17.3. For other prime elements g of B with $g \in Q$, it need not be true that B/gB is a valuation domain. If g is a prime element contained in $\mathbf{m}_B^2$, then the maximal ideal of B/gB is 2-generated but not principal and thus B/gB cannot be a valuation domain. For a specific example over the field $\mathbb{Q}$, let $g = x^2 + y^2\tau$.

17.2. Verification of the three-dimensional examples

In Theorem 17.9 we record and establish the properties asserted in Examples 17.1 and other properties of the ring B. We make some preliminary remarks:

Remarks 17.7. (1) Assume that R is a Noetherian local domain with maximal ideal $\mathbf{m}$, let $z \in R$ be a nonzero nonunit, let R^* denote the z-adic completion of R, and let $\tau_1, \ldots, \tau_s \in zR^*$ be algebraically independent elements over R, as in the setting of Construction 5.3. Then, by Remark 6.4.1, B as defined in Equation 6.1.5 is the same as the ring B defined as a directed union of the localized polynomial rings $B_r := U_{P_r}$, where $P_r := (\mathbf{m}, \tau_{1r}, \ldots, \tau_{sr})U_{\underline{\tau}r}$, with notation as in Section 6.1.

(2) The notation of Examples 17.1 fits that of Construction 5.3 and the modification given in Remarks 6.4.3 of the procedures in Section 6.1, where R is the localized polynomial ring $k[x,y]_{(x,y)}$ over a field k, $R^* = k[y]_{(y)}[[x]]$ is the (x)-adic completion of R and $f \in xR^*$ is transcendental over K. Thus, by Remark 6.4.3, the ring $B = \bigcup B_r$, where $B_r = (U_r)_{P_r}$, $U_r = k[x, y, f_r]$ and $P_r = (x, y, f_r)U_r$, is the same ring B as the ring B described in Equation 6.1.6. A similar remark applies to B_τ with appropriate modifications to $B_{\tau,r}$, $U_{\tau,r}$ and $P_{\tau,r}$. The corresponding rings called B, B_r etc. in Example 18.1 of Chapter 18 also satisfy the relations from Section 6.1. Furthermore B_τ is the same ring B as in the setting in Localized Polynomial Example Theorem 9.7 where $r = 1 = s$.

(3) Thus the results of Noetherian Flatness Theorem 8.8, Construction Properties Theorem 6.19, Proposition 6.20 and Theorem 6.21 hold for the rings B and B_τ of Examples 17.1 and for the rings of Example 18.1 in Chapter 18. Also Localized Polynomial Example Theorem 9.7holds for B_τ. With the rings U_r and U defined as non-localized polynomial rings as in Examples 17.1 and Example 18.1 in Chapter 18, we have the relations $U_0[1/x] = U_r[1/x] = U[1/x]$. We might not, however, have all of the same conclusions for $U = \bigcup U_r$ as for the domains called U or U_r in those results.

In order to examine more closely the prime ideal structure of the ring B of Examples 17.1, we establish in Proposition 17.8 some properties of its overring A and of the map $\operatorname{Spec} A \to \operatorname{Spec} B$.

PROPOSITION 17.8. *With the notation of Examples 17.1, we have*

(1) *$A = B_\tau$ and $A[1/x]$ is a localization of $R[\tau]$.*
(2) *For $P \in \operatorname{Spec} A$ with $x \notin P$, the following are equivalent:*

(a) $A_P = B_{P\cap B}$ **(b)** $\tau \in B_{P\cap B}$ **(c)** $p \notin P$.

PROOF. By Localized Polynomial Example Theorem 9.7 with $r = 1, y = y_1, s = 1$, and $\tau = \tau_1$, as discussed in Remarks 17.7.3, we have $A = B_\tau$ and is Noetherian. By Remark 17.2, the ring $A[1/x]$ is a localization of $R[\tau]$. Thus item 1 holds.

For item 2, since $\tau \in A$, (a) $\implies$ (b) is clear. For (b) $\implies$ (c) we show that $p \in P \implies \tau \notin B_{P\cap B}$. By Remark 17.2, $B[1/x]$ is a localization of $R[f]$. Since $x \notin P$, the ring $B_{P\cap B}$ is a localization of $R[f]$, and thus $B_{P\cap B} = R[f]_{P\cap R[f]}$. The assumption that $p \in P$ implies that some $p_i \in P$, and so $R[f]_{P\cap R[f]}$ is contained in $V := R[f]_{p_iR[f]}$, a DVR. Since $R[f]$ is a polynomial ring over R, f is a unit in V. Hence $\tau = f/p \notin V$ and thus $\tau \notin R[f]_{P\cap R[f]}$. This shows that (b) $\implies$ (c).

For (c) $\implies$ (a), notice that $f = p\tau$ implies that $R[f][1/xp] = R[\tau][1/xp]$. By item 1, $A[1/x]$ is a localization of $R[\tau][1/x]$ and so $A[1/xp]$ is a localization of $R[\tau][1/xp] = R[f][1/xp]$. Thus $A[1/xp]$ is a localization of $R[f]$. By Remark 17.2, $B[1/x]$ is a localization of $R[f]$. Since $xp \notin P$ and $x \notin P \cap B$, we have that A_P and $B_{P\cap B}$ are both localizations of $R[f]$. Thus we have

$$A_P = R[f]_{PA_P\cap R[f]} = R[f]_{(P\cap B)B_{P\cap B}\cap R[f]} = B_{P\cap B}.$$

This completes the proof of Proposition 17.8. □

THEOREM 17.9. *As in the notation of Examples 17.1, let $R := k[x,y]_{(x,y)}$, where k is a field, and x and y are indeterminates. Set $R^* = k[y]_{(y)}[[x]]$, let $\tau \in xk[[x]]$ be transcendental over $k(x)$, and, for each integer i with $1 \le i \le m$, let $p_i \in R \setminus xR$ be such that $p_1R^*, \dots, p_mR^*$ are m prime ideals. Let $p := p_1 \cdots p_m$ and set $f := p\tau$. With the approximation domain B and the intersection domain A defined as in Examples 17.1, $A := A_f = A_\tau$. Set $Q_i := p_iR^* \cap B$, for each i with $1 \le i \le m$. Then:*

(1) *The ring B is a three-dimensional non-Noetherian local UFD with maximal ideal $\mathbf{n} = (x,y)B$, and the $\mathbf{n}$-adic completion of B is the two-dimensional regular local ring $k[[x,y]]$.*
(2) *The rings $B[1/x]$ and B_P, for each nonmaximal prime ideal P of B, are regular Noetherian UFDs, and the ring B/xB is a DVR.*
(3) *The ring A is a two-dimensional regular local domain with maximal ideal $\mathbf{m}_A := (x,y)A$, and $A = B_\tau$. The ring A is excellent if the field k has characteristic zero. If k is a perfect field of characteristic p, then A is not excellent*
(4) *The ideal $\mathbf{m}_A$ is the only prime ideal of A lying over $\mathbf{n}$.*
(5) *The ideals Q_i are the only height-two prime ideals of B.*
(6) *The ideals Q_i are not finitely generated and they are the only nonfinitely generated prime ideals of B.*
(7) *The ring B has saturated chains of prime ideals from (0) to $\mathbf{n}$ of length two and of length three, and hence is not catenary.*

Proof. For item 1, since B is a directed union of three-dimensional regular local domains, $\dim B \leq 3$. By Proposition 6.20, B is local with maximal ideal $(x, y)B$, xB and p_iB are prime ideals, and, by Construction Properties Theorem 6.19.4, the (x)-adic completion of B is equal to R^*, the (x)-adic completion of R. Thus the $\mathbf{n}$-adic completion of B is $k[[x, y]]$. Since each $Q_i = \bigcup_{i=1}^{\infty} Q_{in}$, where $Q_{in} = p_iR^* \cap B_n$, we see that each Q_i is a prime ideal of B with $p_i, f \in Q_i$ and $x \notin Q_i$. Since $p_iB = \bigcup p_iB_n$, we have $f \notin p_iB$. Thus

$$(0) \subsetneq p_iB \subsetneq Q_i \subsetneq (x, y)B.$$

This chain of prime ideals of length at least three yields that $\dim B = 3$ and that the height of each Q_i is 2.

The prime ideal $p_iR^*[1/x]$ has height one, whereas $p_iR^*[1/x] \cap S = (p_i, f)S$ has height two. Since flat extensions satisfy the going-down property, by Remark 2.21.10, the map $S = R[f] \to R^*[1/x]$ is not flat. Therefore Noetherian FlatnessTheorem 8.8 implies that the ring B is not Noetherian. By Theorem 6.21, B is a UFD, and so item 1 holds.

For item 2, by Construction Properties Theorem 6.19.3, $B/xB = R/xR$, and so B/xB is a DVR. By Theorem 6.21, $B[1/x]$ is a regular Noetherian UFD. If $x \in P$ and P is nonmaximal, then, again by Theorem 6.19.3, $P = xB$ and so B_P is a DVR and a regular Noetherian UFD. If $x \notin P$, the ring B_P is a localization of $B[1/x]$ and so is a regular Noetherian UFD. Thus item 2 holds.

The statements in item 3 that A is a two-dimensional regular local domain with maximal ideal $\mathbf{m}_A = (x, y)A$ and $A = B_\tau$ follow from Localized Polynomial Example Theorem 9.7. If the field k has characteristic zero, then A is also excellent by Theorem 9.2 (if the non-localized ring is excellent, so is the localization).

If the field k is perfect with characteristic $p > 0$, then the ring A is not excellent by Remark‘9.4. This completes the proof of item 3.

By Theorem 6.19.3, $A/xA = R/xR$, and so $\mathbf{m}_A = (x, y)A$ is the unique prime ideal of A lying over $\mathbf{n} = (x, y)B$. Thus item 4 holds and for item 5 we see that x is not in any height-two prime ideal of B.

To complete the proof of item 5, it remains to consider $P \in \operatorname{Spec} B$ with $x \notin P$ and $\operatorname{ht} P > 1$. By Proposition 6.20.3, we have $x^n \notin PR^*$ for each $n \in \mathbb{N}$. Thus $\operatorname{ht}(PR^*) \leq 1$. Since $A \hookrightarrow R^*$ is faithfully flat, $\operatorname{ht}(PA) \leq 1$. Let P' be a height-one prime ideal of A containing PA. Since $\dim B = 3$, $\operatorname{ht} P > 1$ and $x \notin P' \cap B$, it follows that $P = P' \cap B$. If $p \notin P$, then Proposition 17.8 implies that $A_{P'} = B_P$. Since P' is a height-one prime ideal of A, it follows that P is a height-one prime ideal of B in case $x \notin P$ and $\mathbf{p} \notin P$.

Now suppose that $p_i \in P$ for some i. Then p_iR^* is a height-one prime ideal contained in PR^* and so $p_iR^* = PR^*$. Hence P is squeezed between p_iB and $Q_i = p_iR^* \cap B \neq (x, y)B$. Since $\dim B = 3$, either P has height one or $P = Q_i$ for some i. This completes the proof of item 5.

For item 6, we show that each Q_i is not finitely generated by showing that $f_{n+1} \notin (p_i, f_n)B$ for each $n \geq 0$. We have $f = p\tau$ and thus $f_n = p\tau_n$. It follows that $f_n = xf_{n+1} + pxc_{n+1}$, by Equation 6.1.2. Assume that $f_{n+1} \in (p_i, f_n)B$. Then

$$(p_i, f_n)B = (p_i, xf_{n+1} + pxc_{n+1})B \implies f_{n+1} = ap_i + b(xf_{n+1} + pxc_{n+1}),$$

for some $a, b \in B$. Thus $f_{n+1}(1 - xb) \in p_iB$. Since $1 - xb$ is a unit of B, it follows that $f_{n+1} \in p_iB$, and thus $f_{n+1} \in p_iB_{n+r}$, for some $r \geq 1$. By Equation 6.1.2, we have

$$f_{n+1} = x^{r-1} f_{n+r} + p\alpha,$$

where $\alpha \in R$. Thus $x^{r-1} f_{n+r} \in (p, f_{n+1})B_{n+r}$. Since $f_{n+1} \in p_i B_{n+r}$, we have $x^{r-1} f_{n+r} \in p_i B_{n+r}$. This implies $f_{n+r} \in p_i B_{n+r}$, a contradiction because the ideal $(p_i, f_{n+r})B_{n+r}$ has height two. We conclude that Q_i is not finitely generated.

Since B is a UFD, the height-one primes of B are principal and since the maximal ideal of B is two-generated, every nonfinitely generated prime ideal of B has height two and thus is in the set $\{Q_1, \dots, Q_m\}$. This completes the proof of item 6.

For item 7, the chain $(0) \subset xB \subset (x, y)B = \mathbf{m}_B$ is saturated and has length two, while the chain $(0) \subset p_1 B \subset Q_1 \subset \mathbf{m}_B$ is saturated and has length three. $\square$

REMARK 17.10. With the notation of Examples 17.1 and Theorem 17.9, we obtain the following additional details about the prime ideals of B.

(1) If $P \in \operatorname{Spec} B$, $P \neq (0)$ and $P \neq \mathbf{m}_B$, then $\operatorname{ht}(PR^*) = 1$ and $\operatorname{ht}(PA) = 1$. Thus every nonmaximal prime ideal of B is contained in a nonmaximal prime ideal of A.
(2) If $P \in \operatorname{Spec} B$ is such that $P \cap R = (0)$, then $\operatorname{ht}(P) \leq 1$ and P is principal.
(3) If $P \in \operatorname{Spec} B$, $\operatorname{ht} P = 1$ and $P \cap R \neq 0$, then $P = (P \cap R)B$.
(4) Let p_i be one of the prime factors of p. Then $p_i B$ is prime in B. Moreover the ideals $p_i B$ and $Q_i := p_i A \cap B = (p_i, f_1, f_2, \dots)B$ are the only nonmaximal prime ideals of B that contain p_i. Thus they are the only prime ideals of B that lie over $p_i R$ in R.
(5) The constructed ring B has Noetherian spectrum.

PROOF. For the proof of item 1, if $P = Q_i$ for some i, then $PR^* \subseteq p_i R^*$ and $\operatorname{ht} PR^* = 1$. If P is not one of the Q_i, then P is a principal height-one prime and $\operatorname{ht} PR^* = 1$ by Theorem 17.9 parts 5 and 1. Since A is Noetherian and local, R^* is faithfully flat over A and hence $\operatorname{ht} PA = 1$. The proof that $\operatorname{ht}(PR^*) \leq 1$ is contained in the proof of item 5 of Theorem 17.9.

For item 2, $\operatorname{ht} P \leq 1$ because the field of fractions $K(f)$ of B has transcendence degree one over the field of fractions K of R; see Cohen's Theorem 2.9. Since B is a UFD, P is principal.

For item 3, if $x \in P$, then $P = xB$ and the statement is clear. Assume $x \notin P$. By Remark 17.2, $B[1/x]$ is a localization of B_n, and so $\operatorname{ht}(P \cap B_n) = 1$ for all integers $n \geq 0$. Thus $(P \cap R)B_n = P \cap B_n$, for each n, and so $P = (P \cap R)B$.

For item 4, $p_i B$ is prime by Proposition 6.20.2. By Theorem 17.9, $\dim B = 3$ and the Q_i are the only height-two primes of B. Since the ideal $p_i R + p_j R$ is $\mathbf{m}_R$-primary for $i \neq j$, it follows that $p_i B + p_j B$ is $\mathbf{n}$-primary, and hence $p_i B$ and Q_i are the only nonmaximal prime ideals of B that contain p_i.

Item 5 follows from Theorem 17.9, since the prime spectrum is Noetherian if it satisfies the ascending chain condition and if, for each finite set in the spectrum, there are only finitely many points minimal with respect to containing all of them. Thus the proof is complete. $\square$

REMARK 17.11. Rotthaus and Sega prove that the approximation domains B in the setting of Theorems 17.9 and 18.5 are coherent and regular; they show that every finitely generated submodule of a free module over B has a finite free resolution [**124**]. For the ring $B = \bigcup_{n=1}^{\infty} B_n$ of these constructions, it is stated in [**124**] that $B_n[1/x] = B_{n+k}[1/x] = B[1/x]$ and that B_{n+k} is generated over B_n by a single element for all positive integers n and k. This is not correct for the local rings B_n. However, if instead of asserting these statements for the localized polynomial rings B_n and their union B of the construction, one makes the statements for the underlying polynomial rings U_n and their union U defined in Equation 6.1.5, or those defined in Examples 17.1, then one does have that $U_n[1/x] = U_{n+k}[1/x] = U[1/x]$ and that U_{n+k} is generated over U_n by a single element for all positive integers n and k; see Remark 17.2.

We use the following lemma.

LEMMA 17.12. *Let the notation be as in Examples 17.1 and Theorem 17.9.*

(1) *For every element $c \in \mathbf{m}_R \setminus xR$ and every $t \in \mathbb{N}$, the element $c + x^t f$ is a prime element of the UFD B.*
(2) *For every fixed element $c \in \mathbf{m}_R \setminus xR$, the set $\{c+x^t f\}_{t \in \mathbb{N}}$ consists of infinitely many nonassociate prime elements of B, and so there exist infinitely many distinct height-one primes of B of the form $(c + x^t f)B$.*

PROOF. For the first item, since $f = p\tau$, Equation 17.3.1 implies that

$$f_r = pc_{r+1}x + xf_{r+1}$$

for each $r \geq 0$. In $B_0 = k[x, y, f]_{(x,y,f)}$, the polynomial $c + x^t f$ is linear in the variable $f = f_0$ and the coefficient x^t of f is relatively prime to the constant term c. Thus $c+x^t f$ is irreducible in B_0. Since $f = f_0 = pc_1x+xf_1$ in $B_1 = k[x, y, f_1]_{(x,y,f_1)}$, the polynomial $c + x^t f = c + x^t pc_1 x + x^{t+1} f_1$ is linear in the variable f_1 and the coefficient x^{t+1} of f_1 is relatively prime to the constant term c. Thus $c + x^t f$ is irreducible in B_1. To see that this pattern continues, observe that in B_2, we have

$$\begin{aligned} f \ &= pc_1x + xf_1 \ = \ pc_1x + pc_2x^2 + x^2 f_2 \implies \\ c + x^t f \ &= \ c + pc_1x^{t+1} + pc_2x^{t+2} + x^{t+2} f_2, \end{aligned}$$

a linear polynomial in the variable f_2. Thus $c + x^t f$ is irreducible in B_2 and a similar argument shows that $c+x^t f$ is irreducible in B_r for each positive integer r. Therefore for each $t \in \mathbb{N}$, the element $c + x^t f$ is prime in B.

For item 2, we prove that $(c+x^t f)B \neq (c+x^m f)B$, for positive integers $t > m$. Assume that $\mathbf{q} := (c+x^t f)B = (c+x^m f)B$ is a height-one prime ideal of B. Then

$$(x^t - x^m)f \ = \ x^m(x^{t-m} - 1)f \ \in \ \mathbf{q}.$$

Since $c \notin xB$ we have $\mathbf{q} \neq xB$. Thus $x^m \notin \mathbf{q}$. Since B is local, $x^{t-m} - 1$ is a unit of B. It follows that $f \in \mathbf{q}$ and thus $(c, f)B \subseteq \mathbf{q}$. By Remark 17.2, $B[1/x]$ is a localization of $R[f] = S$, and $x \notin \mathbf{q}$ implies that $B_{\mathbf{q}} = S_{\mathbf{q} \cap S}$. This is a contradiction since the ideal $(c, f)S$ has height two.

We conclude that there exist infinitely many distinct height-one primes of the form $(c + x^t f)B$. □

Lemma 17.13 is useful for giving a more precise description of Spec B for B as in Examples 17.1. For each nonempty finite subset H of $\{Q_1, \ldots, Q_m\}$, we show there exist infinitely many height-one prime ideals contained in each $Q_i \in H$, but not contained in Q_j if $Q_j \notin H$. Recall that "lost" is defined in Definition 17.4.

LEMMA 17.13. *With the notation of Example 17.1 and Theorem 17.9, let G be a nonempty subset of $\{1,\dots,m\}$, let $H = \{Q_i \mid i \in G\}$, and let $p_G = \prod\{p_i \mid i \in G\}$. Then we have, for each $t \in \mathbb{N}$:*

(1) *$(p_G + x^t f)B$ is a prime ideal of B that is lost in A.*
(2) *$(p_G^2 + x^t f)B$ is a prime ideal of B that is not lost in A; see Definition 17.4.*

The sets $\{(p_G + x^t f)B\}_{t\in\mathbb{N}}$ and $\{(p_G^2 + x^t f)B\}_{t\in\mathbb{N}}$ are both infinite. Moreover, the prime ideals in both item 1 and item 2 are contained in each Q_i such that $Q_i \in H$, but are not contained in Q_j if $Q_j \notin H$.

PROOF. For item 1, we have

$$(p_G + x^t f)A \cap B = p_G(1 + x^t\tau \prod_{j\notin G} p_j)A \cap B = p_G A \cap B = \bigcap_{i\in G} Q_i. \tag{17.13.1}$$

Thus each prime ideal of B of the form $(p_G + x^t f)B$ is lost in A and R^*. By the second item of Lemma 17.12, there exist infinitely many height-one primes $(p_G + x^t f)B$ of B that are lost in A and R^*.

For item 2, we have

$$\begin{aligned}(p_G^2 + x^t f)A \cap B &= (p_G^2 + x^t p_G(\prod_{j\notin G} p_j)\tau)A \cap B \\ &= p_G(p_G + x^t(\prod_{j\notin G} p_j)\tau)A \cap B \subsetneq p_G A \cap B = \bigcap_{i\in G} Q_i.\end{aligned} \tag{17.13.2}$$

The strict inclusion is because $p_G + x^t(\prod_{j\notin G} p_j)\tau \in \mathbf{m}_A$. This implies that prime ideals of B of form $(p_G^2 + x^t f)B$ are not lost. By Lemma 17.12 there are infinitely many distinct prime ideals of that form.

The "moreover" statement for the prime ideals in item 1 follows from Equation 17.13.1. Equation 17.13.2 implies that the prime ideals in item 2 are contained in each $Q_i \in H$. For $j \notin G$, if $p_G^2 + x^t f \in Q_j$, then $p_j + x^t f \in Q_j$ implies that $p_G^2 - p_j \in Q_j$ by subtraction. Since $p_j \in Q_j$, this would imply that $p_G^2 \in Q_j$, a contradiction. This completes the proof of Lemma 17.13. □

REMARK 17.14. With the notation of Examples 17.1, consider the birational inclusion $B \hookrightarrow A$ and the faithfully flat map $A \hookrightarrow R^*$. The following statements hold concerning the inclusion maps $R \hookrightarrow B \hookrightarrow A \hookrightarrow R^*$, and the associated maps in the opposite direction of their spectra: (See Discussion 3.21 for information concerning the spectral maps.)

(1) The map $\operatorname{Spec} R^* \to \operatorname{Spec} A$ is surjective, since every prime ideal of A is contracted from a prime ideal of R^*, while the maps $\operatorname{Spec} R^* \to \operatorname{Spec} B$ and $\operatorname{Spec} A \to \operatorname{Spec} B$ are not surjective. All the induced maps to $\operatorname{Spec} R$ are surjective since the map $\operatorname{Spec} R^* \to \operatorname{Spec} R$ is surjecive.
(2) By Lemma 17.13, each of the prime ideals Q_i of B contains infinitely many height-one primes of B that are the contraction of prime ideals of A and infinitely many that are not.

    An ideal contained in a finite union of prime ideals is contained in one of the prime ideals; see [**9**, Prop. 1.11, page 8] or [**96**, Ex. 1.6, page 6]. Thus there are infinitely many non-associate prime elements of the UFD B that are not contained in the union $\bigcup_{i=1}^m Q_i$. We observe that for each prime element q of B with $q \notin \bigcup_{i=1}^m Q_i$ the ideal qA is contained in a height-one prime $\mathbf{q}$ of A and $\mathbf{q} \cap B$ is properly contained in $\mathbf{m}_B$ since $\mathbf{m}_A$

is the unique prime ideal of A lying over $\mathbf{m}_B$. Hence $\mathbf{q} \cap B = qB$. Thus each qB is contracted from A and R^*.

In the four-dimensional example B of Theorem 18.5, each height-one prime of B is contracted from R^*, but there are infinitely many height-two primes of B that are lost in R^*, in the sense of Definition 17.4; see Section 18.2.

(3) Among the prime ideals of the domain B of Examples 17.1 that are not contracted from A are the p_iB. Since $p_iA \cap B = Q_i$ properly contains p_iB, the prime ideal p_iB is lost in A.

(4) Since x and y generate the maximal ideals of B and A, and since B is integrally closed, a version of Zariski's Main Theorem [**112**], [**33**], implies that A is not essentially finitely generated as a B-algebra. ("Essentially finitely generated" is defined in Section 2.1.)

Using the information above, we display below a picture of Spec(B) in the case $m = 2$.

$$\mathbf{m}_B := (x, y)B$$

$$Q_1 \qquad\qquad\qquad\qquad Q_2$$

$$xB \in \boxed{\text{NOT Lost}} \qquad \boxed{\text{NL}} \quad \boxed{\text{L}} \qquad \boxed{\text{NL}} \quad \boxed{\text{L}} \qquad \boxed{\text{NL}} \quad \boxed{\text{L}}$$

$$(0)$$

Diagram 17.14.0

Comments on Diagram 17.14.0. Here we have $Q_1 = p_1R^* \cap B$ and $Q_2 = p_2R^* \cap B$, and each box represents an infinite set of height-one prime ideals. We label a box "NL" for "not lost" and "L" for "lost". An argument similar to that given for the Type I primes in Example 17.3 shows that the height-one primes q such that $q \notin Q_1 \cup Q_2$ are not lost. That the other boxes are infinite follows from Lemma 17.13.

Exercises

(1) Let $R = k[x, y]_{(x,y)}$ be the localized polynomial ring in the variables x, y over a field k. Consider the local quadratic transformation $S := R[\frac{y}{x}]_{x,\frac{y}{x})R[\frac{y}{x}]}$ of the 2-dimensional RLR R. Using the terminology of Definition 17.4
  (a) Prove that there are infinitely many height-one primes of R that are lost in S.
  (b) Prove that there are infinitely many height-one primes of R that are not lost in S.
  (c) Describe precisely the height-one primes of R that are lost in S, and the prime ideals of R that are not lost in S.

(2) Prove the assertion in Remark 17.14 that each of the prime ideals Q_i of B contains infinitely many height-one primes of B that are the contraction of

prime ideals of A and infinitely many that are not, i.e., there exist infinitely many height-one primes of B contained in Q_i that are lost in A and infinitely many that are not lost in A.

Suggestion: A solution for this exercise can be patterned along the lines of the arguments given in Example 17.3. Since $A[1/x]$ is a localization of the polynomial ring $R[\tau]$, for every nonzero element $c \in (x, y)R$, the ideal $(\tau - c)A$ is a height-one prime in A, and $a\tau - ac$ is a nonzero element in each of the prime ideals Q_i of B. Since p_iA is the only prime ideal of A lying over Q_i in B, the ideal $(\tau - c)A \cap B$ is a height-one prime of B. Also consider elements of the form $p_i + x^n f \in B$.

(3) In connection with Remarks 17.6.2, let $(R, \mathbf{m})$ be a local domain with principal maximal ideal $\mathbf{m} = aR$.
  (a) Prove that $\bigcap_{n=1}^{\infty} \mathbf{m}^n = P$, where P is a prime ideal properly contained in $\mathbf{m}$.
  (b) Prove that every prime ideal of R properly contained in $\mathbf{m}$ is contained in P.
  (c) Prove that R/P is a DVR.
  (d) Prove that $P = PR_P$.
  (e) Prove that R is a valuation domain if and only if R_P is a valuation domain [**72**, Prop. 3.5(iv)].
  (f) Construct an example of a local domain $(R, \mathbf{m})$ with principal maximal ideal $\mathbf{m}$ such that R is not a valuation domain.

Suggestion: To construct an example for part f, let x, y be indeterminates over a field k, let $U = k(x)[y]$, let W be the DVR U_{yU}, and let $P := yW$ denote the maximal ideal of W. Then $W = k(x) + P$. Let $R = k[x^2]_{(x^2k[x^2])} + P$. This is an example of a "$D + M$" construction, as outlined in Remark 18.13.

CHAPTER 18

Non-Noetherian insider examples of dimension ≥ 4

In this chapter we extend the methods of Chapter 17 to construct a four-dimensional local domain that is not Noetherian, but is very close to being Noetherian. We use Insider Construction 13.1 of Section 13.1. This four-dimensional non-catenary non-Noetherian local unique factorization domain has exactly one prime ideal Q of height three; the ideal Q is not finitely generated.

Section 18.1 contains a description of the example. In Section 18.2 we verify that the example has the stated properties.

18.1. A 4-dimensional prime spectrum

In Example 18.1, we present a four-dimensional example analogous to Example 17.3.

EXAMPLE 18.1. Let k be a field, let x, y and z be indeterminates over k. Set

$$R := k[x,y,z]_{(x,y,z)} \qquad \text{and} \qquad R^* := k[y,z]_{(y,z)}[[x]],$$

and let $\mathbf{m}_R$ and $\mathbf{m}_{R^*}$ denote the maximal ideals of R and R^*, respectively. The power series ring R^* is the xR-adic completion of R. Consider τ and σ in $xk[[x]]$

$$\tau := \sum_{n=1}^{\infty} c_n x^n \quad \text{and} \quad \sigma := \sum_{n=1}^{\infty} d_n x^n,$$

where the c_n and d_n are in k and τ and σ are algebraically independent over $k(x)$. Define

$$f := y\tau + z\sigma \quad \text{and} \quad A := A_f = R^* \cap k(x,y,z,f),$$

that is, A is the intersection domain associated with f. For each integer $n \geq 0$, let τ_n and σ_n be the n^{th} endpieces of τ and σ as in Equation 17.1.a. Then the n^{th} endpiece of f is $f_n = y\tau_n + z\sigma_n$. As in Equation 17.1.b, we have

$$\tau_n = x\tau_{n+1} + xc_{n+1} \quad \text{and} \quad \sigma_n = x\sigma_{n+1} + xd_{n+1},$$

where c_{n+1} and d_{n+1} are in the field k. Therefore

$$\begin{aligned} f_n &= y\tau_n + z\sigma_n = yx\tau_{n+1} + yxc_{n+1} + zx\sigma_{n+1} + zxd_{n+1} \\ &= xf_{n+1} + yxc_{n+1} + zxd_{n+1}. \end{aligned} \tag{18.1.1}$$

The approximation domains U_n, B_n, U and B for A are as follows:

$$\begin{aligned} &\text{For } n \geq 0, \quad U_n := k[x,y,z,f_n] \qquad B_n := k[x,y,z,f_n]_{(x,y,z,f_n)} \\ &U := \bigcup_{n=0}^{\infty} U_n \quad \text{and} \qquad B := B_f = \bigcup_{n=0}^{\infty} B_n. \end{aligned} \tag{18.1.2}$$

Thus B is the directed union of 4-dimensional localized polynomial rings. It follows that $\dim B \leq 4$.

The rings A and B are constructed inside the intersection domain $A_{\tau,\sigma} := R^* \cap k(x,y,z,\tau,\sigma)$. By Localized Polynomial Example Theorem 9.7, the domain $A_{\tau,\sigma}$ is Noetherian and equals the approximation domain $B_{\tau,\sigma}$ associated to τ,σ and is a three-dimensional RLR that is a directed union of 5-dimensional RLRs and the extension $T := R[\tau,\sigma] \hookrightarrow R^*[1/x]$ is flat.

Before we list and establish the other properties of Example 18.1 in Theorem 18.5, we prove the following proposition concerning the Jacobian ideal and flatness in Example 18.1. The Jacobian ideal is defined and discussed in Definition and Remarks 11.8.1.

PROPOSITION 18.2. *With the notation of Example 18.1, we have*

(1) *For the extension $\varphi : S = R[f] \hookrightarrow T = R[\tau,\sigma]$, the Jacobian ideal J is the ideal $(y,z)T$. Thus the nonflat locus F of φ contains J.*
(2) *For every $P \in \operatorname{Spec}(R^*[1/x])$, the ideal $(y,z)R^*[1/x] \not\subseteq P \iff$ the map $B_{P\cap B} \hookrightarrow (R^*[1/x])_P$ is flat. Thus the ideal $F_1 := (y,z)R^*[1/x]$ defines the nonflat locus of the map $B \hookrightarrow R^*[1/x]$.*
(3) *For every height-one prime ideal $\mathbf{p}$ of R^*, we have $\operatorname{ht}(\mathbf{p} \cap B) \leq 1$.*
(4) *For every prime element w of B, $wR^* \cap B = wB$.*

PROOF. For item 1, the Jacobian ideal is the ideal of T generated by the 1×1 minors of the matrix $(y \quad z)$ by (11.8.1), and so $J = (y,z)T$. By Theorem 11.11.2, $(y,z)T \subseteq F$.

For item 2, the two statements are equivalent by the definition of nonflat locus in Definition and Remarks 11.8.2. To compute the nonflat locus of $B \hookrightarrow R^*[1/x]$, we use that $T := R[\tau,\sigma] \hookrightarrow R^*[1/x]$ is flat as noted in Example 18.1. Let $P \in \operatorname{Spec}(R^*[1/x])$ and let $Q := P \cap T$. The map $B \hookrightarrow R^*[1/x]_P$ is flat $\iff$ the composition

$$k[x,y,z,f] \hookrightarrow k[x,y,z,\tau,\sigma] \hookrightarrow R^*[1/x]_P \text{ is flat } \iff$$
$$S := k[x,y,z,f] \overset{\varphi}{\hookrightarrow} T_Q = k[x,y,z,\tau,\sigma]_Q \text{ is flat.}$$

By item 1, the Jacobian ideal of φ is the ideal $J = (y,z)T$. Since $(y,z)T \cap S = (y,z,f)S$ has height 3, φ_Q is not flat for every $Q \in \operatorname{Spec}(T)$ such that $(y,z)T \subseteq Q$. Thus the nonflat locus of $B \hookrightarrow R^*[1/x]$ is defined by $F_1 = (y,z)R^*[1/x]$ as stated in item 2.

Item 3 is clear if $\mathbf{p} = xR^*$. Let $\mathbf{p}$ be a height-one prime of R^* other than xR^*. Since $\mathbf{p}$ does not contain $(y,z)R^*$, the map $B_{\mathbf{p}\cap B} \hookrightarrow (R^*)_{\mathbf{p}}$ is faithfully flat. Thus $\operatorname{ht}(\mathbf{p} \cap B) \leq 1$. This establishes item 3.

Item 4 is clear if $wB = xB$. Assume that $wB \neq xB$ and let $\mathbf{p}$ be a height-one prime ideal of R^* that contains wR^*. Then $\mathbf{p}R^*[1/x] \cap R^* = \mathbf{p}$, and by item 3, $\mathbf{p} \cap B$ has height at most one. We have $\mathbf{p} \cap B \supseteq wR^* \cap B \supseteq wB$. Thus item 4 follows. $\square$

Next we prove a proposition about homomorphic images of the constructed ring B. This result enables us in Corollary 18.4 to relate the ring B of Example 18.1 to the ring B of Example 17.3.

PROPOSITION 18.3. *Assume the notation of Example 18.1, and let w be a prime element of $R = k[x,y,z]_{(x,y,z)}$ with $wR \neq xR$. Let $\pi : R^* \to R^*/wR^*$ be the natural homomorphism, and let $^-$ denote image in R^*/wR^*. Let B' be the approximation* *domain formed by considering* $\overline{R}$ *and the endpieces* $\overline{f}_n$ *of* $\overline{f}$*, defined analogously to Equation 17.1.a. That is,* B' *is defined by setting*

$$U'_n \;=\; \overline{k[x,y]}[\overline{f}_n],\;\; B'_n \;=\; (U'_n)_{\mathbf{n}'_n},\;\; U' \;=\; \bigcup_{n=1}^{\infty} U'_n,\;\; \textit{and}\;\; B' \;=\; \bigcup_{n=1}^{\infty} B'_n,$$

where $\mathbf{n}'_n$ *is the maximal ideal of* U'_n *that contains* $\overline{f}_n$ *and the image of* $\mathbf{m}_R$*. Then* $B' = \overline{B}$*.*

Proof. By Proposition 6.20.2, wB is a prime ideal of B. By Proposition 18.2.4, $wR^* \cap B = wB$. Hence $\overline{B} = B/(wR^* \cap B) = B/wB$. We have

$$\overline{R}/x\overline{R} \;=\; \overline{B}/x\overline{B} \;=\; \overline{R^*}/x\overline{R^*},$$

and the ring $\overline{R^*}$ is the $(\overline{x})$-adic completion of $\overline{R}$. Since the ideal $(y,z)R$ has height 2 and the kernel of π has height 1, at least one of $\overline{y}$ and $\overline{z}$ is nonzero. Since τ and σ are algebraically independent over $k(x,y,z)$, the element $\overline{f} = \overline{y}\cdot\overline{\tau} + \overline{z}\cdot\overline{\sigma}$ of the integral domain $\overline{B}$ is transcendental over $\overline{R}$. Similarly the endpieces $\overline{f}_n$ are transcendental over $\overline{R}$. The fact that $\overline{R^*}$ may fail to be an integral domain does not affect the algebraic independence of these elements that are inside the integral domain $\overline{B}$.

By Construction Properties Theorem 6.19.2, with adjustments using Remark 6.4, parts 3 and 4, we have $U_0[1/x] = U_n[1/x] = U[1/x]$, and thus $wU \cap U_n = wU_n$ for each $n \in \mathbb{N}$. Since B_n is a localization of U_n, we also have $wB \cap B_n = wB_n$. Since $wR^* \cap B = wB$, it follows that $wR^* \cap B_n = wB_n$. Thus we have

$$\overline{R} \;\subseteq\; \overline{B_n} \;=\; B_n/wB_n \;\subseteq\; \overline{B} \;=\; B/wB \;\subseteq\; \overline{R^*} \;=\; R^*/wR^*.$$

We conclude that $\overline{B} = \bigcup_{n=0}^{\infty} \overline{B_n}$. Since $B'_n = \overline{B}_n$, we have $B' = \overline{B}$. □

Corollary 18.4. *The homomorphic image* B/zB *of the ring* B *of Example 18.1 is isomorphic to the three-dimensional ring* B *of Example 17.3.*

Proof. Assume the notation of Example 18.1 and Proposition 18.3 and let $w = z$. We show that the ring $B/zB \cong C$, where C is the ring called B in Example 17.3. By Proposition18.3, we have $B' = \overline{B/zB}$, where B' is the approximation domain over $\overline{R} = R/zR$ using the element $\overline{f}$, transcendental over $\overline{R}$. Let R_C denote the base ring $k[x,y]_{(x,y)}$ for C in Example 17.3, and let $\psi_0 : \overline{R} \to R_C$ denote the k-isomorphism defined by $\overline{x} \mapsto x$ and $\overline{y} \mapsto y$. Then, as in the proof of Proposition 18.3, $\overline{R^*}$ is the $(\overline{x})$-adic completion of $\overline{R}$. Thus ψ_0 extends to an isomorphism $\psi : \overline{R^*} \to (R_C)^*$ that agrees with ψ_0 on $\overline{R}$ and such that $\psi(\overline{\tau}) = \tau$. Furthermore $\psi(\overline{f}) = \psi(\overline{y}\cdot\overline{\tau} + \overline{z}\cdot\overline{\sigma}) = y\tau$, which is the transcendental element f used in the construction of C. Thus ψ is an isomorphism from $\overline{B} = B/zB$ to C, the ring constructed in Example 17.3. □

18.2. Verification of the example

We record in Theorem 18.5 properties of the ring B and its prime spectrum.

Theorem 18.5. *As in Example 18.1,* $R := k[x,y,z]_{(x,y,z)}$ *with* k *a field,* x*,* y *and* z *indeterminates, and* $R^* := k[y,z]_{(y,z)}[[x]]$*, the* xR*-adic completion of* R*. Let* τ *and* $\sigma \in xk[[x]]$ *be algebraically independent over* $k(x)$*. Set* $f := y\tau + z\sigma$*,* $A := R^* \cap k(x,y,z,f)$*, and* $B := \bigcup_{n=0}^{\infty} B_n = \bigcup_{n=0}^{\infty} k[x,y,z,f_n]_{(x,y,z,f_n)}$ *as in (18.1.2). Let* $Q := (y,z)R^* \cap B$*. Then*

(1) *The rings A and B are equal.*
(2) *The ring B is a four-dimensional non-Noetherian local UFD with maximal ideal $\mathbf{m}_B = (x, y, z)B$, and the $\mathbf{m}_B$-adic completion of B is the three-dimensional RLR $k[[x, y, z]]$.*
(3) *The ring $B[1/x]$ is a Noetherian regular UFD, the ring B/xB is a two-dimensional RLR, and, for every nonmaximal prime ideal P of B, the ring B_P is an RLR.*
(4) *The ideal Q is the unique prime ideal of B of height 3.*
(5) *The ideal Q equals $\bigcup_{n=0}^{\infty} Q_n$ where $Q_n := (y, z, f_n)B_n$, Q is a nonfinitely generated prime ideal, and $QB_Q = (y, z, f)B_Q$.*
(6) *There exist infinitely many height-two prime ideals of B not contained in Q and each of these prime ideals is contracted from R^*.*
(7) *For certain height-one primes p contained in Q, there exist infinitely many height-two primes between p and Q that are contracted from R^*, and infinitely many that are not contracted from R^*. Hence the map $\operatorname{Spec} R^* \to \operatorname{Spec} B$ is not surjective.*
(8) *Every saturated chain of prime ideals of B has length either 3 or 4, and there exist saturated chains of prime ideals of lengths both 3 and 4. Thus B is not catenary.*
(9) *Each height-one prime ideal of B is the contraction of a height-one prime ideal of R^*.*
(10) *B has Noetherian spectrum.*

We prove Theorem 18.5 below. First, assuming Theorem 18.5, we display a picture of Spec(B) and make comments about the diagram.

$$\mathbf{m}_B := (x, y, z)B$$

$$Q := (y, z, \{f_i\})B$$

$(x, y - \delta z)B \in \boxed{\text{ht. } 2, \not\subset Q}$ $\quad\quad$ $\boxed{\text{ht. } 2, \text{ contr. } R^*}$ $\quad\quad$ $(y, z)B \in \boxed{\text{ht. } 2, \text{ Not contr. } R^*}$

$xB \in \boxed{\text{ht. } 1, \not\subset Q}$ $\quad\quad$ $yB, zB \in \boxed{\text{ht. } 1, \subset Q}$

$$(0)$$

Diagram 18.5.0

Comments on Diagram 18.5.0. A line going from a box at one level to a box at a higher level indicates that every prime ideal in the lower level box is contained in at least one prime ideal in the higher level box. Thus as indicated in the diagram, every height-one prime gB of B is contained in a height-two prime of B that contains x and so is not contained in Q. This is obvious if $gB = xB$ and can be seen by considering minimal primes of $(g, x)B$ otherwise. Thus B has no maximal saturated chain of length 2. We have not drawn any lines from the lower level righthand box to higher boxes that are contained in Q because we are uncertain about what inclusion relations exist for these primes. We discuss this situation in Remarks 18.12.

PROOF. (of Theorem 18.5.) By Proposition 18.2.1, $(y,z)T \subseteq F$ where F is the nonflat locus F of the extension $S \hookrightarrow T$. Hence $\operatorname{ht}(FR^*[1/x]) > 1$. Since $R[\tau]$ is a UFD, Proposition 13.4 implies equality of the approximation and intersection domains B and A corresponding to the element f of R^*. This completes item 1.

For item 2, since B is a directed union of four-dimensional RLRs, we have $\dim B \le 4$. By Corollary 18.4 and Theorem 17.9, $\dim(B/zB) = 3$. Thus $\dim B \ge 4$, and so $\dim B = 4$. By Proposition 6.20.5, the ring B is local with maximal ideal $\mathbf{m}_B = (x,y,z)B$. By Krull's Altitude Theorem 2.6, B is not Noetherian. The ring B is a UFD by Theorem 6.21. Since the (x)-adic completion of B is R^*, the $\mathbf{m}_B$-adic completion of B is $k[[x,y,z]]$.

For item 3, by Theorem 6.21, the ring $B[1/x]$ is a Noetherian regular UFD. By Construction Properties Theorem 6.19.3, we have $R/xR = B/xB$. Thus B/xB is a two-dimensional RLR.

For the last part of item 3, if $x \notin P$, then B_P is a localization of $B[1/x]$, which is Noetherian and regular, and so B_P is a regular local ring. In particular, this proves that B_Q is a regular local ring. If $x \in P$ and $\operatorname{ht} P = 1$, then $P = (x)$ and B_{xB} is a DVR. If $x \in P$ and $\operatorname{ht}(P) = 2$, the ideal P is finitely generated since B/xB is an RLR. Since B is a UFD from item 2, it follows that B_P is a local UFD of dimension 2 with finitely generated maximal ideal. Thus B_P is Noetherian by Cohen's Theorem 2.8. This, combined with B/xB a regular local ring, implies that B_P is a regular local ring. Since $\operatorname{ht} P \le 2$ for every nonmaximal prime ideal P of R with $x \in P$, this completes the proof of item 3.

For item 4, since $(y,z)R^*$ is a prime ideal of R^*, the ideal $Q = (y,z)R^* \cap B$ is prime. By Proposition 6.20.2, the ideals yB and $(y,z)B$ are prime. Consider the chain of prime ideals

$$(0) \subset yB \subset (y,z)B \subset Q \subset \mathbf{m}_B.$$

The list y,z,f,x shows that each of the inclusions is strict; for example, we have $f \in Q \setminus (y,z)B$ since $f \notin (y,z)B_n$ for every $n \in \mathbb{N}$. By item 2 we have $\operatorname{ht} \mathbf{m}_B = 4$. Thus $\operatorname{ht} Q = 3$. This also implies that $(y,z)B$ is a height-two prime ideal of B.

For the uniqueness in item 4, let P be a nonmaximal prime ideal of B. We first consider the case that $x \notin P$. Then, by Proposition 6.20.3, $x^n \notin PR^*$ for each positive integer n. Hence $PR^*[1/x] \neq R^*[1/x]$. Let P_1 be a prime ideal of $R^*[1/x]$ such that $P \subseteq P_1$. If both y and z are in P_1, then $(y,z)R^*[1/x] \subseteq P_1$. Since $(y,z)R^*[1/x]$ is maximal, we have $(y,z)R^*[1/x] = P_1$. Therefore, $P \subseteq (y,z)R^*[1/x] \cap B = Q$, and so either $\operatorname{ht}(P) \le 2$ or $P = Q$.

Next suppose that $x \notin P$ and y or z is not in P_1. Then the map $\psi : B \to R^*[1/x]_{P_1}$ is flat by Proposition 18.2.2. Since $\dim R^*[1/x] = 2$ we have $\operatorname{ht}(P_1) \le 2$. Flatness of ψ implies $\operatorname{ht}(P_1 \cap B) \le 2$, by Remark 2.2110. Hence $\operatorname{ht} P \le 2$.

To complete the proof of item 4, we consider the case that $x \in P$. We have $\operatorname{ht} P \le 3$, since $\dim B = 4$ and P is not maximal. If $\operatorname{ht} P \ge 3$, there exists a chain of primes of the form

$$(0) \subsetneq P_1 \subsetneq P_2 \subsetneq P \subsetneq (x,y,z)B. \tag{18.5.1}$$

By Construction Properties Theorem 6.19.3, $B/xB \cong R/xR$; thus $\dim(B/xB) = 2$. If $x \in P_2$, then $\operatorname{ht} P_2 \geq 2$ implies that $(0) \subsetneq xB \subsetneq P_2 \subsetneq P \subsetneq (x,y,z)B$, a contradiction to $\dim(B/xB) = 2$. Thus $x \notin P_2$. Since $x \in P$ and P is nonmaximal, we have that y or z is not in P. Hence y or z is not in P_2.

By Theorem 6.19.3, P corresponds to a nonmaximal prime ideal P' of R^* containing PR^*. Let P_2' be a prime ideal of R^* inside P' that is minimal over P_2R^*. If both y and z are in P_2', then, $(x,y,z)R^* \subseteq P'$, a contradiction to P' nonmaximal. By Proposition 6.20.4, P_2' does not contain x. Thus $P_2' \subsetneq P' \subsetneq (x,y,z)R^*$. Also $P_2' = P_2'' \cap R^*$, where P_2'' is a prime ideal of $R^*[1/x]$, and one of y and z is not an element of P_2''.

Since the Jacobian ideal of $\varphi : S \to T[1/x]$ is $(y,z)T[1/x]$, Proposition 18.2.2 implies the map $\psi : B \to R^*[1/x]_{P_2''}$ is flat. This implies $\operatorname{ht}(P_2'') \geq \operatorname{ht}(P_2'' \cap B) \geq \operatorname{ht} P_2 \geq 2$; that is, $\operatorname{ht}(P_2'') \geq 2$. Also P_2'' intersects R^* in P_2', and so $\operatorname{ht} P_2' \geq 2$. Thus in R^* we have a chain of primes $P_2' \subsetneq P' \subsetneq (x,y,z)R^*$, where $\operatorname{ht} P_2' \geq 2$, a contradiction, since R^*, a localization of $k[y,z][[x]]$, has dimension 3. This proves item 4.

For item 5, let $Q' = \bigcup_{n=0}^{\infty} Q_n$, where each $Q_n = (y,z,f_n)B_n$. Each Q_n is a prime ideal of height 3 in the 4-dimensional RLR B_n. Therefore Q' is a prime ideal of B of height ≤ 3 that is contained in Q. The ideal $(y,z)B$ is a prime ideal of height 2 strictly contained in Q by the proof of item 3. Hence $\operatorname{ht}(Q') = 3$ and we have $Q' = Q$.

To show the ideal Q is not finitely generated, we show for each positive integer n that $f_{n+1} \notin (y,z,f_n)B$. By Equation 18.1.1, $f_n = xf_{n+1} + yxc_{n+1} + zxd_{n+1}$. If $f_{n+1} \in (y,z,f_n)B$, then $f_{n+1} = ay + bz + c(xf_{n+1} + yxc_{n+1} + zxd_{n+1})$, where $a,b,c \in B$. This implies $f_{n+1}(1-cx)$ is in the ideal $(y,z)B$. By Proposition 6.20.1, $x \in \mathcal{J}(B)$, and so $1 - cx$ is a unit of B. This implies $f_{n+1} \in (y,z)B \cap B_{n+1}$. By Proposition 6.20.2, we have $(y,z)B \cap B_{n+1} = (y,z)B_{n+1}$ Thus $f_{n+1} \in (y,z)B_{n+1}$. Since the ring $B_{n+1} = k[x,y,z,f_{n+1}]_{(x,y,z,f_{n+1})}$, where x,y,z and f_{n+1} are algebraically independent variables over k, this is a contradiction. We conclude that Q is not finitely generated.

We show above for item 3 that B_Q is a three-dimensional regular local ring. Since $Q = (y,z,f,f_1,f_2,\dots)B$ and, since x is a unit of B_Q, it follows from Remark 17.2 that $QB_Q = (y,z,f)B_Q$. This establishes item 5.

For item 6, since $x \notin Q$ and $B/xB \cong R/xR$ is a Noetherian ring of dimension two, there are infinitely many height-two primes of B containing xB; see Exercise 5 of Chapter 2. This proves there are infinitely many height-two primes of B not contained in Q. If P is a height-two prime of B not contained in Q, then $\operatorname{ht}(\mathbf{m}_B/P) = 1$, by item 4 above, and so, by Proposition 6.20.5, P is contracted from R^*. This completes the proof of item 6.

For item 7 we show that $p = zB$ has the stated properties. By Corollary 18.4, the ring B/zB is isomorphic to the ring called B in Example 17.3. For convenience we relabel the ring of Example 17.3 as B'. By Theorem 17.9, B' has exactly one non-finitely generated prime ideal, which we label Q', and $\operatorname{ht} Q' = 2$. It follows that $Q/zB = Q'$. By Discussion 17.5, there are infinitely many height-one primes contained in Q' of Type II (that is, primes that are contracted from R^*/zR^*) and infinitely many height-one primes contained in Q' of Type III (that is, primes that are not contracted from R^*/zR^*). The preimages in R^* of these primes are height-two primes of B that are contained in Q and contain zB. It follows that there are infinitely many contracted from R^* and there are infinitely many not contracted from R^*, as desired for item 7.

For item 8, we have a saturated chain of prime ideals

$$(0) \subset xB \subset (x,y)B \subset (x,y,z)B = \mathbf{m}_B$$

of length 3 since $B/xB = R/xR$ by Theorem 6.19.3. We have a saturated chain of prime ideals

$$(0) \subset yB \subset (y,z)B \subset Q \subset \mathbf{m}_B$$

of length 4 from the proof of item 4. Hence B is not catenary. By item 2, $\dim B = 4$, and so there is no saturated chain of prime ideals of B of length greater than 4. By Comments 18.5.0, there is no saturated chain of prime ideals of B of length less than 3.

For item 9, since R^* is a Krull domain and $B = A = \mathcal{Q}(B) \cap R^*$, it follows that B is a Krull domain and each height-one prime of B is the contraction of a height-one prime of R^*. Item 10 follows since B/xB and $B[1/x]$ are Noetherian [**51**]. $\square$

Remarks 18.6. Let the notation be as in Theorem 18.5.

(1) It follows from Theorem 18.5 that the localization $B[1/x]$ has a unique maximal ideal $QB[1/x] = (y,z,f)B[1/x]$ of height three and has infinitely many maximal ideals of height two. We observe that $B[1/x]$ has no maximal ideal of height one. To show this last statement it suffices to show for each irreducible element p of B with $pB \neq xB$ there exists $P \in \operatorname{Spec} B$ with $pB \subsetneq P$ and $x \notin P$. Assume there does not exist such a prime ideal P. Consider the ideal $(p,x)B$. This ideal has height two and has only finitely many minimal primes since B/xB is Noetherian. Let g be an element of $\mathbf{m}_B$ not contained in any of the minimal primes of $(p,x)B$. Every prime ideal of B that contains $(g,p)B$ also contains x and hence has height greater than two. Since $x \notin Q$, it follows that $(g,p)B$ is $\mathbf{m}_B$-primary, and hence that $(g,p)R^*$ is $\mathbf{m}_{R^*}$-primary. Since R^* is Noetherian and $\operatorname{ht} \mathbf{m}_{R^*} = 3$, this contradicts Krull's Altitude Theorem 2.6.

2) Let I be an ideal of B. Then IR^* is $\mathbf{m}_{R^*}$-primary $\iff$ I is $\mathbf{m}_B$-primary, by Proposition 6.20.5.

(3) Define

$$C_n := \frac{B_n}{(y,z)B_n} \quad \text{and} \quad C := \frac{B}{(y,z)B}.$$

We have $C = \bigcup_{n=0}^{\infty} C_n$ by item 1. We show that C is a rank 2 valuation domain with principal maximal ideal generated by the image of x. For each positive integer n, let $g_n \in C_n$ denote the image in C_n of the element $f_n \in B_n$ and let x denote the image of x. Then $C_n = k[x, g_n]_{(x,g_n)}$ is a 2-dimensional RLR. By (18.1.1), $f_n = xf_{n+1} + x(c_n y + d_n z)$. It follows that $g_n = xg_{n+1}$ for each $n \in \mathbb{N}$. Thus C is an infinite directed union of quadratric transformations of 2-dimensional regular local rings. Thus C is a valuation domain of dimension at most 2 by [**2**]. By items 2 and 4 of Theorem 18.5, $\dim C \geq 2$, and therefore C is a valuation domain of rank 2. The maximal ideal of C is xC.

By Corollary 18.4, $B/zB \cong D$, where D is the ring B of Example 17.3. By an argument similar to that of Proposition 18.3 and by Corollary 18.4, we see that the above ring C is isomorphic to D/yD.

QUESTION 18.7. For the ring B constructed as in Example 18.1, we ask: Is Q the only prime ideal of B that is not finitely generated?

Theorem 18.5 implies that the only possible nonfinitely generated prime ideals of B other than Q have height two. We do not know whether every height-two prime ideal of B is finitely generated. We show in Corollary 18.10 and Theorem 18.11 that certain of the height-two primes of B are finitely generated.

We recall Lemma 8.2, which was the key to the proof of Theorem 8.3. For convenience we repeat two parts of the lemma that are useful in this chapter:

LEMMA 18.8. *Let S be a subring of a ring T and let $b \in S$ be a regular element of both S and T. Assume that $bS = bT \cap S$ and $S/bS = T/bT$. Then*

(1) *$T[1/b]$ is flat over $S \iff T$ is flat over S.*
(2) *If T and $S[1/b]$ are both Noetherian and T is flat over S, then S is Noetherian*

The following theorem shows that the nonflat locus of the map $\varphi : B \to R^*[1/a]$ yields flatness for certain homomorphic images of B, if R, a, R^* and B are as in the general construction outlined in Inclusion Construction 5.3.

THEOREM 18.9. *Let R be a Noetherian integral domain with field of fractions $K := \mathcal{Q}(R)$, let $a \in R$ be a nonzero nonunit, and let R^* denote the (a)-adic completion of R. Let s be a positive integer and let $\underline{\tau} = \{\tau_1, \dots, \tau_s\}$ be a set of elements of R^* that are algebraically independent over K, so that $R[\underline{\tau}]$ is a polynomial ring in s variables over R. Define $A = A_{inc} := K(\underline{\tau}) \cap R^*$, as in Inclusion Construction 5.3. Let U_n, B_n, B and U be defined as in Equations 6.1.4 and 6.1.5. Assume that F is an ideal of $R^*[1/a]$ that defines the nonflat locus of the map $\varphi : B \to R^*[1/a]$. Let I be an ideal in B such that $IR^* \cap B = I$ and a is regular on R^*/IR^*.*

(1) *If $IR^*[1/a] + F = R^*[1/a]$, then $\varphi \otimes_B (B/I)$ is flat.*
(2) *If $R^*[1/a]/IR^*[1/a]$ is flat over B/I, then R^*/IR^* is flat over B/I.*
(3) *If $\varphi \otimes_B (B/I)$ is flat, then B/I is Noetherian.*

PROOF. For item 1, φ_P is flat for each $P \in \operatorname{Spec} R^*[1/a]$ with $I \subseteq P$ by hypothesis. Hence for each such P we have $\varphi_P \otimes_B (B/I)$ is flat. Since flatness is a local property, it follows that $\varphi \otimes_B (B/I)$ is flat.

For items 2 and 3, apply Lemma 18.8 with $S = B/I$ and $T = R^*/IR^*$; the element b of Lemma 18.8 is the image in B/IB of the element a from the setting of Theorem 8.8. Since $IR^* \cap B = I$, the ring B/I embeds into R^*/IR^*, and since $B/aB = R^*/aR^*$, the ideal $a(R^*/IR^*) \cap (B/I) = a(B/I)$. Thus item 2 and item 3 of Theorem 18.9 follow from item 1 and item 2, respectively, of Lemma 18.8. $\square$

COROLLARY 18.10. *Assume the notation of Example 18.1. Let w be a prime element of B. Then B/wB is Noetherian if and only if $w \notin Q$. Thus every every nonfinitely generated ideal of B is contained in Q.*

PROOF. If $w \in Q$, then B/wB is not Noetherian since Q is not finitely generated. Assume $w \notin Q$. Since B/xB is known to be Noetherian, we may assume that $wB \neq xB$. By Proposition 18.2.1, $QR^*[1/x] = (y, z)R^*[1/x]$ defines the nonflat locus of $\varphi : B \to R^*[1/x]$. Since $wR^*[1/x] + (y, z)R^*[1/x] = R^*[1/x]$, Theorem 18.9 with $I = wB$ and $a = x$ implies that B/wB is Noetherian.

For the second statement, we use that every nonfinitely generated ideal is contained in an ideal maximal with respect to not being finitely generated and the latter ideal is prime. Thus it suffices to show every prime ideal P not contained in Q is finitely generated. If $P \not\subseteq Q$, then, since B is a UFD, there exists a prime element $w \in P \setminus Q$. By the first statement, B/wB is Noetherian, and so P is finitely generated. $\square$

THEOREM 18.11. *Assume the notation of Example 18.1. Thus*

$$R \ := \ k[x,y,z]_{(x,y,z)} \quad \text{and} \quad R^* \ := \ k[y,z]_{(y,z)}[[x]],$$

where k is a field, and x, y and z are indeterminates over k. Also τ and σ are elements of $xk[[x]]$ that are algebraically independent over $k(x)$, and $f = y\tau + z\sigma$. Thus the approximation domain B formed by the procedure outlined above is a non-Noetherian four-dimensional UFD with exactly one height-three prime ideal Q and Q is not finitely generated. We have:

$$R = k[x,y,z]_{(x,y,z)} \subseteq R[f] \subseteq B \subseteq R^* \cap k(x,y,z,f) \subseteq R^* : = \ k[y,z]_{(y,z)}[[x]].$$

Let w be a prime element of R with $w \in (y,z)k[x,y,z]$. If w is linear in either y or z, then Q/wB is the unique nonfinitely generated prime ideal of B/wB. Thus Q is the unique nonfinitely generated prime ideal of B that contains w.

PROOF. Let $^-$ denote image under the canonical map $\pi : R^* \to R^*/wR^*$. We may assume that w is linear in z, that the coefficient of z is 1 and therefore that $w = z - yg(x,y)$, where $g(x,y) \in k[x,y]$. Thus $\overline{R} \cong k[x,y]_{(x,y)}$. By Proposition 18.3, $\overline{B}$ is the approximation domain over $\overline{R}$ with respect to the transcendental element

$$\overline{f} = \overline{y} \cdot \overline{\tau} + \overline{z} \cdot \overline{\sigma} = \overline{y} \cdot \overline{\tau} + \overline{y} \cdot \overline{g(x,y)} \cdot \overline{\sigma}.$$

The setting of Theorem 6.21 applies with B replaced by $\overline{B}$, the underlying ring R replaced by $\overline{R}$, and $z = \overline{x}$. Thus the ring $\overline{B}$ is a UFD, and so every height-one prime ideal of $\overline{B}$ is principal. Since $w \in Q$ and Q is not finitely generated, it follows that $\operatorname{ht}(\overline{Q}) = 2$ and that $\overline{Q}$ is the unique nonfinitely generated prime ideal of $\overline{B}$. Hence the theorem holds. $\square$

REMARKS 18.12. It follows from Proposition 6.20.5 that every height two prime of B that is not contained in Q is contracted from a prime ideal of R^*. As we state in item 7 of Theorem 18.5, there are infinitely many height-two prime ideals of B that are contained in Q and are contracted from R^* and there are infinitely many height-two prime ideals of B that are contained in Q and are *not* contracted from R^*. In particular infinitely many of each type exist between zB and Q by Corollary 18.4, and similarly infinitely many of each type exist between yB and Q.

Since B_Q is a three-dimensional regular local ring, for each height-one prime p of B with $p \subset Q$, the set

$$\mathcal{S}_p \ = \ \{P \in \operatorname{Spec} B \mid p \subset P \subset Q \text{ and } \operatorname{ht} P = 2\}$$

is infinite. The infinite set $\mathcal{S}_p$ is the disjoint union of the sets $\mathcal{S}_{pc}$ and $\mathcal{S}_{pn}$, where the elements of $\mathcal{S}_{pc}$ are contracted from R^* and the elements of $\mathcal{S}_{pn}$ are not contracted from R^*.

We do not know whether there exists a height-one prime p contained in Q having the property that one of the sets $\mathcal{S}_{pc}$ or $\mathcal{S}_{pn}$ is empty. Furthermore if one of these sets is empty, which one is empty? If there are some such height-one primes p

with one of the sets $\mathcal{S}_{pc}$ or $\mathcal{S}_{pn}$ empty, which height-one primes are they? It would be interesting to know the answers to these questions.

REMARK 18.13. A natural question related to Example 18.1 is to ask how it compares to a ring constructed using the three-dimensional ring of Example 17.3 and applying the popular "$D + M$" technique of multiplicative ideal theory; see for example the work of Gilmer in [**38**, p. 95], [**39**] or the paper of Brewer and Rutter [**14**]. The "$D + M$" construction involves an integral domain D and a prime ideal M of an extension domain E of D such that $D \cap M = (0)$. Then $D + M = \{a + b \mid a \in D,\ b \in M\}$. Moreover, for $a, a' \in D$ and $b, b' \in M$, if $a + b = a' + b'$, then $a = a'$ and $b = b'$. Since D embeds in E/M, the ring $D + M$ may be regarded as a pullback as in the paper of Gabelli and Houston [**40**] or the book of Leuschke and R. Wiegand [**84**, p. 42].

In Example 18.14, we consider a "$D + M$" construction that contrasts nicely with Example 18.1.

EXAMPLE 18.14. Let $(B, \mathbf{m}_B)$ be the ring of Example 17.3. Thus k is a coefficient field of B and $B = k + \mathbf{m}_B$. Assume the field k is the field of fractions of a DVR V, and let t be a generator of the maximal ideal of V. Define

$$C \ := \ V \ + \ \mathbf{m}_B \ = \ \{\ a \ + \ b \mid a \ \in \ V,\ b \ \in \ \mathbf{m}_B\ \}.$$

The integral domain C has the following properties:

(1) The maximal ideal $\mathbf{m}_B$ of B is also a prime ideal of C, and $C/\mathbf{m}_B \cong V$.
(2) C has a unique maximal ideal $\mathbf{m}_C$; moreover, $\mathbf{m}_C = tC$.
(3) $\mathbf{m}_B = \bigcap_{n=1}^{\infty} t^n C$, and $B = C_{\mathbf{m}_B} = C[1/t]$.
(4) Each $P \in \operatorname{Spec} C$ with $P \neq \mathbf{m}_C$ is contained in $\mathbf{m}_B$; thus $P \in \operatorname{Spec} B$.
(5) $\dim C = 4$ and C has a unique prime ideal of height h, for $h = 2, 3$ or 4.
(6) $\mathbf{m}_C$ is the only nonzero prime ideal of C that is finitely generated. Indeed, every nonzero proper ideal of B is an ideal of C that is not finitely generated.

Thus C is a non-Noetherian non-catenary four-dimensional local domain.

PROOF. Since C is a subring of B, $\mathbf{m}_B \cap V = (0)$ and $V\mathbf{m}_B = \mathbf{m}_B$, item 1 holds. We have $C/(tV + \mathbf{m}_B) = V/tV$. Thus $tV + \mathbf{m}_B$ is a maximal ideal of C. Let $b \in \mathbf{m}_B$. Since $1 + b$ is a unit of the local ring B, we have

$$\frac{1}{1+b} \ = \ 1 - \frac{b}{1+b} \quad \text{and} \quad \frac{b}{1+b} \ \in \ \mathbf{m}_B.$$

Hence $1+b$ is a unit of C. Let $a+b \in C \setminus (tV + \mathbf{m}_B)$, where $a \in V \setminus tV$ and $b \in \mathbf{m}_B$. Then a is a unit of V and thus a unit of C. Moreover, $a^{-1}(a + b) = 1 + a^{-1}b$ and $a^{-1}b \in \mathbf{m}_B$. Therefore $a + b$ is a unit of C. We conclude that $\mathbf{m}_C := tV + \mathbf{m}_B$ is the unique maximal ideal of C. Since t is a unit of B, we have $\mathbf{m}_B = t\mathbf{m}_B$. Hence $\mathbf{m}_C = tV + \mathbf{m}_B = tC$. This proves item 2.

For item 3, since t is a unit of B, we have $\mathbf{m}_B = t^n\mathbf{m}_B \subseteq t^n C$ for all $n \in \mathbb{N}$. Thus $\mathbf{m}_B \subseteq \bigcap_{n=1}^{\infty} t^n C$. If $a + b \in \bigcap_{n=1}^{\infty} t^n C$ with $a \in V$ and $b \in \mathbf{m}_B$, then

$$b \ \in \ \bigcap_{n=1}^{\infty} t^n C \ \implies \ a \ \in \ (\bigcap_{n=1}^{\infty} t^n C) \cap V \ = \ \bigcap_{n=1}^{\infty} t^n V \ = \ (0).$$

Hence $\mathbf{m}_B = \bigcap_{n=1}^{\infty} t^n C$. Again using $t\mathbf{m}_B = \mathbf{m}_B$, we obtain

$$C[1/t] \ = \ V[1/t] \ + \ \mathbf{m}_B \ = \ k \ + \ \mathbf{m}_B \ = \ B.$$

Since $t \notin \mathbf{m}_B$, we have $B = C[1/t] \subseteq C_{\mathbf{m}_B} \subseteq B_{\mathbf{m}_B} = B$. This proves item 3.

For P as in item 4, we have $P \subsetneq tC$. Since P is a prime ideal of C, it follows that $P = t^n P$ for each $n \in \mathbb{N}$. By item 3, $P \subseteq \mathbf{m}_B$, and it follows that $P \in \operatorname{Spec} B$. Item 5 now follows from item 4 and the structure of $\operatorname{Spec} B$.

For item 6, let J be a nonzero proper ideal of B. Since t is a unit of B, we have $J = tJ$. This implies by Nakayama's Lemma that J as an ideal of C is not finitely generated; see [**14**, Lemma 1]. Thus item 6 follows from item 4.

By item 6, C is non-Noetherian. Since $(0) \subsetneq xB \subsetneq \mathbf{m}_B \subsetneq tC$ is a saturated chain of prime ideals of C of length 3, and $(0) \subsetneq yB \subsetneq Q \subsetneq \mathbf{m}_B \subsetneq tC$ is a saturated chain of prime ideals of C of length 4, the ring C is not catenary. □

REMARK 18.15. An integral domain R is said to be a *finite conductor domain* if for elements a, b in the field of fractions of R the R-module $aR \cap bR$ is finitely generated. This concept is considered in the paper of McAdam [**97**].

A ring R is said to be *coherent* if every finitely generated ideal of R is finitely presented. By a theorem of Chase [**21**, Theorem 2.2], this condition is equivalent to each of the following:

(1) For each finitely generated ideal I and element a of R, the ideal $(I :_R a) = \{b \in R \mid ba \in I\}$ is finitely generated.
(2) For each $a \in R$ the ideal $(0 :_R a) = \{b \in R \mid ba = 0\}$ is finitely generated, and the intersection of two finitely generated ideals of R is again finitely generated.

Thus a coherent integral domain is a finite conductor domain. Examples of finite conductor domains that are not coherent are given by Glaz in [**42**, Example 4.4] and by Olberding and Saydam in [**109**, Proposition 3.7].

As noted in Remark 17.11, the rings of Examples 17.3 and 18.1 are coherent. On the other hand, by a result of Brewer and Rutter [**14**, Prop. 2], the ring of Example 18.14 is not a finite conductor domain and thus is not coherent.

18.3. Non-Noetherian examples in higher dimension

We show in Theorem 18.16 that the rings constructed in Examples 13.8 have many of the properties of Examples 17.1 and 18.1.

THEOREM 18.16. *Let k be a field, let d be a positive integer, and let $x, y_1, \dots, y_d$ be indeterminates over k. For every positive integer m, there exists a non-Noetherian local integral domain $(B, \mathbf{n})$ with*

$$R := k[x, y_1, \dots, y_d]_{(x, y_1, \dots, y_d)} \subset B \subset R^* := k[y_1, \dots, y_d]_{(y_1, \dots, y_d)}[[x]]$$

having the following properties:

(1) $\dim B = d + 2$.
(2) *The maximal ideal $\mathbf{n}$ of B is generated by $x, y_1, \dots, y_d$, and the $\mathbf{n}$-adic completion of B is the formal power series ring $k[[x, y_1, \dots, y_d]]$, a regular local domain of dimension $d + 1$.*
(3) *The ring B has exactly m prime ideals $Q_1, \dots, Q_m$ of height $d + 1$.*
(4) *Each Q_i is not finitely generated.*
(5) *The ring $B[1/x]$ is a regular Noetherian UFD.*
(6) *The ring B is a UFD that is not catenary.*
(7) *The local ring $(B, \mathbf{n})$ birationally dominates a localized polynomial ring in $d + 2$ variables over the field k.*

Proof. We first define the extension domain B of $R = k[x, y_1, \ldots, y_d]_{(x,y_1,\ldots,y_d)}$. Let $\tau_1, \ldots, \tau_d \in xk[[x]]$ be algebraically independent over R. For $1 \leq i \leq m$, let $p_i = y_1 - x^i$. Set $q = p_1 p_2 \ldots p_m$, and consider the element

$$f = q\tau_1 + y_2\tau_2 + \ldots y_d\tau_d,$$

and let f_n denote the n^{th}-endpiece of f as in Equation 17.1.a. Define B to be the nested union of localized polynomial rings of dimension $d+2$:

$$B = \bigcup_{n=1}^{\infty} B_n, \quad \text{where} \quad B_n = k[x, y_1, \ldots, y_d, f_n]_{(x,y_1,\ldots,y_d,f_n)}.$$

Thus B is local, $\dim B \leq d+2$, and $\mathbf{n} = \bigcup_{n=1}^{\infty}(x, y_1, \ldots, y_d, f_n)B_n$. We have $\mathbf{n} = (x, y_1, \ldots, y_d)B$ because $f_n \in (x, y_1, \ldots, y_d)B$ for each n. By Construction Properties Theorem 6.19.4, the (x)-adic completion of B is R^*. Hence the $\mathbf{n}$-adic completion of B is the same as the $\mathbf{m}$-adic completion of R, that is $\widehat{B} = k[[x, y_1, \ldots, y_d]]$. This proves item 2.

The inclusion map:

$$\varphi : \ S \ := \ R[f] \ \hookrightarrow \ T \ := \ R[\tau_1, \ldots, \tau_d]$$

is not flat because the prime ideal $P = (p_1, y_2, y_3, \ldots, y_d)T$ has height d, while $P \cap S$ has height $d+1$; see Remark 2.21.10. Thus by Theorem 13.3.2 the ring B is non-Noetherian.

For item 1, it remains to show $\dim B \geq d+2$. Define $Q_i = (p_i, y_2, \ldots, y_d)R^* \cap B$, for i with $1 \leq i \leq m$. Since $(p_i, y_2, \ldots, y_d)R^*$ is a prime ideal of R^*, the ideal Q_i is prime. By Proposition 6.20.2, the ideals p_iB and $(p_i, y_2, \ldots, y_j)B$ are prime, for every j with $2 \leq j \leq d$. The inclusions in the chain of prime ideals

$$(0) \ \subset p_iB \ \subset (p_i, y_2)B \ \subset \cdots \ \subset (p_i, y_2, \ldots, y_d)B \subset \ Q_i \ \subset \ \mathbf{n}.$$

are strict because the contractions to B_n are strict for each n; to verify this, consider the list $p_i, y_2, y_3, \ldots, y_d, f, x$, and use that $f \in Q_i \setminus (p_i, y_2, \ldots, y_d)B_n$ for each n. Thus $\dim B = d+2$, each Q_i has height $d+1$, and $(p_i, y_2, \ldots, y_d))B$ has height d for each i. This proves item 1 and part of item 3.

To complete the proof of item 3, we show that $Q_1, \ldots, Q_m$ are the only prime ideals of B of height $d+1$. Let P be a nonmaximal prime ideal of B. We first consider the case where $x \notin P$. By Proposition 6.20.3, $x^n \notin PR^*$ for each positive integer n. Hence $PR^*[1/x] \neq R^*[1/x]$. Let P^* be a prime ideal of $R^*[1/x]$ such that $P \subseteq P^*$. If $q, y_2, \ldots, y_d$ are all in P^*, then for some i with $1 \leq i \leq m$ we have $(p_i, y_2, \ldots, y_d)R^*[1/x] \subseteq P^*$. Since $(p_i, y_2, \ldots, y_d)R^*[1/x]$ is a maximal ideal, we have $(p_i, y_2, \ldots, y_d)R^*[1/x] = P^*$. Therefore, $P \subseteq (p_i, y_2, \ldots, y_d)R^*[1/x] \cap B = Q_i$, and so either $\text{ht}(P) \leq d$ or $P = Q_i$.

By Theorem 13.6.2, the nonflat locus of $\beta : B \hookrightarrow R^*[1/x]$ is defined by $LR^*[1/x]$, where $L = (q, y_2, \ldots, y_d)R$. Hence if $x \notin P$ and some element of $\{q, y_2, \ldots, y_d\}$ is not in P^*, then the map $\beta_{P^*} : B \to R^*[1/x]_{P^*}$ is flat. Since $\dim(R^*[1/x]) = d$ we have $\text{ht}(P^*) \leq d$. Flatness of β_{P^*} implies $\text{ht}(P^* \cap B) \leq d$, by Remark 2.21.10. Hence $\text{ht}\, P \leq d$. This completes the proof of item 3 in the case where $x \notin P$.

Assume $x \in P$. We have $\text{ht}\, P \leq d+1$, since $\dim B = d+2$ and P is not maximal. Suppose $\text{ht}\, P = d+1$. Then there exists a saturated chain of prime ideals of B:

$$(0) \subsetneq P_1 \subsetneq P_2 \ldots \subsetneq P_d \ \subsetneq \ P_{d+1} = P \ \subsetneq (x, y_1, y_2, \ldots, y_d)B = \mathbf{m}_B. \tag{18.5.1}$$

By Construction Properties Theorem 6.19.3, we have $R/xR = B/xB = R^*/xR^*$. It follows that $d = \dim(R/xR) = \dim(B/xB) = \dim(R^*/xR^*)$. If $x \in P_1$, then $P_1 = xB$, and we get a chain of prime ideals in B/xB of length $d+1$, a contradiction.

Thus we have $x \notin P_1$, and there exists an integer i with $1 \le i \le d$ such that $x \in P_{i+1} \setminus P_i$. Using $B/xB = R^*/xR^*$ and $x \in P_{i+1}$, the righthand part of Equation 18.5.1 extends to a chain of prime ideals

$$P_{i+1}R^* \ \subsetneq\ \cdots\ \subsetneq\ P_{d+1}R^* \ =\ PR^* \ \subsetneq \mathbf{m}_{R^*}$$

of R^*. Also $P_iR^* \subseteq P_{i+1}R^*$, and so there exists a prime ideal $P_i' \in \operatorname{Spec} R^*$ such that $P_i' \subseteq P_{i+1}R^*$ and P_i' is minimal over P_iR^*. By Proposition 6.20.4, P_i' does not contain x. Thus $P_i' \subsetneq P_{i+1}R^*$. Moreover $x \notin P_i'$ implies $P_i'R^*[1/x]$ is a prime ideal of $R^*[1/x]$, and $P_i'R^*[1/x] \cap B = P_i' \cap B = P_i$.

Since $x \in P$ and $P \subsetneq \mathbf{m}_B$, there exists an integer j with $1 \le j \le d$ such that $y_j \notin P$; thus $y_j \notin P_i$. It follows that $y_j \notin P_i'' := P_i'R^*[1/x]$, and so P_i'' does not contain $LR^*[1/x] = (q, y_1, \ldots, y_d)R^*[1/x]$. By Theorem 13.6.2 again, the map $\beta_{P_i''} : B \hookrightarrow R^*[1/x]_{P''}$ is flat. Thus $P_i'' \cap B = P_i' \cap B = P_i$ and $\operatorname{ht}(P_i) = i$ together imply that $\operatorname{ht} P_i' = \operatorname{ht} P_i'' \ge \operatorname{ht} P_i = i$. We now have the following chain of prime ideals of R^* of length $d + 2 - i$:

$$P_i' \ \subsetneq\ P_{i+1}R^* \ \subsetneq\ \cdots\ \subsetneq\ P_{d+1}R^* \ \subsetneq\ \mathbf{m}_{R^*},$$

as well as the information that $\operatorname{ht} P_i' \ge i$, a contradiction to $\dim R^* = d + 1$. This contradiction completes the proof of item 3 of Theorem 18.16.

For item 4, we first show for each i with $1 \le i \le m$:

$$Q_i \ =\ Q_i' \ :=\ \bigcup_{n \in \mathbb{N}} Q_{in}, \qquad \text{where} \qquad Q_{in} \ =\ (p_i, y_2, \ldots, y_d, f_n)B_n.$$

Each Q_{in} is a prime ideal of height $d + 1$ in the $(d + 2)$-dimensional RLR B_n. Thus Q_i' is a prime ideal of B of height $\le d + 1$ that is contained in Q_i. Since $f \in Q_i' \setminus (p_i, y_2, \ldots, y_d)B$ and $(p_i, y_2, \ldots, y_d)B$ has height d, we have $\operatorname{ht} Q_i' = d$, and hence $Q_i' = Q_i$.

To show Q_i is not finitely generated, define $P_i := (p_i, y_2, \ldots, y_d)R$. It suffices to show for each $n \in \mathbb{N}$ that $f_{n+1} \notin P_iB + f_nB$. Suppose

$$f_{n+1} = \alpha + f_n\beta, \tag{18.5.2}$$

where $\alpha \in P_iB$ and $\beta \in B$. By definition $f = q\tau_1 + y_2\tau_2 + \ldots + y_d\tau_d \in P_iR[\tau_1, \ldots, \tau_d]$. Since each of the τ_i has an expression as a "power series" in x where the coefficients are in R, it follows that there is an expression for f as a "power series" in x where the coefficients are in P_i. Thus, with $f = \gamma$ in Equation 6.1.2, we have $f_n = ax + xf_{n+1}$, where $a \in P_i$. By replacing f_n in Equation 18.5.2, we get

$$f_{n+1} = \alpha + ax\beta + xf_{n+1}\beta \implies f_{n+1}(1 - \beta x) \in P_iB.$$

Since Proposition 6.20.1 implies that x is in the Jacobson radical of B, that is, $1 - x\beta$ is a unit of B, we have

$$f_{n+1} \in P_iB \cap B_{n+1} = P_iB_{n+1},$$

where the last equality is by Proposition 6.20.2. This is a contradiction, since the set $\{p_i = y_1 - x^i, x, y_2, \ldots, y_d, f_{n+1}\}$ is a minimal generating set for the maximal ideal of the regular local ring B_{n+1}. This contradiction shows that Q_i is not finitely generated.

Theorem 6.21 implies item 5, and the assertion in item 5 that B is a UFD. There always exists a saturated chain of prime ideals of B between (0) and $\mathbf{n}$ that contains the height-one prime xB and $B/xB = R^*/xR^*$ implies that this chain has length equal to $\dim R^* = d+1$. Since $\dim B > \dim R^*$, there also exists a saturated chain of prime ideals in B of length $d+2 = \dim B$. Hence B is not catenary. Thus item 5 holds. Item 6 follows from the construction of B. $\square$

QUESTION 18.17. For the ring B constructed in Theorem 18.16, are the prime ideals

$$Q_i \;:=\; (p_i, y_2, \ldots, y_{d-1}, f_1, \ldots, f_i, \ldots)B$$

the only prime ideals of B that are not finitely generated?

Exercises

(1) Let K denote the field of fractions of the integral domain B of Example 17.3, let t be an indeterminate over K and let V denote the DVR $K[t]_{(t)}$. Let M denote the maximal ideal of V. Thus $V = K + M$. Define $C := B + M$. Show that the integral domain C has the following properties:
   (a) $\mathbf{m}_B C$ is the unique maximal ideal of C, and is generated by two elements.
   (b) For every nonzero element $a \in \mathbf{m}_B$, we have $M \subset aC$.
   (c) M is the unique prime ideal of C of height one; moreover M is not finitely generated as an ideal of C.
   (d) $\dim C = 4$ and C has a unique prime ideal of height h, for $h = 1, 3$ or 4.
   (e) For each $P \in \operatorname{Spec} C$ with $\operatorname{ht} P \geq 2$, the ring C_P is not Noetherian.
   (f) C has precisely two prime ideals that are not finitely generated.
   (g) C is non-catenary.

(2) Let $C = V + \mathbf{m}_B$ be as in Example 18.14. Assume that V has a coefficient field L, and that L is the field of fractions of a DVR V_1. Define $C_1 := V_1 + tC$. Let s be a generator of V_1. Show that the integral domain C_1 has the following properties:
   (a) The maximal ideal $\mathbf{m}_C$ of C is also a prime ideal of C_1, and $C_1/\mathbf{m}_C \cong V_1$.
   (b) The principal ideal sC_1 is the unique maximal ideal of C_1.
   (c) $\mathbf{m}_C = \bigcap_{n=1}^{\infty} s^n C_1$, and $C = C_1[1/s]$.
   (d) Each $P \in \operatorname{Spec} C_1$ with $P \neq sC_1$ is contained in $\mathbf{m}_C$; thus $P \in \operatorname{Spec} C$.
   (e) $\dim C_1 = 5$.
   (f) C_1 has a unique prime ideal of height h for $h = 2, 3, 4$, or 5.
   (g) The maximal ideal of C_1 is the only nonzero prime ideal of C_1 that is finitely generated. Indeed, every nonzero proper ideal of C is an ideal of C_1 that is not finitely generated.
   (h) C_1 is non-catenary.

   Comment: For item h, exhibit two saturated chains of prime ideals from (0) to sC_1 of different lengths.

CHAPTER 19

Idealwise algebraic independence I, int

Let $(R, \mathbf{m})$ be an excellent normal local domain with field of fractions K and completion $(\widehat{R}, \widehat{\mathbf{m}})$. We consider three concepts of independence over R for elements $\tau_1, \ldots, \tau_n \in \widehat{\mathbf{m}}$ that are algebraically independent over K. The first of these, *idealwise independence*, is that $K(\tau_1, \ldots, \tau_n) \cap \widehat{R}$ equals the localized polynomial ring $R[\tau_1, \ldots, \tau_n]_{(\mathbf{m}, \tau_1, \ldots, \tau_n)}$. If R is countable with $\dim(R) > 1$, we show the existence of an infinite sequence of elements $\tau_1, \tau_2, \ldots$ of $\widehat{\mathbf{m}}$ such that $\tau_1, \ldots, \tau_n$ are idealwise independent over R for each positive integer n. This implies that the subfield $K(\tau_1, \tau_2, \ldots)$ of $\mathcal{Q}(\widehat{R})$ has the property that the intersection domain $A = K(\tau_1, \tau_2, \ldots) \cap \widehat{R}$ is a localized polynomial ring in infinitely many variables over R. In particular, this intersection domain A is not Noetherian. These topics are continued in Chapter 20.

19.1. Idealwise independence, weakly flat and PDE extensions

We use the following setting throughout this chapter and Chapter 20.

Setting and Notation 19.1. Let $(R, \mathbf{m})$ be an excellent normal local domain with field of fractions K and completion $(\widehat{R}, \widehat{\mathbf{m}})$. Let $t_1, \ldots, t_n, \ldots$ be indeterminates over R, and assume that $\tau_1, \tau_2, \ldots, \tau_n, \ldots \in \widehat{\mathbf{m}}$ are algebraically independent over K. For each integer $n \geq 0$ and ∞, we consider the following localized polynomial rings:

$S_n := R[t_1, \ldots, t_n]_{(\mathbf{m}, t_1, \ldots, t_n)}$,
$R_n := R[\tau_1, \ldots, \tau_n]_{(\mathbf{m}, \tau_1, \ldots, \tau_n)}$,
$S_\infty := R[t_1, \ldots, t_n, \ldots]_{(\mathbf{m}, t_1, \ldots, t_n, \ldots)}$ and
$R_\infty := R[\tau_1, \ldots, \tau_n, \ldots]_{(\mathbf{m}, \tau_1, \ldots, \tau_n, \ldots)}$.

For $n = 0$, we define $R_0 = R = S_0$. Of course, S_n is R-isomorphic to R_n and S_∞ is R-isomorphic to R_∞ with respect to the R-algebra homomorphism taking $t_i \to \tau_i$ for each i. When working with a particular n or ∞, we sometimes define S to be R_n or R_∞.

The completion $\widehat{S_n}$ of S_n is $\widehat{R}[[t_1, \ldots, t_n]]$, and we have the following commutative diagram:

$$\begin{tikzcd} & S_n = R[t_1, \ldots, t_n]_{(\mathbf{m}, t_1, \ldots, t_n)} \arrow[r, "\subseteq"] \arrow[d, "\cong"'] & \widehat{S_n} = \widehat{R}[[t_1, \ldots, t_n]] \arrow[d, "\lambda"'] \\ R \arrow[r, "\subseteq"] & S = R_n = R[\tau_1, \ldots, \tau_n]_{(\mathbf{m}, \tau_1, \ldots, \tau_n)} \arrow[r, "\varphi"] & \widehat{R}. \end{tikzcd}$$

Here the first vertical isomorphism is the R-algebra map taking $t_i \to \tau_i$, the restriction of the R-algebra surjection $\lambda : \widehat{S_n} \to \widehat{R}$ where

$$\mathfrak{p} := \ker(\lambda) = (t_1 - \tau_1, \dots, t_n - \tau_n)\widehat{S_n}.$$

Note that $\mathfrak{p} \cap S_n = (0)$.

The central definition of this chapter is the following:

DEFINITION 19.2. Let $(R, \mathbf{m})$ and $\tau_1, \dots, \tau_n \in \widehat{\mathbf{m}}$ be as in Setting 19.1. We say that $\tau_1, \dots, \tau_n$ are *idealwise independent over* R if

$$\widehat{R} \ \cap \ K(\tau_1, \dots, \tau_n) \ = \ R_n.$$

Similarly, an infinite sequence $\{\tau_i\}_{i=1}^{\infty}$ of algebraically independent elements of $\widehat{\mathbf{m}}$ is *idealwise independent over* R if $\widehat{R} \cap K(\{\tau_i\}_{i=1}^{\infty}) = R_{\infty}$.

REMARKS 19.3. Assume Setting and Notation 19.1.

(1) A subset of an idealwise independent set $\{\tau_1, \dots, \tau_n\}$ over R is also idealwise independent over R. For example, to see that $\tau_1, \dots, \tau_m$ are idealwise independent over R for $m \le n$, let K denote the field of fractions of R and observe that

$$\begin{aligned}\widehat{R} \cap K(\tau_1, \dots, \tau_m) &= \widehat{R} \cap K(\tau_1, \dots, \tau_n) \cap K(\tau_1, \dots, \tau_m)\\ &= R[\tau_1, \dots, \tau_n]_{(\mathbf{m}, \tau_1, \dots, \tau_n)} \cap K(\tau_1, \dots, \tau_m) = R[\tau_1, \dots, \tau_m]_{(\mathbf{m}, \tau_1, \dots, \tau_m)}.\end{aligned}$$

(2) Idealwise independence is a strong property of the elements $\tau_1, \dots, \tau_n$ and of the embedding map $\varphi : R_n \hookrightarrow \widehat{R}$. It is often difficult to compute $\widehat{R} \cap L$ for an intermediate field L between the field K and the field of fractions of $\widehat{R}$. In order for $\widehat{R} \cap L$ to be the localized polynomial ring R_n, there can be no new quotients in $\widehat{R}$ other than those in $\varphi(R_n)$; that is, if $f/g \in \widehat{R}$ and $f, g \in R_n$, then $f/g \in R_n$. This does not happen, for example, if one of the τ_i is in the completion of R with respect to a principal ideal; in particular, if $\dim(R) = 1$, then there do not exist idealwise independent elements over R.

The following example, considered in Chapter 4, illustrates Remark 19.3.2. This is Example 4.4; other details are given in Remarks 4.5.

EXAMPLE 19.4. Let $R = \mathbb{Q}[x, y]_{(x,y)}$, the localized ring of polynomials in two variables over the rational numbers. The elements $\tau_1 = e^x - 1, \tau_2 = e^y - 1$, and $e^x - e^y = \boldsymbol{\tau_1} - \boldsymbol{\tau_2}$ of $\widehat{R} = \mathbb{Q}[[x, y]]$ belong to completions of R with respect to principal ideals (and so are not idealwise independent). If $S = R_2 = \mathbb{Q}[x, y, \tau_1, \tau_2]_{(x,y,\tau_1,\tau_2)}$ and L is the field of fractions of S, then the elements $(e^x - 1)/x, (e^y - 1)/y$, and $(e^x - e^y)/(x - y)$ are certainly in $L \cap \widehat{R}$ but not in S. Theorem 4.2 implies that $L \cap \widehat{R}$ is a two-dimensional regular local ring with completion $\widehat{R}$.

Recall the concepts PDE, weakly flat and height-one preserving from Definitions 2.3.3 in Chapter 2 and 12.1 in Chapter 12. We state the definitions again here for Krull domains.

DEFINITIONS 19.5. Let $S \hookrightarrow T$ be an extension of Krull domains.

- T is a PDE extension of S if for every height-one prime ideal Q in T, the height of $Q \cap S$ is at most one.
- T is a *height-one preserving* extension of S if for every height-one prime ideal P of S with $PT \neq T$ there exists a height-one prime ideal Q of T with $PT \subseteq Q$.

- T is *weakly flat* over S if every height-one prime ideal P of S with $PT \neq T$ satisfies $PT \cap S = P$.

We summarize the results of this chapter.

SUMMARY 19.6. Let $(R, \mathbf{m})$ be an excellent normal local domain of dimension d with field of fractions K and completion $(\widehat{R}, \widehat{\mathbf{m}})$. In Section 19.1 we consider idealwise independent elements as defined in Definition 19.2. We show in Theorem 19.11 that $\tau_1, \ldots, \tau_n \in \widehat{\mathbf{m}}$ are idealwise independent over R if and only if the extension $R[\tau_1, \ldots, \tau_n] \hookrightarrow \widehat{R}$ is weakly flat in the sense of Definition 19.5. If R has the additional property that every height-one prime of R is the radical of a principal ideal, we show in Section 19.1 that a sufficient condition for $\tau_1, \ldots, \tau_n$ to be idealwise independent over R is that the extension $R[\tau_1, \ldots, \tau_n] \hookrightarrow \widehat{R}$ satisfies PDE ("pas d'éclatement", or in English "no blowing up"), defined in Definitions 19.5. At the end of Section 19.1 we display in a schematic diagram the relationships among these concepts and some others, for extensions of Krull domains.

In Sections 19.2 and 19.3 we present two methods for obtaining idealwise independent elements over a countable ring R. The method in Section 19.2 is to find elements $\tau_1, \ldots, \tau_n \in \widehat{\mathbf{m}}$ so that (1) $\tau_1, \ldots, \tau_n$ are algebraically independent over R, and (2) for every prime ideal P of $S = R[\tau_1, \ldots, \tau_n]_{(\mathbf{m}, \tau_1, \ldots, \tau_n)}$ with $\dim(S/P) = n$, the ideal $P\widehat{R}$ is $\widehat{\mathbf{m}}$-primary. In this case, we say that $\tau_1, \ldots, \tau_n$ are *primarily independent* over R. If R is countable and $\dim(R) > 2$, we show in Theorem 19.28 the existence over R of idealwise independent elements that fail to be primarily independent.

The main theorem of this chapter is Theorem 19.20: For every countable excellent normal local domain R of dimension at least two, there exists an infinite sequence $\tau_1, \tau_2, \ldots$ of elements of $\widehat{\mathbf{m}}$ that are primarily independent over R. It follows that $A = K(\tau_1, \tau_2, \ldots) \cap \widehat{R}$ is an infinite-dimensional non-Noetherian local domain. Thus, for the example $R = k[x, y]_{(x,y)}$ with k a countable field, there exists for every positive integer n and $n = \infty$, an extension $A_n = L_n \cap \widehat{R}$ of R such that $\dim(A_n) = \dim(R) + n$. In particular, the canonical surjection $\widehat{A_n} \to \widehat{R}$ has a nonzero kernel.

In Section 19.3 we define $\tau \in \widehat{\mathbf{m}}$ to be *residually algebraically independent* over R if τ is algebraically independent over R and, for each height-one prime ideal P of $\widehat{R}$ such that $P \cap R \neq 0$, the image of τ in $\widehat{R}/P$ is algebraically independent over $R/(P \cap R)$. We extend the concept of residual algebraic independence to a finite or infinite number of elements $\tau_1, \ldots, \tau_n, \ldots \in \widehat{\mathbf{m}}$ and observe the equivalence of residual algebraic independence to the extension $R[\tau_1, \ldots, \tau_n] \hookrightarrow \widehat{R}$ satisfying PDE.

We show that primary independence implies residual algebraic independence and that primary independence implies idealwise independence. If every height-one prime of R is the radical of a principal ideal, we show that residual algebraic independence implies idealwise independence

For R of dimension two, we show that primary independence is equivalent to residual algebraic independence. Hence residual algebraic independence implies idealwise independence if $\dim R = 2$. As remarked above, if R has dimension greater than two, then primary independence is stronger than residual algebraic independence. We show in Theorems 19.33 and 19.35 the existence of idealwise independent elements that fail to be residually algebraically independent.

The following diagram summarizes some relationships among the independence concepts for one element τ of $\widehat{\mathbf{m}}$, over a local normal excellent domain $(R, \mathbf{m})$. In the diagram we use "ind." and "resid." to abbreviate "independent" and "residually algebraic".

R Henselian

$\dim(R) = 2$

τ primarily ind.

τ resid. ind.

τ idealwise ind.*

* In order to conclude that the idealwise independent set contains the residually algebraically independent set for $\dim R > 2$, we assume that every height-one prime of R is the radical of a principal ideal.

In Section 20.4 we include a diagram that displays many more relationships among the independence concepts and other related properties.

In the remainder of this section we discuss some properties of extensions of Krull domains related to idealwise independence. A diagram near the end of this section displays the relationships among these properties.

Remark 19.7. Let $S \hookrightarrow T$ be an extension of Krull domains. If S is a UFD, or more generally, if every height-one prime ideal of S is the radical of a principal ideal, then T is a height-one preserving extension of S. This is clear from the fact that every minimal prime of a principal ideal in a Krull domain is of height one.

Remark 19.8. Let $(R, \mathbf{m})$ and $\tau_1, \ldots, \tau_n \in \widehat{\mathbf{m}}$ be as in Setting 19.1. Also assume that each height-one prime of R is the radical of a principal ideal. Since this property is preserved in a polynomial ring extension, Remark 19.7 implies that the embedding

$$\varphi : R_n = R[\tau_1, \ldots, \tau_n]_{(\mathbf{m}, \tau_1, \ldots, \tau_n)} \hookrightarrow \widehat{R}$$

is a height-one preserving extension.

Corollary 19.9 is immediate from Remark 19.8 and Proposition 12.13.

Corollary 19.9. *Let* $(R, \mathbf{m})$ *and* $\tau_1, \ldots, \tau_n \in \widehat{\mathbf{m}}$ *be as in Setting 19.1. Assume that each height-one prime of* R *is the radical of a principal ideal. Let* $S = R[\tau_1, \ldots, \tau_n]_{(\mathbf{m}, \tau_1, \ldots, \tau_n)}$. *If* $S \hookrightarrow \widehat{R}$ *satisfies PDE, then* $\widehat{R}$ *is weakly flat over* S.

Let $S \hookrightarrow T$ be an extension of Krull domains, and let F be the field of fractions of S. Throughout the diagram "Q" denotes a prime ideal $Q \in \operatorname{Spec}(T)$ with $\operatorname{ht}(Q) = 1$, and "P" denotes $P \in \operatorname{Spec}(S)$ with $\operatorname{ht}(P) = 1$. The following diagram illustrates the relationships among the terms in Definitions 19.5 using the results (12.13), (12.4), (12.6), and (12.9):

$\boxed{S \hookrightarrow T \text{ flat}}$

(12.6.b)

$S \hookrightarrow T$ ht-1 pres., PDE,
(19.5) *and* $PT \neq T, \forall P$

(12.13)

$S \hookrightarrow T$ w.f. (19.5)
and $PT \neq T, \forall P$

$S \hookrightarrow T$ ht-1 pres.,
and PDE (19.5)

(12.4)

$\boxed{T \cap F = S}$ (12.6.a) $\boxed{\forall P, \exists Q | Q \cap S = P}$

(12.13)

(12.6a)

$\boxed{S \hookrightarrow T \text{ w.f. (19.5)}}$ (19.5) $\boxed{PT \neq T \implies PT \cap S = P}$

$\boxed{S \hookrightarrow T \text{ PDE (19.5)}}$

(12.9)

(12.9) $\boxed{PT \neq T \implies \exists Q | Q \cap S = P}$

(19.5)

$\boxed{\forall Q, \operatorname{ht}(Q \cap S) \leq 1}$ $\boxed{S \hookrightarrow T \text{ ht-1 pres. (19.5)}}$ (19.5) $\boxed{PT \neq T \implies \exists Q | PT \subseteq Q}$

The relationships among properties of an extension $S \hookrightarrow T$ of Krull domains

Remark 19.10. Let S be a Krull domain, and let $S \hookrightarrow T$ be an extension of commutative rings such that every nonzero element of S is regular on T. In Corollary 12.4 the condition that $PT \neq T$ for every height-one prime ideal P of S relates weak flatness to $S = \mathcal{Q}(S) \cap T$. This condition holds if S and T are local Krull domains with T dominating S, and so it holds for $R_n \hookrightarrow \widehat{R}$ as in Setting 19.1.

Summarizing from Corollaries 19.9 and 12.4, we have the following implications among the concepts of weakly flat, PDE and idealwise independence in Setting 19.1:

THEOREM 19.11. *Let $(R, \mathbf{m})$ be an excellent normal local domain with $\mathbf{m}$-adic completion $(\widehat{R}, \widehat{\mathbf{m}})$ and let $\tau_1, \ldots, \tau_n \in \widehat{\mathbf{m}}$ be algebraically independent elements over R. Then:*

(1) *$\tau_1, \ldots, \tau_n$ are idealwise independent over $R \iff R[\tau_1, \ldots, \tau_n] \hookrightarrow \widehat{R}$ is weakly flat.*

(2) *If $R[\tau_1, \ldots, \tau_n] \hookrightarrow \widehat{R}$ satisfies PDE and each height-one prime of R is the radical of a principal ideal, then $R[\tau_1, \ldots, \tau_n] \hookrightarrow \widehat{R}$ is weakly flat.*

In view of Remark 12.6.b, these assertions also hold with $R[\tau_1, \ldots, \tau_n]$ replaced by its localization $R[\tau_1, \ldots, \tau_n]_{(\mathbf{m}, \tau_1, \ldots, \tau_n)}$.

In order to demonstrate idealwise independence we develop in the next two sections the concepts of primary independence and residual algebraic independence. Primary independence implies idealwise independence. If we assume that every height-one prime ideal of the base ring R is the radical of a principal ideal, then residual algebraic independence implies idealwise independence.

19.2. Primarily independent elements

In this section we introduce primary independence, a concept we show in Proposition 19.15 implies idealwise independence. We construct in Theorem 19.20 infinitely many primarily independent elements over any countable excellent normal local domain of dimension at least two.

DEFINITION 19.12. Let $(R, \mathbf{m})$ be an excellent normal local domain. We say that elements $\tau_1, \ldots, \tau_n \in \widehat{\mathbf{m}}$ that are algebraically independent over R are *primarily independent over* R, if for every prime ideal P of $S = R[\tau_1, \ldots, \tau_n]_{(\mathbf{m}, \tau_1, \ldots, \tau_n)}$ such that $\dim(S/P) \leq n$, the ideal $P\widehat{R}$ is $\widehat{\mathbf{m}}$-primary. A countably infinite sequence $\{\tau_i\}_{i=1}^{\infty}$ of elements of $\widehat{\mathbf{m}}$ is *primarily independent over* R if $\tau_1, \ldots, \tau_n$ are primarily independent over R for each n.

REMARKS 19.13. (1) Referring to the diagram in Setting 19.1, primary independence of $\tau_1, \ldots, \tau_n$ as defined in (19.12) is equivalent to the statement that for every prime ideal P of S with $\dim(S/P) \leq n$, the ideal $\lambda^{-1}(P\widehat{R}) = P\widehat{S_n} + \ker(\lambda)$ is primary for the maximal ideal of $\widehat{S_n}$.

(2) A subset of a primarily independent set is again primarily independent. For example, if $\tau_1, \ldots, \tau_n$ are primarily independent over R, to see that $\tau_1, \ldots, \tau_{n-1}$ are primarily independent, let P be a prime ideal of R_{n-1} with $\dim(R_{n-1}/P) \leq n-1$. Then PR_n is a prime ideal of R_n with $\dim(R_n/PR_n) \leq n$, and so $P\widehat{R}$ is primary for the maximal ideal of $\widehat{R}$.

LEMMA 19.14. *Let $(R, \mathbf{m})$ be an excellent normal local domain of dimension at least 2, let n be a positive integer, and let $S = R_n = R[\tau_1, \ldots, \tau_n]_{(\mathbf{m}, \tau_1, \ldots, \tau_n)}$, where $\tau_1, \ldots, \tau_n$ are primarily independent over R. Let P be a prime ideal of S such that $\dim(S/P) \geq n+1$. Then*

(1) *the ideal $P\widehat{R}$ is not $\widehat{\mathbf{m}}$-primary, and*

(2) *$P\widehat{R} \cap S = P$.*

PROOF. For item 1, if $\dim(S/P) \geq n+1$ and if $P\widehat{R}$ is primary for $\widehat{\mathbf{m}}$, then the diagram in Setting 19.1 shows that $\lambda^{-1}(P\widehat{R}) = P\widehat{S}_n + \ker(\lambda)$ is primary for the maximal ideal of $\widehat{S}$. Hence the maximal ideal of $\widehat{S}/P\widehat{S}$ is the radical of an

n-generated ideal. We also have $\widehat{S_n}/P\widehat{S_n} \cong \widehat{(S/P)}$ is the completion of S/P, and $\dim(S/P) \geq n+1$ implies that $\dim(\widehat{S/P}) \geq n+1$. This is a contradiction by Theorem 2.6.

For item 2, if $\dim(S/P) = n+1$, and $P \subsetneq (P\widehat{R} \cap S)$, then $\dim(\frac{S}{(P\widehat{R}\cap S)}) \leq n$. Thus $P\widehat{R} = (P\widehat{R} \cap S)\widehat{R}$ is primary for $\widehat{\mathbf{m}}$, a contradiction to item 1. Therefore $P\widehat{R} \cap S = P$ for each P such that $\dim(S/P) = n+1$.

Assume that $\dim(S/P) > n+1$ and let

$$\mathcal{A} \ := \ \{Q \in \operatorname{Spec} S \mid P \subset Q \text{ and } \dim(S/Q) \ = \ n+1\}.$$

Proposition 3.20 implies that $P = \bigcap_{Q\in\mathcal{A}} Q$. Since for each prime ideal $Q \in \mathcal{A}$, we have $Q\widehat{R} \cap S = Q$, it follows that

$$P \subseteq P\widehat{R} \cap S = (\bigcap_{Q\in\mathcal{A}} Q)\widehat{R} \cap S \subseteq \bigcap_{Q\in\mathcal{A}} Q = P.$$

□

Proposition 19.15. *Let $(R, \mathbf{m})$ be an excellent normal local domain of dimension at least 2.*

(1) *Let n be a positive integer, and let $S = R[\tau_1, \ldots, \tau_n]_{(\mathbf{m},\tau_1,\ldots,\tau_n)}$, where $\tau_1, \ldots, \tau_n$ are primarily independent over R. Then $S = L \cap \widehat{R}$, where L is the field of fractions of S. Thus $\tau_1, \ldots, \tau_n$ are idealwise independent elements of $\widehat{R}$ over R.*
(2) *If $\{\tau_i\}_{i=1}^{\infty}$ is a countably infinite sequence of primarily independent elements of $\widehat{\mathbf{m}}$ over R, then $\{\tau_i\}_{i=1}^{\infty}$ are idealwise independent over R.*

Proof. Item 2 is a consequence of item 1. Thus it suffices to prove item 1. Let P be a height-one prime of S. Since S is catenary and $\dim R \geq 2$, $\dim(S/P) \geq n+1$. Lemma 19.14.2 implies that $P\widehat{R} \cap S = P$. Therefore $\widehat{R}$ is weakly flat over S. Hence by Theorem 19.11.1, we have $S = L \cap \widehat{R}$. □

Proposition 19.16. *Let $(R, \mathbf{m})$ and $\tau_1, \ldots, \tau_n \in \widehat{\mathbf{m}}$ be as in Setting 19.1. Let $R_n = R[\tau_1, \ldots, \tau_n]_{(\mathbf{m},\tau_1,\ldots,\tau_n)} \cong S_n = R[t_1, \ldots, t_n]_{(\mathbf{m},t_1,\ldots,t_n)}$, where $t_1, \ldots, t_n$ are indeterminates over R. Then the following are equivalent:*

(1) *For each prime ideal P of S_n such that $\dim(S_n/P) \geq n$ and each prime ideal $\widehat{P}$ of $\widehat{S_n}$ minimal over $P\widehat{S_n}$, the images of $t_1 - \tau_1, \ldots, t_n - \tau_n$ in $\widehat{S_n}/\widehat{P}$ generate an ideal of height n in $\widehat{S_n}/\widehat{P}$.*
(2) *For each prime ideal P of S_n with $\dim(S_n/P) \geq n$ and each nonnegative integer $i \leq n$, the element $t_i - \tau_i$ is outside every prime ideal $\widehat{Q}$ of $\widehat{S}_n$ minimal over $(P, t_1 - \tau_1, \ldots, t_{i-1} - \tau_{i-1})\widehat{S}_n$.*
(3) *For each prime ideal P of S_n such that $\dim(S_n/P) = n$, the images of $t_1 - \tau_1, \ldots, t_n - \tau_n$ in $\widehat{S_n}/P\widehat{S_n}$ generate an ideal primary for the maximal ideal of $\widehat{S}_n/P\widehat{S}_n$.*
(4) *The elements $\tau_1, \ldots, \tau_n$ are primarily independent over R*

Proof. It is clear that item 1 and item 2 are equivalent, that item 1 and item 2 imply item 3 and that item 3 is equivalent to item 4. It remains to observe that item 3 implies item 1. For this, let P be a prime ideal of S_n such that $\dim(S_n/P) = n + h$, where $h \geq 0$. There exist $s_1, \ldots, s_h \in S_n$ so that if $I = (P, s_1, \ldots, s_h)S_n$, then for each minimal prime Q of I we have $\dim(S_n/Q) = n$. Item 3 implies that the images of $t_1 - \tau_1, \ldots, t_n - \tau_n$ in $\widehat{S_n}/Q\widehat{S_n}$ generate an ideal primary for the maximal ideal of $\widehat{S}_n/Q\widehat{S_n}$. It follows that the images of $t_1-\tau_1, \ldots, t_n-\tau_n$ in $\widehat{S_n}/I\widehat{S_n}$ generate an ideal primary for the maximal ideal of $\widehat{S}_n/I\widehat{S_n}$, and therefore that the images of $s_1, \ldots, s_h, t_1 - \tau_1, \ldots, t_n - \tau_n$ in $\widehat{S_n}/P\widehat{S_n}$ are a system of parameters for the $(n+h)$-dimensional local ring $\widehat{S_n}/P\widehat{S_n}$. Let $\widehat{P}$ be a minimal prime of $P\widehat{S_n}$. Then $\dim(\widehat{S_n}/\widehat{P}) = n + h$, and the images of $s_1, \ldots, s_h, t_1 - \tau_1, \ldots, t_n - \tau_n$ in the complete local domain $\widehat{S_n}/\widehat{P}$ are a system of parameters. It follows that the images of $t_1 - \tau_1, \ldots, t_n - \tau_n$ in $\widehat{S_n}/\widehat{P}$ generate an ideal of height n in $\widehat{S_n}/\widehat{P}$. Therefore item 1 holds. $\square$

Corollary 19.17. *With the notation of Setting 19.1 and Proposition 19.16, assume that $\tau_1, \ldots, \tau_n$ are primarily independent over R.*

(1) *Let I be an ideal of S_n such that $\dim(S/I) = n$. It follows that the ideal $(I, t_1 - \tau_1, \ldots, t_n - \tau_n)\widehat{S_n}$ is primary to the maximal ideal of S_n.*
(2) *Let $P \in \operatorname{Spec}(S_n)$ be a prime ideal with $\dim(S_n/P) > n$. Then the ideal $\widehat{W} = (P, t_1 - \tau_1, \ldots, t_n - \tau_n)\widehat{S}_n$ has $\operatorname{ht}(\widehat{W}) = \operatorname{ht}(P) + n$ and $\widehat{W} \cap S_n = P$.*

Proof. Part 1 is an immediate corollary of Proposition 19.16.3, and it follows from (19.16.1) that $\operatorname{ht}(\widehat{W}) = \operatorname{ht}(P) + n$. Let λ_n be the restriction to S_n of the canonical homomorphism $\lambda : \widehat{S_n} \to \widehat{R}$ from (19.1) so that $\lambda_n : S_n \overset{\cong}{\to} \to R_n$. Then $\dim(R_n/\lambda_n(P)) > n$, and so by (19.14.2), $\lambda_n(P)\widehat{R} \cap R_n = \lambda_n(P)$. Now

$$\widehat{W} \cap S_n = \lambda^{-1}(\lambda_n(P)\widehat{R}) \cap \lambda_n^{-1}(R_n) = \lambda_n^{-1}(\lambda_n(P)\widehat{R} \cap R_n) = \lambda_n^{-1}(\lambda_n(P)) = P. \qquad \square$$

To establish the existence of primarily independent elements, we use the following prime avoidance lemma over a complete local ring. (This is similar to [**19**, Lemma 3],[**143**, Lemma 10], [**131**] and [**84**, Lemma 14.2].) We also use this result in two constructions given in Section 19.3.

Lemma 19.18. *Let $(T, \mathbf{n})$ be a complete local ring of dimension at least 2, and let $t \in \mathbf{n} - \mathbf{n}^2$. Assume that I is an ideal of T containing t, and that $\mathcal{U}$ is a countable set of prime ideals of T each of which fails to contain I. Then there exists an element $a \in I \cap \mathbf{n}^2$ such that $t - a \notin \bigcup\{Q : Q \in \mathcal{U}\}$.*

Proof. Let $\{P_i\}_{i=1}^{\infty}$ be an enumeration of the prime ideals of $\mathcal{U}$. We may assume that there are no containment relations among the primes of $\mathcal{U}$. Choose $f_1 \in \mathbf{n}^2 \cap I$ so that $t - f_1 \notin P_1$. Then choose $f_2 \in P_1 \cap \mathbf{n}^3 \cap I$ so that $t - f_1 - f_2 \notin P_2$. Note that $f_2 \in P_1$ implies $t - f_1 - f_2 \notin P_1$. Successively, by induction, choose

$$f_n \in P_1 \cap P_2 \cap \cdots \cap P_{n-1} \cap \mathbf{n}^{n+1} \cap I$$

so that $t - f_1 - \ldots - f_n \notin \bigcup_{i=1}^{n} P_i$ for each positive integer n. Then we have a Cauchy sequence $\{f_1 + \cdots + f_n\}_{n=1}^{\infty}$ in T that converges to an element $a \in \mathbf{n}^2$. Now

$$t - a = (t - f_1 - \ldots - f_n) + (f_{n+1} + \ldots),$$

where $(t - f_1 - \ldots - f_n) \notin P_n, (f_{n+1} + \ldots) \in P_n$. Therefore $t - a \notin P_n$, for all n, and so $t - a \in I$. $\square$

Remark 19.19. Let $A \hookrightarrow B$ be an extension of Krull domains. If α is a nonzero nonunit of B such that $\alpha \notin Q$ for each height-one prime Q of B such that $Q \cap A \neq (0)$, then $\alpha B \cap A = (0)$. In particular, such an element α is algebraically independent over A.

THEOREM 19.20. *Let $(R, \mathbf{m})$ be a countable excellent normal local domain of dimension at least 2, and let $(\widehat{R}, \widehat{\mathbf{m}})$ be the completion of R. Then:*

(1) *There exists $\tau \in \widehat{\mathbf{m}}$ that is primarily independent over R.*
(2) *If $\tau_1, \ldots, \tau_{n-1} \in \widehat{\mathbf{m}}$ are primarily independent over R, then there exists $\tau_n \in \widehat{\mathbf{m}}$ such that $\tau_1, \ldots, \tau_{n-1}, \tau_n$ are primarily independent over R.*
(3) *There exists an infinite sequence $\tau_1, \ldots, \tau_n, \ldots \in \widehat{\mathbf{m}}$ of elements that are primarily independent over R.*

PROOF. Item 2 implies item 1 and item 3. To prove item 2, let $t_1, \ldots, t_n$ be indeterminates over R, and let the notation be as in Setting 19.1. Thus we have $S_{n-1} \cong R_{n-1}$, under the R-algebra isomorphism taking $t_i \to \tau_i$. Let $\widehat{\mathbf{n}}$ denote the maximal ideal of $\widehat{S_n}$. We show the existence of $a \in \widehat{\mathbf{n}}^2$ such that, if λ denotes the $\widehat{R}$-algebra surjection $\widehat{S_n} \to \widehat{R}$ with kernel $(t_1 - \tau_1, \ldots, t_{n-1} - \tau_{n-1}, t_n - a)\widehat{S_n}$, then $\tau_1, \ldots, \tau_{n-1}$ together with the image τ_n of t_n under the map λ are primarily independent over R.

Since S_n is countable and Noetherian we can enumerate as $\{P_j\}_{j=1}^{\infty}$ the prime ideals of S_n such that $\dim(S_n/P_j) \geq n$. Let $\widehat{I} = (t_1 - \tau_1, \ldots, t_{n-1} - \tau_{n-1})\widehat{S_{n-1}}$, and let $\mathcal{U}$ be the set of all prime ideals of $\widehat{S_n} = \widehat{R}[[t_1, \ldots, t_n]]$ minimal over ideals of the form $(P_j, \widehat{I})\widehat{S_n}$ for some P_j; then $\mathcal{U}$ is countable and $\widehat{\mathbf{n}} \notin \mathcal{U}$ since $(P_j, \widehat{I})\widehat{S_n}$ is generated by $n-1$ elements over $P_j\widehat{S_n}$ and $\dim(\widehat{S_n}/P_j\widehat{S_n}) \geq n$. By Lemma 19.18 with the ideal I of that lemma taken to be $\widehat{\mathbf{n}}$, there exists an element $a \in \widehat{\mathbf{n}}^2$ so that $t_n - a$ is not in $\widehat{Q}$, for every prime ideal $\widehat{Q} \in \mathcal{U}$. Let $\tau_n \in \widehat{R}$ denote the image of t_n under the $\widehat{R}$-algebra surjection $\lambda : \widehat{S_n} \to \widehat{R}$ with kernel $(\widehat{I}, t_n - a)\widehat{S_n}$. The kernel of λ is also generated by $(\widehat{I}, t_n - \tau_n)\widehat{S}_n$. Therefore the setting will be as in the diagram of Setting 19.1 after we establish Claim 19.21.

CLAIM 19.21. $(\widehat{I}, t_n - \tau_n)\widehat{S_n} \cap S_n = (0)$.

PROOF. (of Claim 19.21) Since $\tau_1, \ldots, \tau_{n-1}$ are algebraically independent over R, we have $\widehat{I} \cap S_{n-1} = (0)$. Let $R'_n = R_{n-1}[t_n]_{(\max(R_{n-1}), t_n))}$. Consider the diagram:

$$\begin{array}{ccc}
S_n = S_{n-1}[t_n]_{(\max(S_{n-1}), t_n)} & \xrightarrow{\subset} & \widehat{S_n} = \widehat{S_{n-1}}[[t_n]] \\
\cong \Big\downarrow & & \lambda_1 \Big\downarrow \\
R'_n = R_{n-1}[t_n]_{(\max(R_{n-1}), t_n)} & \xrightarrow{\subset} & \widehat{R}[[t_n]] \cong (\widehat{S_{n-1}}/\widehat{I})[[t_n]],
\end{array}$$

where $\lambda_1 : \widehat{S_n} \to \widehat{S_n}/(\widehat{I}\widehat{S_n}) \cong \widehat{R}$ is the canonical projection.

For $\widehat{Q}$ a prime ideal of $\widehat{S_n}$, we have $\widehat{Q} \in \mathcal{U} \iff \lambda_1(\widehat{Q}) = \widehat{P}$, where $\widehat{P}$ is a prime ideal of $\widehat{R}[[t_n]] \cong (\widehat{S_{n-1}}/\widehat{I})[[t_n]]$ minimal over $\lambda_1(P_j)\widehat{R}[[t_n]]$ for some prime ideal P_j of S_n such that $\dim(S_n/P_j) \leq n$. Since $t_n - a$ is outside every $\widehat{Q} \in \mathcal{U}$, $t_n - \lambda_1(a) = \lambda_1(t_n - a)$ is outside every prime ideal $\widehat{P}$ of $\widehat{R}[[t_n]]$, such that $\widehat{P}$ is minimal over $\lambda_1(P_j)\widehat{R}[[t_n]]$. Since S_n is catenary and $\dim(S_n) = n + \dim(R)$, a prime ideal P_j of S_n is such that $\dim(S_n/P_j) \geq n \iff \mathrm{ht}(P_j) \leq \dim(R)$. Suppose $\widehat{P}$ is a height-one prime ideal of $\widehat{R}[[t_n]]$ such that $\widehat{P} \cap R'_n = P \neq (0)$. Then $\widehat{P}$ is a minimal prime ideal of $P\widehat{R}[[t_n]]$. But also $P = \lambda_1(Q)$, where Q is a height-one prime of S_n and $\dim(S_n/Q) = n + \dim(R) - 1 \geq n$. Therefore $Q \in \{P_j\}_{j=1}^{\infty}$. Hence by choice of a, we have $t_n - \lambda_1(a) \notin \widehat{P}$. By Remark 19.19, $(t_n - \lambda_1(a))\widehat{R}[[t_n]]) \cap R'_n = (0)$. Hence $(\widehat{I}, t_n - \tau_n)\widehat{S_n} \cap S_n = (0)$. $\square$

CLAIM 19.22. *Let P be a prime ideal of S_n such that* $\dim(S_n/P) = n$. *Then the ideal* $(P, \widehat{I}, t_n - \tau_n)\widehat{S_n}$ *is* $\widehat{\mathbf{n}}$*-primary.*

PROOF. (of Claim 19.22) Let $Q = P \cap S_{n-1}$. Either $QS_n = P$, or $QS_n \subsetneq P$. If $QS_n = P$, then $\dim(S_{n-1}/Q) = n-1$ and the primary independence of $\tau_1, \dots, \tau_{n-1}$ implies that $(Q, \widehat{I})\widehat{S_{n-1}}$ is primary for the maximal ideal of $\widehat{S_{n-1}}$. Therefore $(Q, \widehat{I}, t_n - \tau_n)\widehat{S_n} = (P, \widehat{I}, t_n - \tau_n)\widehat{S_n}$ is $\widehat{\mathbf{n}}$-primary in this case. On the other hand, if $QS_n \subsetneq P$, then $\dim(S_{n-1}/Q) = n$. Let $\widehat{Q}'$ be a minimal prime of $(Q, \widehat{I})\widehat{S_{n-1}}$. By Proposition 19.16, $\dim(\widehat{S_{n-1}}/\widehat{Q}') = 1$, and hence $\dim(\widehat{S_n}/\widehat{Q}'\widehat{S_n}) = 2$. The primary independence of $\tau_1, \dots \tau_{n-1}$ implies that $\widehat{Q}' \cap S_{n-1} = Q$. Therefore $\widehat{Q}'\widehat{S_{n-1}}[[t_n]] \cap S_n = QS_n \subsetneq P$, so P is not contained in $\widehat{Q}'\widehat{S_n}$. Therefore $\dim(\widehat{S_n}/(P, \widehat{I})\widehat{S_n})) = 1$ and our choice of a implies that $(P, \widehat{I}, t_n - \tau_n)\widehat{S_n}$ is $\widehat{\mathbf{n}}$-primary. $\square$

This completes the proof of Theorem 19.20. $\square$

COROLLARY 19.23. *Let $(R, \mathbf{m})$ be a countable excellent normal local domain of dimension at least 2, and let K denote the field of fractions of R. Then there exist $\tau_1, \dots, \tau_n, \dots \in \widehat{\mathbf{m}}$ such that $A = K(\tau_1, \tau_2, \dots) \cap \widehat{R}$ is an infinite-dimensional non-Noetherian local domain. In particular, for k a countable field, the localized polynomial ring $R = k[x, y]_{(x,y)}$ has such extensions inside $\widehat{R} = k[[x, y]]$.*

PROOF. By Theorem 19.20.3, there exist $\tau_1, \dots, \tau_n, \dots \in \widehat{\mathbf{m}}$ that are primarily independent over R. It follows that $A = K(\tau_1, \tau_2, \dots) \cap \widehat{R}$ is an infinite-dimensional local domain. In particular, A is not Noetherian. $\square$

19.3. Residually algebraically independent elements

We introduce in this section a third concept, that of residual algebraic independence. Residual algebraic independence is weaker than primary independence. In Theorem 19.28 we show that over every countable normal excellent local domain $(R, \mathbf{m})$ of dimension at least three there exists an element residually algebraically independent over R that is not primarily independent over R. In Theorems 19.33 and 19.35 we show the existence of idealwise independent elements that fail to be residually algebraically independent.

DEFINITION 19.24. Let $(\widehat{R}, \widehat{\mathbf{m}})$ be a complete normal Noetherian local domain and let A be a Krull subdomain of $\widehat{R}$ such that $A \hookrightarrow \widehat{R}$ satisfies PDE.

(1) An element $\tau \in \widehat{\mathbf{m}}$ is *residually algebraically independent with respect to* $\widehat{R}$ over A if τ is algebraically independent over A and for each height-one prime $\widehat{P}$ of $\widehat{R}$ such that $\widehat{P} \cap A \neq (0)$, the image of τ in $\widehat{R}/\widehat{P}$ is algebraically independent over the integral domain $A/(\widehat{P} \cap A)$.
(2) Elements $\tau_1, \dots \tau_n \in \widehat{\mathbf{m}}$ are said to be *residually algebraically independent* over A if for each $0 \leq i < n$, τ_{i+1} is residually algebraically independent over $A[\tau_1, \dots, \tau_i]$.
(3) An infinite sequence $\{\tau_i\}_{i=1}^{\infty}$ of elements of $\widehat{\mathbf{m}}$ is *residually algebraically independent* over A, if $\tau_1, \dots \tau_n$ are residually algebraically independent over A for each positive integer n.

Proposition 19.25 relates residual algebraic independence for τ over A to the PDE property of Definition 19.5 for $A[\tau] \hookrightarrow \widehat{R}$. By Proposition 12.12, for an extension of Krull domains, the PDE property is equivalent to the LF_1 property of Definition 12.1.3.

Proposition 19.25. *Let $(R, \mathbf{m})$ and $\tau \in \widehat{\mathbf{m}}$ be as in Setting 19.1. Let A be a Krull subdomain of $\widehat{R}$ such that $A \hookrightarrow \widehat{R}$ satisfies PDE. Then τ is residually algebraically independent with respect to $\widehat{R}$ over A $\iff$ $A[\tau] \hookrightarrow \widehat{R}$ satisfies PDE.*

Proof. Assume $A[\tau] \hookrightarrow \widehat{R}$ does not satisfy PDE. Then there exists a prime ideal $\widehat{P}$ of $\widehat{R}$ of height one such that $\mathrm{ht}(\widehat{P} \cap A[\tau]) \geq 2$. Now $\mathrm{ht}(\widehat{P} \cap A) = 1$, since PDE holds for $A \hookrightarrow \widehat{R}$. Thus, with $\mathbf{p} = \widehat{P} \cap A$, we have $\mathbf{p}A[\tau] \subsetneq \widehat{P} \cap A[\tau]$; that is, there exists $f(\tau) \in (\widehat{P} \cap A[\tau]) \setminus \mathbf{p}A[\tau]$, or equivalently there is a nonzero polynomial $\bar{f}(x) \in (A/\mathbf{p})[x]$ so that $\bar{f}(\bar{\tau}) = \bar{0}$ in $A[\tau]/(\widehat{P} \cap A[\tau])$, where $\bar{\tau}$ denotes the image of τ in $\widehat{R}/\widehat{P}$. This means that $\bar{\tau}$ is algebraic over $A/(\widehat{P} \cap A)$. Hence τ is not residually algebraically independent with respect to $\widehat{R}$ over A.

For the converse, assume that $A[\tau] \hookrightarrow \widehat{R}$ satisfies PDE and let $\widehat{P}$ be a height-one prime of $\widehat{R}$ such that $\widehat{P} \cap A = \mathbf{p} \neq 0$. Since $A[\tau] \hookrightarrow \widehat{R}$ satisfies PDE, $\widehat{P} \cap A[\tau] = \mathbf{p}A[\tau]$ and $A[\tau]/(\mathbf{p}A[\tau])$ canonically embeds in $\widehat{R}/\widehat{P}$. Hence the image of τ in $A[\tau]/\mathbf{p}A[\tau]$ is algebraically independent over $A/\mathbf{p}$. It follows that τ is residually algebraically independent with respect to $\widehat{R}$ over A. $\square$

Theorem 19.26. *Let $(R, \mathbf{m})$ be an excellent normal local domain with completion $(\widehat{R}, \widehat{\mathbf{m}})$ and let $\tau_1, \dots, \tau_n \in \widehat{\mathbf{m}}$ be algebraically independent over R. The following statements are equivalent:*

1. *The elements $\tau_1, \dots, \tau_n$ are residually algebraically independent with respect to $\widehat{R}$ over R.*
2. *For each integer i with $1 \leq i \leq n$, if $\widehat{P}$ is a height-one prime ideal of $\widehat{R}$ such that $\widehat{P} \cap R[\tau_1, \dots, \tau_{i-1}] \neq 0$, then $\mathrm{ht}(\widehat{P} \cap R[\tau_1, \dots, \tau_i]) = 1$.*
3. *$R[\tau_1, \dots, \tau_n] \hookrightarrow \widehat{R}$ satisfies PDE.*

If each height-one prime of R is the radical of a principal ideal, then these equivalent conditions imply the map $R[\tau_1, \dots, \tau_n] \hookrightarrow \widehat{R}$ is weakly flat.

Proof. The equivalence of the three items follows from Proposition 19.25. The last sentence follows from Theorem 19.11) $\square$

Theorem 19.27. *Let $(R, \mathbf{m})$ and $\{\tau_i\}_{i=1}^m \subseteq \widehat{\mathbf{m}}$ be as in Setting 19.1, where $\dim(R) \geq 2$ and m is either a positive integer or $m = \infty$.*

1. *If $\{\tau_i\}_{i=1}^m$ is primarily independent over R, then $\{\tau_i\}_{i=1}^m$ is residually algebraically independent over R.*
2. *If $\dim(R) = 2$, then $\{\tau_i\}_{i=1}^m$ is primarily independent over R if and only if it is residually algebraically independent over R.*
3. *If each height-one prime of R is the radical of a principal ideal and $\{\tau_i\}_{i=1}^m$ is residually algebraically independent over R, then $\{\tau_i\}_{i=1}^m$ is idealwise independent over R.*

Proof. To prove item 1, it suffices by Theorem 19.26 to show that for each positive integer $n \leq m$, if $\tau_1, \dots, \tau_n$ are primarily independent over R, then the extension $R[\tau_1,\dots,\tau_n] \hookrightarrow \widehat{R}$ satisfies PDE. Let $S = R[\tau_1,\dots,\tau_n]_{(\mathbf{m},\tau_1,\dots,\tau_n)}$ and let the notation be as in the diagram of Setting 19.1.

Let $\widehat{P}$ be a height-one prime ideal of $\widehat{R}$ with $\mathbf{p} := \widehat{P} \cap R \neq (0)$. Consider the ideal $\widehat{W} := (\mathbf{p}, t_1 - \tau_1, \dots, t_n - \tau_n)\widehat{S}_n$. Using the diagram of Setting 19.1, we see that $\lambda(\widehat{W}) = \mathbf{p}\widehat{R} \subseteq \widehat{P}$. By Corollary 19.17.2, $\operatorname{ht}(\widehat{W}) = \operatorname{ht}(\mathbf{p}) + n$. However, $\widehat{W} \subseteq (\widehat{P}, t_1 - \tau_1, \dots, t_n - \tau_n) = \lambda^{-1}(\widehat{P})$ and thus

$$1 + n \le \operatorname{ht}(\mathbf{p}) + n = \operatorname{ht}(\widehat{W}) \le \operatorname{ht}(\lambda^{-1}(\widehat{P})) \le \operatorname{ht}(\widehat{P}) + n = 1 + n.$$

Therefore $\operatorname{ht}(\mathbf{p}) = 1$.

In view of item 1, to prove item 2, we assume that $\dim R = 2$ and $n \le m$ is a positive integer such that $\tau_1,\dots,\tau_n$ are residually algebraically independent over R. Let $S = R[\tau_1,\dots,\tau_n]_{(\mathbf{m},\tau_1,\dots,\tau_n)}$. By Theorem 19.26, $S \hookrightarrow \widehat{R}$ satisfies PDE. Let P be a prime ideal of S such that $\dim(S/P) \le n$. Since $\dim(S) = n+2$ and S is catenary, it follows that $\operatorname{ht}(P) \ge 2$. To show $\tau_1,\dots,\tau_n$ are primarily independent over R, it suffices to show that $P\widehat{R}$ is primary for the maximal ideal of $\widehat{R}$. Since $\dim(\widehat{R}) = 2$, this is equivalent to showing P is not contained in a height-one prime of $\widehat{R}$, and this last statement holds since $S \hookrightarrow \widehat{R}$ satisfies PDE.

The proof of item 3 follows from Theorems 19.26 and 19.11. $\square$

Theorem 19.28. *Let $(R, \mathbf{m})$ be a countable excellent normal local domain of dimension d and let $(\widehat{R}, \widehat{\mathbf{m}})$ be the completion of R. If $d \ge 3$, then there exists an element $\tau \in \widehat{\mathbf{m}}$ that is residually algebraically independent over R, but not primarily independent over R.*

Proof. We use techniques similar to those in the proof of Theorem 19.20. Let t be an indeterminate over R and set $S_1 = R[t]_{(\mathbf{m},t)}$. Thus $\widehat{S_1} = \widehat{R}[[t]]$. Let $\widehat{\mathbf{n}_1}$ denote the maximal ideal of $\widehat{S_1}$. We have the top line

$$S_1 \;=\; R[t]_{(\mathbf{m},t)} \;\hookrightarrow\; \widehat{S_1} = \widehat{R}[[t]]$$

of a diagram as in Setting 19.1 for $n = 1$. We seek an appropriate mapping

$$\lambda : \widehat{S_1} \;\longrightarrow\; \widehat{R}.$$

Then we use the element $\tau := \lambda(t) \in \widehat{\mathbf{m}}$ to complete the diagram of Setting 19.1. Let $\widehat{Q_0}$ be a prime ideal of $\widehat{S_1}$ of height d that contains t and is such that $Q_0 := \widehat{Q_0} \cap S_1$ also has height d. Let

$$\mathcal{U} \;=\; \{\widehat{Q} \;\in\; \operatorname{Spec}\widehat{S_1} \mid \operatorname{ht}\widehat{Q} \;\le\; d,\ \operatorname{ht}(\widehat{Q} \cap S_1) = \operatorname{ht}\widehat{Q} \text{ and } \;\widehat{Q} \neq \widehat{Q_0}\;\}.$$

Since S_1 is countable, the set $\mathcal{U}$ is countable. We apply Lemma 19.18 with $T = \widehat{S_1}, \mathbf{n} = \widehat{\mathbf{n}_1}$ and $I = \widehat{Q_0}$, to obtain an element $a \in \widehat{Q_0} \;\cap\; \widehat{\mathbf{n}_1}^2$ so that $t - a \in \widehat{Q}_0$ but $t - a$ is not in any prime ideal in $\mathcal{U}$. Since $a \in \widehat{\mathbf{n}_1}^2$, we have $\widehat{R}[[t]] = \widehat{R}[[t-a]]$. Define λ to be the natural surjection

$$\lambda : \widehat{S_1} \;\longrightarrow\; \widehat{S_1}/(t-a)\widehat{S_1} \;=\; \widehat{R}.$$

We have $\tau = \lambda(t) = \lambda(a)$.

Since λ restricted to S_1 is an isomorphism from S_1 onto $S := R[\tau]_{(\mathbf{m},\tau)}$, the prime ideal $\lambda(Q_0)$ in $S = R[\tau]_{(\mathbf{m},\tau)}$ is of height d. Thus $\dim(S/\lambda(Q_0)) = 1$. Since the diagram of Setting 19.1 is commutative, we have $\lambda(Q_0)\widehat{R} \subseteq \lambda(\widehat{Q_0})$. Since $(t-\tau)\widehat{S_1} = (t-a)\widehat{S_1} \subseteq \widehat{Q_0}$, the prime ideal $\lambda(\widehat{Q_0})$ is of height $d-1$. Therefore $\lambda(Q_0)\widehat{R}$ is not $\widehat{\mathbf{m}}$-primary. Hence τ is not primarily independent.

To prove that τ is residually algebraically independent over R, by Theorem 19.26, it suffices to show the extension

$$S \;=\; R[\tau]_{(\mathbf{m},\tau)} \;\hookrightarrow\; \widehat{R}$$

satisfies PDE.

If $\widehat{P}$ is a height-one prime ideal of $\widehat{R}$ with $\widehat{P}\cap R \neq 0$, then the height of $\widehat{P}\cap R$ is 1, and so the height of $\widehat{P}\cap S$ is at most 2. Let $\widehat{Q_2} := \lambda^{-1}(\widehat{P})$ in $\widehat{S_1}$. Then $\operatorname{ht}(\widehat{Q_2}) = 2$—since it is generated by the inverse images of the generators of $\widehat{P}$ and $\ker(\lambda) = (t-a)\widehat{S_1}$.

Suppose that the height of $\widehat{P}\cap S = 2$. Then under the R-isomorphism of S_1 to S taking t to τ, $\widehat{P}\cap S$ corresponds to a height-two prime P of S_1. Since $\widehat{S_1}$ is flat over S_1, the height of $\widehat{Q_2}\cap S_1$ is at most two. We have $P \subseteq \widehat{Q_2}\cap S_1$. Hence $P = \widehat{Q_2}\cap S_1$. The following diagram illustrates this situation:

$$\begin{array}{ccccc}
P = \widehat{Q}_2 \cap S_1(\text{ht } 2) & \xrightarrow{\subseteq} & \widehat{Q}_2 = \lambda^{-1}(\widehat{P}) & \xleftarrow{\supseteq} & (\widehat{P},(t-a))\widehat{S_1} \text{ (ht 2)} \\
\cong\big\downarrow & & \lambda\big\downarrow & & \\
\widehat{P}\cap S \text{ (ht 2)} & \xrightarrow{\subseteq} & \widehat{P} & & (\text{ht 1 in } \widehat{R}).
\end{array}$$

Since $\operatorname{ht}\widehat{Q_2} = 2 < d = \operatorname{ht}\widehat{Q_0}$, we have $\widehat{Q_2} \in \mathcal{U}$. However, $t-a \in \widehat{Q_2}$, a contradiction.

We conclude that $\operatorname{ht}(\widehat{P}\cap S) = 1$. Thus τ is residually algebraically independent over R. $\square$

Example 19.29. The following construction, similar to that in Theorem 19.28, shows that condition 2 in Definition 19.24 is stronger than

($2'$) For each height-one prime ideal $\widehat{P}$ of $\widehat{R}$ with $\widehat{P}\cap R \neq 0$, the images of $\tau_1,\dots,\tau_n$ in $\widehat{R}/\widehat{P}$ are algebraically independent over $R/(\widehat{P}\cap R)$.

Construction 19.30. Let $(R,\mathbf{m})$ be a countable excellent local unique factorization domain (UFD) of dimension two and let $(\widehat{R},\widehat{\mathbf{m}})$ be the completion of R, for example $R = \mathbb{Q}[x,y]_{(x,y)}$ and $\widehat{R} = \mathbb{Q}[[x,y]]$. As in Theorem 19.20, construct $\tau_1 \in \widehat{\mathbf{m}}$ primarily independent over R (or equivalently residually algebraically independent in this context). Let t_1, t_2 be variables over R and let $S_2 := R[t_1,t_2]_{(\mathbf{m},t_1,t_2)}$. Consider the ideal $I := (t_1, t_2, t_1-\tau_1)\widehat{S_2}$ and define

$$\begin{aligned}\mathcal{U} = \{\widehat{Q} \in \operatorname{Spec}(\widehat{S_2}) \mid I \not\subseteq \widehat{Q}\ ,\ \operatorname{ht}(\widehat{Q}) = \operatorname{ht}(\widehat{Q}\cap S_2) \text{ and } \ \widehat{Q}\cap S_2 = (P, t_1-\tau_1),\\ \text{for some } P \in \operatorname{Spec} S_2 \text{ with } \ \operatorname{ht}(P) \le 2\}.\end{aligned}$$

Thus $P \neq (t_1,t_2)S_2$. Let $\widehat{\mathbf{n}}$ denote the maximal ideal of $\widehat{S_2}$. By Lemma 19.18, there exists $a \in \widehat{\mathbf{n}}^2 \cap I$ so that $t_2 - a \notin \bigcup\{\widehat{Q} : \widehat{Q} \in \mathcal{U}\}$. We have $\widehat{S_2} = \widehat{R}[[t_1,t_2]] = \widehat{R}[[t_1-\tau_1, t_2-a]]$. Let λ denote the canonical R-algebra surjection

$$\lambda : \ \widehat{S_2} \ \longrightarrow\ \widehat{S_2}/(t_1-\tau_1, t_2-a)\widehat{S_2} \ = \widehat{R},$$

and $\tau_2 = \lambda(t_2)$. Notice that $\ker(\lambda)$ has height two.

Claim 19.31. *The element τ_2 is not residually algebraically independent over $R[\tau_1]$; thus τ_1, τ_2 do not satisfy item 2 of Definition 19.24.*

Proof. (of Claim 19.31) Let $\widehat{W}$ be a prime ideal of $\widehat{S_2}$ that is minimal over I. Then $\operatorname{ht}\widehat{W} \le 3$. Also we have $t_2 \in I$ and $a \in I$, and so $t_2 - a \in I \subseteq \widehat{W}$.

Thus $\ker(\lambda) \subseteq \widehat{W}$. Let $\widehat{P} = \lambda(\widehat{W}) \subset \widehat{R}$. Thus $\operatorname{ht} \widehat{P} \leq 1$. In fact $\operatorname{ht} \widehat{P} = 1$, since $0 \neq \tau_1 = \lambda(t_1) \in \widehat{P}$. Since τ_1 is residually algebraically independent over R, the extension $R[\tau_1] \hookrightarrow \widehat{R}$ satisfies PDE by Proposition 19.25. Therefore $\operatorname{ht}(\widehat{P} \cap R[\tau_1]) \leq 1$. But $\tau_1 \in \widehat{P} \cap R[\tau_1]$, and so $\operatorname{ht}(\widehat{P} \cap R[\tau_1]) = 1$ and $\widehat{P} \cap R = (0)$. Also $\tau_2 = \lambda(t_2) \in \widehat{P}$; thus $\tau_1, \tau_2 \in \widehat{P} \cap R[\tau_1, \tau_2]$, and so $\operatorname{ht}(\widehat{P} \cap R[\tau_1, \tau_2]) \geq 2$. Thus the extension $R[\tau_1, \tau_2] \hookrightarrow \widehat{R}$ does not satisfy PDE. By Proposition 19.25, τ_2 is not residually algebraically independent over $R[\tau_1]$. $\square$

CLAIM 19.32. *For each height-one prime ideal $\widehat{P}$ of $\widehat{R}$ with $\widehat{P} \cap R \neq 0$, the images of τ_1 and τ_2 in $\widehat{R}/\widehat{P}$ are algebraically independent over $R/(\widehat{P} \cap R)$. That is, τ_1, τ_2 satisfy item $2'$ above.*

PROOF. (of Claim 19.32) Suppose $\widehat{P}$ is a height-one prime ideal of $\widehat{R}$ with $\widehat{P} \cap R \neq (0)$ and let $\widehat{Q} = \lambda^{-1}(\widehat{P})$. Then $\operatorname{ht}(\widehat{Q}) = 3$ and $\operatorname{ht}(\widehat{P} \cap R) = 1$. Set $R_1 := R[\tau_1]_{(\mathbf{m}, \tau_1)}$ and $R_2 := R[\tau_1, \tau_2]_{(\mathbf{m}, \tau_1, \tau_2)}$. By Proposition 19.25 and the residual algebraic independence of τ_1 over R, we have $\operatorname{ht}(\widehat{P} \cap R_1) = 1$, and so $\operatorname{ht}(\widehat{P} \cap R_2) \leq 2$. If $\operatorname{ht}(\widehat{P} \cap R_2) = 1$, we are done by Proposition 19.25. Suppose $\operatorname{ht}(\widehat{P} \cap R_2) = 2$. The following diagram illustrates this situation:

$$\begin{array}{ccccccccc}
 & & \widehat{Q} \cap S_1 & \xrightarrow{\subseteq} & \widehat{Q} \cap S_2 & \xrightarrow{\subseteq} & \widehat{Q} = \lambda^{-1}(\widehat{P}) & \xrightarrow{\subseteq} & \widehat{S_2} \\
 & & \cong \big\downarrow & & \cong \big\downarrow & & \lambda \big\downarrow & & \lambda \big\downarrow \\
\widehat{P} \cap R & \xrightarrow{\subseteq} & \widehat{P} \cap R_1 & \xrightarrow{\subseteq} & \widehat{P} \cap R_2 & \xrightarrow{\subseteq} & \widehat{P} & \xrightarrow{\subseteq} & \widehat{R}.
\end{array}$$

Thus $\widehat{Q} \cap S_2 = P$ is a prime ideal of height 2, and $\operatorname{ht}(\widehat{Q} \cap S_1) = 1$. Also $P \neq (t_1, t_2)S_2$ because $(t_1, t_2)S_2 \cap R = (0)$. But this means that $\widehat{Q} \in \mathcal{U}$ since $\widehat{Q}$ is minimal over $(P, t_1 - \tau_1)\widehat{S_2}$ where P is a prime of S_2 with $\dim(S_2/P) = 2$ and $P \neq (t_1, t_2)S_2$. This contradicts the choice of a and establishes that item $2'$ holds. $\square$

We present in Theorem 19.33 a method to obtain an idealwise independent element that fails to be residually algebraically independent.

THEOREM 19.33. *Let $(R, \mathbf{m})$ be a countable excellent local UFD of dimension at least two. Assume there exists a height-one prime P of R such that P is contained in at least two distinct height-one primes $\widehat{P}$ and $\widehat{Q}$ of $\widehat{R}$. Also assume that $\widehat{P}$ is not the radical of a principal ideal in $\widehat{R}$. Then there exists $\tau \in \mathbf{m}\widehat{R}$ that is idealwise independent but not residually algebraically independent over R.*

PROOF. Let t be an indeterminate over R and set $S_1 = R[t]_{(\mathbf{m}, t)}$ so that $\widehat{S_1} = \widehat{R}[[t]]$. Let $\widehat{\mathbf{n}_1}$ denote the maximal ideal of $\widehat{S_1}$. By Lemma 19.18 with $I = (\widehat{P}, t)\widehat{S_1}$ and

$$\mathcal{U} = \{\mathbf{p} \in \operatorname{Spec}(\widehat{S_1}) \mid \mathbf{p} \neq I, \operatorname{ht}(\mathbf{p}) \leq 2, \text{ and } \mathbf{p} \text{ minimal over } \mathbf{p} \cap S_1\},$$

there exists $a \in (\widehat{P}, t)\widehat{S_1} \cap \widehat{\mathbf{n}_1}^2$, such that $t - a \notin \bigcup\{\mathbf{p} \mid \mathbf{p} \in \mathcal{U}\}$, but $t - a \in (\widehat{P}, t)\widehat{S_1}$. That is, if $t - a \in \mathbf{p}$, for some prime ideal $\mathbf{p} \neq (\widehat{P}, t)\widehat{S_1}$ of $\widehat{S_1}$ with $\operatorname{ht}(\mathbf{p}) \leq 2$, then $\operatorname{ht}(\mathbf{p}) > \operatorname{ht}(\mathbf{p} \cap S_1)$. Let λ be the surjection $\widehat{S_1} \to \widehat{R}$ with kernel $(t - a)\widehat{S_1}$. By construction, $(t - a)\widehat{S_1} \cap S_1 = (0)$. Therefore the restriction of λ to S_1 maps S_1 isomorphically onto $S = R[\tau]_{(\mathbf{m}, \tau)}$, where $\lambda(t) = \tau \in \mathbf{m}\widehat{R}$ is algebraically independent over R.

That τ is not residually algebraically independent over R follows because the prime ideal $\lambda((P,t)S_1) = (P,\tau)S$ has height two and is the contraction to S of the prime ideal $\lambda((\widehat{P},t)\widehat{S_1}) = \widehat{P}$ of $\widehat{R}$. Since $(t-\tau)\widehat{S_1} = (t-a)\widehat{S_1} \subseteq (\widehat{P},t)\widehat{S_1}$, $\lambda((\widehat{P},t)\widehat{S_1})$ has height one and equals $\widehat{P}$. Therefore τ is not residually algebraically independent over R.

Our choice of $t-a$ insures that each height-one prime $\widehat{\mathbf{q}}$ other than $\widehat{P}$ of $\widehat{R}$ has the property that $\operatorname{ht}(\widehat{\mathbf{q}} \cap S) \leq 1$. We show that τ is idealwise independent over R by showing each height-one prime of S is the contraction of a height-one prime of $\widehat{R}$. Let $\varphi : S_1 \to S$ denote the restriction of λ. For $\mathbf{q}$ a height-one prime of S, let $\mathbf{q}_1 := \varphi^{-1}(\mathbf{q})$ denote the corresponding height-one prime of S_1. Then $(\mathbf{q}_1, t-a)\widehat{S_1}$ is an ideal of height two. Let $\mathbf{w}_1$ be a height-two prime of $\widehat{S_1}$ containing $(\mathbf{q}_1, t-a)$. If $\mathbf{q}_1$ is not contained in $(\widehat{P},t)\widehat{S_1}$, then by the choice of $t-a$, $\mathbf{w}_1 \cap S_1$ has height at most one. Therefore $\mathbf{w}_1 \cap S_1 = \mathbf{q}_1$. Let $\mathbf{w} = \lambda(\mathbf{w}_1)$. Then $\mathbf{w}$ is a height-one prime of $\widehat{R}$ and $\mathbf{w} \cap S = \mathbf{q}$.

Therefore each height-one prime $\mathbf{q}$ of S such that $\mathbf{q}_1 := \varphi^{-1}(\mathbf{q})$ is not contained in $(\widehat{P},t)\widehat{S_1}$ is the contraction of a height-one prime of $\widehat{R}$. Since $\lambda((\widehat{P},t)\widehat{S_1}) \cap S = (P,\tau)S$, it remains to consider height-one primes $\mathbf{q}$ of S such that $\mathbf{q} \subseteq (P,\tau)S$. By hypothesis we have $PS = \widehat{Q} \cap S$. Let $\mathbf{q}$ be a height-one prime of S such that $\mathbf{q} \neq PS$ and $\mathbf{q} \subseteq (P,\tau)S$. Since R is a UFD, S is a UFD and $\mathbf{q} = fS$ for an element $f \in \mathbf{q}$. Since $\widehat{P}$ is not the radical of a principal ideal, there exists a height-one prime $\widehat{\mathbf{q}} \neq \widehat{P}$ of $\widehat{R}$ such that $f \in \widehat{\mathbf{q}}$. Since $\operatorname{ht}(\widehat{\mathbf{q}} \cap S) \leq 1$, we have $\widehat{\mathbf{q}} \cap S = fS = \mathbf{q}$. Therefore τ is idealwise independent over R. $\square$

EXAMPLE 19.34. An example of a countable excellent local UFD having a height-one prime P satisfying the conditions in Theorem 19.33 is $R = k[x,y,z]_{(x,y,z)}$, where k is the algebraic closure of the field $\mathbb{Q}$ and $z^2 = x^3 + y^7$. That R is a UFD is shown in [**126**, page 32]. Since $z - xy$ is an irreducible element of R, the ideal $P = (z-xy)R$ is a height-one prime of R. It is observed in [**49**, pages 300-301] that in the completion $\widehat{R}$ of R there exist distinct height-one primes $\widehat{P}$ and $\widehat{Q}$ lying over P. Moreover, the blowup of $\widehat{P}$ has a unique exceptional prime divisor and this exceptional prime divisor is not the unique exceptional prime divisor on the blowup of an $\widehat{\mathbf{m}}$-primary ideal. Therefore $\widehat{P}$ is not the radical of a principal ideal of $\widehat{R}$.

In Theorem 19.35 we present an alternative method to obtain idealwise independent elements that are not residually algebraically independent.

THEOREM 19.35. *Let $(R,\mathbf{m})$ be a countable excellent local UFD of dimension at least two. Assume there exists a height-one prime P_0 of R such that P_0 is contained in at least two distinct height-one primes $\widehat{P}$ and $\widehat{Q}$ of $\widehat{R}$. Also assume that the Henselization $(R^h, \mathbf{m}^h)$ of R is a UFD. Then there exists $\tau \in \mathbf{m}\widehat{R}$ that is idealwise independent but not residually algebraically independent over R.*

PROOF. Since R is excellent, $P := \widehat{P} \cap R^h$ and $Q := \widehat{Q} \cap R^h$ are distinct height-one primes of R^h with $P\widehat{R} = \widehat{P}$, and $Q\widehat{R} = \widehat{Q}$. Let $x \in R^h$ be such that $xR^h = P$. Theorem 19.20 implies there exists $y \in \mathbf{m}\widehat{R}$ that is primarily independent and hence residually algebraically independent over R^h.

We show that $\tau = xy$ is idealwise independent but not residually algebraically independent over R. Since x is nonzero and algebraic over R, xy is algebraically independent over R. Let $S = R[xy]_{(\mathbf{m},xy)}$. Then S is a UFD and $\widehat{P} \cap S = x\widehat{R} \cap S \supseteq (P_0, xy)S$ has height at least two in S. Therefore by Theorem 19.26, xy is not residually algebraically independent over R.

Since y is idealwise independent over R^h, every height-one prime of the polynomial ring $R^h[y]$ contained in the maximal ideal $\mathbf{n} = (\mathbf{m}^h, y)R^h[y]$ is the contraction of a height-one prime of $\widehat{R}$. To show xy is idealwise independent over R, it suffices to show every prime element $w \in (\mathbf{m}, xy)R[xy]$ is such that $wR[xy]$ is the contraction of a height-one prime of $R^h[y]$ contained in $\mathbf{n}$. If $w \notin (P, xy)R^h[xy]$, then the constant term of w as a polynomial in $R^h[xy]$ is in $\mathbf{m}^h \setminus P$. Thus $w \in \mathbf{n}$ and $w \notin xR^h[y]$. Since $R^h[xy][1/x] = R^h[y][1/x]$ and $xR^h[y] \cap R^h[xy] = (x, xy)R^h[xy]$, it follows that there is a prime factor u of w in $R^h[xy]$ such that $u \in \mathbf{n} \setminus xR^h[y]$. Then $uR^h[y]$ is a height-one prime of $R^h[y]$ and $uR^h[x] \cap R^h[xy] = uR^h[xy]$. Since $R^h[xy]$ is faithfully flat over $R[xy]$, it follows that $uR^h[y] \cap R[xy] = wR[xy]$.

We have $QR^h[xy] = QR^h[y] \cap R^h[xy]$ and $QR^h[xy] \cap R[xy] = P_0R[xy]$. Thus it remains to show, for a prime element $w \in (\mathbf{m}, xy)R[xy]$ such that $w \in (P, xy)R^h[xy]$ and $wR[xy] \neq P_0R[xy]$, that $wR[xy]$ is the contraction of a height-one prime of R^h contained in $\mathbf{n}$. Since $(P, xy)R^h[xy] \cap R[xy] = (P_0, xy)R[xy]$, it follows that w is a nonconstant polynomial in $R[xy]$ and the constant term w_0 of w is in P_0. In the polynomial ring $R^h[y]$ we have $w = x^n v$, where $v \notin xR^h[y]$. If v_0 denotes the constant term of v as a polynomial in $R^h[y]$, then $x^n v_0 = w_0 \in P_0 \subseteq R$ implies $x^n v_0 \in Q \subseteq R^h$. Since $x \in R^h \setminus Q$, we must have $v_0 \in Q$ and hence $v \in \mathbf{n}$. Also $v \notin xR^h[y]$ implies there is a height-one prime ideal $\mathbf{v}$ of $R^h[y]$ with $v \in \mathbf{v}$ and $x \notin \mathbf{v}$. Then, since $R^h[y]_{\mathbf{v}}$ is a localization of $R^h[xy]$, $\mathbf{v} \cap R^h[xy]$ is a height-one prime of $R^h[xy]$ that is contained in $(\mathbf{m}^h, xy)R^h[xy]$. It follows that $\mathbf{v} \cap R^h[xy] = wR^h[xy]$, which completes the proof of Theorem 19.35. $\square$

Example 4.6 is a specific example with the hypothesis of Theorem 19.35. In more generality, we have:

EXAMPLE 19.36. Let $R = k[s,t]_{(s,t)}$ be a localized polynomial ring in two variables s and t over a countable field k where k has characteristic not equal to 2. Let $P_0 = (s^2 - t^2 - t^3)R$. Then P_0 is a height-one prime of R and $P_0\widehat{R} = (s^2 - t^2 - t^3)k[[s,t]]$ is the product of two distinct height-one primes of $\widehat{R}$.

REMARK 19.37. Let $(R, \mathbf{m})$ be excellent normal local domain and let $(\widehat{R}, \widehat{\mathbf{m}})$ be its completion. Assume that $\tau \in \widehat{\mathbf{m}}$ is algebraically independent over R. By Theorem 19.11, the extension $R[\tau] \hookrightarrow \widehat{R}$ is weakly flat if and only if τ is idealwise independent over R. By Theorem 19.26, this extension satisfies PDE (or equivalently LF_1) if and only if τ is residually independent over R. Thus Examples 19.34 and 19.36 give extensions of Krull domains $R[\tau] \hookrightarrow \widehat{R}$, that are weakly flat, but do not satisfy PDE. In fact, in these examples the ring $R[\tau]$ is a 3-dimensional excellent UFD.

Exercises

(1) As in Remark 19.19, let $A \hookrightarrow B$ be an extension of Krull domains, and let α be a nonzero nonunit of B such that $\alpha \notin Q$ for each height-one prime Q of B such that $Q \cap A \neq (0)$.
   (a) Prove that $\alpha B \cap A = (0)$ as asserted in Remark 19.19.
   (b) Prove that α is algebraically independent over A.

(2) Let $R = k[s,t]_{(s,t)}$ and the field k be as in Example 19.36.
   (a) Prove as asserted in Example 19.36 that $(s^2 - t^2 - t^3)R$ is a prime ideal.
   (b) Prove that $s^2 - t^2 - t^3$ factors in the power series ring $k[[s,t]]$ as the product of two nonassociate prime elements.

CHAPTER 20

Idealwise algebraic independence II

We relate the three concepts of independence from Chapter 19 to flatness conditions of extensions of Krull domains, establish implications among them, and draw some conclusions concerning their equivalence in special situations. We also investigate their stability under change of base ring.

We use Setting 19.1, from Chapter 19, for this chapter. Thus $(R, \mathbf{m})$ is an excellent normal local domain with field of fractions K and completion $(\widehat{R}, \widehat{\mathbf{m}})$, and $t_1, \dots, t_n$ are indeterminates over R. The elements $\tau_1, \dots, \tau_n \in \widehat{\mathbf{m}}$ are algebraically independent over R, and we have embeddings:

$$R \hookrightarrow S = R_n = R[\tau_1, \dots, \tau_n]_{(\mathbf{m}, \tau_1, \dots, \tau_n)} \overset{\varphi}{\hookrightarrow} \widehat{R}$$

Using this setting and other terminology of Chapter 19, we summarize the results of this chapter.

Summary 20.1. In Section 20.1 we describe the three concepts of idealwise independence, residual algebraic independence, and primary independence defined in Definitions 19.2,19.24 and 19.12 of Chapter 19 in terms of certain flatness conditions on the embedding

$$\varphi : R[\tau_1, \dots, \tau_n]_{(\mathbf{m}, \tau_1, \dots, \tau_n)} \hookrightarrow \widehat{R},$$

where $(R, \mathbf{m})$ is an excellent normal local ring and $(\widehat{R}, \widehat{\mathbf{m}})$ is the $\mathbf{m}$-adic completion of R. In Section 20.2 we investigate the stability of these independence concepts under base change, composition and polynomial extension. We prove in Corollary 20.19 the existence of uncountable excellent normal local domains R such that $\widehat{R}$ contains infinite sets of primarily independent elements.

We show in Section 20.3 that both residual algebraic independence and primary independence hold for elements over the original ring R exactly when they hold over the Henselization R^h of R (20.21). Also idealwise independence descends from the Henselization to the ring R.

A large diagram in Section 20.4 displays the relationships among the independence concepts and many other related properties.

20.1. Primary independence and flatness

In this section we describe the concept of primary independence in terms of flatness of certain localizations of the canonical embedding of Setting 19.1

$$\varphi : R_n = R[\tau_1, \dots, \tau_n]_{(\mathbf{m}, \tau_1, \dots, \tau_n)} \hookrightarrow \widehat{R}.$$

We establish in Chapter 19 flatness conditions for φ that are equivalent to idealwise independence and residual algebraic independence. We summarize these conditions in Remark 20.2.

REMARK 20.2. Let $(R, \mathbf{m})$ and $\tau_1, \ldots, \tau_n \in \widehat{\mathbf{m}}$ be as in Setting 19.1, and let $\varphi : R_n = R[\tau_1, \ldots, \tau_n]_{(\mathbf{m}, \tau_1, \ldots, \tau_n)} \hookrightarrow \widehat{R}$ denote the canonical embedding. Then:

(1) $\tau_1, \ldots, \tau_n$ are idealwise independent over R if and only if the map $R_n \hookrightarrow \widehat{R}$ is weakly flat; see Definitions 19.2 and 19.5 and Theorem 19.11..
(2) The elements $\tau_1, \ldots, \tau_n$ are residually algebraically independent over R $\iff$ $\varphi : R_n = R[\tau_1, \ldots, \tau_n]_{(\mathbf{m}, \tau_1, \ldots, \tau_n)} \longrightarrow \widehat{R}$ satisfies LF_1; see Definition 12.1.3, Proposition 12.12 and Theorem 19.26.
(3) If each height-one prime of R is the radical of a principal ideal and the elements $\tau_1, \ldots, \tau_n$ are residually algebraically independent over R, then the elements $\tau_1, \ldots, \tau_n$ are idealwise independent over R. See Theorem 19.27.3.

These items follow from Theorem 19.11, Proposition 12.12 and Theorem 19.26.

To describe primary independence in terms of flatness of certain localizations of the embedding $\varphi : R_n \longrightarrow \widehat{R}$, we use the following lemma.

LEMMA 20.3. *Let $d \in \mathbb{N}$ and $n \in \mathbb{N}_0$, and let $(S, \mathbf{m}) \hookrightarrow (T, \mathbf{n})$ be a local embedding of catenary Noetherian local domains with $\dim T = d$ and $\dim S = d+n$. Assume the extension $S \hookrightarrow T$ satisfies the following property for every $P \in \operatorname{Spec} S$:*

$$(20.3.0) \qquad \operatorname{ht} P \ \geq \ d \quad \Longrightarrow \quad PT \quad \text{is} \quad \mathbf{n}\text{-primary.}$$

Then, for every $Q \in \operatorname{Spec} T$ with $\operatorname{ht} Q \leq d-1$, we have $\operatorname{ht}(Q \cap S) \leq \operatorname{ht} Q$.

PROOF. If $Q \in \operatorname{Spec} T$ is such that $\operatorname{ht}(Q \cap S) \geq d$, then, by Property 20.3.0, $(Q \cap S)T$ is $\mathbf{n}$-primary, and so $Q = \mathbf{n}$ and $\operatorname{ht} Q = d$. Thus, for every $Q \in \operatorname{Spec} T$ with $\operatorname{ht} Q \leq d-1$, we have $\operatorname{ht}(Q \cap S) \leq d-1$. In particular, if $\operatorname{ht} Q = d-1$, then $\operatorname{ht}(Q \cap S) \leq \operatorname{ht} Q$.

We proceed by induction on $s \geq 1$: Assume $s \geq 2$ and $\operatorname{ht}(P \cap S) \leq \operatorname{ht} P$, for every $P \in \operatorname{Spec} T$ with $d > \operatorname{ht} P \geq d-s+1$. Let $Q \in \operatorname{Spec} T$ with $\operatorname{ht} Q = d-s$. Suppose $\operatorname{ht}(Q \cap S) \geq d-s+1$; choose $b \in \mathbf{m} \setminus Q$ and let $Q_1 \in \operatorname{Spec} T$ be minimal over $(b, Q)T$. Since T is catenary and Noetherian, we have $\operatorname{ht} Q_1 = d-s+1$. By the inductive hypothesis, $\operatorname{ht}(Q_1 \cap S) \leq d-s+1$. Since $b \in Q_1 \cap S$, the ideal $Q_1 \cap S$ properly contains $Q \cap S$. But this implies

$$d-s+1 \ \geq \operatorname{ht}(Q_1 \cap S) \ > \ \operatorname{ht}(Q \cap S) \ \geq d-s+1,$$

a contradiction. Thus $\operatorname{ht}(Q \cap S) \leq \operatorname{ht} Q$, for every $Q \in \operatorname{Spec} T$ with $\operatorname{ht} Q \leq d-1$. $\square$

We use the LF_d notation of Definition 12.1.3 and Remark 12.2 in the following theorem.

THEOREM 20.4. *Let $(R, \mathbf{m})$ be an excellent normal local domain, and let the elements $\tau_1, \ldots, \tau_n \in \widehat{\mathbf{m}}$ be as in Setting 19.1. Assume that $\dim R = d$. Then the elements $\tau_1, \ldots, \tau_n$ are primarily independent over R if and only if*

$$\varphi : R_n = R[\tau_1, \ldots, \tau_n]_{(\mathbf{m}, \tau_1, \ldots, \tau_n)} \longrightarrow \widehat{R}$$

satisfies LF_{d-1}.

PROOF. To prove the $\Longrightarrow$ direction: Since R_n is a localized polynomial ring over R, the map $R \hookrightarrow R_n$ has regular fibers. Since R is excellent, the map $R \hookrightarrow \widehat{R}$ has regular, hence Cohen-Macaulay, fibers. Consider the sequence

$$R \ \longrightarrow \ R_n \ \xrightarrow{\varphi} \ \widehat{R}.$$

To show φ satisfies LF_{d-1}, we show that $\varphi_{\widehat{Q}}$ is flat for every $\widehat{Q} \in \operatorname{Spec} \widehat{R}$ with $\operatorname{ht} \widehat{Q} \le d-1$. For this, by (2) $\implies$ (1) of Theorem 11.3, it suffices to show $\operatorname{ht}(\widehat{Q} \cap R_n) \le \operatorname{ht} \widehat{Q}$ for every $\widehat{Q} \in \operatorname{Spec} \widehat{R}$ with $\operatorname{ht} \widehat{Q} \le d-1$. This holds by Lemma 20.3, since primary independence implies Property 20.3.0.

For $\Longleftarrow$, let $P \in \operatorname{Spec} R_n$ be a prime ideal with $\dim(R_n/P) \le n$. Suppose that $P\widehat{R}$ is not $\widehat{\mathbf{m}}$-primary and let $\widehat{Q} \supseteq P\widehat{R}$ be a minimal prime of $P\widehat{R}$. Then $\operatorname{ht}(\widehat{Q}) \le d-1$. Set $Q = \widehat{Q} \cap R_n$, then LF_{d-1} implies that the map

$$\varphi_{\widehat{Q}} : (R_n)_Q \longrightarrow \widehat{R}_{\widehat{Q}}$$

is faithfully flat. Hence by going-down (Remark 2.21.10), $\operatorname{ht} Q \le d-1$. But $P \subseteq Q$ and R_n is catenary, so $d-1 \ge \operatorname{ht} Q \ge \operatorname{ht} P \ge d$, a contradiction. We conclude that $\tau_1, \ldots, \tau_n$ are primarily independent. $\square$

Remark 20.5. Theorem 20.4 yields a different proof of statements (1) and (3) of Theorem 19.27, that primarily independent elements are residually algebraically independent and that in dimension two, the two concepts are equivalent. Considering again our basic setting from (19.1) , with $d = \dim(R)$, Theorem 20.4 equates the LF_{d-1} condition on the extension $R_n = R[\tau_1, \ldots, \tau_n]_{(\mathbf{m}, \tau_1, \ldots, \tau_n)} \longrightarrow \widehat{R}$, to the primary independence of the τ_i. Also Proposition 12.12 and Theorem 19.26 yield that residual algebraic independence of the τ_i is equivalent to the extension $R_n = R[\tau_1, \ldots, \tau_n]_{(\mathbf{m}, \tau_1, \ldots, \tau_n)} \longrightarrow \widehat{R}$ satisfying LF_1. Clearly $\mathrm{LF}_i \implies \mathrm{LF}_{i-1}$, for $i > 1$, and if $d = \dim(R) = 2$, then $\mathrm{LF}_{d-1} = \mathrm{LF}_1$.

Remark 20.6. In Setting 19.1, if $\tau_1, \ldots, \tau_n$ are primarily independent over R and $\dim(R) = d$, then $\varphi : R_n \longrightarrow \widehat{R}$ satisfies LF_{d-1}, but *not* LF_d, that is, φ fails to be faithfully flat; for faithful flatness would imply going-down and hence that $\dim(R_n) \le d = \dim(\widehat{R})$.

Example 20.7. By a modification of Example 12.14, it is possible to obtain, for each integer $d \ge 2$, an injective local map $\varphi : (A, \mathbf{m}) \longrightarrow (B, \mathbf{n})$ of normal Noetherian local domains with B essentially of finite type over A, $\varphi(\mathbf{m})B = \mathbf{n}$, and $\dim(B) = d$ such that φ satisfies LF_{d-1}, but fails to be faithfully flat over A. Let k be a field and let $x_1, \ldots, x_d, y$ be indeterminates over k. Let A be the localization of $k[x_1, \ldots, x_d, x_1y, \ldots, x_dy]$ at the maximal ideal generated by $(x_1, \ldots, x_d, x_1y, \ldots, x_dy)$, and let B be the localization of $A[y]$ at the prime ideal $(x_1, \ldots, x_d)A[y]$. Then A is an $d+1$-dimensional normal Noetherian local domain and B is an d-dimensional regular local domain birationally dominating A. For any nonmaximal prime Q of B we have $B_Q = A_{Q \cap A}$. Hence $\varphi : A \longrightarrow B$ satisfies LF_{d-1}, but φ is not faithfully flat since $\dim(B) < \dim(A)$.

The local injective map $\varphi : (A, \mathbf{m}) \longrightarrow (B, \mathbf{n})$ of Example 20.7 is not height-one preserving. Remark 19.8 shows that if each height-one prime ideal of R is the radical of a principal ideal then the maps studied in this chapter are height-one preserving. We have the following question:

Question 20.8. Let $\varphi : (A, \mathbf{m}) \longrightarrow (B, \mathbf{n})$ be a local injective map of normal Noetherian local integral domains. Assume that B is essentially of finite type over A with $\dim B = d \ge 2$. If φ is both LF_{d-1} and height-one preserving, does it follow that φ is faithfully flat?

20.2. Composition, base change and polynomial extensions

In this section we investigate idealwise independence, residual algebraic independence, and primary independence under polynomial ring extensions and localizations of these polynomial extensions.

We start with a more general situation. Let

$$\begin{tikzcd} & C \\ A \arrow[ur, "\psi\varphi"] \arrow[r, "\varphi"] & B \arrow[u, "\psi"] \end{tikzcd}$$

be a commutative diagram of commutative rings and injective maps. Proposition 20.9 implies that many of the properties of injective maps that we consider are stable under composition:

Proposition 20.9. *Let $\varphi : A \longrightarrow B$ and $\psi : B \longrightarrow C$ be injective maps of commutative rings, and let $s \in \mathbb{N}$.*

(1) *If φ and ψ satisfy LF_s, then $\psi\varphi$ satisfies LF_s.*
(2) *If C is Noetherian, ψ is faithfully flat and the composite map $\psi\varphi$ satisfies LF_s, then φ satisfies LF_s.*
(3) *Assume that A, B and C are Krull domains, and that $QC \neq C$, for each height-one prime Q of B. If φ and ψ are height-one preserving (respectively weakly flat), then $\psi\varphi$ is height-one preserving (respectively weakly flat).*

Proof. The first item follows because a flat map satisfies going-down, see Remark 2.21.10. For item , since C is Noetherian and ψ is faithfully flat, B is Noetherian; see Remark 2.21.8. Let $Q \in \operatorname{Spec}(B)$ with $\operatorname{ht}(Q) = d \leq k$. We show $\varphi_Q : A_{Q\cap A} \longrightarrow B_Q$ is faithfully flat. By localization of B and C at $B \setminus Q$, we may assume that B is local with maximal ideal Q. Since C is faithfully flat over B, $QC \neq C$. Let $Q' \in \operatorname{Spec}(C)$ be a minimal prime of QC. Since C is Noetherian and B is local with maximal ideal Q, we have $\operatorname{ht}(Q') \leq d$ and $Q' \cap B = Q$. Since the composite map $\psi\varphi$ satisfies LF_k, the composite map

$$A_{Q'\cap A} = A_{Q\cap A} \xrightarrow{\varphi_Q} B_Q = B_{Q'\cap B} \xrightarrow{\psi_{Q'}} C_{Q'}$$

is faithfully flat. This and the faithful flatness of $\psi_{Q'} : B_{Q'\cap B} \longrightarrow C_{Q'}$ implies that φ_Q is faithfully flat [**94**, (4.B) page 27].

For item 3, let P be a height-one prime of A such that $PC \neq C$. Then $PB \neq B$ so if φ and ψ are height-one preserving then there exists a height-one prime Q of B such that $PB \subseteq Q$. By assumption, $QC \neq C$ (and ψ is height-one preserving), so there exists a height-one prime Q' of C such that $QC \subseteq Q'$. Hence $PC \subseteq Q'$.

If φ and ψ are weakly flat, then by Proposition 12.9 there exists a height-one prime Q of B such that $Q \cap A = P$. Again by assumption, $QC \neq C$, thus weakly flatness of ψ implies $QC \cap B = Q$. Now

$$P \subseteq PC \cap A \subseteq QC \cap A = QC \cap B \cap A = Q \cap A = P.$$

$\square$

REMARKS 20.10. If in Proposition 20.9.3 the Krull domains B and C are local, but not necessarily Noetherian and ψ is a local map, then clearly $QC \neq C$ for each height-one prime Q of B.

If a map λ of Krull domains is faithfully flat, then λ is height-one preserving, weakly flat and satisfies condition LF_k for every integer $k \in \mathbb{N}$. Thus if $\varphi : A \longrightarrow B$ and $\psi : B \longrightarrow C$ are injective maps of Krull domains, such that one of φ or ψ is faithfully flat and the other is weakly flat (respectively height-one preserving or satisfies LF_k), then the composition $\psi\varphi$ is again weakly flat (respectively height-one preserving or satisfies LF_k). Moreover, if the map ψ is faithfully flat, we also obtain the following converse to Proposition 20.9.3:

PROPOSITION 20.11. *Let $\varphi : A \longrightarrow B$ and $\psi : B \longrightarrow C$ be injective maps of Krull domains. Assume that ψ is faithfully flat. If $\psi\varphi$ is height-one preserving (respectively weakly flat), then φ is height-one preserving (respectively weakly flat).*

PROOF. Let P be a height-one prime ideal of A such that $PB \neq B$. Since ψ is faithfully flat, $PC \neq C$; so if $\psi\varphi$ is height-one preserving, then there exists a height-one prime ideal Q' of C containing PC. Now $Q = Q' \cap B$ has height one by going-down for flat extensions, and $PB \subseteq Q' \cap B = Q$, so φ is height-one preserving. The proof of the weakly flat statement is similar, using Proposition 12.9. $\square$

Next we consider a commutative square of commutative rings and injective maps:

$$\begin{array}{ccc} A' & \xrightarrow{\varphi'} & B' \\ {\scriptstyle\mu}\uparrow & & {\scriptstyle\nu}\uparrow \\ A & \xrightarrow{\varphi} & B \end{array}$$

PROPOSITION 20.12. *In the diagram above, assume that μ and ν are faithfully flat, and let $k \in \mathbb{N}$. Then:*

1. *(Ascent) Assume that $B' = B \otimes_A A'$, or that B' is a localization of $B \otimes_A A'$. Let ν denote the canonical map associated with this tensor product. If $\varphi : A \longrightarrow B$ satisfies LF_k, then $\varphi' : A' \longrightarrow B'$ satisfies LF_k.*
2. *(Descent) If B' is Noetherian and $\varphi' : A' \longrightarrow B'$ satisfies LF_k, then $\varphi : A \longrightarrow B$ satisfies LF_k.*
3. *(Descent) Assume that the rings A, A', B and B' are Krull domains. If $\varphi' : A' \longrightarrow B'$ is height-one preserving (respectively weakly flat), then $\varphi : A \longrightarrow B$ is height-one preserving (respectively weakly flat).*

PROOF. For (1), assume that φ satisfies LF_k and let $Q' \in \operatorname{Spec}(B')$ with $\operatorname{ht}(Q') \leq k$. Put $Q = (\nu)^{-1}(Q')$, $P' = (\varphi')^{-1}(Q')$, and $P = \mu^{-1}(P') = \varphi^{-1}(Q)$ and consider the commutative diagrams:

$$\begin{array}{ccc} A' & \xrightarrow{\varphi'} & B' \\ {\scriptstyle\mu}\uparrow & & {\scriptstyle\nu}\uparrow \\ A & \xrightarrow{\varphi} & B \end{array} \qquad\qquad \begin{array}{ccc} A'_{P'} & \xrightarrow{\varphi'_{Q'}} & B'_{Q'} \\ {\scriptstyle\mu_{P'}}\uparrow & & {\scriptstyle\nu_{Q'}}\uparrow \\ A_P & \xrightarrow{\varphi_Q} & B_Q \end{array}$$

The flatness of ν implies that $\operatorname{ht}(Q) \leq k$ and so by assumption, φ_Q is faithfully flat. The ring $B'_{Q'}$ is a localization of $B_Q \otimes_{A_P} A'_{P'}$ and B_Q is faithfully flat over A_P implies $B'_{Q'}$ is faithfully flat over $A'_{P'}$.

For (2), by Proposition 20.9.1, $\varphi'\mu = \nu\varphi$ satisfies LF_k. Now by Proposition 20.9 .2, φ satisfies LF_k.

Item 3 follows immediately from the assumption that μ and ν are faithfully flat maps and hence going-down holds [**94**, Theorem 4, page 33]. □

Next we examine the situation for polynomial extensions.

PROPOSITION 20.13. *Let $(R, \mathbf{m})$ and $\{\tau_i\}_{i=1}^m \subseteq \widehat{\mathbf{m}}$ be as in Setting 19.1, where m is either an integer or $m = \infty$, and the dimension of R is at least 2. Let z be an indeterminate over $\widehat{R}$. Then:*

(1) *$\{\tau_i\}_{i=1}^m$ is residually algebraically independent over $R \iff \{\tau_i\}_{i=1}^m$ is residually algebraically independent over $R[z]_{(\mathbf{m},z)}$.*
(2) *If $\{\tau_i\}_{i=1}^m$ is idealwise independent over $R[z]_{(\mathbf{m},z)}$, then $\{\tau_i\}_{i=1}^m$ is idealwise independent over R.*

PROOF. Let $n \in \mathbb{N}$ be an integer with $n \leq m$. Set $R_n = R[\tau_1, \ldots, \tau_n]_{(\mathbf{m},\tau_1,\ldots,\tau_n)}$. Let $\varphi : R_n \longrightarrow \widehat{R}$ and $\mu : R_n \longrightarrow R_n[z]$ be the inclusion maps. We have the following commutative diagram:

$$\begin{array}{ccccc}
R_n[z]_{(\max(R_n),z)} & \xrightarrow{\varphi'} & R' = \widehat{R}[z]_{(\widehat{\mathbf{m}},z)} & \xrightarrow{\psi} & \widehat{R'} = \widehat{R}[[z]] \\
{\scriptstyle\mu}\uparrow & & {\scriptstyle\mu'}\uparrow & & \\
R_n & \xrightarrow{\varphi} & \widehat{R} & &
\end{array}$$

The ring R' is a localization of the tensor product $\widehat{R} \otimes_{R_n} R_n[z]$ and Proposition 20.12 applies. Thus, for (1), φ satisfies LF_1 if and only if φ' satisfies LF_1. Since the inclusion map ψ of $R' = \widehat{R}[z]_{(\widehat{\mathbf{m}},z)}$ to its completion $\widehat{R}[[z]]$ is faithfully flat, we obtain equivalences:

φ satisfies $LF_1 \iff \varphi'$ satisfies $LF_1 \iff \psi\varphi'$ satisfies LF_1.

(2) If the τ_i are idealwise independence over $R[z]_{(\mathbf{m},z)}$, the map $\psi\varphi'$ is weakly flat. Thus φ' is weakly flat and the statement follows by Proposition 20.9 . □

We also obtain:

PROPOSITION 20.14. *Let $A \hookrightarrow B$ be an extension of Krull domains such that for each height-one prime $P \in \operatorname{Spec}(A)$ we have $PB \neq B$, and let Z be a (possibly uncountable) set of indeterminates over A. Then $A \hookrightarrow B$ is weakly flat if and only if $A[Z] \hookrightarrow B[Z]$ is weakly flat.*

PROOF. Let F denote the field of fractions of A. By Corollary 12.4, the extension $A \hookrightarrow B$ is weakly flat if and only if $F \cap B = A$. Thus the assertion follows from $F \cap B = A \iff F(Z) \cap B[Z] = A[Z]$. □

It would be interesting to know whether the converse of Proposition 20.13.2 is true. In this connection we have:

REMARKS 20.15. Let $\varphi : A \longrightarrow B$ be a weakly flat map of Krull domains, and let P be a height-one prime in A.

(1) Let Q be a minimal prime of the extended ideal PB. If the map $\varphi_Q : A \longrightarrow B_Q$ is weakly flat, then $\operatorname{ht} Q = 1$. To see this, observe that QB_Q is the unique minimal prime of PB_Q, so QB_Q is the radical of PB_Q. If φ_Q is weakly flat, then $PB_Q \cap A = P$ and hence $QB_Q \cap A = P$. It follows

that $A_P \hookrightarrow B_Q$. Since A_P is a DVR and its maximal ideal PA_P extends to an ideal primary for the maximal ideal QB_Q of the Krull domain B_Q, we must have that B_Q is a DVR and hence $\operatorname{ht} Q = 1$.

(2) Thus if there exists a weakly flat map of Krull domains $\varphi : A \longrightarrow B$ and a minimal prime Q of PB such that $\operatorname{ht} Q > 1$, then the map $\varphi_Q : A \longrightarrow B_Q$ fails to be weakly flat,

(3) If P is the radical of a principal ideal, then each minimal prime of PB has height one.

Question 20.16. Let $\varphi : A \longrightarrow B$ be a weakly flat map of Krull domains, and let P be a height-one prime in A, as in Remarks 20.15. Is it possible that the extended ideal PB has a minimal prime Q with $\operatorname{ht} Q > 1$?

Remark 20.17. Primary independence never lifts to polynomial rings. To see that $\tau \in \widehat{\mathbf{m}}$ fails to be primarily independent over $R[z]_{(\mathbf{m},z)}$, observe that $\mathbf{m}R[z]_{(\mathbf{m},z)}$ is a dimension-one prime ideal that extends to $\widehat{\mathbf{m}}[[z]]$, which also has dimension one and is *not* $(\mathbf{m}, z)$-primary in $\widehat{R}[[z]]$. Alternatively, in the language of locally flat maps, if the elements $\{\tau_i\}_{i=1}^m \subseteq \widehat{\mathbf{m}}$ are primarily independent over R, then Proposition 20.9 implies that the map

$$\varphi' : R_n[z]_{(\max(R_n),z)} \longrightarrow \widehat{R}[[z]]$$

satisfies condition LF_{d-1}, where $d = \dim(R)$. For $\{\tau_i\}_{i=1}^m$ to be primarily independent over $R[z]_{(\mathbf{m},z)}$, however, the map φ' has to satisfy LF_d, since $\dim R[z]_{(\mathbf{m},z)} = d+1$. Using Proposition 20.9 again this forces $\varphi : R_n \longrightarrow \widehat{R}$ to satisfy condition LF_d and thus φ is flat, which can happen only if $n = 0$. This is an interesting phenomenon; the construction of primarily independent elements involves all parameters of the ring R.

In the remainder of this section we consider localizations of polynomial extensions so that the dimension does not increase. Theorem 20.18 gives a method to obtain residually algebraically independent and primarily independent elements over an uncountable excellent local domain. In Theorem 20.18 we make use of the fact that if A is a Noetherian ring and Z is a set of indeterminates over A, then the ring $A(Z)$ obtained by localizing the polynomial ring $A[Z]$ at the multiplicative system of polynomials whose coefficients generate the unit ideal of A is again a Noetherian ring [**40**, Theorem 6].

Theorem 20.18. *Let $(R, \mathbf{m})$ and $\{\tau_i\}_{i=1}^m \subset \widehat{\mathbf{m}}$ be as in Setting 19.1, where m is either an integer or $m = \infty$, and $\dim(R) = d \geq 2$. Let Z be a set (possibly uncountable) of indeterminates over $\widehat{R}$ and let $R(Z) = R[Z]_{(\mathbf{m}R[Z])}$. Then:*

(1) *$\{\tau_i\}_{i=1}^m$ is primarily independent over R $\iff$ $\{\tau_i\}_{i=1}^m$ is primarily independent over $R(Z)$.*

(2) *$\{\tau_i\}_{i=1}^m$ is residually algebraically independent over R $\iff$ $\{\tau_i\}_{i=1}^m$ is residually algebraically independent over $R(Z)$.*

(3) *If $\{\tau_i\}_{i=1}^m$ is idealwise independent over $R(Z)$, then $\{\tau_i\}_{i=1}^m$ is idealwise independent over R.*

Proof. Let $n \in \mathbb{N}$ be an integer with $n \leq m$, put $R_n = R[\tau_1, \ldots, \tau_n]_{(\mathbf{m},\tau_1,\ldots,\tau_n)}$ and let $\mathbf{n}$ denote the maximal ideal of R_n. Let $\varphi : R_n \longrightarrow \widehat{R}$ and $\mu : R_n \longrightarrow R_n(Z) = R_n[Z]_{\mathbf{n}R_n[Z]}$ be the inclusion maps. We have the following commutative diagram:

$$\begin{array}{ccc} R_n(Z) & \xrightarrow{\varphi'} & \widehat{R}(Z) \\ {\scriptstyle\mu}\uparrow & & {\scriptstyle\mu'}\uparrow \\ R_n & \xrightarrow{\varphi} & \widehat{R} \end{array}$$

The ring $\widehat{R}(Z)$ is a localization of the tensor product $\widehat{R} \otimes_{R_n} R_n[Z]$ and Proposition 20.12 applies. Thus, for item 1, φ satisfies LF_{d-1} if and only if φ' satisfies LF_{d-1}. Similarly, for item 2, φ satisfies LF_1 if and only if φ' satisfies LF_1.

Since the inclusion map ψ taking $\widehat{R}(Z)$ to its completion is faithfully flat, we obtain equivalences:

$$\varphi \text{ satisfies } LF_k \iff \varphi' \text{ satisfies } LF_k \iff \psi\varphi' \text{ satisfies } LF_k.$$

Since primary independence is equivalent to LF_{d-1} by Theorem 20.4 and residual algebraic independence is equivalent to LF_1 by Proposition 12.12, statements 1 and 2 follow.

For item 3, if the τ_i are idealwise independence over $R(Z)$, the morphism $\psi\varphi'$ is weakly flat. Thus φ' is weakly flat. The statement follows by Proposition 20.9. $\square$

COROLLARY 20.19. *Let k be a countable field, let Z be an uncountable set of indeterminates over k and let x, y be additional indeterminates. Then $R := k(Z)[x,y]_{(x,y)}$ is an uncountable excellent normal local domain of dimension two, and, for m a positive integer or $m = \infty$, there exist m primarily independent elements (and hence also residually algebraically and idealwise independent elements) over R.*

PROOF. Apply Proposition 19.15 and Theorems 19.20, 19.27 and 20.18. $\square$

20.3. Passing to the Henselization

In this section we investigate idealwise independence, residual algebraic independence, and primary independence as we pass from R to the Henselization R^h of R. In particular, we show in Proposition 20.24 that for a single element $\tau \in \mathbf{m}\widehat{R}$ the notions of idealwise independence and residual algebraic independence coincide if $R = R^h$. This implies that for every excellent normal local Henselian domain of dimension 2 all three concepts coincide for an element $\tau \in \widehat{\mathbf{m}}$; that is, τ is idealwise independent $\iff$ τ is residually algebraically independent $\iff$ τ is primarily independent.

We use the commutative square of Propostion 20.12 and obtain the following result for Henselizations:

PROPOSITION 20.20. *Let $\varphi : (A, \mathbf{m}) \hookrightarrow (B, \mathbf{n})$ be an injective local map of normal Noetherian local domains, and let $\varphi^h : A^h \longrightarrow B^h$ denote the induced map of the Henselizations. Then:*

(1) *For each k with $1 \le k \le \dim(B)$, φ satisfies LF_k $\iff$ φ^h satisfies LF_k. Thus φ satisfies PDE $\iff$ φ^h satisfies PDE.*
(2) *(Descent) If φ^h is height-one preserving (respectively weakly flat), then φ is height-one preserving (respectively weakly flat).*

Using shorthand and diagrams, we show Proposition 20.20 schematically:

$$\boxed{\varphi \text{ is } LF_k} \iff \boxed{\varphi^h \text{ is } LF_k} \quad ; \quad \boxed{\varphi \text{ is PDE}} \iff \boxed{\varphi^h \text{ is PDE}}$$

$$\boxed{\varphi \text{ ht-1 pres}} \Longleftarrow \boxed{\varphi^h \text{ ht-1 pres}} \quad ; \quad \boxed{\varphi \text{ w.f.}} \Longleftarrow \boxed{\varphi^h \text{ w.f.}}.$$

PROOF. (of Proposition 20.20) Consider the commutative diagram:

$$\begin{tikzcd} A^h \arrow[r, "\varphi^h"] & B^h \\ A \arrow[u, "\mu"] \arrow[r, "\varphi"] & B \arrow[u, "\nu"'] \end{tikzcd}$$

where μ and ν are the faithfully flat canonical injections [**104**, (43.8), page 182]. Since φ is injective and A is normal, φ^h is injective by [**104**, (43.5)]. By Proposition 19.25, Proposition 12.12 and Proposition 19.26), we need only show "$\Rightarrow$" in (1).

Let $Q' \in \operatorname{Spec}(B^h)$ with $\operatorname{ht}(Q') \le k$. Put $Q = Q' \cap B$, $P' = Q' \cap A^h$, and $P = P' \cap A$. We consider the localized diagram:

$$\begin{tikzcd} A^h_{P'} \arrow[r, "\varphi^h_{Q'}"] & B^h_{Q'} \\ A_P \arrow[u, "\mu_{P'}"] \arrow[r, "\varphi_Q"] & B_Q \arrow[u, "\nu_{Q'}"'] \end{tikzcd}$$

The faithful flatness of ν implies $\operatorname{ht}(Q) \le k$.

In order to show that $\varphi^h_{Q'} : A^h_{P'} \longrightarrow B^h_{Q'}$ is faithfully flat, we apply Remark 11.2.2 with $M = B^h_{Q'}$ and $I = PB^h_{Q'}$.

First note that P' is a minimal prime divisor of PA^h and that $(A^h/PA^h)_{P'} = (A^h/P')_{P'}$ is a field [**104**, (43.20)]. Thus

$$\overline{\varphi^h_{Q'}} : (A^h/PA^h)_{P'} \longrightarrow (B^h/PB^h)_{Q'}$$

is faithfully flat and it remains to show that

$$PA^h_{P'} \otimes_{A^h_{P'}} B^h_{Q'} \cong PB^h_{Q'}.$$

This can be seen as follows:

$$\begin{aligned} PA^h_{P'} \otimes_{A^h_{P'}} B^h_{Q'} &\cong (P \otimes_{A_P} A^h_{P'}) \otimes_{A^h_{P'}} B^h_{Q'} && \text{by flatness of} \quad \mu \\ &\cong P \otimes_{A_P} B^h_{Q'} \\ &\cong (P \otimes_{A_P} B_Q) \otimes_{B_Q} B^h_{Q'} \\ &\cong PB_Q \otimes_{B_Q} B^h_{Q'} && \text{by flatness of} \quad \varphi_Q \\ &\cong PB^h_{Q'} && \text{by flatness of} \quad \nu. \end{aligned}$$

$\square$

COROLLARY 20.21. *Let $(R, \mathbf{m})$ and $\{\tau_i\}_{i=1}^m$ be as in Setting 19.1, where m is either a positive integer or $m = \infty$ and $\dim(R) = d \ge 2$. Then:*

(1) *$\{\tau_i\}_{i=1}^m$ is primarily independent over R $\iff$ $\{\tau_i\}_{i=1}^m$ is primarily independent over R^h.*

(2) *$\{\tau_i\}_{i=1}^m$ is residually algebraically independent over R $\iff$ $\{\tau_i\}_{i=1}^m$ is residually algebraically independent over R^h.*
(3) *(Descent) If $\{\tau_i\}_{i=1}^m$ is idealwise independent over R^h then $\{\tau_i\}_{i=1}^m$ is idealwise independent over R.*

Proof. For (1) and (2) it suffices to show the equivalence for every positive integer $n \le m$. By [**104**, (43.5)]. the local rings $R_n = R[\tau_1, \dots, \tau_n]_{(\mathbf{m},\tau_1,\dots,\tau_n)}$ and $\tilde{R}_n = R^h[\tau_1, \dots, \tau_n]_{(\mathbf{m},\tau_1,\dots,\tau_n)}$ have the same Henselization R_n^h. Also $R_n \subseteq \tilde{R}_n$. By Theorem 20.4 and Proposition 20.20 we have:

$$\begin{aligned} \tau_1, \dots, \tau_n \text{ are primarily (respectively residually algebraically)}& \\ \text{independent over } R \quad & \iff \\ R_n \longrightarrow \widehat{R} \text{ satisfies } LF_{d-1} \text{ (respectively } LF_1) & \iff \\ R_n^h \longrightarrow \widehat{R} = \widehat{R}^h \text{ satisfies } LF_{d-1} \text{ (respectively } LF_1) & \iff \\ \tilde{R}_n \longrightarrow \widehat{R} \text{ satisfies } LF_{d-1} \text{ (respectively } LF_1).& \end{aligned}$$

The third statement on idealwise independence follows from Theorem 20.12.3 by considering

$$\begin{array}{ccc} \tilde{R}_n & \xrightarrow{\varphi'} & \widehat{R} \\ {\scriptstyle\mu}\uparrow & & \| \\ R_n & \xrightarrow{\varphi} & \widehat{R}\,. \end{array}$$

□

Remark 20.22. The examples given in Theorems 19.33 and 19.35 show the converse to part (3) of (20.21) fails: weak flatness need not lift to the Henselization. With the notation of Proposition 20.20, if φ is weakly flat, then for every $P \in \operatorname{Spec}(A)$ of height one with $PB \neq B$ there exists by Proposition 12.9, $Q \in \operatorname{Spec}(B)$ of height one such that $P = Q \cap A$. In the Henselization A^h of A, the ideal PA^h is a finite intersection of height-one prime ideals P_i' of A^h [**104**, (43.20)]. Only one of the P_i' is contained in Q. Thus as in Theorems 19.33 and 19.35, one of the minimal prime divisors P_i' may fail the condition for weak flatness.

Let R be an excellent normal local domain with Henselization R^h and let K, and K^h denote the fields of fractions of R and R^h respectively. Let L be an intermediate field with $K \subseteq L \subseteq K^h$. It is shown in [**122**] that the intersection ring $T = L \cap \widehat{R}$ is an excellent normal local domain with Henselization $T^h = R^h$. Henselian excellent normal local domains are algebraically closed in their completion; see Remark 14.16.2. Thus we have:

Corollary 20.23. *Let $(R, \mathbf{m})$ and $\{\tau_i\}_{i=1}^m$ be as in Setting 19.1, where m denotes a positive integer or $m = \infty$. Let T be a Noetherian local domain dominating R and algebraic over R and dominated by $\widehat{R}$ with $\widehat{R} = \widehat{T}$. Then:*
(1) *$\{\tau_i\}_{i=1}^m$ is primarily independent over R $\iff$ $\{\tau_i\}_{i=1}^m$ is primarily independent over T.*
(2) *$\{\tau_i\}_{i=1}^m$ is residually algebraically independent over R $\iff$ $\{\tau_i\}_{i=1}^m$ is residually algebraically independent over T.*
(3) *If $\{\tau_i\}_{i=1}^m$ is idealwise independent over T, then $\{\tau_i\}_{i=1}^m$ is idealwise independent over R.*

PROOF. By [**122**], R and T have a common Henselization, and the statements follow from Corollary 20.21. $\square$

We have seen in Theorem 19.27 that, if R has the property that every height-one prime ideal is the radical of a principal ideal and $\tau \in \widehat{\mathbf{m}}$ is residually algebraically independent over R, then τ is idealwise independent over R. In Proposition 20.24 we describe a situation in which idealwise independence implies residual algebraic independence.

PROPOSITION 20.24. *Let $(R, \mathbf{m})$ and $\tau \in \widehat{m}$ be as in Setting 19.1. Suppose R has the property that, for each $P \in \operatorname{Spec}(R)$ with $\operatorname{ht}(P) = 1$, the ideal $P\widehat{R}$ is prime.*

(1) *If τ is idealwise independent over R, then τ is residually algebraically independent over R.*
(2) *If R has the additional property that every height-one prime ideal is the radical of a principal ideal, then τ is idealwise independent over $R \iff \tau$ is residually algebraically independent over R.*

PROOF. For item 1, let $\widehat{P} \in \operatorname{Spec}(\widehat{R})$ be such that $\operatorname{ht}(\widehat{P}) = 1$ and $\widehat{P} \cap R \neq 0$. Then $\operatorname{ht}(\widehat{P} \cap R) = 1$ and $(\widehat{P} \cap R)R[\tau]$ is a prime ideal of $R[\tau]$ of height 1. Idealwise independence of τ implies that $(\widehat{P} \cap R)R[\tau] = (\widehat{P} \cap R)\widehat{R} \cap R[\tau]$. Since $(\widehat{P} \cap R)\widehat{R}$ is nonzero and prime, we have $\widehat{P} = (\widehat{P} \cap R)\widehat{R}$ and $\widehat{P} \cap R[\tau] = (\widehat{P} \cap R)R[\tau]$. Therefore $\operatorname{ht}(\widehat{P} \cap R[\tau]) = 1$ and Theorem 19.26 implies that τ is residually algebraically independent over R.

Item 1 implies item 2 by Theorem 19.27.3. $\square$

REMARK 20.25. If R is Henselian, or if R/P is Henselian for each height-one prime P of R, then R has the property that, for each $P \in \operatorname{Spec}(R)$ with $\operatorname{ht}(P) = 1$, the ideal $P\widehat{R}$ is prime, as in the hypothesis of Proposition 20.24. To see this, let P be a height-one prime of R such that R/P is Henselian. Then the integral closure of the domain R/P in its field of fractions is again local, in fact an excellent normal local domain and so analytically normal. This implies that the extended ideal $P\widehat{R}$ is prime, because of the behavior of completions of finite integral extensions [**104**, (17.7), (17.8)]. There is an example in [**6**] of a normal Noetherian local domain R that is not Henselian but, for each prime ideal P of R of height-one, the domain R/P is Henselian.

It is unclear whether Proposition 20.24 extends to more than one algebraically independent element $\tau \in \widehat{\mathbf{m}}$, because even if R is Henselian, the localized polynomial ring $R[\tau]_{(\mathbf{m},\tau)}$ fails to be Henselian.

COROLLARY 20.26. *Let R be an excellent Henselian normal local domain of dimension 2, and assume the notation of Setting 19.1. Then:*

(1) *τ is residually algebraically independent over $R \iff \tau$ is primarily independent over R.*
(2) *Either of these equivalent conditions implies τ is idealwise independent over R.*
(3) *If R has the additional property that every height-one prime ideal is the radical of a principal ideal, then the three conditions are equivalent.*

PROOF. This follows from Theorem 19.27, Proposition 19.15.1 and Proposition 20.24. $\square$

20.4. Summary diagram for the independence concepts

With the notation of Setting 19.1 for $R, \mathbf{m}, R_n, \tau_1 \dots, \tau_n$, let $d = \dim(R)$, L the field of fractions of R_n, $\mathbf{p} \in \operatorname{Spec}(R_n)$ such that $\dim(R_n/\mathbf{p}) \le d-1$, $P \in \operatorname{Spec}(R_n)$ with $\operatorname{ht}(P) = 1$, $\widehat{P} \in \operatorname{Spec}(\widehat{R})$ with $\operatorname{ht}(\widehat{P}) = 1$, $R^h =$ the Henselization of R in $\widehat{R}$, T a local Noetherian domain dominating and algebraic over R and dominated by $\widehat{R}$ with $\widehat{R} = \widehat{T}$, z an indeterminate over the field of fractions of $\widehat{R}$ and Z a possibly uncountable set of set of indeterminates over the field of fractions of $\widehat{R}$. Then we have the implications shown below. We use the abbreviations "prim. ind.", "res. ind." and "idw. ind" for "primarily independent", "residually independent" and "idealwise independent".

NOTE 20.27. $R_n \hookrightarrow \widehat{R}$ is always height-one preserving by Proposition 19.8.

$\boxed{R_n \hookrightarrow \widehat{R}\, LF_{d-1}(12.13)}$	(20.4.2)	$\boxed{\tau_i \text{ prim. ind. } /R\,(19.12)}$	(20.18.1)	$\boxed{\tau_i \text{ prim. ind. } /R(Z)\,(19.12)}$
		(19.12)		
(12.13)		$\boxed{\forall \mathbf{p}, \mathbf{p}\widehat{R} \text{ is } \widehat{\mathbf{m}}\text{-primary}}$	(20.23.1)	$\boxed{\tau_i \text{ prim. ind.}/T\,(19.12)}$
		(19.27.1)		
$\boxed{R_n \hookrightarrow \widehat{R}\, LF_1(12.13)}$	(20.4.1)	$\boxed{\tau_i \text{ res. ind. } /R\,(19.24)}$	(20.18.2)	$\boxed{\tau_i \text{ res. ind. } /R(Z)\,(19.24)}$
(12.12)		(19.26)		
$\boxed{R_n \hookrightarrow \widehat{R} \text{ PDE } (19.5)}$	(19.26)	$\boxed{\operatorname{ht}(\widehat{P} \cap R_i) \le 1, \forall \widehat{P} \forall i}$	(20.23.2)	$\boxed{\tau_i \text{ res. ind. } /T\,(19.24)}$
(19.5)		(19.26)		
$\boxed{\operatorname{ht}(\widehat{P} \cap R_n) \le 1, \forall \widehat{P}}$	(19.26)	$\boxed{\tau_i \text{ res. ind. } /R\,(19.24)}$	(20.13.1)	$\boxed{\tau_i \text{ res. ind. } /R[z]\,(19.24)}$
(19.9) *		(19.27.2)*		(19.27.2)*
$\boxed{R_n \hookrightarrow \widehat{R} \text{ wf. } (19.5)}$	(19.11.1)	$\boxed{\tau_i \text{ idw. ind. } /R(19.2)}$	(20.13.2)	$\boxed{\tau_i \text{ idw. ind. } /R[z]_{(\mathbf{m},z)}(19.24)}$
(19.5)		(19.2)		
$\boxed{P\widehat{R} \cap R_n = P, \forall P}$		$\boxed{\widehat{R} \cap L = R_n}$	(20.18.3)	$\boxed{\tau_i \text{ idw. ind. } /R(Z)\,(19.24)}$

* We assume that every height-one prime ideal of R is a principal ideal in order to have these arrows.

CHAPTER 21

Rings between excellent normal local domains and their completions I

In this chapter we continue the investigation of Chapters 4 through 13. We adjust the focus to include as base rings certain Krull domains that are not necessarily Noetherian. We take our working setting here to be Krull domains because in the setting of Krull domains it is possible to iterate the construction. The intersection of a normal Noetherian domain with a subfield of its field of fractions is always a Krull domain, but may fail to be Noetherian. As in Chapters 4 to 13 we consider completions with respect to a principal ideal.

For an excellent normal local domain $(R, \mathbf{m})$, the construction in Chapters 19 and 20 uses the entire $\mathbf{m}$-adic completion rather than a completion with respect to a principal ideal. With $(S, \mathbf{n})$ a localized polynomial ring in several variables over R, Chapters 19 and 20 contain non-trivial examples of ideals $\mathbf{a}$ of the $\mathbf{n}$-adic completion $\widehat{S}$ of S such that the constructed ring $D := \mathcal{Q}(S) \cap (\widehat{S}/\mathbf{a})$ of Homomorphic Image Construction 5.6 results in the ring $D = \mathcal{Q}(S) \cap \widehat{S}/\mathbf{a} = S$. Chapters 19 and 20 also contain examples of subfields L of the field of fractions of $\widehat{R}$ such that the ring $D := L \cap \widehat{R}$ of Inclusion Construction 5.3 is a localized polynomial ring over R in finitely many or infinitely many variables. In particular, this gives examples where the intersection ring is a non-Noetherian Krull domain.

We use the completion with respect to a principal ideal again in this chapter because in most examples of new Noetherian domains produced using Homomorphic Image Construction 5.6, that is, $D = \mathcal{Q}(S) \cap (\widehat{S}/\mathbf{a})$, the ideal $\mathbf{a}$ of $\widehat{S}$ is extended from a completion of R with respect to a principal ideal. Furthermore the completion with respect to a larger ideal can be obtained by appropriately iterating the procedure with the completion with respect to a principal ideal.

Let z be a nonzero nonunit of a Noetherian integral domain R and let R^* denote the (z)-adic completion of R. Let $\tau_1, \dots, \tau_s$ be elements of zR^* that are algebraically independent over R. Assume that every nonzero element of the polynomial ring $R[\tau_1, \dots, \tau_s]$ is a regular element of R^*. In Definition 6.5, we define $\tau_1, \dots, \tau_s$ to be *limit-intersecting* over R if the intersection domain A is equal to the approximation domain B as defined in Section 6.1.

We also investigate here two stronger forms of the limit-intersecting condition, given in Definitions 21.9; these are useful for constructing examples and for determining if A is Noetherian or excellent. We give criteria for $\tau_1, \tau_2, \dots, \tau_s$ to have these properties. These properties are analogs to types of "idealwise independence" over R defined in Chapter 19. These modified independence conditions enable us to produce concrete examples illustrating the concepts.

Many concepts from our earlier chapters are useful in this study, including several flatness conditions for extensions of Krull domains. For convenience, we recall some definitions that are relevant for this chapter:

DEFINITIONS 21.1. Let $S \hookrightarrow T$ be an extension of Krull domains.

- T is a *PDE* extension of S if for every height-one prime ideal Q in T, the height of $Q \cap S$ is at most one.
- T is a *height-one preserving* extension of S if for every height-one prime ideal P of S with $PT \neq T$ there exists a height-one prime ideal Q of T with $PT \subseteq Q$.
- T is *weakly flat* over S if every height-one prime ideal P of S with $PT \neq T$ satisfies $PT \cap S = P$.
- Let $r \in \mathbb{N}$ be an integer with $1 \leq r \leq d = \dim(T)$ where d is an integer or $d = \infty$. Then φ is called *locally flat in height* r, abbreviated LF_r, if, for every prime ideal Q of T with $\operatorname{ht}(Q) \leq r$, the induced map on the localizations $\varphi_Q : S_{Q \cap S} \longrightarrow T_Q$ is faithfully flat.

We recall the following proposition, a restatement of Corollary 12.4 of Chapter 12.

PROPOSITION 21.2. *Let $\varphi : S \hookrightarrow T$ be an extension of Krull domains and let F denote the field of fractions of S.*

1. *Assume that $PT \neq T$ for every height-one prime ideal P of S. Then $S \hookrightarrow T$ is weakly flat $\iff$ $S = F \cap T$.*
2. *If $S \hookrightarrow T$ is weakly flat, then φ is height-one preserving and, moreover, for every height-one prime ideal P of S with $PT \neq T$, there is a height-one prime ideal Q of T with $Q \cap S = P$.*

REMARK 21.3. By Proposition 21.2 weakly flat extensions are height-one preserving. Example 12.10 of Chapter 12 shows that the height-one preserving condition does not imply weakly flat.

21.1. Intersections and directed unions

In general the intersection of a normal Noetherian domain with a subfield of its field of fractions is a Krull domain, but need not be Noetherian. The Krull domain B in the motivating example of Section 4.3 and Chapter 7 (in the case where $B \neq A$) illustrates that a directed union of normal Noetherian domains may be a non-Noetherian Krull domain. Thus, in order to apply an iterative procedure in Section 21.2, we consider a local Krull domain $(T, \mathbf{n})$ that is not assumed to be Noetherian, but is assumed to have a Noetherian completion. To distinguish from the earlier Noetherian hypothesis on R, we let T denote the base domain.

SETTING AND NOTATION 21.4. Let $(T, \mathbf{n})$ be a local Krull domain with field of fractions F. Assume there exists a nonzero element $y \in \mathbf{n}$ such that the y-adic completion $(\widehat{T, (y)}) := (T^*, \mathbf{n}^*)$ of T is an analytically normal Noetherian local domain. It then follows that the $\mathbf{n}$-adic completion $\widehat{T}$ of T is also a normal Noetherian local domain, since the $\mathbf{n}$-adic completion of T is the same as the $\mathbf{n}^*$-adic completion of T^*. Let F^* denote the field of fractions of T^*. Since T^* is Noetherian, $\widehat{T}$ is faithfully flat over T^* and we have $T^* = \widehat{T} \cap F^*$. Therefore $F \cap T^* = F \cap \widehat{T}$.

Let d denote the dimension of the Noetherian domain T^*. It follows that d is also the dimension of $\widehat{T}$. [1]

(1) Assume that $T = F \cap T^* = F \cap \widehat{T}$, or equivalently by (21.2.1), that T^* and $\widehat{T}$ are weakly flat over T.

(2) Let $\widehat{T}[1/y]$ denote the localization of $\widehat{T}$ at the powers of y, and similarly, let $T^*[1/y]$ denote the localization of T^* at the powers of y. The domains $\widehat{T}[1/y]$ and $T^*[1/y]$ have dimension $d-1$.

(3) Let $\tau_1, \dots, \tau_s \in \mathbf{n}^*$ be algebraically independent over F.

(4) For each i with $1 \le i \le s$, we have an expansion $\tau_i := \Sigma_{j=1}^{\infty} c_{ij} y^j$ where $c_{ij} \in T$.

(5) For each $n \in \mathbb{N}$ and each i with $1 \le i \le s$, we define the n^{th}-endpiece of τ_i with respect to y as in Notation 6.1, so that

$$\tau_{in} := \Sigma_{j=n+1}^{\infty} c_{ij} y^{j-n}, \qquad \tau_{in} = y\tau_{i,n+1} + c_{i,n+1} y.$$

(6) For each $n \in \mathbb{N}$, we define $B_n := T[\tau_{1n}, \dots, \tau_{sn}]_{(\mathbf{n}, \tau_{1n}, \dots, \tau_{sn})}$. In view of (5), we have $B_n \subseteq B_{n+1}$ and B_{n+1} dominates B_n for each n. We define

$$B := \varinjlim_{n \in \mathbb{N}} B_n = \bigcup_{n=1}^{\infty} B_n, \qquad \text{and} \qquad A := F(\tau_1, \dots, \tau_s) \cap \widehat{T}.$$

Thus B and A are local domains. We show that B and A are local Krull domains and that A birationally dominates B in Theorem 21.8. We are especially interested in conditions which imply that $B = A$.

(7) Let A^* denote the y-adic completion of A and let B^* denote the y-adic completion of B.

Remark 21.5. The iterative example of Section 4.3 and Chapter 7 as given in Theorem 7.2 with $T := B \neq A$ (from the notation of (7.1) and Example 7.3 continued in Example 8.11) shows that $T \longrightarrow T^*[1/y]$ can satisfy the other conditions of (21.4) but not satisfy the assumption (21.4.1); that is, such an extension is not in general weakly flat.

We show that the definitions of B and B_n are independent of the representations for $\tau_1, \dots, \tau_s$ as power series in y with coefficients in T; the proof is analogous to that of Proposition 6.3.

Proposition 21.6. *The definitions of B and B_n are independent of representations for $\tau_1, \dots, \tau_s$ as power series in y with coefficients in T.*

Proof. For $1 \le i \le s$, assume that τ_i and $\omega_i = \tau_i$ have representations

$$\tau_i = \sum_{j=1}^{\infty} a_{ij} y^j \qquad \text{and} \qquad \omega_i = \sum_{j=1}^{\infty} b_{ij} y^j,$$

where each $a_{ij}, b_{ij} \in T$. We define the n^{th}-endpieces τ_{in} and ω_{in} as in Section 6.1:

$$\tau_{in} = \sum_{j=n+1}^{\infty} a_{ij} y^{j-n} \qquad \text{and} \qquad \omega_{in} = \sum_{j=n+1}^{\infty} b_{ij} y^{j-n}.$$

[1] If T is Noetherian, then d is also the dimension of T. However, if T is not Noetherian, then the dimension of T may be greater than d. This is illustrated by taking T to be the ring B of Example 7.3.

Then we have

$$\tau_i = \Sigma_{j=1}^{\infty} a_{ij} y^j = \Sigma_{j=1}^{n} a_{ij} y^j + y^n \tau_{in} = \Sigma_{j=1}^{\infty} b_{ij} y^j = \Sigma_{j=1}^{n} b_{ij} y^j + y^n \omega_{in} = \omega_i.$$

Therefore, for $1 \le i \le s$ and each positive integer n,

$$y^n \tau_{in} - y^n \omega_{in} = \Sigma_{j=1}^{n} b_{ij} y^j - \Sigma_{j=1}^{n} a_{ij} y^j, \quad \text{and so} \quad \tau_{in} - \omega_{in} = \frac{\Sigma_{j=1}^{n}(b_{ij} - a_{ij}) y^j}{y^n}.$$

Since $\Sigma_{j=1}^{n}(b_{ij} - a_{ij}) y^j \in T$ is divisible by y^n in T^* and $T = F \cap T^*$, it follows that y^n divides $\Sigma_{j=1}^{n}(b_{ij} - a_{ij}) y^j$ in T. Therefore $\tau_{in} - \omega_{in} \in T$. It follows that B_n and $B = \cup_{n=1}^{\infty} B_n$ are independent of the representation of the τ_i. $\square$

Theorems 21.7 and 21.8 are adaptations of Construction Properties Theorem 6.19, Theorem 6.21 and Noetherian Flatness Theorem 8.8 of Chapters 6 and 8 that hold in Setting 21.4 even though the base ring T might not be Noetherian.

THEOREM 21.7. *Assume the setting and notation of (21.4). Then the intermediate rings B_n, B and A have the following properties:*

1. *$yA = yT^* \cap A$ and $yB = yA \cap B = yT^* \cap B$. More generally, for every $t \in \mathbb{N}$, we have $y^tA = y^tT^* \cap A$ and $y^tB = y^tA \cap B = y^tT^* \cap B$.*
2. *$T/y^tT = B/y^tB = A/y^tA = T^*/y^tT^*$, for each positive integer t.*
3. *Every ideal of T, B or A that contains y is finitely generated by elements of T. In particular, the maximal ideal $\mathbf{n}$ of T is finitely generated, and the maximal ideals of B and A are $\mathbf{n}B$ and $\mathbf{n}A$.*
4. *For every $n \in \mathbb{N}$: $yB \cap B_n = (y, \tau_{1n}, \dots, \tau_{sn})B_n$, an ideal of B_n of height $s+1$.*
5. *Let $P \in \operatorname{Spec}(A)$ be minimal over yA, and let $Q = P \cap B$ and $W = P \cap T$. Then $T_W \subseteq B_Q = A_P$, and all three localizations are DVRs.*
6. *For every $n \in \mathbb{N}$, $B[1/y]$ is a localization of B_n, i.e., for each $n \in \mathbb{N}$, there exists a multiplicatively closed subset S_n of B_n such that $B[1/y] = S_n^{-1}B_n$.*
7. *$B = B[1/y] \cap B_{\mathbf{q}_1} \cap \dots \cap B_{\mathbf{q}_r}$, where $\mathbf{q}_1, \dots, \mathbf{q}_r$ are the prime ideals of B minimal over yB.*

PROOF. Let $K := F(\tau_1, \dots \tau_s)$, the field of fractions of A and B. Then $A = T^* \cap K$ implies $yA \subseteq yT^* \cap A$. Let $g \in yT^* \cap A \subseteq yT^* \cap K$. Then $g/y \in T^* \cap K = A \implies g \in yA$. Since $B = \bigcup_{n=1}^{\infty} B_n$, we have $yB = \bigcup_{n=1}^{\infty} yB_n$. It is clear that $yB \subseteq yA \cap B \subseteq yT^* \cap B$. We next show $yT^* \cap B = yB$. Let $g \in yT^* \cap B$. Then there is an $n \in \mathbf{N}$ with $g \in B_n$ and, multiplying g by a unit of B_n if necessary, we may assume that $g \in T[\tau_{1n}, \dots, \tau_{sn}]$. Write $g = r_0 + g_0$ where $g_0 \in (\tau_{1n}, \dots, \tau_{sn})T[\tau_{1n}, \dots, \tau_{sn}]$ and $r_0 \in T$. Substituting $\tau_{jn} = y\tau_{jn+1} + c_{jn}y \in yT^*$ from (21.4.5) yields that $g_0 \in yT^*$ and so $r_0 \in yT^* \cap T = yT$. Since by (21.4.5), $(\tau_{1n}, \dots, \tau_{sn})B_n \subseteq yB_{n+1}$, it follows that $g \in yB$. Now $yB = yT^* \cap B$ implies $y^2B = y(yT^* \cap B) = y^2T^* \cap yB = y^2T^* \cap B$. Similarly $y^tB = y^tT^* \cap B$ for every $t \in \mathbb{N}$.

Since $y^tT^* \cap T = y^tT$, $T/y^tT = T^*/y^tT^*$, and $T/(y^tT) \hookrightarrow B/(y^tB) \hookrightarrow A/(y^tA) \hookrightarrow T^*/y^tT^*$, the assertions in item 2 follow.

Since T^* is Noetherian, item 3 follows from item 2.

For item 4, let $f \in yB \cap B_n$. After multiplication by a unit of B_n, we may assume that $f \in T[\tau_{1n}, \dots, \tau_{sn}]$, and hence f is of the form

$$f = \sum_{(i) \in \mathbf{N}^s} a_{(i)} \tau_{1n}^{i_1} \dots \tau_{sn}^{i_s}$$

with $a_{(i)} \in T$. Since $\tau_{jn} \in yB$, we see that $a_{(0)} \in yB \cap T \subseteq yT^* \cap T$, and we can write $a_{(0)} = y\widehat{b}$ for some element $\widehat{b} \in T^*$. This implies that $\widehat{b} \in F \cap T^* = T$; the last equality uses (21.4.1). Therefore $a_{(0)} \in yT$ and $f \in (y, \tau_{1n}, \dots, \tau_{sn})B_n$. Furthermore if $g \in (y, \tau_{1n}, \dots, \tau_{sn})B_n$, then $\tau_{in} \subseteq yB \cap B_n$, so $g \in yB \cap B_n$.

For item 5, since T^* and hence A is Krull, P has height one and A_P is a DVR. Also A_P has the same fraction field as B_Q. By (2), W is a minimal prime of yT. Since T is a Krull domain, T_W is a DVR and the maximal ideal of T_W is generated by $u \in T$. Thus by item 2 the maximal ideal of B_Q is generated by u and so B_Q is a DVR dominated by A_P. Therefore they must be the same DVR.

Item 6 follows from (21.4.5).

For item 7, suppose $\beta \in B[1/y] \cap B_{\mathbf{q}_1} \cap \cdots \cap B_{\mathbf{q}_r}$. Now $B_{\mathbf{q}_1} \cap \cdots \cap B_{\mathbf{q}_r} = (B \setminus (\cup\, \mathbf{q}_i))^{-1}B$. There exist $t \in \mathbb{N}, a, b, c \in B$ with $c \notin \mathbf{q}_1 \cup \cdots \cup \mathbf{q}_r$ such that $\beta = a/y^t = b/c$. We may assume that either $t = 0$ (and we are done) or that $t > 0$ and $a \notin yB$. Since $yB = yA \cap B$, it follows that $\mathbf{q}_1, \dots, \mathbf{q}_r$ are the contractions to B of the minimal primes $\mathbf{p}_1, \dots, \mathbf{p}_r$ of yA in A. Since A is a Krull domain, $A = A[1/y] \cap A_{\mathbf{p}_1} \cap \cdots \cap A_{\mathbf{p}_n}$. Thus $\beta \in A$, and $a = y^t\beta \in yA \cap B = yB$, a contradiction. Thus $t = 0$ and $\beta = a \in B$. $\square$

THEOREM 21.8. *With the setting and notation of (21.4), the intermediate rings A and B have the following properties:*

1. *A and B are local Krull domains.*
2. *$B \subseteq A$, with A dominating B.*
3. *$A^* = B^* = T^*$.*
4. *If B is Noetherian, then $B = A$.*

Moreover, if T is a unique factorization domain (UFD) and y is a prime element of T, then B is a UFD.

PROOF. As noted in the proof of Theorem 21.7, A is a Krull domain. By (21.7.6), $B[1/y]$ is a localization of B_0. Since B_0 is a Krull domain, it follows that $B[1/y]$ is a Krull domain. By (21.7.7), B is the intersection of $B[1/y]$ and the DVR's $B_{\mathbf{q}_1}, \dots, B_{\mathbf{q}_r}$. Therefore B is a Krull domain. Items 2 and 3 are immediate from Theorem 21.7. If B is Noetherian, then B^* is faithfully flat over B, and hence $B = F(\tau_1, \dots, \tau_s) \cap B^* = A$. For the last statement, if T is a UFD, so is the localized polynomial ring B_0. By (21.7.6), $B[1/y] = S_0^{-1}B_0[1/y]$, which implies that $B[1/y]$ is also a UFD. By (21.7.2), y is a prime element of B; hence it follows from Theorem 2.10 that B is a UFD. $\square$

21.2. Limit-intersecting elements

Let $(R, \mathbf{m})$ be an excellent normal local domain and let $\widehat{R}$ be the $\mathbf{m}$-adic completion of R. We are interested in the structure of $L \cap \widehat{R}$, for intermediate fields L between the fields of fractions of R and $\widehat{R}$. This structure is difficult to determine in general. We show in Theorem 21.14 that each of the *limit-intersecting* properties of Definitions 21.9 implies $L \cap \widehat{R}$ is a directed union of localized polynomial ring extensions of R. These limit-intersecting properties are related to the idealwise independence concepts defined in Chapter 19 and to the LF_d properties defined in Definitions 21.1.

DEFINITIONS 21.9. Let $(T, \mathbf{n})$ be a local Krull domain, let $0 \neq y \in \mathbf{n}$ be such that the y-adic completion $(\widehat{T, (y)}) := (T^*, \mathbf{n}^*)$ of T is an analytically normal Noetherian local domain of dimension d. Assume that T^* and $\widehat{T}$ are weakly flat over T. Let $\tau_1, \dots, \tau_s \in \mathbf{n}^*$ be algebraically independent over T as in Setting 21.4.

(1) The elements $\tau_1, \dots, \tau_s$ are said to be *limit-intersecting* in y over T provided the approximation domain B and the intersection domain A defined in Notation 21.4.6 are equal.

(2) The elements $\tau_1 \dots, \tau_s$ are said to be *residually limit-intersecting* in y over T provided the inclusion map

$$B_0 := T[\tau_1, \dots, \tau_s]_{(\mathbf{n}, \tau_1, \dots, \tau_s)} \longrightarrow T^*[1/y] \quad \text{is } LF_1 \qquad (21.9.2).$$

(3) The elements $\tau_1 \dots, \tau_s$ are said to be *primarily limit-intersecting* in y over T provided the inclusion map

$$B_0 := T[\tau_1, \dots, \tau_s]_{(\mathbf{n}, \tau_1, \dots, \tau_s)} \longrightarrow T^*[1/y] \quad \text{is flat.} \qquad (21.9.3).$$

Since $T^*[1/y]$ and $\widehat{T}[1/y]$ have dimension $d-1$, the condition LF_{d-1} is equivalent to primarily limit-intersecting, that is, to the flatness of the map $B_0 \longrightarrow T^*[1/y]$.

REMARKS 21.10. We show in Theorem 21.14 that the elements $\tau_1, \cdots, \tau_s$ are limit-intersecting in the sense of Definition 21.9.1 if and only if the inclusion map

$$B_0 := T[\tau_1, \dots, \tau_s]_{(\mathbf{n}, \tau_1, \dots, \tau_s)} \longrightarrow T^*[1/y] \quad \text{is weakly flat} \qquad (21.10.0).$$

Here are some other remarks concerning Definitions 21.9.

(1) The terms "residually" and "primarily" come from Chapter 19. We justify this terminology in Proposition 21.16) and Theorem 22.3. It is clear that primarily limit-intersecting implies residually limt-intersecting. By Theorem 19.8 if T is an excellent normal local domain, then the extension $B_0 \hookrightarrow T^*$ is height-one preserving. By Proposition 12.13 an extension of Krull domains that is height-one preserving and satisfies PDE is weakly flat.

(2) Since $\widehat{T}[1/y]$ is faithfully flat over $T^*[1/y]$, the statements obtained by replacing $T^*[1/y]$ by $\widehat{T}[1/y]$ give equivalent definitions to those of Definitions 21.9; see Propositions 20.9 and 20.11 of Chapter 20).

(3) We remark that

$$B \longrightarrow T^*[1/y] \text{ is weakly flat } \iff B \longrightarrow T^* \text{ is weakly flat .}$$

To see this, observe that by Theorem 21.7.2, every height-one prime of B containing y is the contraction of a height-one prime of T^*. If $\mathbf{p}$ is a height-one prime of B with $y \notin \mathbf{p}$, then $\mathbf{p}T^* \cap B = \mathbf{p}$ if and only if $\mathbf{p}T^*[1/y] \cap B = \mathbf{p}$.

(4) The ring $B[1/y]$ is a localization $S_0^{-1}B_0$ of B by Theorem 21.7.6. Since S_0 consists of units of $T^*[1/y]$, Remark 12.6.b implies the extension $B_0 \hookrightarrow T^*[1/y]$ is weakly flat if and only if the canonical map

$$S_0^{-1}B_0 = B[1/y] \longrightarrow T^*[1/y]$$

is weakly flat. In view of Proposition 21.11 below, we have $\tau_1, \dots, \tau_s$ are residually (resp. primarily) limit-intersecting in y over T if and only if the canonical map

$$S_0^{-1}B_0 = B[1/y] \longrightarrow T^*[1/y]$$

is LF_1 (resp. LF_{d-1} or equivalently flat).

(5) If $d = 2$, then obviously $LF_1 = LF_{d-1}$. Hence in this case primarily limit-intersecting is equivalent to residually limit-intersecting.

(6) Since $T \longrightarrow B_n$ is faithfully flat for every n, it follows [**12**, Chap.1, Sec.2.3, Prop.2, p.14] that $T \longrightarrow B$ is always faithfully flat. Thus if residually limit-intersecting elements exist over T, then $T \longrightarrow T^*[1/y]$ must be LF_1. If primarily limit-intersecting elements exist over T, then $T \longrightarrow T^*[1/y]$ must be flat.

(7) The examples of Remarks 12.10 and 21.5 show that in some situations T^* contains no limit-intersecting elements. Indeed, if T is complete with respect to some nonzero ideal I, and y is outside every minimal prime over I, then every element $\tau = \sum a_i y^i$ of T^* that is transcendental over T fails to be limit-intersecting in y. To see this, choose an element $x \in I$, x outside every minimal prime ideal of yT; define $\sigma := \sum a_i x^i \in T$. Then $\tau - \sigma \in (x-y)T^* \cap T[\tau]$. Thus a minimal prime over $x - y$ in T^* intersects $T[\tau]$ in an ideal of height greater than one, because it contains $x - y$ and $\tau - \sigma$.

Proposition 21.11. *Assume the notation and setting of (21.4) and let k be a positive integer with $1 \le k \le d-1$. Then the following are equivalent:*

(1) *The canonical injection $\varphi : B_0 := T[\tau_1, \dots, \tau_s]_{(\mathbf{m}, \tau_1, \dots, \tau_s)} \longrightarrow T^*[1/y]$ is LF_k.*
(1′) *The canonical injection $\varphi_1 : B_0 := T[\tau_1, \dots, \tau_s]_{(\mathbf{m}, \tau_1, \dots, \tau_s)} \longrightarrow \widehat{T}[1/y]$ is LF_k.*
(2) *The canonical injection $\varphi' : U_0 := T[\tau_1, \dots, \tau_s] \longrightarrow T^*[1/y]$ is LF_k.*
(2′) *The canonical injection $\varphi_1' : U_0 := T[\tau_1, \dots, \tau_s] \longrightarrow \widehat{T}[1/y]$ is LF_k.*
(3) *The canonical injection $\theta : B_n := R[\tau_{1n}, \dots, \tau_{sn}]_{(\mathbf{m}, \tau_{1n}, \dots, \tau_{sn})} \longrightarrow T^*[1/y]$ is LF_k.*
(3′) *The canonical injection $\theta_1 : B_n := R[\tau_{1n}, \dots, \tau_{sn}]_{(\mathbf{m}, \tau_{1n}, \dots, \tau_{sn})} \longrightarrow \widehat{T}[1/y]$ is LF_k.*
(4) *The canonical injection $\psi : B \longrightarrow T^*[1/y]$ is LF_k.*
(4′) *The canonical injection $\psi : B \longrightarrow \widehat{T}[1/y]$ is LF_k.*

Moreover, these statements are also all equivalent to LF_k of the corresponding canonical injections obtained by replacing B_0, U_0 and B by $B_0[1/y]$, $U_0[1/y]$ and $B[1/y]$.

Proof. We have:

$$U_0 \xrightarrow{\text{loc.}} B_0 \xrightarrow{\varphi} T^*[1/y] \xrightarrow{\text{f.f.}} \widehat{T}[1/y].$$

The injection $\varphi_1' : U_0 \longrightarrow \widehat{T}[1/y]$ factors as $\varphi' : U_0 \longrightarrow T^*[1/y]$ followed by the faithfully flat injection $T^*[1/y] \longrightarrow \widehat{T}[1/y]$. Therefore φ' is LF_k if and only if φ_1' is LF_k. The injection φ' factors through the localization $U_0 \longrightarrow B_0$ and so φ is LF_k if and only if φ' is LF_k.

Now set $U_n := T[\tau_{1n}, \dots, \tau_{sn}]$ for each $n > 1$ and $U := \bigcup_{n=0}^{\infty} U_n$. For each positive integer i, $\tau_i = y^n \tau_{in} + \sum_{i=0}^{n} a_i y^i$. Thus $U_n \subseteq U_0[1/y]$, and $U_0[1/y] = \bigcup U_n[1/y] = U[1/y]$. Moreover, for each n, B_n is a localization of U_n, and hence B is a localization of U.

We have:

$$\begin{aligned} B[1/y] \longrightarrow T^*[1/y] \text{ is } LF_k &\iff U[1/y] \longrightarrow T^*[1/y] \text{ is } LF_k \\ \iff U_0[1/y] \longrightarrow T^*[1/y] \text{ is } LF_k &\iff B_n[1/y] \longrightarrow T^*[1/y] \text{ is } LF_k \\ &\iff B_0[1/y] \longrightarrow T^*[1/y] \text{ is } LF_k\,. \end{aligned}$$

Thus

$$\begin{aligned} \psi : B \longrightarrow T^*[1/y] \text{ is } LF_k &\iff U \longrightarrow T^*[1/y] \text{ is } LF_k \\ \iff \varphi' : U_0 \longrightarrow T^*[1/y] \text{ is } LF_k &\iff \theta : B_n \longrightarrow T^*[1/y] \text{ is } LF_k \\ &\iff \varphi : B_0 \longrightarrow T^*[1/y] \text{ is } LF_k\,. \end{aligned}$$

□

REMARKS 21.12. (1) If $(T, \mathbf{n})$ is a one-dimensional local Krull domain, then T is a rank-one discrete valuation domain (DVR). Hence T^* is also a DVR and $T^*[1/y]$ is flat over $U_0 = T[\tau_1, \dots, \tau_s]$. Therefore, in this situation, $\tau_1, \dots, \tau_s$ are primarily limit-intersecting in y over T if and only if $\tau_1, \dots, \tau_s$ are algebraically independent over T.

(2) Let $\tau_1, \dots, \tau_s \in k[[y]]$ be transcendental over $k(y)$, where k is a field. Then $\tau_1, \dots, \tau_s$ are primarily limit-intersecting in y over $k[y]_{(y)}$ by (1) above. If $x_1, \dots, x_m$ are additional indeterminates over $k(y)$, then by Polynomial Example Theorem 9.2 and Noetherian Flatness Theorem 8.8, the elements $\tau_1, \dots, \tau_s$ are primarily limit-intersecting in y over $k[x_1, \dots, x_m, y]_{(x_1, \dots, x_m, y)}$.

(3) Assume the notation of Setting 21.4, and also assume that B is Noetherian. We show that $\tau_1, \dots, \tau_s$ are primarily limit-intersecting in y over T. Since T^* is the (y)-adic completion of B and B is Noetherian, it follows that T^* is flat over B. Hence $T^*[1/y]$ is also flat over B, and it follows from Proposition 21.11 that $\tau_1, \dots, \tau_s$ are primarily limit-intersecting in y over T.

(4) By the equivalence of (1) and (2) of Proposition 21.11, we see that $\tau_1, \dots, \tau_s$ are primarily limit-intersecting in y over T if and only if the endpiece power series $\tau_{1n}, \dots, \tau_{sn}$ are primarily limit-intersecting in y over T.

THEOREM 21.13. *Assume the notation of Setting 21.4. Thus $(T, \mathbf{n})$ is a local Krull domain with field of fractions F, and $y \in \mathbf{n}$ is such that the (y)-adic completion $(T^*, \mathbf{n}^*)$ of T is an analytically normal Noetherian local domain and $T = T^* \cap F$. For elements $\tau_1, \dots, \tau_s \in \mathbf{n}^*$ that are algebraically independent over T, the following are equivalent:*

(1) *The extension $T[\tau_1, \dots, \tau_s] \hookrightarrow T^*[1/y]$ is flat.*
(2) *The elements $\tau_1, \dots, \tau_s$ are primarily limit-intersecting in y over T.*
(3) *The intermediate rings A and B are equal and are Noetherian.*
(4) *The constructed ring B is Noetherian.*

Moreover, if these equivalent conditions hold, then the Krull domain T is Noetherian.

PROOF. By Theorem 21.7, we have $T/y^tT = B/y^tB = A/y^tA = T^*/y^tT^*$, for each positive integer t. By Definition 21.9.3, the elements $\tau_1 \dots, \tau_s$ are primarily limit-intersecting in y over T if and only if the inclusion map

$$B_0 := T[\tau_1, \dots, \tau_s]_{(\mathbf{n}, \tau_1, \dots, \tau_s)} \longrightarrow T^*[1/y]$$

is flat. Thus item 1 is equivalent to item 2. By Theorem 21.7.6, $B[1/y]$ is a localization of B_0. Hence flatness of the map $B_0 \hookrightarrow T^*[1/y]$ implies flatness of the map $B \hookrightarrow T^*[1/y]$. Applying Lemma 8.2 to the extension $B \hookrightarrow T^*$, we conclude that flatness of the map $B \hookrightarrow T^*[1/y]$ implies that T^* is flat over B and B is Noetherian. Therefore item 1 implies item 4. On the other hand, if B is Noetherian, then T^* is faithfully flat over B since T^* is the (y)-adic completion of B. Therefore $B = A$ and $B \hookrightarrow T^*[1/y]$ is flat. Thus item 4 is equivalent to item 3 and implies item 1. If these equivalent conditions hold, then $T \hookrightarrow T^*[1/y]$ is flat, and Lemma 8.2 implies that $T \hookrightarrow T^*$ is flat and T is Noetherian. $\square$

THEOREM 21.14. *Assume the notation of Setting 21.4. Thus $(T, \mathbf{n})$ is a local Krull domain with field of fractions F, and $y \in \mathbf{n}$ is such that the (y)-adic completion $(T^*, \mathbf{n}^*)$ of T is an analytically normal Noetherian local domain and $T = T^* \cap F$. For elements $\tau_1, \ldots, \tau_s \in \mathbf{n}^*$ that are algebraically independent over T, the following are equivalent:*

(1) *The elements $\tau_1, \ldots, \tau_s$ are limit-intersecting in y over T, that is, the intermediate rings A and B are equal.*
(2) *$B_0 \longrightarrow T^*[1/y]$ is weakly flat.*
(3) *$B \longrightarrow T^*[1/y]$ is weakly flat.*
(4) *$B \longrightarrow T^*$ is weakly flat.*

PROOF. (2)$\Rightarrow$(1): Since A and B are Krull domains with the same field of fractions and $B \subseteq A$ it is enough to show that every height-one prime ideal $\mathbf{p}$ of B is the contraction of a (height-one) prime ideal of A. By Theorem 21.7.3, each height-one prime of B containing yB is the contraction of a height-one prime of A.

Let $\mathbf{p}$ be a height-one prime of B which does not contain yB. Consider the prime ideal $\mathbf{q} = T[\tau_1, \ldots, \tau_s] \cap \mathbf{p}$. Since $B[1/y]$ is a localization of the ring $T[\tau_1, \ldots, \tau_s]$, we see that $B_{\mathbf{p}} = T[\tau_1, \ldots, \tau_s]_{\mathbf{q}}$ and so $\mathbf{q}$ has height one in $T[\tau_1, \ldots, \tau_s]$. The weakly flat hypothesis implies $\mathbf{q}T^* \cap T[\tau_1, \ldots, \tau_s] = \mathbf{q}$, and there is a height-one prime ideal $\mathbf{w}$ of T^* with $\mathbf{w} \cap T[\tau_1, \ldots, \tau_s] = \mathbf{q}$. This implies that $\mathbf{w} \cap B = \mathbf{p}$ and thus also $(\mathbf{w} \cap A) \cap B = \mathbf{p}$. Hence every height-one prime ideal of B is the contraction of a prime ideal of A. Since A is birational over B, this prime ideal of A can be chosen to have height one.

(3) $\iff$ (4): This is shown in Remark 21.10.3.

(1)$\Rightarrow$(4): If $B = A = F \cap T^*$, then by Proposition 21.2 every height-one prime ideal of B is the contraction of a height-one prime ideal of T^*.

(4)$\Rightarrow$(2): If $B \hookrightarrow T^*$ is weakly flat so is the localization $B[1/y] \hookrightarrow T^*[1/y]$. Since $B[1/y] = S_0^{-1}B_{0y}$ for a suitable multiplicative subset $S_0 \subseteq B_{0y}$ the embedding $B_{0y} \hookrightarrow T^*[1/y]$ is weakly flat. Now (2) holds by Remark 21.10.4. $\square$

REMARKS 21.15. (1) If an injective map of Krull domains is weakly flat, then it is height-one preserving by Proposition 21.2.2. Thus any of the equivalent conditions of Theorem 21.14 imply that $B \longrightarrow T^*$ is height-one preserving.

(2) In Theorem 21.14 if B is Noetherian, then by Theorem 21.8.4, $A = B$ and so all the conclusions of Theorem 21.14 hold.

(3) Example 13.8 yields the existence of a three-dimensional regular local domain $R = k[x, y, z]_{(x,y,z)}$, over an arbitrary field k, and an element $f = y\tau_1 + z\tau_2$ in the (x)-adic completion of R such that f is residually limit-intersecting in x over R, but fails to be primarily limit-intersecting in x over R. In particular, the rings A and B constructed using f are equal, yet A and B are not Noetherian. Here the elements τ_1 and τ_2 are chosen to be elements of $xk[[x]]$ that are algebraically independent over $k(x)$.

Proposition 21.16 gives criteria for elements to be residually limit-intersecting similar to those in Chapter 19 for elements to be residually algebraically independent. The corresponding result for primarily limit-intersecting is given in Theorem 22.3.

PROPOSITION 21.16. *With the setting and notation of (21.4) and $s = 1$, the following are equivalent:*

(1) *The element $\tau = \tau_1$ is residually limit-intersecting in y over T.*
(2) *If $\widehat{P}$ is a height-one prime ideal of $\widehat{T}$ such that $y \notin \widehat{P}$ and $\widehat{P} \cap T \neq 0$, then $\operatorname{ht}(\widehat{P} \cap T[\tau]_{(\mathbf{n},\tau)}) = 1$.*
(3) *For every height-one prime ideal P of T such that $y \notin P$ and for every minimal prime divisor $\widehat{P}$ of $P\widehat{T}$ in $\widehat{T}$, the image $\bar{\tau}$ of τ in $\widehat{T}/\widehat{P}$ is algebraically independent over T/P.*
(4) *$B \longrightarrow T^*[1/y]$ is LF_1.*

PROOF. For (1)$\Rightarrow$ (2), suppose (2) fails; that is, there exists a prime ideal $\widehat{P}$ of $\widehat{T}$ of height one such that $y \notin \widehat{P}$, $\widehat{P} \cap T \neq 0$, but $\operatorname{ht}(\widehat{P} \cap T[\tau]) \geq 2$. Let $\widehat{Q} := \widehat{P}\widehat{T}[1/y]$. Then $Q := \widehat{Q} \cap T[\tau]_{(\mathbf{n},\tau)}$ has height greater than or equal to 2. But by the definition of residually limit-intersecting in (21.9.2), the injective morphism $T[\tau]_{(\mathbf{n},\tau)} \longrightarrow \widehat{T}[1/y]$ is LF_1 and so by Definition 21.1), $(T[\tau]_{(\mathbf{n},\tau)})_Q \longrightarrow (\widehat{T[1/y]})_{\widehat{Q}}$ is faithfully flat, a contradiction to $\operatorname{ht}(Q) > \operatorname{ht}(\widehat{P}) = \operatorname{ht}(\widehat{Q})$.

For (2)$\Rightarrow$ (1), the argument of (1) $\Rightarrow$ (2) can be reversed since $(T[\tau]_{(\mathbf{n},\tau)})_Q \longrightarrow (\widehat{T}[1/y])_{\widehat{Q}}$ is faithfully flat.

For (3)$\Rightarrow$ (2), again suppose (2) fails; that is, there exists a prime ideal $\widehat{P}$ of $\widehat{T}$ of height one such that $y \notin \widehat{P}$, $\widehat{P} \cap T \neq 0$, but $\operatorname{ht}(\widehat{P} \cap T[\tau]) \geq 2$. Now $\operatorname{ht}(\widehat{P} \cap T) = 1$, since LF_1 holds for $T \hookrightarrow \widehat{T}$. Thus, with $P = \widehat{P} \cap T$, we have $PT[\tau] < \widehat{P} \cap T[\tau]$; that is, there exists $f(\tau) \in (\widehat{P} \cap T[\tau]) - PT[\tau]$, or equivalently there is a nonzero polynomial $\bar{f}(x) \in (T/(\widehat{P} \cap T))[x]$ so that $\bar{f}(\bar{\tau}) = \bar{0}$ in $T[\tau]/(\widehat{P} \cap T[\tau])$, where $\bar{\tau}$ denotes the image of τ in $\widehat{T}/\widehat{P}$. This means that $\bar{\tau}$ is algebraic over the field of fractions of $T/(\widehat{P} \cap T)$, a contradiction to (3).

For (2)$\Rightarrow$ (3), let $\widehat{P}$ be a height-one prime of $\widehat{R}$ such that $\widehat{P} \cap T = P \neq 0$. Since $\operatorname{ht}(\widehat{P} \cap T[\tau]) = 1$, $\widehat{P} \cap T[\tau] = PT[\tau]$ and $T[\tau]/(PT[\tau])$ canonically embeds in $\widehat{T}/\widehat{P}$. Thus the image of τ in $T[\tau]/PT[\tau]$ is algebraically independent over T/P.

For (1) $\iff$ (4), we see by (21.11) that (1) is equivalent to the embedding $\psi : B \longrightarrow T^*[1/y]$ being LF_1. $\square$

REMARK 21.17. Assume the notation of Setting 21.4. If T has the property that every height-one prime of T is the radical of a principal ideal, and τ is residually limit-intersecting in y over T, then the extension $B \hookrightarrow T^*[1/y]$ is height-one preserving by Remark 12.6,c, and hence weakly flat by Propositions 21.16, 12.12 and 12.13. Thus with these assumptions, if τ is residually limit-intersecting, then τ is limit-intersecting.

We have the following transitive property of limit-intersecting elements.

PROPOSITION 21.18. *Assume the setting and notation of (21.4). Also assume that $s > 1$ and for all $j \in \{1, \ldots s\}$, set $A(j) := F(\tau_1, \ldots, \tau_j) \cap \widehat{T}$. Then the following statements are equivalent:*

(1) *$\tau_1, \ldots, \tau_s$ are limit-intersecting, respectively, residually limit-intersecting, respectively, primarily limit-intersecting in y over T*
(2) *$\forall j \in \{1, \ldots, s\}$, the elements $\tau_1, \ldots, \tau_j$ are limit-intersecting, respectively, residually limit-intersecting, respectively, primarily limit-intersecting in y over T and the elements $\tau_{j+1}, \ldots, \tau_s$ are limit-intersecting, respectively,*

residually limit-intersecting, respectively, primarily limit-intersecting in y over $A(j)$.

(3) *There exists a $j \in \{1, \dots, s\}$, such that the elements $\tau_1, \dots, \tau_j$ are limit-intersecting, respectively, residually limit-intersecting, respectively, primarily limit-intersecting in y over T and the elements $\tau_{j+1}, \dots, \tau_s$ are limit-intersecting, respectively, residually limit-intersecting, respectively, primarily limit-intersecting in y over $A(j)$.*

PROOF. Set $B(j) := \bigcup_{n=1}^{\infty} T[\tau_{1n}, \dots, \tau_{jn}]_{(\mathbf{n}, \tau_{1n}, \dots, \tau_{jn})}$. That (2) implies (3) is clear.

For (3) $\Longrightarrow$ (1), items (21.14) and (21.10.1) imply that $A(j) = B(j)$ under each of the conditions on $\tau_1, \dots, \tau_j$. The definitions of $\tau_{j+1}, \dots, \tau_s$ being limit-intersecting, respectively, residually limit-intersecting, respectively, primarily limit-intersecting in y over $A(j)$ together with (21.10.4) imply the equivalence of the stated flatness properties for each of the maps

$$\begin{aligned}
\varphi_1 :& A(j)[\tau_{j+1}, \dots, \tau_s]_{(-)} && \longrightarrow A(j)^*[1/y] = T^*[1/y] \\
\varphi_2 :& (A(j)[\tau_{j+1}, \dots, \tau_s]_{(-)})[1/y] && \longrightarrow T^*[1/y] \\
\varphi_3 :& (B(j)[\tau_{j+1}, \dots, \tau_s]_{(-)})[1/y] && \longrightarrow T^*[1/y] \\
\varphi_4 :& (T[\tau_1, \dots, \tau_s]_{(-)})[1/y] && \longrightarrow T^*[1/y] \\
\varphi_5 :& T[\tau_1, \dots, \tau_s]_{(\mathbf{n}, \tau_1, \dots, \tau_s)} && \longrightarrow T^*[1/y].
\end{aligned}$$

Thus

$$\begin{aligned}
\psi : B \longrightarrow T^*[1/y] \text{ is } LF_k &\iff U \longrightarrow T^*[1/y] \text{ is } LF_k \\
&\iff \varphi' : U_0 \longrightarrow T^*[1/y] \text{ is } LF_k \\
&\iff \theta : B_n \longrightarrow T^*[1/y] \text{ is } LF_k \\
&\iff \varphi : B_0 \longrightarrow T^*[1/y] \text{ is } LF_k\,.
\end{aligned}$$

The respective flatness properties for φ_5 are equivalent to the conditions that $\tau_1, \dots, \tau_s$ be limit-intersecting, respectively, residually limit-intersecting, respectively, primarily limit-intersecting in y over T. Thus (3) $\Longrightarrow$ (1).

For (1) $\Longrightarrow$ (2), we go backwards: The statement of (1) for $\tau_1, \dots, \tau_s$ is equivalent to the respective flatness property for φ_5. This is equivalent to φ_4 and thus φ_3 having the respective flatness property. By (21.10.4), $B(j)[\tau_{j+1}, \dots, \tau_s]_{(-)} \longrightarrow T^*[1/y]$ has the appropriate flatness property. Also $B(j) \longrightarrow B(j)[\tau_{j+1}, \dots, \tau_s]_{(-)}$ is flat, and so $B(j) \longrightarrow T^*[1/y]$ has the appropriate flatness property. Thus $\tau_1, \dots, \tau_j$ are limit-intersecting, respectively, residually limit-intersecting, respectively, primarily limit-intersecting in y over T. Therefore $A(j) = B(j)$, and so $A(j) \longrightarrow T^*[1/y]$ has the appropriate flatness property. It follows that $\tau_{j+1}, \dots, \tau_s$ are limit-intersecting, respectively, residually limit-intersecting, respectively, primarily limit-intersecting in y over $A(j)$. □

21.3. A specific example where $B = A$ is non-Noetherian

Theorem 13.6 and Examples 13.8 yield examples where the constructed domains A and B are equal and are not Noetherian. We present in Theorem 21.19 a specific example of an excellent regular local domain $(R, \mathbf{m})$ of dimension three with $\mathbf{m} = (x, y, z)R$ and $\widehat{R} = \mathbb{Q}[[x, y, z]]$ such that there exists an element $\tau \in yR^*$, where

R^* is the (y)-adic completion of R, with τ limit-intersecting and residually limit-intersecting, but *not* primarily limit-intersecting in y over R. In this example we have $B = A$ and B is non-Noetherian.

THEOREM 21.19. *There exist an excellent regular local three-dimensional domain $(R, \mathbf{m})$ contained in $\mathbb{Q}[[x, y, z]]$, a power series ring in the indeterminates x, y, z over $\mathbb{Q}$, the rational numbers, with $\mathbf{m} = (x, y, z)R$, and an element τ in the (y)-adic completion R^* of R such that*

(21.19.1) *τ is residually limit-intersecting in y over R.*
(21.19.2) *τ is not primarily limit-intersecting in y over R.*
(21.19.3) *τ is limit-intersecting in y over R.*

In particular, the rings A and B constructed using τ and Notation 21.4.6 are equal, yet A and B fail to be Noetherian.

PROOF. We use the following elements of $\mathbb{Q}[[x, y, z]]$:

$$\gamma := e^x - 1 \in x\mathbb{Q}[[x]], \qquad \delta := e^{x^2} - 1 \in x\mathbb{Q}[[x]],$$
$$\sigma := \gamma + z\delta \in \mathbb{Q}[z]_{(z)}[[x]] \qquad \text{and} \qquad \tau := e^y - 1 \in y\mathbb{Q}[[y]].$$

For each n, we define the endpieces $\gamma_n, \delta_n, \sigma_n$ and τ_n as in (6.1), considering γ, δ, σ as series in x and τ as a series in y. Thus, for example,

$$\gamma = \sum_{i=1}^{\infty} a_i x^i; \qquad \gamma_n = \sum_{i=n+1}^{\infty} a_i x^{i-n}, \qquad \text{and} \quad x^n\gamma_n + \sum_{i=1}^{n} a_i x^i = \gamma.$$

(Here $a_i := 1/i!$.) The δ_n, σ_n satisfy similar relations. Therefore for each positive integer n,

(1) $\mathbb{Q}[x, \gamma_{n+1}, \delta_{n+1}]_{(x,\gamma_{n+1},\delta_{n+1})}$ birationally dominates $\mathbb{Q}[x, \gamma_n, \delta_n]_{(x,\gamma_n,\delta_n)}$,
(2) $\mathbb{Q}[x, \gamma_n, \delta_n]_{(x,\gamma_n,\delta_n)}$ birationally dominates $\mathbb{Q}[x, \gamma, \delta]_{(x,\gamma,\delta)}$, and
(3) $\mathbb{Q}[x, z, \sigma_{n+1}]_{(x,z,\sigma_{n+1})}$ birationally dominates $\mathbb{Q}[x, z, \sigma_n]_{(x,z,\sigma_n)}$. □

For our proof of Theorem 21.19 we first establish that certain subrings of $\mathbb{Q}[[x, y, z]]$ can be expressed as directed unions:

CLAIM 21.20. *For $V := \mathbb{Q}(x, \gamma, \delta) \cap \mathbb{Q}[[x]]$ and $D := \mathbb{Q}(x, z, \sigma) \cap \mathbb{Q}[z]_{(z)}[[x]]$, the equalities (*1)-(*5) of the diagram below hold. Furthermore the ring $V[z]_{(x,z)}$ is excellent and the canonical local embedding $\psi : D \longrightarrow V[z]_{(x,z)}$ is a direct limit of the maps $\psi_n : \mathbb{Q}[x, z, \sigma_n]_{(x,z,\sigma_n)} \longrightarrow \mathbb{Q}[x, z, \gamma_n, \delta_n]_{(x,z,\gamma_n,\delta_n)}$, where $\psi(\sigma_n) = \gamma_n + z\delta_n$.*

$$\mathbb{Q}[[x, y, z]]$$

$$\mathbb{Q}[[x, y]] \qquad\qquad A := \mathbb{Q}(x, y, z, \sigma, \tau) \cap D[[y]]$$

$$V[y, z]_{(x,y,z)} \qquad B := \cup D[y, \tau_n]_{(x,y,z,\tau_n)}$$

$$V[z]_{(x,z)}$$
$$(*2) = \mathbb{Q}(x, z, \gamma, \delta) \cap \mathbb{Q}[[x, z]]$$
$$(*3) = \cup \mathbb{Q}[x, z, \gamma_n, \delta_n]_{(x,z,\gamma_n,\delta_n)}$$

$$V[y]_{(x,y)} \qquad\qquad R := D[y]_{(x,y,z)}$$
$$(*5) = \cup \mathbb{Q}[x, y, z, \sigma_n]_{(x,y,z,\sigma_n)}$$

$$S := \mathbb{Q}[z, x, \gamma, \delta]_{(x,z,\gamma,\delta)}$$

$$V := \mathbb{Q}(x, \gamma, \delta) \cap \mathbb{Q}[[x]] \qquad\qquad D := \mathbb{Q}(x, z, \sigma) \cap \mathbb{Q}[z]_{(z)}[[x]]$$
$$(*1) = \cup \mathbb{Q}[x, \gamma_n, \delta_n]_{(x,\gamma_n,\delta_n)} \qquad\qquad (*4) = \cup \mathbb{Q}[x, z, \sigma_n]_{(x,z,\sigma_n)}$$

$$F[\sigma]_{(x,z,\sigma)}$$

$$\mathbb{Q}[x, \gamma, \delta]_{(x,\gamma,\delta)}$$

$$F := \mathbb{Q}[x, z]_{(x,z)}$$

The rings of the example

PROOF. (of Claim 21.20) The Noetherian Flatness Theorem 8.8 implies that the elements γ and δ are primarily limit-intersecting in x over $\mathbb{Q}[x]_{(x)}$ and thus we have (*1):

$$V := \mathbb{Q}(x, \gamma, \delta) \cap \mathbb{Q}[[x]] = \varinjlim(\mathbb{Q}[x, \gamma_n, \delta_n]_{(x,\gamma_n,\delta_n)}) = \bigcup \mathbb{Q}[x, \gamma_n, \delta_n]_{(x,\gamma_n,\delta_n)}.$$

Since V is a DVR of characteristic zero, V is excellent and (*2) holds:

$$V[z]_{(x,z)} = \mathbb{Q}(z, x, \gamma, \delta) \cap \mathbb{Q}[[z, x]].$$

Also $V[z]_{(x,z)}$ is excellent since it is a localization of a finitely generated extension of V. Item (*3) is clear from (*1), and so $V[z]_{(x,z)}$ is a directed union of the four-dimensional regular local domains given.

To establish (*4), observe that for each positive integer n, the map

$$\mathbb{Q}[x, z, \sigma_n]_{(x,z,\sigma_n)} \longrightarrow \mathbb{Q}[x, z, \gamma_n, \delta_n]_{(x,z,\gamma_n,\delta_n)}$$

is faithfully flat. Thus the induced map on the direct limits:

$$\psi_n : \varinjlim \mathbb{Q}[x, z, \sigma_n]_{(x,z,\sigma_n)} \longrightarrow \varinjlim \mathbb{Q}[x, z, \gamma_n, \delta_n]_{(x,z,\gamma_n,\delta_n)}$$

is also faithfully flat. Since $V = \varinjlim \mathbb{Q}[x, \gamma_n, \delta_n]_{(x,\gamma_n,\delta_n)}$, it follows that $V[z]_{(x,z)} = \varinjlim \mathbb{Q}[x, z, \gamma_n, \delta_n]_{(x,z,\gamma_n,\delta_n)}$ is faithfully flat over the limit $\varinjlim \mathbb{Q}[x, z, \sigma_n]_{(x,z,\sigma_n)}$. Since $V[z]_{(x,z)}$ is Noetherian, we conclude that $\varinjlim \mathbb{Q}[x, z, \sigma_n]_{(x,z,\sigma_n)}$ is Noetherian. Therefore ψ is a direct limit of ψ_n and

$$D = \varinjlim \mathbb{Q}[x, z, \sigma_n]_{(x,z,\sigma_n)} = \bigcup \mathbb{Q}[x, z, \sigma_n]_{(x,z,\sigma_n)}.$$

Now item (*5) follows:

$$R := D[y]_(x, y, z) = \varinjlim \mathbb{Q}[x, y, z, \sigma_n]_{(x,y,z,\sigma_n)} = \bigcup \mathbb{Q}[x, y, z, \sigma_n]_{(x,y,z,\sigma_n)}.$$

$\square$

CLAIM 21.21. *The ring $D := \mathbb{Q}(x, z, \sigma) \cap \mathbb{Q}[z]_{(z)}[[x]]$ is excellent and $R := D[y]_{(x,y,z)}$ is a three-dimensional excellent regular local domain with maximal ideal $\mathbf{m} = (x, y, z)R$ and $\mathbf{m}$-adic completion $\widehat{R} = \mathbb{Q}[[x, y, x]]$.*

PROOF. (of Claim 21.21) By Theorem 4.2 of Valabrega, the ring

$$D := \mathbb{Q}(x, z, \sigma) \cap \mathbb{Q}[z]_{(z)}[[x]]$$

is a two-dimensional regular local domain and the completion $\widehat{D}$ of D with respect to the powers of its maximal ideal is canonically isomorphic to $\mathbb{Q}[[x, z]]$.

We observe that with an appropriate change of notation, Theorem 13.11 applies to prove Claim 21.21.

Let $F = \mathbb{Q}[x, z]_{(x,z)}$ and let F^* denote the (x)-adic completion of F. Consider the local injective map

$$F[\sigma]_{(x,z,\sigma)} \xrightarrow{\ \phi\ } F[\gamma, \delta]_{(x,z,\gamma,\delta)} := S.$$

Let $\phi_x : F[\sigma]_{(x,z,\sigma)} \longrightarrow S_x$ denote the composition of ϕ followed by the canonical map of S to S_x. We have the setting of (13.9) and (13.11) where F plays the role of R and $V[z]_{(x,z)}$ plays the role of B.

By Theorem 13.11, to show D is excellent, it suffices to show that ϕ_x is a regular morphism. The map ϕ_x may be identified as the inclusion map

$$\begin{array}{ccc} \mathbb{Q}[z, x, t_1 + zt_2]_{(z,x,t_1+zt_2)} & \xrightarrow{\ \phi_x\ } & \mathbb{Q}[z, x, t_1, t_2]_{(z,x,t_1,t_2)}[1/x] \\ \big\downarrow{\scriptstyle \mu} & & \big\downarrow{\scriptstyle \nu} \\ \mathbb{Q}[z, x, \gamma + z\delta]_{(z,x,\gamma+z\delta)} & \xrightarrow{\ \phi_x\ } & \mathbb{Q}[z, x, \gamma, \delta]_{(z,x,\gamma,\delta)}[1/x] \end{array}$$

where μ and ν are the isomorphisms mapping $t_1 \to \gamma$ and $t_2 \to \delta$. Since $\mathbb{Q}[z, x, t_1, t_2] = \mathbb{Q}[z, x, t_1 + zt_2][t_2]$ is isomorphic to a polynomial ring in one variable over its subring $\mathbb{Q}[z, x, t_1 + zt_2]$, ϕ_x is a regular morphism, so by Theorem 13.11, D is excellent. This completes the proof of Claim 21.21. $\square$

CLAIM 21.22. *The element $\tau := e^y - 1$ is in the (y)-adic completion R^* of $R := D[y]_{(x,y,z)}$, but τ is not primarily limit-intersecting in y over R and the ring B (constructed using τ) is not Noetherian.*

PROOF. (of Claim 21.22) Consider the height-two prime ideal $\widehat{P} := (z, y - x)\widehat{R}$ of $\widehat{R}$. Now $y \notin \widehat{P}$, so $\widehat{P}\widehat{R}_y$ is a height-two prime ideal of $\widehat{R}_y$. Moreover, the ideal $Q := \widehat{P} \cap R[\tau]_{(\mathbf{m},\tau)}$ contains the element $\sigma - \tau$. Thus $\text{ht}(Q) = 3$ and the canonical map $R[\tau]_{(\mathbf{m},\tau)} \longrightarrow \widehat{R}_y$ is not flat. The Noetherian Flatness Theorem 8.8 implies that τ is not primarily limit-intersecting in y over R, and B is not Noetherian. $\square$

For the completion of the proof of Theorem 21.19, it remains to show that τ is residually limit-intersecting in y over R. We first establish the following claim.

CLAIM 21.23. *Let $\widehat{P}$ be a height-one prime ideal of $\widehat{R} = \mathbb{Q}[[x, y, z]]$, and suppose $\widehat{P} \cap \mathbb{Q}[[x, y]] \neq (0)$. Then the prime ideal $P_0 := \widehat{P} \cap R$ is extended from $\mathbb{Q}[x, y] \subseteq R$.*

PROOF. We may assume P_0 is distinct from (0), $x\widehat{R}$ and $y\widehat{R}$ since these are obviously extended. Since $\widehat{R}$ is faithfully flat over R, P_0 has height one. Similarly $P_1 := \widehat{P} \cap V[y, z]_{(x,y,z)}$ has height at most one since $\widehat{R}$ is also the completion of $V[y, z]_{(x,y,z)}$. We also have $P_1 \cap R = P_0$, so P_1 is nonzero and hence has height one. Since, for every $n \in \mathbb{N}$, $\sigma_n \in \mathbb{Q}[x, y, z, \sigma][1/x]$, the ring $R[1/x]$ is a localization of $\mathbb{Q}[x, y, z, \sigma][1/x]$. Thus $\widehat{P} \cap \mathbb{Q}[x, y, z, \sigma]$ has height one and contains an element f which generates P_0. Similarly, for every $n \in \mathbb{N}$, γ_n and δ_n are in $\mathbb{Q}[x, y, z, \gamma, \delta][1/x]$, which implies the ring $V[y, z]_{(x,y,z)}[1/x]$ is a localization of $\mathbb{Q}[x, y, z, \gamma, \delta][1/x]$. Thus $\widehat{P} \cap \mathbb{Q}[x, y, z, \gamma, \delta]$ has height one and contains a generator g for P_1. Let $\hat{h} \in \mathbb{Q}[[x, y]]$ be a generator of $\widehat{P} \cap \mathbb{Q}[[x, y]]$. Then $\hat{h}\widehat{R} = \widehat{P}$. The following diagram illustrates this situation:

$$\widehat{P} \subset \mathbb{Q}[[x, y, z]]$$

$$P_1 := \widehat{P} \cap V[y, z]_{(x,y,z)}$$

$$\hat{h} \in \widehat{P} \cap \mathbb{Q}[[x, y]] \qquad\qquad g \in \widehat{P} \cap \mathbb{Q}[x, y, z, \gamma, \delta]$$

$$P_0 := \widehat{P} \cap R$$

$$\widehat{P} \cap \mathbb{Q}[x, y, \gamma_n] \qquad\qquad f \in \widehat{P} \cap \mathbb{Q}[x, y, z, \sigma]$$

$$\widehat{P} \cap \mathbb{Q}[x, y]$$

Picture for proof of (21.23)

Subclaim 1: $g \in \widehat{P} \cap \mathbb{Q}[x, y, \gamma, \delta]$.

PROOF. (of Subclaim 1) : Write $g = g_0 + g_1 z + \cdots + g_r z^r$, where the $g_i \in \mathbb{Q}[x, y, \gamma, \delta]$. Since $g \in \widehat{P}$, we have $g = \hat{h}(x, y)\phi(x, y, z)$, for some $\phi(x, y, z) \in \mathbb{Q}[[x, y, z]]$. Since g is irreducible and $P_1 \neq zV[y, z]_{(x,y,z)}$, we have $g_0 \neq (0)$.

Setting $z = 0$, we have $g_0 = g(0) = \hat{h}(x, y)\phi(x, y, 0) \in \mathbb{Q}[[x, y]]$. Thus $g_0 \in \hat{h}\mathbb{Q}[[x, y]] \cap \mathbb{Q}[x, y, \gamma, \delta] \neq (0)$. Therefore $g\mathbb{Q}[x, y, \gamma, \delta, z] = g_0\mathbb{Q}[x, y, \gamma, \delta]$ is extended from $\mathbb{Q}[x, y, \gamma, \delta]$ and $g = g_0 \in \mathbb{Q}[x, y, \gamma, \delta]$, $\square$

Subclaim 2: If we express f as $f = f_0 + f_1 z + \cdots + f_r z^r$, where the $f_i \in \mathbb{Q}[x, y, \gamma, \delta]$, then $f_0 \in \mathbb{Q}[x, y, \gamma]$.

Proof. (of Subclaim 2) Since f is an element of $\mathbb{Q}[x, y, \sigma, z]$, we can write f as a polynomial

$$f = \sum a_{ij} z^i \sigma^j = \sum a_{ij} z^i (\gamma + z\delta)^j, \text{ where } a_{ij} \in \mathbb{Q}[x, y].$$

Setting $z = 0$, we have $f_0 = f(0) = \sum a_{0j}(\gamma)^j \in \mathbb{Q}[x, y, \gamma]$. $\square$

Proof. Completion of proof of Claim 21.23. Since $f \in \widehat{P} \cap \mathbb{Q}[x, y, z, \gamma, \delta]$, we have $f = dg$, for some $d \in \mathbb{Q}[x, y, \gamma, \delta, z]$. Regarding d as a polynomial in z with coefficients in $\mathbb{Q}[x, y, \gamma, \delta]$ and setting $z = 0$, gives $f_0 = f(0) = d(0)g \in \mathbb{Q}[x, y, \gamma, \delta]$. Thus f_0 is a multiple of g. Hence $g \in \mathbb{Q}[x, y, \gamma]$, by Subclaim 2.

Now again using that $f = dg$ and setting $z = 1$, we have $d(1)g = f(1) \in \mathbb{Q}[x, y, \gamma + \delta]$. This says that $f(1)$ is a multiple of the polynomial $g \in \mathbb{Q}[x, y, \gamma]$. Since γ and δ are algebraically independent over $\mathbb{Q}[x, y]$, this implies $f(1)$ has degree 0 in $\gamma + \delta$ and g has degree 0 in γ. Therefore $g \in \mathbb{Q}[x, y]$, $d \in \mathbb{Q}$, $0 \neq f \in \mathbb{Q}[x, y]$, and $P_0 = fR$ is extended from $\mathbb{Q}[x, y]$. $\square$

It follows that τ is residually limit-intersecting provided we show the following:

Claim 21.24. *Suppose $\widehat{P}$ is a height-one prime ideal of $\widehat{R}$ with $y \notin \widehat{P}$ and $\operatorname{ht}(\widehat{P} \cap R) = 1$. Then the image $\bar{\tau}$ of τ in $\widehat{R}/\widehat{P}$ is algebraically independent over $R/(\widehat{P} \cap R)$.*

Proof. (of Claim 21.24) Let $P_0 := \widehat{P} \cap R$ and let $\pi : \mathbb{Q}[[x, y, z]] \longrightarrow \mathbb{Q}[[x, y, z]]/\widehat{P}$; we use $^-$ to denote the image under π. If $\widehat{P} = x\widehat{R}$, then we have the commutative diagram:

$$\begin{array}{ccccc}
R/P_0 & \longrightarrow & (R/P_0)[\bar{\tau}] & \longrightarrow & \mathbb{Q}[[x, y, z]]/\widehat{P} \\
\cong\big\uparrow & & \cong\big\uparrow & & \cong\big\uparrow \\
\mathbb{Q}[y, z]_{(y,z)} & \longrightarrow & \mathbb{Q}[y, z]_{(y,z)}[\tau] & \longrightarrow & \mathbb{Q}[[y, z]] \ .
\end{array}$$

Since τ is transcendental over $\mathbb{Q}[y, z]$, the result follows in this case. $\square$

For the other height-one primes $\widehat{P}$ of $\widehat{R}$, we distinguish two cases:
case 1: $\widehat{P} \cap \mathbb{Q}[[x, y]] = (0)$.
Let $P_1 := V[y, z]_{(x,y,z)} \cap \widehat{P}$. We have the following commutative diagram of local injective morphisms:

$$\begin{array}{ccccc}
R/P_0 & \longrightarrow & V[y, z]_{(x,y,z)}/P_1 & \longrightarrow & \mathbb{Q}[[x, y, z]]/\widehat{P} \\
 & & \big\uparrow & & \big\uparrow \\
 & & V[y]_{(x,y)} & \longrightarrow & \mathbb{Q}[[x, y]] \ ,
\end{array}$$

where $V[y, z]_{(x,y,z)}/P_1$ is algebraic over $V[y]_{(x,y)}$. Since $\tau \in \mathbb{Q}[[x, y]]$ is transcendental over $V[y]_{(x,y)}$, its image $\bar{\tau}$ in $\mathbb{Q}[[x, y, z]]/\widehat{P}$ is transcendental over $V[y, z]_{(x,y,z)}/P_1$ and thus is transcendental over R/P_0.
case 2: $\widehat{P} \cap \mathbb{Q}[[x, y]] \neq (0)$.
In this case, by Claim 21.23, the height-one prime $P_0 := \widehat{P} \cap R$ is extended from a prime ideal in $\mathbb{Q}[x, y]$. Let p be a prime element of $\mathbb{Q}[x, y]$ such that $(p) = \widehat{P} \cap \mathbb{Q}[x, y]$. We have the inclusions:

$$G := \mathbb{Q}[x, y]_{(x,y)}/(p) \hookrightarrow R/P_0 \hookrightarrow \widehat{R}/\widehat{P} \ ,$$

where $R/P_0 = \varinjlim \mathbb{Q}[x,y,z,\sigma_n]_{(x,y,z,\sigma_n)}/(p)$ has transcendence degree ≤ 1 over $G[\bar{z}]_{(\bar{x},\bar{y},\bar{z})}$. It suffices to show that $\bar{\sigma}$ and $\bar{\tau}$ are algebraically independent over the field of fractions $\mathbb{Q}(\bar{x},\bar{y},\bar{z})$ of $G[\bar{z}]$.

Let $\tilde{G}$ be the integral closure of G in its field of fractions and let $H := \tilde{G}_{\mathbf{n}}$ be a localization of $\tilde{G}$ at a maximal ideal $\mathbf{n}$ such that the completion $\widehat{H}$ of H is dominated by the integral closure $(\widehat{R}/\widehat{P})'$ of $\widehat{R}/\widehat{P}$.

Now $H[\bar{z}]$ has transcendence degree at least one over $\mathbb{Q}[\bar{z}]$. Also since $\widehat{P} \cap \mathbb{Q}[x,y] \neq 0$, the transcendence degree of $\mathbb{Q}[\bar{x},\bar{y}]$ and so of H is at most one over $\mathbb{Q}$. Thus $H[\bar{z}]$ has transcendence degree exactly one over $\mathbb{Q}(\bar{z})$. There exists an element $t \in H$ that is transcendental over $\mathbb{Q}[\bar{z}]$ and is such that t generates the maximal ideal of the DVR H. Then H is algebraic over $\mathbb{Q}[t]$ and H may be regarded as a subring of $\mathbb{C}[[t]]$, where $\mathbb{C}$ is the complex numbers. In order to show that $\bar{\sigma}$ and $\bar{\tau}$ are algebraically independent over $G[\bar{z}]$, it suffices to show that $\bar{\sigma}$ and $\bar{\tau}$ are algebraically independent over $H[\bar{z}]$ and thus it suffices to show that these elements are algebraically independent over $\mathbb{Q}(t,\bar{z})$. Thus it suffices to show that $\bar{\sigma}$ and $\bar{\tau}$ are algebraically independent over $\mathbb{C}(t,\bar{z})$.

We have the setup shown in the following diagram:

$$\begin{array}{ccccccc}
 & & & \mathbb{C}(\bar{z})[[t]] & & & (\widehat{R}/\widehat{P})' \\
\mathbb{C}[[t]] & & \mathbb{C}(\bar{z})[t] & & H[\bar{z}] & & \\
 & H := \tilde{G}_{\mathbf{n}} & & \mathbb{Q}[t,\bar{z}] & & G[\bar{z}]_{(\bar{x},\bar{y},\bar{z})} & \\
 & \mathbb{Q}[t] & & G := \mathbb{Q}[\bar{x},\bar{y}]_{(\bar{x},\bar{y})} & & \mathbb{Q}[\bar{z}] &
\end{array}$$

By [**11**], if $\bar{x}, \bar{x}^2, \bar{y} \in t\mathbb{C}[[t]]$ are linearly independent over $\mathbb{Q}$, then:

$$\operatorname{trdeg}_{\mathbb{C}(t)}(\mathbb{C}(t)(\bar{x},\bar{x}^2,\bar{y},e^{\bar{x}},e^{\bar{x}^2},e^{\bar{y}})) \geq 3.$$

Since $\bar{x}, \bar{x}^2$ and $\bar{y}$ are in H, these elements are algebraic over $\mathbb{Q}(t)$. Therefore if $\bar{x}, \bar{x}^2, \bar{y}$ are linearly independent over $\mathbb{Q}$, then the exponential functions $e^{\bar{x}}, e^{\bar{x}^2}, e^{\bar{y}}$ are algebraically independent over $\mathbb{Q}(t)$ and hence $\bar{\sigma}$ and $\bar{\tau}$ are algebraically independent over $G(\bar{z})$.

We observe that if $\bar{x}, \bar{x}^2, \bar{y} \in tH$ are linearly dependent over $\mathbb{Q}$, then there exist $a, b, c \in \mathbb{Q}$ such that

$$lp = ax + bx^2 + cy \quad \text{in} \quad \mathbb{Q}[x,y].$$

where $l \in \mathbb{Q}[x,y]$. Since $(p) \neq (x)$, we have $c \neq 0$. Hence we may assume $c = 1$ and $lp = y - ax - bx^2$ with $a, b \in \mathbb{Q}$. Since $y - ax - bx^2$ is irreducible in $\mathbb{Q}[x,y]$, we may assume $l = 1$. Also a and b cannot both be 0 since $y \notin \widehat{P}$. Thus if $\bar{x}, \bar{x}^2, \bar{y} \in tH$ are linearly dependent over $\mathbb{Q}$, then we may assume

$$p = y - ax - bx^2 \qquad \text{for some } a, b \in \mathbb{Q} \qquad \text{not both } 0.$$

It remains to show that $\bar\sigma$ and $\bar\tau$ are algebraically independent over $G(\bar z)$ provided that $p = y - ax - bx^2$, that is

$$\bar y = a\bar x + b\bar x^2, \qquad \text{for } a, b \in \mathbb{Q}, \text{ not both } 0.$$

Suppose $h \in G[\bar z][u, v]$, where u, v are indeterminates and that $h(\bar\sigma, \bar\tau) = 0$. This implies

$$h(e^{\bar x} + \bar z e^{\bar x^2}, e^{a\bar x + b\bar x^2}) = 0.$$

We have $e^{a\bar x} = (e^{\bar x})^a$ and $e^{b\bar x^2}$ are algebraic over $G(\bar z, e^{\bar x}, e^{\bar x^2})$ since a and b are rational. By substituting $\bar z = 0$ we obtain an equation over G:

$$h(e^{\bar x}, e^{a\bar x + b\bar x^2}) = 0$$

which implies that $b = 0$ since $\bar x$ and $\bar x^2$ are linearly independent over $\mathbb{Q}$. Now the only case to consider is the case where $p = y + ax$. The equation we obtain then is:

$$h(e^{\bar x} + \bar z e^{\bar x^2}, e^{a\bar x}) = 0$$

which implies that h must be the zero polynomial, since $e^{\bar x^2}$ is transcendental over the algebraic closure of the field of fractions of $G[\bar z, e^{\bar x}]$. This completes the proof of Claim 21.24. Thus τ is residually limit-intersecting over R.

Since R is a UFD, the element τ is limit-intersecting over R by Remark 21.17. This completes the proof of Theorem 21.19. $\square$

Remark 21.25. With notation as in Theorem 21.19, let u, v be indeterminates over $\mathbb{Q}[[x, y, z]]$. Then the height-one prime ideal $\widehat{Q} = (u - \tau)$ in $\mathbb{Q}[[x, y, z, u]]$ is in the generic formal fiber of the excellent regular local ring $R[u]_{(x,y,z,u)}$ and the intersection domain

$$K(u) \cap \mathbb{Q}[[x, y, z, u]]/\widehat{Q} \cong K(\tau) \cap \widehat{R}\ ,$$

where K is the fraction field of R, fails to be Noetherian. In a similar fashion this intersection ring $K(\tau) \cap \widehat{R}$ may be identified with the following ring: Let $\widehat{U} = (u - \tau, v - \sigma)$ be the height-two prime ideal in $\mathbb{Q}[[x, y, z, u, v]]$ which is in the generic formal fiber of the polynomial ring $\mathbb{Q}[x, y, z, u, v]_{(x,y,z,u,v)}$. Then we have:

$$\mathbb{Q}(x, y, z, u, v) \cap ((\mathbb{Q}[[x, y, z, u, v]])/\widehat{U}) \cong K(\tau) \cap \widehat{R}\ ,$$

and as shown in Theorem 21.19, this ring is not Noetherian. We do not know an example of a height-one prime ideal $\widehat{W}$ in the generic formal fiber of a polynomial ring T for which the intersection ring $A = \mathcal{Q}(T) \cap (\widehat{T}/\widehat{W})$ fails to be Noetherian. In Chapter intsec we present an example of such an intersection ring A whose completion is not equal to $\widehat{T}$, however in this example the ring A is still Noetherian.

21.4. Several additional examples

Let $R = \mathbb{Q}[x, y]_{(x,y)}$, the localized polynomial ring in two variables x and y over the field $\mathbb{Q}$ of rational numbers. Then $\widehat{R} = \mathbb{Q}[[x, y]]$, the formal power series ring in x and y, is the $\mathbf{m} = (x, y)R$-adic completion of R. In Chapter 19, an element $\tau \in \widehat{\mathbf{m}} = (x, y)\widehat{R}$ is defined to be *residually algebraically independent over* R if τ is algebraically independent over R and for each height-one prime $\widehat{P}$ of $\widehat{R}$ such that $\widehat{P} \cap R \neq (0)$, the image of τ in $\widehat{R}/\widehat{P}$ is algebraically independent over the fraction field of $R/(\widehat{P} \cap R)$. It is shown in Theorem 19.27 of Chapter 19, that if τ is residually algebraically independent over R and L is the field of fractions of $R[\tau]$, then $L \cap \widehat{R}$ is the localized polynomial ring $R[\tau]_{(\mathbf{m},\tau)}$.

In this section we present several examples of residually algebraically independent elements.

EXAMPLE 21.26. For $S := \mathbb{Q}[x,y,z]_{(x,y,z)}$, the construction of Theorem 7.12 yields an example of a height-one prime ideal $\widehat{P}$ of $\widehat{S} = \mathbb{Q}[[x,y,z]]$ in the generic formal fiber of S such that

$$\mathcal{Q}(S) \cap (\widehat{S}/\widehat{P}) = S.$$

PROOF. Let $\widehat{P} := (z-\tau) \subseteq \mathbb{Q}[[x,y,z]]$, where τ is as in Theorem 7.12. Then $\mathbb{Q}(x,y,z) \cap (\widehat{S}/\widehat{P})$ can be identified with the intersection $\mathbb{Q}(x,y,\tau) \cap \mathbb{Q}[[x,y]]$ of (6.1). Therefore

$$\mathbb{Q}(x,y,z) \cap (\widehat{S}/\widehat{P}) = S = \mathbb{Q}[x,y,z]_{(x,y,z)}.$$

□

With $S = \mathbb{Q}[x,y,z]_{(x,y,z)}$, every prime ideal of $\widehat{S} = \mathbb{Q}[[x,y,z]]$ that is maximal in the generic formal fiber of S has height 2. Thus the prime ideal $\widehat{P}$ is not maximal in the generic formal fiber of $S = \mathbb{Q}[x,y,z]_{(x,y,z)}$.

REMARK 21.27. Let $(R,\mathbf{m})$ be a localized polynomial ring over a field and let $\widehat{R}$ denote the $\mathbf{m}$-adic completion of R. It is observed in [**53**, Theorem 2.5] that there exists a one-to-one correspondence between prime ideals $\widehat{\mathbf{p}}$ of $\widehat{R}$ that are maximal in the generic formal fiber of R and DVRs C such that C birationally dominates R and $C/\mathbf{m}C$ is a finitely generated R-module. Example 21.26 demonstrates that this strong connection between the maximal ideals of the generic formal fiber of a localized polynomial ring R and certain birational extensions of R does not extend to prime ideals nonmaximal in the generic formal fiber R.

EXAMPLE 21.28. Again let $S = \mathbb{Q}[x,y,z]_{(x,y,z)}$. With a slight modification of Example 21.26, we exhibit a prime ideal $\widehat{P}$ in the generic formal fiber of S which *does* correspond to a nontrivial birational extension; that is, the intersection ring

$$A := \mathcal{Q}(S) \cap \widehat{S}/\widehat{P}$$

is essentially finitely generated over S.

PROOF. Let τ be the element from Theorem 7.12. Let $\widehat{P} = (z - x\tau) \subseteq \mathbb{Q}[[x,y,z]]$. Since τ is transcendental over $\mathbb{Q}(x,y,z)$, the prime ideal $\widehat{P}$ is in the generic formal fiber of S. The ring S can be identified with a subring of $\widehat{S}/\widehat{P} \cong \mathbb{Q}[[x,y]]$ by considering $S = \mathbb{Q}[x,y,x\tau]_{(x,y,x\tau)}$. By reasoning similar to that of Example 21.26,

$$\mathcal{Q}(S) \cap \mathbb{Q}[[x,y]] = \mathbb{Q}(x,y,\tau) \cap \mathbb{Q}[[x,y]] = \mathbb{Q}[x,y,\tau]_{(x,y,\tau)}.$$

The ring $\mathbb{Q}[x,y,\tau]_{(x,y,\tau)}$ is then the essentially finitely generated birational extension of S defined as $S[z/x]_{(x,y,z/x)}$. □

EXAMPLE 21.29. Let $\sigma \in x\mathbb{Q}[[x]]$ and $\rho \in y\mathbb{Q}[[y]]$ be as in Theorem 7.12. If $D := \mathbb{Q}(x,\sigma) \cap \mathbb{Q}[[x]] = \bigcup_{n=1}^{\infty} \mathbb{Q}[x,\sigma_n]_{(x,\sigma_n)}$ and $T := D[y]_{(x,y)}$, so T is regular local with completion $\widehat{T} = \mathbb{Q}[[x,y]]$, then the element ρ is primarily limit-intersecting in y over T.

Proof. We show that the map $\varphi_y : T[\rho] \longrightarrow \mathbb{Q}[[x,y]][1/y]$ is LF_1; that is, the induced map $\varphi_{\widehat{P}} : T[\rho]_{\widehat{P}\cap T[\rho]} \longrightarrow \mathbb{Q}[[x,y]]_{\widehat{P}}$ is flat for every height-one prime ideal $\widehat{P}$ of $\mathbb{Q}[[x,y]]$ with $y \notin \widehat{P}$. It is equivalent to show for every height-one prime $\widehat{P}$ of $\mathbb{Q}[[x,y]]$ that $\widehat{P} \cap T[\rho]$ has height ≤ 1. If $\widehat{P} = (x)$, the statement is immediate, since ρ is algebraically independent over $\mathbb{Q}(y)$. Next we consider the case $\widehat{P} \cap Q[x,y,\sigma] = (0)$. Since $\mathbb{Q}(x,y,\sigma) = \mathbb{Q}(x,y,\sigma_n)$ for every positive integer n, $\widehat{P} \cap Q[x,y,\sigma] = (0)$ if and only if $\widehat{P} \cap Q[x,y,\sigma_n] = (0)$. Moreover, if this is true, then since the fraction field of $T[\rho]$ has transcendence degree one over $\mathbb{Q}(x,y,\sigma)$, then $\widehat{P} \cap T[\rho]$ has height ≤ 1. The remaining case is where $P := \widehat{P} \cap Q[x,y,\sigma] \neq (0)$ and $xy \notin \widehat{P}$. By Proposition 6.3, $\bar{\rho}$ is transcendental over $\bar{T} = T/(\widehat{P} \cap T)$, and this is equivalent to $\mathrm{ht}(\widehat{P} \cap T[\tau]) = 1$. $\square$

Still referring to ρ, σ, σ_n as in Theorem 7.12 and Example 21.29, and using that σ is primarily limit-intersecting in y over T, we have:

$$A := \mathcal{Q}(T)(\rho) \cap \mathbb{Q}[[x,y]] = \varinjlim T[\rho_n]_{(x,y,\rho_n)} = \varinjlim \mathbb{Q}[x,y,\sigma_n,\rho_n]_{(x,y,\sigma_n,\rho_n)}$$

where the endpieces ρ_n are defined as in Section 6.1; viz., $\rho := \sum_{n=1}^{\infty} b_i y^i$ and $\rho_n = \sum_{i=n+1}^{\infty} b_i y^{i-n}$. The philosophy here is that sufficient "independence" of the algebraically independent elements σ and ρ allows us to explicitly describe the intersection ring A.

The previous examples have been over localized polynomial rings, where we are free to exchange variables. The next example shows, over a different regular local domain, that an element in the completion with respect to one regular parameter x may be residually limit-intersecting with respect to x whereas the corresponding element in the completion with respect to another regular parameter y may be transcendental but fail to be residually limit-intersecting.

Example 21.30. There exists a regular local ring R with $\widehat{R} = \mathbb{Q}[[x,y]]$ such that $\sigma = e^x - 1$ is residually limit-intersecting in x over R, whereas $\gamma = e^y - 1$ fails to be limit-intersecting in y over R.

Proof. Let $\{\omega_i\}_{i\in I}$ be a transcendence basis of $\mathbb{Q}[[x]]$ over $\mathbb{Q}(x)$ such that:

$$\{e^{x^n}\}_{n\in\mathbb{N}} \subseteq \{\omega_i\}_{i\in I}.$$

Let D be the discrete valuation ring:

$$D = \mathbb{Q}(x, \{\omega_i\}_{i\in I, \omega_i \neq e^x}) \cap \mathbb{Q}[[x]].$$

Obviously, $\mathbb{Q}[[x]]$ has transcendence degree 1 over D. The set $\{e^x\}$ is a transcendence basis of $\mathbb{Q}[[x]]$ over D. Let $R = D[y]_{(x,y)}$.

By Remark 21.12.1, the element $\sigma = e^x - 1$ is primarily limit-intersecting and hence residually limit-intersecting in x over D. Moreover, by Remark 21.12.2, σ is also primarily and hence residually limit-intersecting over $R := D[y]_{(x,y)}$. However, the element $\gamma = e^y - 1$ is not residually limit-intersecting in y over R. To see this, consider the height-one prime ideal $P := (y - x^2)\mathbb{Q}[[x,y]]$. The prime ideal $W := P \cap R[\tau]_{(x,y,\tau)}$ contains the element $\gamma - e^{x^2} - 1 = e^y - e^{x^2}$. Therefore W has height greater than one and γ is not residually limit-intersecting in y over R. $\square$

Note that the intersection ring $\mathcal{Q}(R)(\tau) \cap \mathbb{Q}[[x,y]]$ is a regular local ring with completion $\mathbb{Q}[[x,y]]$ by Theorem 4.2, a theorem of Valabrega.

CHAPTER 22

Rings between excellent normal local domains and their completions II

Let $(R, \mathbf{m})$ be an excellent normal local domain. Let y be a nonzero element in $\mathbf{m}$ and let R^* denote the (y)-adic completion of R. In this chapter we consider certain extension domains A inside R^* arising from Inclusion Construction 5.3 and Homomorphic Image Construction 5.4. We use test criteria given in Theorem 11.3, Theorem 11.4 and Corollary 11.5, involving the heights of certain prime ideals to determine flatness for the map φ defined in Equation 22.1.0. These characterizations of flatness involve the condition that certain fibers are Cohen-Macaulay and other fibers are regular.

We give in Theorem 22.12 and Remarks 22.14 necessary and sufficient conditions for an element $\tau \in yR^*$ to be primarily limit-intersecting in y over R; see Remark 22.2. If R is countable, we prove in Theorem 22.19 the existence of an infinite sequence of elements of yR^* that are primarily limit-intersecting in y over R. Using this result we establish the existence of a normal Noetherian local domain B such that: B dominates R; B has (y)-adic completion R^*; and B contains a height-one prime ideal $\mathbf{p}$ such that $R^*/\mathbf{p}R^*$ is not reduced. Thus B is not a Nagata domain and hence is not excellent; see Remark 3.32.

In Section 22.3 we observe that every Noetherian local ring containing an excellent local subring R and having the same completion as R has Cohen-Macaulay formal fibers. This applies to examples obtained by Inclusion Construction 5.3; see Corollary 22.23. It does not apply to examples obtained by Homomorphic Image Construction 5.4. In Remark 22.25, we discuss connections with a famous example of Ogoma.

We present in Section 22.4 integral domains B and A arising from Inclusion Construction 5.3 and C arising from Homomorphic Image Construction 5.4. In Theorems 22.27 and 22.28 we show that A and B are non-Noetherian and $B \subsetneq A$. We establish in Theorem 22.30 that the domain C is a two-dimensional Noetherian local domain, C is a homomorphic image of B and C has the property that its generic formal fiber is not Cohen-Macaulay.

22.1. Primarily limit-intersecting extensions and flatness

In this section, we consider properties of Inclusion Construction 5.3 under the assumptions of Setting 22.1.

SETTING 22.1. Let $(R, \mathbf{m})$ be an excellent normal local domain and let y be a nonzero element in $\mathbf{m}$. Let $(R^*, \mathbf{m}^*)$ be the (y)-adic completion of R and let $(\widehat{R}, \widehat{\mathbf{m}})$ be the $\mathbf{m}$-adic completion of R. Thus R^* and $\widehat{R}$ are normal Noetherian local domains and $\widehat{R}$ is the $\mathbf{m}^*$-adic completion of R^*. Let $\tau_1, \ldots, \tau_s$ be elements of yR^* that are algebraically independent over R, and set $U_0 = S := R[\tau_1, \ldots, \tau_s]$.

The field of fractions L of S is a subfield of the field of fractions $\mathcal{Q}(R^*)$ of R^*. Define $A := L \cap R^*$.

REMARK 22.2. The Noetherian Flatness Theorem 8.8 implies that $A = L \cap R^*$ is both Noetherian and a localization of a subring of $S[1/y]$ if and only if the extension φ is flat, where

$$\varphi : S \longrightarrow R^*[1/y] \tag{22.1.0}$$

By Definition 21.9.3, the elements $\tau_1, \dots, \tau_s$ are *primarily limit-intersecting* in y over R if and only if φ is flat.

THEOREM 22.3. *Assume notation as in Setting 22.1. That is, $(R, \mathbf{m})$ is an excellent normal local domain, y is a nonzero element in $\mathbf{m}$, $(R^*, \mathbf{m}^*)$ is the (y)-adic completion of R, and the elements $\tau_1, \dots, \tau_s \in yR^*$ are algebraically independent over R. Then the following statements are equivalent:*

1. *$S := R[\tau_1, \dots, \tau_s] \hookrightarrow R^*[1/y]$ is flat. Equivalently, $\tau_1, \dots, \tau_s$ are primarily limit-intersecting in y over R^*.*
2. *For P a prime ideal of S and Q^* a prime ideal of R^* minimal over PR^*, if $y \notin Q^*$, then $\operatorname{ht}(Q^*) = \operatorname{ht}(P)$.*
3. *If Q^* is a prime ideal of R^* with $y \notin Q^*$, then $\operatorname{ht}(Q^*) \geq \operatorname{ht}(Q^* \cap S)$.*

Moreover, if any of (1)-(3) hold, then $S \hookrightarrow R^*[1/y]$ has Cohen-Macaulay fibers.

PROOF. By Remark 22.2, we have the equivalence in item 1.

(1) $\Rightarrow$ (2): Let P be a prime ideal of S and let Q^* be a prime ideal of R^* that is minimal over PR^* and is such that $y \notin Q^*$. The assumption of item 1 implies flatness of the map:

$$\varphi_{Q^*} : S_{Q^* \cap S} \longrightarrow R^*_{Q^*}.$$

By Remark 2.21.10, we have $Q^* \cap S = P$, and by [**96**, Theorem 15.1], $\operatorname{ht} Q^* = \operatorname{ht} P$.

(2) $\Rightarrow$ (3): Let Q^* be a prime ideal of R^* with $y \notin Q^*$. Set $Q := Q^* \cap S$ and let $\mathbf{w}^*$ be a prime ideal of R^* that is minimal over QR^* and is contained in Q^*. Then $\operatorname{ht}(Q) = \operatorname{ht}(\mathbf{w}^*)$ by (2) since $y \notin \mathbf{w}^*$ and therefore $\operatorname{ht}(Q^*) \geq \operatorname{ht}(Q)$.

(3) $\Rightarrow$ (1): Let Q^* be a prime ideal of R^* with $y \notin Q^*$. Then for every prime ideal $\mathbf{w}^*$ of R^* contained in Q^*, we also have $y \notin \mathbf{w}^*$, and by (3), $\operatorname{ht}(\mathbf{w}^*) \geq \operatorname{ht}(\mathbf{w}^* \cap S)$. Therefore, by Theorem 11.4, $\varphi_{Q^*} : S_{Q^* \cap S} \longrightarrow R^*_{Q^*}$ is flat with Cohen-Macaulay fibers. □

With notation as in Setting 22.1, the map $R^* \hookrightarrow \widehat{R}$ is flat. Hence the corresponding statements in Theorem 22.3 with R^* replaced by $\widehat{R}$ also hold. We record this as

COROLLARY 22.4. *Assume notation as in Setting 22.1. Then the following statements are equivalent:*

1. *$S := R[\tau_1, \dots, \tau_s] \hookrightarrow \widehat{R}[1/y]$ is flat.*
2. *For P a prime ideal of S and $\widehat{Q}$ a prime ideal of $\widehat{R}$ minimal over PR, if $y \notin \widehat{Q}$, then $\operatorname{ht}(\widehat{Q}) = \operatorname{ht}(P)$.*
3. *If $\widehat{Q}$ is a prime ideal of $\widehat{R}$ with $y \notin \widehat{Q}$, then $\operatorname{ht}(\widehat{Q}) \geq \operatorname{ht}(\widehat{Q} \cap S)$.*

Moreover, if any of (1)-(3) hold, then $S \hookrightarrow \widehat{R}[1/y]$ has Cohen-Macaulay fibers.

As another corollary to Theorem 22.3, we have the following result:

COROLLARY 22.5. *With the notation of Theorem 22.3, assume that* $\widehat{R}[1/y]$ *is flat over* S. *Let* $P \in \operatorname{Spec} S$ *with* $\operatorname{ht}(P) \geq \dim(R)$. *Then*

(1) *For every* $\widehat{Q} \in \operatorname{Spec} \widehat{R}$ *minimal over* $P\widehat{R}$ *we have* $y \in \widehat{Q}$.
(2) *Some power of* y *is in* $P\widehat{R}$.

PROOF. Clearly items 1 and 2 are equivalent. To prove these hold, suppose that $y \notin \widehat{Q}$. By Theorem 22.3.2, $\operatorname{ht}(P) = \operatorname{ht}(\widehat{Q})$. Since $\dim(R) = \dim(\widehat{R})$, we have $\operatorname{ht}(\widehat{Q}) \geq \dim(\widehat{R})$. But then $\operatorname{ht}(\widehat{Q}) = \dim(\widehat{R})$ and $\widehat{Q}$ is the maximal ideal of $\widehat{R}$. This contradicts the assumption that $y \notin \widehat{Q}$. We conclude that $y \in \widehat{Q}$. □

Theorem 22.3, together with results from Chapter 8, gives the following corollary.

COROLLARY 22.6. *Assume notation as in Setting 22.1, and consider the following conditions:*

(1) A *is Noetherian and is a localization of a subring of* $S[1/y]$.
(2) $S \hookrightarrow R^*[1/y]$ *is flat.*
(3) $S \hookrightarrow R^*[1/y]$ *is flat with Cohen-Macaulay fibers.*
(4) *For every* $Q^* \in \operatorname{Spec}(R^*)$ *with* $y \notin Q^*$, *we have* $\operatorname{ht}(Q^*) \geq \operatorname{ht}(Q^* \cap S)$.
(5) A *is Noetherian.*
(6) $A \hookrightarrow R^*$ *is flat.*
(7) $A \hookrightarrow R^*[1/y]$ *is flat.*
(8) $A \hookrightarrow R^*[1/y]$ *is flat with Cohen-Macaulay fibers.*

Conditions (1)-(4) are equivalent, conditions (5)-(8) are equivalent and (1)-(4) imply (5)-(8).

PROOF. Item 1 is equivalent to item 2 by Noetherian Flatness Theorem 8.3, item 2 is equivalent to item 3 and item 7 is equivalent to item 8 by Theorem 11.4, and item 2 is equivalent to item 4 by Theorem 22.3.

It is obvious that item 1 implies item 5. By Construction Properties Theorem 6.19.4, the ring R^* is the y-adic completion of A, and so item 5 is equivalent to item 6. By Lemma 8.2).1, item 6 is equivalent to item 7. □

REMARKS 22.7. **(i)** With the notation of Corollary 22.6, if $\dim A = 2$, it follows that condition (7) of Corollary 22.6 holds. Since R^* is normal, so is A. Thus if $Q^* \in \operatorname{Spec} R^*$ with $y \notin Q^*$, then $A_{Q^* \cap A}$ is either a DVR or a field. The map $A \to R^*_{Q^*}$ factors as $A \to A_{Q^* \cap A} \to R^*_{Q^*}$. Since $R^*_{Q^*}$ is a torsionfree and hence flat $A_{Q^* \cap A}$-module, it follows that $A \to R^*_{Q^*}$ is flat. Therefore $A \hookrightarrow R^*[1/y]$ is flat and A is Noetherian.
(ii) There exist examples where $\dim A = 2$ and conditions (5)-(8) of Corollary 22.6 hold, but yet conditions (1)-(4) fail to hold; see Theorem 4.12.

QUESTION 22.8. With the notation of Corollary 22.6, suppose for every prime ideal Q^* of R^* with $y \notin Q^*$ that $\operatorname{ht}(Q^*) \geq \operatorname{ht}(Q^* \cap A)$. Does it follow that R^* is flat over A or, equivalently, that A is Noetherian?

Theorem 22.3 also extends to give equivalences for the *locally flat in height* k property; see Definitions 21.1.

THEOREM 22.9. *Assume notation as in Setting 22.1. That is,* $(R, \mathbf{m})$ *is an excellent normal local domain,* y *is a nonzero element in* $\mathbf{m}$, $(R^*, \mathbf{m}^*)$ *is the* (y)*-adic* *completion of R, and the elements $\tau_1, \dots, \tau_s \in yR^*$ are algebraically independent over R. Then the following statements are equivalent:*

(1) $S := R[\tau_1, \dots, \tau_s] \hookrightarrow \widehat{R}[1/y]$ *is* LF_k.

(2) *If P is a prime ideal of S and $\widehat{Q}$ is a prime ideal of $\widehat{R}$ minimal over $P\widehat{R}$ and if, moreover, $y \notin \widehat{Q}$ and* $\mathrm{ht}(\widehat{Q}) \le k$, *then* $\mathrm{ht}(\widehat{Q}) = \mathrm{ht}(P)$.

(3) *If $\widehat{Q}$ is a prime ideal of $\widehat{R}$ with $y \notin \widehat{Q}$ and* $\mathrm{ht}(\widehat{Q}) \le k$, *then* $\mathrm{ht}(\widehat{Q}) \ge \mathrm{ht}(\widehat{Q} \cap S)$.

PROOF. (1) $\Rightarrow$ (2): Let P be a prime ideal of S and let $\widehat{Q}$ be a prime ideal of $\widehat{R}$ that is minimal over $P\widehat{R}$ with $y \notin \widehat{Q}$ and $\mathrm{ht}(\widehat{Q}) \le k$. The assumption of item 1 implies flatness for the map:

$$\varphi_{\widehat{Q}} : S_{\widehat{Q} \cap S} \longrightarrow \widehat{R}_{\widehat{Q}},$$

and we continue as in Theorem 22.3.

(2) $\Rightarrow$ (3): Let $\widehat{Q}$ be a prime ideal of $\widehat{R}$ with $y \notin \widehat{Q}$ and $\mathrm{ht}(\widehat{Q}) \le k$. Set $Q := \widehat{Q} \cap S$ and let $\widehat{W}$ be a prime ideal of $\widehat{R}$ which is minimal over $Q\widehat{R}$, and so that $\widehat{W} \subseteq \widehat{Q}$. Then $\mathrm{ht}(Q) = \mathrm{ht}(\widehat{W})$ by item 2 since $y \notin \widehat{W}$ and therefore $\mathrm{ht}(\widehat{Q}) \ge \mathrm{ht}(Q)$.

(3) $\Rightarrow$ (1): Let $\widehat{Q}$ be a prime ideal of $\widehat{R}$ with $y \notin \widehat{Q}$ and $\mathrm{ht}(\widehat{Q}) \le k$. Then for every prime ideal $\widehat{W}$ contained in $\widehat{Q}$, we also have $y \notin \widehat{W}$ and $\mathrm{ht}(\widehat{W}) \ge \mathrm{ht}(\widehat{W} \cap S)$, by item 3. To complete the proof it suffices to show that $\varphi_{\widehat{Q}} : S_{\widehat{Q} \cap S} \longrightarrow \widehat{R}_{\widehat{Q}}$ is flat, and this is a consequence of Theorem 11.4. □

22.2. Existence of primarily limit-intersecting extensions

In this section, we establish the existence of primary limit-intersecting elements over countable excellent normal local domains. To do this, we use the following prime avoidance lemma that is analogous to Lemma 19.18, but avoids the hypothesis of Lemma 19.18 that T is complete in its $\mathbf{n}$-adic topology. See the articles [**19**], [**131**], [**143**] and the book [**84**, Lemma 14.2] for other prime avoidance results involving countably infinitely many prime ideals.

LEMMA 22.10. *Let $(T, \mathbf{n})$ be a Noetherian local domain that is complete in the (y)-adic topology, where y is a nonzero element of $\mathbf{n}$. Let $\mathcal{U}$ be a countable set of prime ideals of T such that $y \notin P$ for each $P \in \mathcal{U}$, and fix an arbitrary element $t \in \mathbf{n} \setminus \mathbf{n}^2$. Then there exists an element $a \in y^2T$ such that $t - a \notin \bigcup\{P : P \in \mathcal{U}\}$.*

PROOF. We may assume there are no inclusion relations among the $P \in \mathcal{U}$. We enumerate the prime ideals in $\mathcal{U}$ as $\{P_i\}_{i=1}^{\infty}$. We choose $b_2 \in T$ so that $t - b_2y \notin P_1$ as follows: (i) if $t \in P_1$, let $b_2 = 1$. Since $y \notin P_1$, we have $t - y^2 \notin P_1$. (ii) if $t \notin P_1$, let b_2 be a nonzero element of P_1. Then $t - b_2y^2 \notin P_1$. Assume by induction that we have found $b_2, \dots, b_n$ in T such that

$$t - cy^2 \;:=\; t - b_2y^2 - \cdots - b_ny^n \;\notin\; P_1 \cup \cdots \cup P_{n-1}.$$

We choose $b_{n+1} \in T$ so that $t - cy^2 - b_{n+1}y^{n+1} \notin \bigcup_{i=1}^{n} P_i$ as follows: (i) if $t - cy^2 \in P_n$, let $b_{n+1} \in (\prod_{i=1}^{n-1} P_i) \setminus P_n$. (ii) if $t - cy^2 \notin P_n$, let b_{n+1} be any nonzero element in $\prod_{i=1}^{n} P_i$. Hence in either case there exists $b_{n+1} \in T$ so that

$$t - b_2y^2 - \cdots - b_{n+1}y^{n+1} \;\notin P_1 \cup \cdots \cup P_n.$$

Since T is complete in the (y)-adic topology, the Cauchy sequence

$$\{b_2y^2 + \cdots + b_ny^n\}_{n=2}^{\infty}$$

has a limit $a \in \mathbf{n}^2$. Since T is Noetherian and local, every ideal of T is closed in the (y)-adic topology. Hence, for each integer $n \geq 2$, we have

$$t - a \ = \ (t - b_2y^2 - \cdots - b_ny^n) \ - \ (b_{n+1}y^{n+1} + \cdots),$$

where $t - b_2y^2 - \cdots - b_ny^n \notin P_{n-1}$ and $(b_{n+1}y^{n+1} + \cdots) \in P_{n-1}$. We conclude that $t - a \notin \bigcup_{i=1}^{\infty} P_i$. $\square$

We use the following setting to describe necessary and sufficient conditions for an element to be primarily limit-intersecting.

Setting 22.11. Let $(R, \mathbf{m})$ be a d-dimensional excellent normal local domain with $d \geq 2$, let y be a nonzero element of $\mathbf{m}$ and let R^* denote the (y)-adic completion of R. Let t be a variable over R, let $S := R[t]_{(\mathbf{m},t)}$, and let S^* denote the I-adic completion of S, where $I := (y, t)S$. Then $S^* = R^*[[t]]$ is a $(d + 1)$-dimensional normal Noetherian local domain with maximal ideal $\mathbf{n}^* := (\mathbf{m}, t)S^*$. For each element $a \in y^2S^*$, we have $S^* = R^*[[t]] = R^*[[t - a]]$. Let $\lambda_a : S^* \to R^*$ denote the canonical homomorphism $S^* \to S^*/(t-a)S^* = R^*$, and let $\tau_a = \lambda_a(t) = \lambda_a(a)$. Consider the set

$$\mathcal{U} \ := \ \{P^* \in \operatorname{Spec} S^* \mid \operatorname{ht}(P^* \cap S) = \operatorname{ht} P^*, \text{ and } y \notin P^* \ \}.$$

Since $S \hookrightarrow S^*$ is flat and thus satisfies the going-down property, the set $\mathcal{U}$ can also be described as the set of all $P^* \in \operatorname{Spec} S^*$ such that $y \notin P^*$ and P^* is minimal over PS^* for some $P \in \operatorname{Spec} S$, see [**96**, Theorem 15.1]

Theorem 22.12. *With the notation of Setting 22.11, the element τ_a is primarily limit-intersecting in y over R if and only if $t - a \notin \bigcup\{P^* \mid P^* \in \mathcal{U}\}$.*

Proof. Consider the commutative diagram:

$$\begin{array}{ccccccc}
 & & S = R[t]_{(\mathbf{m},t)} & \xrightarrow{\subseteq} & S^* = R^*[[t]] & \xrightarrow{\subseteq} & S^*[1/y] \\
 & & \downarrow{\scriptstyle \lambda_0} & & \downarrow{\scriptstyle \lambda_a} & & \\
R & \xrightarrow{\subseteq} & R_1 = R[\tau_a]_{(\mathbf{m},\tau_a)} & \longrightarrow & R^* & \xrightarrow{\subseteq} & R^*[1/y].
\end{array}$$

Diagram 22.12.0

The map λ_0 denotes the restriction of λ_a to S.

Assume that τ_a is primarily limit-intersecting in y over R. Then τ_a is algebraically independent over R and λ_0 is an isomorphism. If $t - a \in P^*$ for some $P^* \in \mathcal{U}$, we prove that $\varphi : R_1 \to R^*[1/y]$ is not flat. Let $Q^* := \lambda_a(P^*)$. We have $\operatorname{ht} Q^* = \operatorname{ht} P^* - 1$, and $y \notin P^*$ implies $y \notin Q^*$. Let $P := P^* \cap S$ and $Q := Q^* \cap R_1$. Commutativity of Diagram 22.12.0 and λ_0 an isomorphism imply that $\operatorname{ht} P = \operatorname{ht} Q$. Since $P^* \in \mathcal{U}$, we have $\operatorname{ht} P = \operatorname{ht} P^*$. It follows that $\operatorname{ht} Q > \operatorname{ht} Q^*$. This implies that $\varphi : R_1 \to R^*[1/y]$ is not flat.

For the converse, assume that $t - a \notin \bigcup\{P^* \mid P^* \in \mathcal{U}\}$. Since $a \in y^2S^*$ and S^* is complete in the (y, t)-adic topology, we have $S^* = R^*[[t]] = R^*[[t - a]]$. Thus

$$\mathfrak{p} \ := \ \ker(\lambda_a) \ = \ (t - \tau_a)S^* \ = \ (t - a)S^*$$

is a height-one prime ideal of S^*. Since $y \in R$ and $\mathfrak{p} \cap R = (0)$, we have $y \notin \mathfrak{p}$.

Since $t - a$ is outside every element of $\mathcal{U}$, we have $\mathfrak{p} \notin \mathcal{U}$. Since $\mathfrak{p}$ does not fit the condition of $\mathcal{U}$, we have $\operatorname{ht}(\mathfrak{p} \cap S) \neq \operatorname{ht} \mathfrak{p} = 1$, and so, by the faithful flatness of

$S \hookrightarrow S^*$, $\mathfrak{p} \cap S = (0)$. Therefore the map $\lambda_0 : S \to R_1$ has trivial kernel, and so λ_0 is an isomorphism. Thus τ_a is algebraically independent over R.

Since R is excellent and R_1 is a localized polynomial ring over R, the hypotheses of Corollary 11.5 are satisfied. It follows that the element τ_a is primarily limit-intersecting in y over R provided that $\operatorname{ht}(Q_1^* \cap R_1) \le \operatorname{ht} Q_1^*$ for every prime ideal $Q_1^* \in \operatorname{Spec}(R^*[1/y])$, or, equivalently, if for every $Q^* \in \operatorname{Spec} R^*$ with $y \notin Q^*$, we have $\operatorname{ht}(Q^* \cap R_1) \le \operatorname{ht} Q^*$. Thus, to complete the proof of Theorem 22.12, it suffices to prove Claim 22.13. $\square$

Claim 22.13. *For every prime ideal $Q^* \in \operatorname{Spec} R^*$ with $y \notin Q^*$, we have*

$$\operatorname{ht}(Q^* \cap R_1) \le \operatorname{ht} Q^*.$$

Proof. (of Claim 22.13) Since $\dim R^* = d$ and $y \notin Q^*$, we have $\operatorname{ht} Q^* = r \le d-1$. Since the map $R \hookrightarrow R^*$ is flat, we have $\operatorname{ht}(Q^* \cap R) \le \operatorname{ht} Q^* = r$. Suppose that $Q := Q^* \cap R_1$ has height at least $r+1$ in $\operatorname{Spec} R_1$. Since R_1 is a localized polynomial ring in one variable over R and $\operatorname{ht}(Q \cap R) \le r$, we have $\operatorname{ht}(Q) = r+1$. Let $P := \lambda_0^{-1}(Q) \in \operatorname{Spec} S$. Then $\operatorname{ht} P = r+1$ and $y \notin P$.

Let $P^* := \lambda_a^{-1}(Q^*)$. Since the prime ideals of S^* that contain $t-a$ and have height $r+1$ are in one-to-one correspondence with the prime ideals of R^* of height r, we have $\operatorname{ht} P^* = r+1$. By the commutativity of the diagram, we also have $y \notin P^*$ and $P \subseteq P^* \cap S$, and so

$$r+1 = \operatorname{ht} P \le \operatorname{ht}(P^* \cap S) \le \operatorname{ht} P^* = r+1,$$

where the last inequality holds because the map $S \hookrightarrow S^*$ is flat. It follows that $P = P^* \cap S$, and so $P^* \in \mathcal{U}$. This contradicts the fact that $t-a \notin P_1^*$ for each $P_1^* \in \mathcal{U}$. Thus we have $\operatorname{ht}(Q^* \cap R_1) \le r = \operatorname{ht} Q^*$, as asserted in Claim 22.13. This completes the proof of Theorem 22.12. $\square$

Theorem 22.12 yields a necessary and sufficient condition for an element of R^* that is algebraically independent over R to be primarily limit-intersecting in y over R.

Remarks 22.14. Assume notation as in Setting 22.11.

(1) For each $a \in y^2 S^*$ as in Setting 22.11, we have $(t-a)S^* = (t-\tau_a)S^*$. Hence $t-a \notin \bigcup\{P^* \mid P^* \in \mathcal{U}\} \iff t-\tau_a \notin \bigcup\{P^* \mid P^* \in \mathcal{U}\}$.
(2) If $a \in R^*$, then the commutativity of Diagram 22.12.0 implies that $\tau_a = a$.
(3) For $\tau \in R^*$, we have $\tau = a_0 + a_1 y + \tau'$, where a_0 and a_1 are in R and $\tau' \in y^2 R^*$.
   (a) The rings $R[\tau]$ and $R[\tau']$ are equal. Hence τ is primarily limit-intersecting in y over R if and only if τ' is primarily limit-intersecting in y over R.
   (b) Assume $\tau \in R^*$ is algebraically independent over R. Then τ is primarily limit-intersecting in y over R if and only if $t-\tau' \notin \bigcup\{P^* \mid P^* \in \mathcal{U}\}$.

   Item 3b follows from Theorem 22.12 by setting $a = \tau'$ and applying item 3a and item 2.

We use Theorem 22.12 and Lemma 22.10 to prove Theorem 22.15.

THEOREM 22.15. *Let $(R, \mathbf{m})$ be a countable excellent normal local domain with dimension $d \geq 2$, and let y be a nonzero element in $\mathbf{m}$. Let R^* denote the (y)-adic completion of R. Then there exists an element $\tau \in yR^*$ that is primarily limit-intersecting in y over R.*

PROOF. As in Setting 22.11, let

$$\mathcal{U} \ := \ \{P^* \in \operatorname{Spec} S^* \mid \operatorname{ht}(P^* \cap S) = \operatorname{ht} P^*, \quad \text{and } y \notin P^* \ \}.$$

Since the ring S is countable and Noetherian, the set $\mathcal{U}$ is countable. Lemma 19.18 implies that there exists an element $a \in y^2 S^*$ such that $t - a \notin \bigcup\{P^* \mid P^* \in \mathcal{U}\}$. By Theorem 22.12, the element τ_a is primarily limit-intersecting in y over R. $\square$

To establish the existence of more than one primarily limit-intersecting element we use the following setting.

SETTING 22.16. Let $(R, \mathbf{m})$ be a d-dimensional excellent normal local domain, let y be a nonzero element of $\mathbf{m}$ and let R^* denote the (y)-adic completion of R. Let $t_1, \ldots, t_{n+1}$ be indeterminates over R, and let S_n and S_{n+1} denote the localized polynomial rings

$$S_n \ := \ R[t_1, \ldots, t_n]_{(\mathbf{m}, t_1, \ldots, t_n)} \quad \text{and} \quad S_{n+1} \ := \ R[t_1, \ldots, t_{n+1}]_{(\mathbf{m}, t_1, \ldots, t_{n+1})}.$$

Let S_n^* denote the I_n-adic completion of S_n, where $I_n := (y, t_1, \ldots, t_n)S_n$. Then $S_n^* = R^*[[t_1, \ldots, t_n]]$ is a $(d+n)$-dimensional normal Noetherian local domain with maximal ideal $\mathbf{n}^* = (\mathbf{m}, t_1, \ldots, t_n)S_n^*$. Assume that $\tau_1, \ldots, \tau_n \in yR^*$ are primarily limit-intersecting in y over R, and define $\lambda : S_n^* \to R^*$ to be the R^*-algebra homomorphism such that $\lambda(t_i) = \tau_i$, for $1 \leq i \leq n$.

Since $S_n^* = R^*[[t_1 - \tau_1, \ldots, t_n - \tau_n]]$, we have $\mathbf{p}_n := \ker \lambda = (t_1 - \tau_1, \ldots, t_n - \tau_n)S_n^*$. Consider the commutative diagram:

$$\begin{tikzcd}
 & S_n = R[t_1, \ldots, t_n]_{(\mathbf{m}, t_1, \ldots, t_n)} \arrow[r, "\subseteq"] \arrow[d, "\lambda_0,\ \cong"'] & S_n^* = R^*[[t_1, \ldots, t_n]] \arrow[r, "\subseteq"] \arrow[d, "\lambda"'] & S_n^*[1/y] \\
R \arrow[r, "\subseteq"] & R_n = R[\tau_1, \ldots, \tau_n]_{(\mathbf{m}, \tau_1, \ldots, \tau_n)} \arrow[r, "\varphi_0"] & R^* \arrow[r, "\alpha"] & R^*[1/y].
\end{tikzcd}$$

Let S_{n+1}^* denote the I_{n+1}-adic completion of S_{n+1}, where $I_{n+1} := (y, t_1, \ldots, t_{n+1})S_{n+1}$. For each element $a \in y^2 S_{n+1}^*$, we have

$$S_{n+1}^* = S_n^*[[t_{n+1}]] = S_n^*[[t_{n+1} - a]]. \tag{22.16.1}$$

Let $\lambda_a : S^* \to R^*$ denote the composition

$$S_{n+1}^* \ = \ S_n^*[[t_{n+1}]] \ \longrightarrow \ \frac{S_n^*[[t_{n+1}]]}{(t_{n+1} - a)} \ = \ S_n^* \ \xrightarrow{\lambda} \ R^*,$$

and let $\tau_a := \lambda_a(t_{n+1}) = \lambda_a(a)$. We have $\ker \lambda_a = (\mathbf{p}_n, t_{n+1} - a)S_{n+1}^*$. Consider the commutative diagram

$$\begin{tikzcd}
 & S_n \arrow[r, "\subseteq"] \arrow[d, "\lambda_0,\ \cong"'] & S_n^* \arrow[r, "\subseteq"] \arrow[d, "\lambda"'] & S_{n+1}^* \arrow[r] \arrow[d, "\lambda_a"'] & S_{n+1}^*[1/y] \arrow[d] \\
R \arrow[r, "\subseteq"] & R_n \arrow[r, "\varphi_0"] & R^* \arrow[r, "="] & R^* \arrow[r] & R^*[1/y].
\end{tikzcd}$$

Diagram 22.16.2

Let

$\mathcal{U} := \{P^* \in \operatorname{Spec} S^*_{n+1} \mid P^* \cap S_{n+1} = P,\ y \notin P$ and P^* is minimal over $(P, \mathbf{p}_n)S^*_{n+1}\}$.

Notice that $y \notin P^*$ for each $P^* \in \mathcal{U}$, since $y \in R$ implies $\lambda_a(y) = y$.

THEOREM 22.17. *With the notation of Setting 22.16, the elements $\tau_1, \dots, \tau_n, \tau_a$ are primarily limit-intersecting in y over R if and only if $t_{n+1} - a \notin \bigcup\{P^* \mid P^* \in \mathcal{U}\}$.*

PROOF. Assume that $\tau_1, \dots, \tau_n, \tau_a$ are primarily limit-intersecting in y over R. Then $\tau_1, \dots, \tau_n, \tau_a$ are algebraically independent over R. Consider the following commutative diagram:

$$\begin{array}{ccccc} & & S_{n+1} = R[t_1, \dots, t_{n+1}]_{(\mathbf{m}, t_1, \dots, t_{n+1})} & \xrightarrow{\subseteq} & S^*_{n+1} = R^*[[t_1, \dots, t_{n+1}]] \\ & & \lambda_1 \Big\downarrow & & \lambda_a \Big\downarrow \\ R & \xrightarrow{\subseteq} & R_{n+1} = R[\tau_1, \dots, \tau_a]_{(\mathbf{m}, \tau_1, \dots, \tau_a)} & \longrightarrow & R^*. \end{array}$$

Diagram 22.17.0

The map λ_1 is the restriction of λ_a to S_{n+1}, and is an isomorphism since $\tau_1, \dots, \tau_n, \tau_a$ are algebraically independent over R.

If $t_{n+1} - a \in P^*$ for some $P^* \in \mathcal{U}$, we prove that $\varphi : R_{n+1} \to R^*[1/y]$ is not flat, a contradiction to our assumption that $\tau_1, \dots, \tau_n, \tau_a$ are primarily limit-intersecting. Since $P^* \in \mathcal{U}$, we have $\mathbf{p}_n \subset P^*$. Then $t_{n+1} - a \in P^*$ implies $\ker \lambda_a \subset P^*$. Let $\lambda_a(P^*) := Q^*$. Then $\lambda_a^{-1}(Q^*) = P^*$ and $\operatorname{ht} P^* = n + 1 + \operatorname{ht} Q^*$. Since $P^* \in \mathcal{U}$, we have $y \notin P^*$. The commutativity of Diagram 22.17.0 implies that $y \notin Q^*$. Let $P := P^* \cap S_{n+1}$ and let $Q := Q^* \cap R_{n+1}$. Commutativity of Diagram 22.17.0 and λ_0 an isomorphism imply that $\operatorname{ht} P = \operatorname{ht} Q$. Since P^* is a minimal prime of $(P, \mathbf{p}_n)S^*_{n+1}$ and $\mathbf{p}_n$ is n-generated and S^*_{n+1} is Noetherian and catenary, we have $\operatorname{ht} P^* \le \operatorname{ht} P + n$. Hence $\operatorname{ht} P \ge \operatorname{ht} P^* - n$. Thus

$$\operatorname{ht} Q \ = \ \operatorname{ht} P \ \ge \ \operatorname{ht} P^* - n \ = \ \operatorname{ht} Q^* + n + 1 - n \ = \ \operatorname{ht} Q^* + 1.$$

The fact that $\operatorname{ht} Q > \operatorname{ht} Q^*$ implies that the map $R_{n+1} \to R^*[1/y]$ is not flat.

For the converse, we have

Assumption 22.17.1: $\quad t_{n+1} - a \ \notin \ \bigcup \{ \ P^* \mid P^* \in \mathcal{U} \ \}$.

Since $\lambda_a : S^*_{n+1} \to R^*$ is an extension of $\lambda : S^*_n \to R^*$ as in Diagram 22.16.2, we have $\ker \lambda_a \cap S_n = (0)$. Let $\mathbf{p} := (t_{n+1} - a)S^*_{n+1} = (t_{n+1} - \tau_a)S^*_{n+1}$. As in Equation 22.16.1, we have

$$S^*_{n+1} \ = \ R^*[[t_1, \dots, t_{n+1}]] \ = \ R^*[[t_1 - \tau_1, \dots, t_n - \tau_n, t_{n+1} - a]].$$

Thus $P^* := (\mathbf{p}_n, \mathbf{p})S^*_{n+1}$ is a prime ideal of height $n+1$ and $P^* \cap R^* = (0)$. It follows that $y \notin P^*$. We show that $P^* \cap S_{n+1} = (0)$. Assume that $P = P^* \cap S_{n+1} \neq (0)$. Since $\operatorname{ht} P* = n+1$, P^* is minimal over $(P, \mathbf{p}_n)S^*_{n+1}$, and so $P^* \in \mathcal{U}$, a contradiction to Assumption 22.17.1. Therefore $P^* \cap S_{n+1} = (0)$. It follows that $\mathbf{p} \cap S_{n+1} = (0)$ since $\mathbf{p} \subset P^*$. Thus $\ker \lambda_1 = (0)$, and so λ_1 in Diagram 22.17.0 is an isomorphism. Therefore τ_a is algebraically independent over R_n.

Since R is excellent and R_{n+1} is a localized polynomial ring in $n+1$ variables over R, the hypotheses of Corollary 11.5 are satisfied. It follows that the elements $\tau_1, \dots, \tau_n, \tau_a$ are primarily limit-intersecting in y over R if for every $Q^* \in \operatorname{Spec} R^*$

with $y \notin Q^*$, we have $\operatorname{ht}(Q^* \cap R_{n+1}) \le \operatorname{ht} Q^*$. Thus, to complete the proof of Theorem 22.17, it suffices to prove Claim 22.18. $\square$

Claim 22.18. *Let $Q^* \in \operatorname{Spec} R^*$ with $y \notin Q^*$ and $\operatorname{ht} Q^* = r$. Then*

$$\operatorname{ht}(Q^* \cap R_{n+1}) \le \ r.$$

Proof. (of Claim 22.18) Let $Q_1 := Q^* \cap R_{n+1}$ and let $Q_0 := Q^* \cap R_n$. Suppose $\operatorname{ht} Q_1 > r$. Notice that $r < d$, since $d = \dim R^*$ and $y \notin Q^*$.

Since $\tau_1, \ldots, \tau_n$ are primarily limit-intersecting in y over R, the extension

$$R_n := R[\tau_1, \ldots, \tau_n]_{(\mathbf{m}, \tau_1, \ldots, \tau_n)} \hookrightarrow R^*[1/y]$$

from Diagram 22.16.2 is flat. Thus $\operatorname{ht} Q_0 \le r$ and $\operatorname{ht} Q_0 \le \operatorname{ht} L^*$ for every prime ideal L^* of R^* with $Q_0 R^* \subseteq L^* \subseteq Q^*$. Since R_{n+1} is a localized polynomial ring in the indeterminate τ_a over R_n, we have that $\operatorname{ht} Q_1 \le \operatorname{ht} Q_0 + 1 = r + 1$. Thus $\operatorname{ht} Q_1 = r + 1$ and $\operatorname{ht} Q_0 = r$. It follows that Q^* is a minimal prime of $Q_0 R^*$.

Let $h(\tau_a)$ be a polynomial in

$$(Q^* \cap R_n[\tau_a]) \ \setminus \ (Q^* \cap R_n) R_{n+1}.$$

It follows that $Q^* \cap R_{n+1} := Q_1$ is a minimal prime of the ideal $(Q^* \cap R_n, h(\tau_a))R_{n+1}$.

With notation from Diagram 22.16.2, define

$$P_0 \ := \ \lambda_0^{-1}(Q_0) \ \text{ and } \ P_0^* \ := \lambda^{-1}(Q^*).$$

Since λ_0 is an isomorphism, P_0 is a prime ideal of S_n with $\operatorname{ht} P_0 = r$. Moreover, we have the following:

(1) $P_0^* \cap S_n = P_0$ (by commutativity in Diagram 22.16.2),
(2) $y \notin P_0^*$ (by item 1),
(3) P_0^* is a minimal prime of $(P_0, \mathbf{p}_n)S_n^*$ (since $S_n^*/\mathbf{p}_n = R^*$ in Diagram 22.16.2, and Q^* is a minimal prime of $Q_0 R^*$),
(4) $\operatorname{ht} P_0^* = n + r$ (by the correspondence between prime ideals of S_n^* containing $\mathbf{p}_n$ and prime ideals of R^*).

Consider the commutative diagram below with the left and right ends identified:

$$\begin{tikzcd}
S_{n+1}^* \arrow[d, "\lambda_a"'] & S_n^* \arrow[l] \arrow[d, "\lambda"'] & S_n \arrow[l] \arrow[r] \arrow[d, "\lambda_0, \cong"'] & S_{n+1} \arrow[r, "\theta"] \arrow[d, "\lambda_1, \cong"'] & S_{n+1}^* \arrow[d, "\lambda_a"'] \\
R^* & R^* \arrow[l] & R_n \arrow[l] \arrow[r] & R_{n+1} \arrow[r] & R^*,
\end{tikzcd}$$

Diagram 22.18.0

where λ, λ_0 and λ_1 are as in Diagrams 22.16.2 and 22.17.0, and so λ_a restricted to S_n^* is λ. Let $h(t_{n+1}) \ = \ \lambda_1^{-1}(h(\tau_a))$ and set

$$P_1 \ := \ \lambda_1^{-1}(Q_1) \ \in \ \operatorname{Spec}(S_{n+1}), \ \text{ and } \ P^* \ := \ \lambda_a^{-1}(Q^*) \in \operatorname{Spec}(S_{n+1}^*).$$

Then P_1 is a minimal prime of $(P_0, h(t_{n+1}))S_{n+1}$, since Q_1 is a minimal prime of $(Q_0, h(\tau_a))R_{n+1}$. Since $Q_1 \subseteq Q^*$, we have $h(t_{n+1}) \in P^*$ and $P_1 S_{n+1}^* \subseteq P^*$ because $\lambda_a(h(t_{n+1})) = \lambda_1(h(t_{n+1})) = h(\tau_a) \in Q_1$ and $\lambda_a(P_1) = \lambda_1(P_1) = Q_1$. By the correspondence between prime ideals of S_{n+1}^* containing $\ker(\lambda_a) = \mathbf{p}_{n+1}$ and prime ideals of R^*, we see

$$\operatorname{ht} P^* = \operatorname{ht} Q^* + n + 1 = r + n + 1.$$

Since $\lambda_a(P_0^*) \subseteq Q^*$, we have $P_0^* \subseteq P^*$, but $h(t_{n+1}) \notin P_0$ implies $h(t_{n+1}) \notin P_0^* S_{n+1}^*$. Therefore

$$(P_0, \mathbf{p}_n)S_{n+1}^* \subseteq P_0^* S_{n+1}^* \subsetneq (P_0^*, h(t_{n+1}))S_{n+1}^* \subseteq P^*.$$

By items 3 and 4 above, $\operatorname{ht} P_0^* = n + r$ and P_0^* is a minimal prime of $(P_0, \mathbf{p}_n)S_n^*$. Since $\operatorname{ht} P^* = n+r+1$, it follows that P^* is a minimal prime of $(P_0, h(t_{n+1}), \mathbf{p}_n)S_{n+1}^*$. Since $(P_0, h(t_{n+1}), \mathbf{p}_n)S_{n+1}^* \subseteq (P_1, \mathbf{p}_n)S_{n+1}^* \subseteq P^*$, we have P^* is a minimal prime of $(P_1, \mathbf{p}_n)S_{n+1}^*$. But then, by Assumption 22.17.1, $t_{n+1} - a \notin P^*$, a contradiction. This contradiction implies that $\operatorname{ht} Q_1 = r$. This completes the proof of Claim 22.18 and thus also the proof of Theorem 22.17. $\square$

We use Theorem 22.15, Theorem 22.17 and Lemma 22.10 to prove in Theorem 22.19 the existence over a countable excellent normal local domain of dimension at least two of an infinite sequence of primarily limit-intersecting elements.

THEOREM 22.19. *Let R be a countable excellent normal local domain with dimension $d \geq 2$, let y be a nonzero element in the maximal ideal $\mathbf{m}$ of R, and let R^* be the (y)-adic completion of R. Let n be a positive integer. Then*

(1) *If the elements $\tau_1, \ldots, \tau_n \in yR^*$ are primarily limit-intersecting in y over R, then there exists an element $\tau_a \in yR^*$ such that $\tau_1, \ldots, \tau_n, \tau_a$ are primarily limit-intersecting in y over R.*
(2) *There exists an infinite sequence $\tau_1, \ldots, \tau_n, \ldots \in yR^*$ of elements that are primarily limit-intersecting in y over R.*

PROOF. Since item 1 implies item 2, it suffices to prove item 1. Theorem 22.15 implies the existence of an element $\tau_1 \in yR^*$ that is primarily limit-intersecting in y over R. As in Setting 22.16, let

$$\mathcal{U} := \{P^* \in \operatorname{Spec} S_{n+1}^* \mid P^* \cap S_{n+1} = P \in \mathcal{S} \text{ and } P^* \text{ is minimal over } (P, \mathbf{p}_n)S_{n+1}^*\}.$$

Since the ring S_{n+1} is countable and Noetherian, the set $\mathcal{U}$ is countable. Lemma 19.18 implies that there exists an element $a \in y^2 S_{n+1}^*$ such that

$$t_{n+1} - a \notin \bigcup\{P^* \mid P^* \in \mathcal{U}\}.$$

By Theorem 22.17, the elements $\tau_1, \ldots, \tau_n, \tau_a$ are primarily limit-intersecting in y over R. $\square$

Using Theorem 22.15, we establish in Theorem 22.20, for every countable excellent normal local domain R of dimension $d \geq 2$, the existence of a primarily limit-intersecting element $\eta \in yR^*$ such that the constructed Noetherian domain

$$B = A = R^* \cap \mathcal{Q}(R[\eta])$$

is not a Nagata domain and hence is not excellent.

THEOREM 22.20. *Let R be a countable excellent normal local domain of dimension $d \geq 2$, let y be a nonzero element in the maximal ideal $\mathbf{m}$ of R, and let R^* be the (y)-adic completion of R. There exists an element $\eta \in yR^*$ such that*

(1) *η is primarily limit-intersecting in y over R.*
(2) *The associated intersection domain $A := R^* \cap \mathcal{Q}(R[\eta])$ is equal to its approximation domain B.*
(3) *The ring A has a height-one prime ideal $\mathbf{p}$ such that $R^*/\mathbf{p}R^*$ is not reduced.*

Thus the integral domain $A = B$ associated to η is a normal Noetherian local domain that is not a Nagata domain and hence is not excellent.

PROOF. Since $\dim R \geq 2$, there exists $x \in \mathbf{m}$ such that $\operatorname{ht}(x,y)R = 2$. By Theorem 22.15, there exists $\tau \in yR^*$ such that τ is primarily limit-intersecting in y over R. Hence the extension $R[\tau] \longrightarrow R^*[1/y]$ is flat. Let $n \in \mathbb{N}$ with $n \geq 2$, and let $\eta := (x+\tau)^n$. Since τ is algebraically independent over R, the element η is also algebraically independent over R. Moreover, the polynomial ring $R[\tau]$ is a free $R[\eta]$-module with $1, \tau, \ldots, \tau^{n-1}$ as a free module basis. Hence the map $R[\eta] \longrightarrow R^*[1/y]$ is flat. It follows that η is primarily limit-intersecting in y over R. Therefore the intersection domain $A := R^* \cap \mathcal{Q}(R[\eta])$ is equal to its associated approximation domain B and is a normal Noetherian domain with (y)-adic completion R^*. Since η is a prime element of the polynomial ring $R[\eta]$ and $B[1/y]$ is a localization of $R[\eta]$, it follows that $\mathbf{p} := \eta B$ is a height-one prime ideal of B. Since $\tau \in R^*$, and $\eta = (x+\tau)^n$, the ring $R^*/\mathbf{p}R^*$ contains nonzero nilpotent elements. Since a Nagata local domain is analytically unramified, it follows that the normal Noetherian domain B is not a Nagata ring, [**96**, page 264] or [**104**, (32.2)]. $\square$

Let d be an integer with $d \geq 2$. In Examples 13.8 we give extensions that satisfy LF_{d-1} but do not satisfy LF_d; see Definition 21.1. These extensions are weakly flat but are not flat. In our setting these examples have the intersection domain A equal to its approximation domain B but A is not Noetherian In Theorem 22.21, we present a more general construction of examples with these properties.

THEOREM 22.21. *Let $(R, \mathbf{m})$ be a countable excellent normal local domain. Assume that $\dim R = d+1 \geq 3$, that $(x_1, \ldots, x_d, y)R$ is an $\mathbf{m}$-primary ideal, and that R^* is the (y)-adic completion of R. Then there exists $f \in yR^*$ such that f is algebraically independent over R and the map $\varphi : R[f] \longrightarrow R^*[1/y]$ is weakly flat but not flat. Indeed, φ satisfies LF_{d-1}, but fails to satisfy LF_d. Thus the intersection domain $A := \mathcal{Q}(R[f]) \cap R^*$ is equal to its approximation domain B, but A is not Noetherian.*

PROOF. By Theorem 22.19, there exist elements $\tau_1, \ldots, \tau_d \in yR^*$ that are primarily limit-intersecting in y over R. Let

$$f \ := \ x_1\tau_1 \ + \cdots + \ x_d\tau_d.$$

Using that $\tau_1, \ldots, \tau_d$ are algebraically independent over R, we regard f as a polynomial in the polynomial ring $T := R[\tau_1, \ldots, \tau_d]$. Let $S := R[f]$. For $Q \in \operatorname{Spec} R^*[1/y]$ and $P := Q \cap T$, consider the composition φ_Q

$$S \ \longrightarrow \ T_P \ \longrightarrow \ R^*[1/y]_Q.$$

Since $\tau_1, \ldots, \tau_d$ are primarily limit-intersecting in y over R, the map $T \hookrightarrow R^*[1/y]$ is flat. Thus the map φ_Q is flat if and only if the map $S \longrightarrow T_P$ is flat. Let $\mathbf{p} := P \cap R$.

Assume that P is a minimal prime of $(x_1, \ldots, x_d)T$. Then $\mathbf{p}$ is a minimal prime of $(x_1, \ldots, x_d)R$. Since T is a polynomial ring over R, we have $P = \mathbf{p}T$ and $\operatorname{ht}(\mathbf{p}) = d = \operatorname{ht} P$. Notice that $(\mathbf{p}, f)S = P \cap S$ and $\operatorname{ht}(\mathbf{p}, f)S = d+1$. Since a flat extension satisfies the going-down property, the map $S \longrightarrow T_P$ is not flat. Hence φ does not satisfy LF_d.

Assume that $\operatorname{ht} P \leq d-1$. Then $(x_1, \ldots, x_d)T$ is not contained in P. Hence $(x_1, \ldots, x_d)R$ is not contained in $\mathbf{p}$. Consider the sequence

$$S \ = \ R[f] \ \hookrightarrow \ R_{\mathbf{p}}[f] \ \xrightarrow{\psi} \ R_{\mathbf{p}}[\tau_1, \ldots, \tau_d] \ \hookrightarrow \ T_P,$$

where the first and last injections are localizations. Since the nonconstant coefficients of f generate the unit ideal of $R_{\mathbf{p}}$, the map ψ is flat; see Theorem 11.20. Thus φ satisfies LF_{d-1}.

We conclude that the intersection domain $A = R^* \cap \mathcal{Q}(R[f])$ is equal to its approximation domain B and is not Noetherian. □

22.3. Cohen-Macaulay formal fibers and Ogoma's example

In Corollary 22.23 we observe that if R is excellent, then every Noetherian example A obtained via Inclusion Construction 5.3 has Cohen-Macaulay formal fibers. We observe in Remark 22.25 that this implies the non-Noetherian property of a certain integral domain B that has Ogoma's example as a homomorphic image.

The following is an analogue of [**96**, Theorem 32.1(ii)]. The distinction is that we are considering regular fibers rather than geometrically regular fibers.

Proposition 22.22. *Suppose R, S, and T are Noetherian commutative rings and suppose we have maps $R \to S$ and $S \to T$ and the composite map $R \to T$. Assume*

(i) *$R \to T$ is flat with regular fibers,*
(ii) *$S \to T$ is faithfully flat.*

Then $R \to S$ is flat with regular fibers.

As an immediate consequence of Theorem 11.4 and Proposition 22.22, we have the following implication concerning Cohen-Macaulay formal fibers.

Corollary 22.23. *Every Noetherian local ring B containing an excellent local subring R and having the same completion as R has Cohen-Macaulay formal fibers. Thus the ring A of Setting 22.1 has Cohen-Macaulay formal fibers whenever A is Noetherian.*

Remark 22.24. (Cohen-Macaulay formal fibers) Corollary 22.23 implies that every Noetherian local ring B that has as its completion $\widehat{B}$ the formal power series ring $k[[x_1, \ldots, x_d]]$ and that contains the polynomial ring $k[x_1, \ldots, x_d]$ has Cohen-Macaulay formal fibers. In connection with Cohen-Macaulay formal fibers, Luchezar Avramov pointed out to us that every homomorphic image of a regular local ring has formal fibers that are complete intersections and therefore Cohen-Macaulay [**44**, (3.6.4), page 118]. Also every homomorphic image of a Cohen-Macaulay local ring has formal fibers that are Cohen-Macaulay [**96**, page 181]. It is interesting that while regular local rings need not have regular formal fibers, they must have Cohen-Macaulay formal fibers.

Remark 22.25. (Ogoma's example) Corollary 22.23 sheds light on Ogoma's famous example [**110**] of a Nagata local domain of dimension three whose generic formal fiber is not equidimensional.

Ogoma's construction begins with a countable field k of infinite but countable transcendence degree over the field $\mathbb{Q}$ of rational numbers. Let x, y, z, w be variables over k, and let $R = k[x,y,z,w]_{(x,y,z,w)}$ be the localized polynomial ring. By a clever enumeration of the prime elements in R, Ogoma constructs three power series $g, h, \ell \in \widehat{R} = k[[x,y,z,w]]$ that satisfy the following conditions:

(a) g, h, ℓ are algebraically independent over $k(x,y,z,w) = \mathcal{Q}(R)$.

(b) g, h, ℓ are part of a regular system of parameters for $\widehat{R} = k[[x,y,z,w]]$.

(c) If $\widehat{P} = (g,h,\ell)\widehat{R}$, then $\widehat{P} \cap R = (0)$, i.e., $\widehat{P}$ is in the generic formal fiber of R.

(d) If $I = (gh, g\ell)\widehat{R}$ and $C = \mathcal{Q}(R) \cap (\widehat{R}/I)$, then C is a Nagata local domain[1] with completion $\widehat{C} = \widehat{R}/I$.

(e) It is then obvious that the completion $\widehat{C} = \widehat{R}/I$ of C has a minimal prime $g\widehat{R}/I$ of dimension 3 and a minimal prime $(h,\ell)\widehat{R}/I$ of dimension 2. Thus C fails to be formally equidimensional. Therefore C is not universally catenary [**96**, Theorem 31.7] and provides a counterexample to the catenary chain condition.

Since C is not universally catenary, C is not a homomorphic image of a regular local ring. There exists a local integral domain B that dominates R, has completion $\widehat{R} = k[[x,y,z,w]]$, and contains an ideal J such that $C = B/J$. If B were Noetherian, then B would be a regular local ring and $C = B/J$ would be universally catenary. Thus B is necessarily non-Noetherian.

Theorem 11.4 provides a different way to deduce that the ring B is non-Noetherian. To see this, we consider more details about the construction of B. The ring B is defined as a nested union of rings:

Let $\lambda_1 = gh$ and $\lambda_2 = g\ell$ and define:

$$B = \bigcup_{n=1}^{\infty} R[\lambda_{1n}, \lambda_{2n}]_{(x,y,z,w,\lambda_{1n},\lambda_{2n})} \subseteq k[[x,y,z,w]]$$

where the λ_{in} are endpieces of the λ_i. The construction is done in such a way that the λ's are in every completion of R with respect to a nonzero principal ideal. By the construction of the power series g, h, ℓ, for every nonzero element $f \in R$ the ring B/fB is essentially of finite type over the field k. This implies that the maximal ideal of B is generated by x,y,z,w and that the completion with respect to the maximal ideal of B is the formal power series ring $\widehat{R} = k[[x,y,z,w]]$. Let $K = k(x,y,z,w)$, then $K \otimes_R B$ is a localization of the polynomial ring in two variables $K[\lambda_1, \lambda_2]$. Recall that $I = (\lambda_1, \lambda_2)\widehat{R}$ and $\widehat{P} = (g,h,\ell)\widehat{R}$. Let $J = I \cap B$. Since $\widehat{P} \cap R = (0)$ we see that $J = \widehat{P} \cap B$ is a prime ideal such that $J(K \otimes_R B)$ is a localization of the prime ideal $(\lambda_1, \lambda_2)K[\lambda_1, \lambda_2]$. Thus

$$J(K \otimes_R \widehat{R}) \;=\; (\lambda_1, \lambda_2)(K \otimes_R \widehat{R})$$

and $\widehat{P}$ is in the formal fiber of B/J. Since $(\widehat{R}/I)_{\widehat{P}}$ is not Cohen-Macaulay, Corollary 22.23 implies that B is not Noetherian.

There is another intermediate ring between R and its completion $k[[x,y,z,w]]$ that carries information about C. This is the intersection ring:

$$A = k(x,y,z,w,\lambda_1,\lambda_2) \cap k[[x,y,z,w]].$$

[1]Ogoma [**110**, page 158] actually constructs C as a directed union of birational extensions of R. He proves that C is Noetherian and that $\widehat{C} = \widehat{R}/I$. It follows that $C = \mathcal{Q}(R) \cap (\widehat{R}/I)$. Heitmann observes in [**77**] that C is already normal.

It is shown in [**53**, Claim 4.3] that the maximal ideal of A is generated by x, y, z, w, and is shown in [**53**, Claim 4.4] that A is non-Noetherian.

22.4. Examples not having Cohen-Macaulay fibers

In this section we adapt the two forms of the basic construction technique to obtain three rings A, B and C that we describe in detail. The setting is somewhat similar to that of Ogoma's example. It is simpler in the sense that it is fairly easy to see that the ring C that corresponds to the ring C in Ogoma's example is Noetherian. Also C is a birational extension of a polynomial ring in 3 variables over a field. On the other hand this setting seems more complicated, since for A and B (which are the two obvious choices of intermediate rings) the ring B maps surjectively onto C, while A does not.

Setting and Notation 22.26. Let k be a field, and x, y, z variables over k. Let $\tau_1, \tau_2 \in xk[[x]]$ be formal power series in x which are algebraically independent over $k(x)$. Suppose that

$$\tau_i = \sum_{n=1}^{\infty} a_{in} x^n, \qquad \text{with } a_{in} \in k, \qquad \text{for } i = 1, 2.$$

The intersection ring $V := k(x, \tau_1, \tau_2) \cap k[[x]]$ is a discrete valuation domain which is a nested union of localized polynomial rings in 3 variables over k: $V = \bigcup_{n=1}^{\infty} k[x, \tau_{1n}, \tau_{2n}]_{(x,\tau_{1n},\tau_{2n})}$, where τ_{1n}, τ_{2n} are the endpieces:

$$\tau_{in} = \sum_{j=n}^{\infty} a_{ij} x^{j-n+1}, \qquad \text{for all } n \in \mathbb{N} \quad \text{and } i = 1, 2.$$

We now define a 3-dimensional regular local ring D such that: (i) D is a localization of a nested union of polynomial rings in 5 variables, (ii) D has maximal ideal $(x, y, z)D$ and completion $\widehat{R} = k[[x, y, z]]$, and (iii) D dominates the localized polynomial ring $R := k[x, y, z]_{(x,y,z)}$:

$$\text{(22.4.1.1)} \quad D := V[y, z]_{(x,y,z)} = U_{(x,y,z)D\cap U}, \qquad \text{where } U := \bigcup_{n=1}^{\infty} k[x, y, z, \tau_{1n}, \tau_{2n}].$$

Moreover, $D = k(x, y, z, \tau_1, \tau_2) \cap \widehat{R}$ (see Polynomial Example Theorem 9.2).

We consider the following elements of $\widehat{R}$:

$$s := y + \tau_1, \quad t := z + \tau_2, \quad \rho := s^2 = (y + \tau_1)^2 \quad \text{and} \quad \sigma := st = (y + \tau_1)(z + \tau_2).$$

The elements s and t are algebraically independent over $k(x, y, z)$ as are also the elements ρ and σ. The endpieces of ρ and σ are given as

$$\rho_n := \frac{1}{x^n}((y + \tau_1)^2 - (y + \sum_{j=1}^{n} a_{1j} x^j)^2)$$

$$\sigma_n := \frac{1}{x^n}((y + \tau_1)(z + \tau_2) - (y + \sum_{j=1}^{n} a_{1j} x^j)(z + \sum_{j=1}^{n} a_{2j} x^j).$$

The ideal $I := (\rho, \sigma)\widehat{R}$ has height 1 and is the product of two prime ideals $I = P_1 P_2$ where $P_1 := s\widehat{R}$ and $P_2 := (s, t)\widehat{R}$. Observe that P_1 and P_2 are the associated prime ideals of I, and that P_1 and P_2 are in the generic formal fiber of R.

We now define rings A and C as follows:

(22.4.1.2) $$A := \mathcal{Q}(R)(\rho,\sigma) \cap k[[x,y,z]], \qquad C := \mathcal{Q}(R) \cap (k[[x,y,z]]/I).$$

In analogy with the rings D and U of (22.4.1.1), we have rings $B \subseteq D$ and $W \subseteq U$ defined as follows:

(22.4.1.3)

$$B := \bigcup_{n=1}^{\infty} R[\rho_n,\sigma_n]_{(x,y,z,\rho_n,\sigma_n)} = W_{(x,y,z)B\cap W}, \text{ where } W := \bigcup_{n=1}^{\infty} k[x,y,z,\rho_n,\sigma_n].$$

It is clear that A, B and C are local domains and $B \subseteq A$ with A birationally dominating B. Moreover $(x,y,z)B$ is the maximal ideal of B.

We show in Theorems 22.27 and 22.28 that A and B are non-Noetherian and that $B \subsetneq A$. In Theorem 22.30, we show that C is a Noetherian local domain with completion $\widehat{C} = \widehat{R}/I$ such that C has a non-Cohen-Macaulay formal fiber.

THEOREM 22.27. *With the notation of (22.26), the local integral domains $B \subseteq A$ both have completion $\widehat{R}$ with respect to the powers of their maximal ideals. Also:*

(1) *We have $P_1 \cap B = P_2 \cap B$,*
(2) *B is a UFD,*
(3) $\operatorname{ht}(P_1 \cap B) > \operatorname{ht}(P_1) = 1$,
(4) *B fails to have Cohen-Macaulay formal fibers, and*
(5) *B is non-Noetherian.*

PROOF. It follows from Construction Properties Theorem 6.19 that $\widehat{R}$ is the completion of both A and B.

For item 1, it suffices to show $P_1 \cap W = P_2 \cap W$. It is clear that $P_1 \cap W \subseteq P_2 \cap W$. Let $v \in P_2 \cap W$. Then there is an integer $n \in \mathbb{N}$ such that $x^n v \in k[x,y,z,\rho,\sigma]$. Thus

$$x^n v = \sum b_{ij}\rho^i\sigma^j, \qquad \text{where } b_{ij} \in k[x,y,z], \qquad \text{for all } i,j \in \mathbb{N}.$$

Since $P_2 \cap k[x,y,z] = (0)$ and since $\rho,\sigma \in P_2$ we have that $b_{00} = 0$. This implies that $v \in P_1$. Thus item 1 holds.

For item 2, since $B/xB = R/xR$, the ideal $xB = \mathbf{q}$ is a principal prime ideal in B. Since B is dominated by $\widehat{R}$, we have $\cap_{n=1}^{\infty}\mathbf{q}^n = (0)$. Hence $B_{\mathbf{q}}$ is a DVR. Moreover, by construction, B_x is a localization of $(B_0)_x$, where $B_0 := R[\rho,\sigma]_{(x,y,z,\rho,\sigma)}$, and $(B_0)_x$ is a UFD. Therefore $B = B_x \cap B_{\mathbf{q}}$ is a UFD by Theorem 2.10.

For item 3, we have B_x is a localization of the ring $(B_0)_x$ and the ideal $J = (\rho,\sigma)B_0$ is a prime ideal of height 2. Let $Q = P_1 \cap B$; then $x \notin Q$ and $B_Q = (B_0)_J$. Therefore $\operatorname{ht} Q = 2$. Since $P_1 = s\widehat{R}$ has height one, this proves item 3.

Item 3 implies item 5, since $\operatorname{ht}(P_1 \cap B) > \operatorname{ht} P_1$ implies that $B \to \widehat{R}$ fails to satisfy the going down property, so $\widehat{R}$ is not flat over B and B is not Noetherian.

For item 4, as we saw above, $Qk[[x,y,z]]_{P_2} = (\rho,\sigma)_{P_2} = I_{P_2}$. Thus $\widehat{R}_{P_2}/I\widehat{R}_{P_2}$ is a formal fiber of B. Since $k[[x,y,z]]/I = k[[x,s,t]]/(s^2,st)$, we see that $P_2/I = (s,t)\widehat{R}/(s^2,st)\widehat{R}$ is an embedded associated prime of the ring $k[[x,y,z]]/I$. Hence $(k[[x,y,z]]/I)_{P_2}$ is not Cohen-Macaulay and the embedding $B \longrightarrow k[[x,y,z]]$ fails to have Cohen-Macaulay formal fibers. This also implies that B is non-Noetherian by Corollary 22.6. $\square$

THEOREM 22.28. *With the notation of Setting 22.26 we have:*

(1) *A is a local Krull domain with maximal ideal $(x,y,z)A$ and completion $\widehat{R}$,*

(2) *$P_1 \cap A \subsetneq P_2 \cap A$, so $B \subsetneq A$,*
(3) *A is non-Noetherian.*

Proof. For item 1, it follows from Construction Properties Theorem 6.19 that $(x, y, z)A$ is the maximal ideal of A. By definition, A is the intersection of a field with the Krull domain $\widehat{R}$; thus A is a Krull domain.

For item 2, let $Q_i := P_i \cap A$, for $i = 1, 2$. Observe that

$$\sigma^2/\rho \ = \ (z + \tau_2)^2 \ = \ t^2 \ \in \ (Q_2 \ \setminus \ B) \ \setminus \ Q_1.$$

For item 3, assume A is Noetherian. Then A is a regular local ring and the embedding $A \longrightarrow \widehat{R} = k[[x, y, z]]$ is flat. In particular, A is a UFD and the ideal $P := s\widehat{R} \cap A = P_1 \cap A$ is a prime ideal of height one in A. Thus P is principal. We have that $\rho = s^2 \in P$ and $\sigma^2 = \rho(\sigma^2/\rho)$, therefore $st = \sigma \in P$. Let v be a generator of P. Then $v = sa$ where a is a unit in $D \subseteq k[[x, y, z]]$. We write:

$$v = sa = h(\rho, \sigma)/g(\rho, \sigma), \qquad \text{where} \qquad h(\rho, \sigma),\ g(\rho, \sigma) \ \in \ k[x, y, z][\rho, \sigma]. \tag{22.28.1}$$

Now $a \in D = U_{(x,y,z)D \cap U}$, so $a = g_1/g_2$, where $g_1, g_2 \in k[x, y, z, \tau_{1n}, \tau_{2n}]$, for some $n \in \mathbb{N}$, and g_2 as a power series in $k[[x, y, z]]$ has nonzero constant term. There exists $m \in \mathbb{N}$ such that $x^m g_1 := f_1$ and $x^m g_2 := f_2$ are in the polynomial ring $k[x, y, z, \tau_1, \tau_2] = k[x, y, z][s, t]$. We regard $f_2(s, t)$ as a polynomial in s and t with coefficients in $k[x, y, z]$. We have $f_2 k[[x, y, z]] = x^m k[[x, y, z]] = x^m k[[x, s, t]]$. Therefore $f_2 \notin (s, t)k[[x, s, t]]$. It follows that the constant term of $f_2(s, t) \in k[x, y, z][s, t]$ is a nonzero element of $k[x, y, z]$. Since we have

$$a = \frac{x^m g_1}{x^m g_2} = \frac{f_1}{f_2}, \tag{22.28.2}$$

and a is a unit of D, the constant term of $f_1(s, t) \in k[x, y, z][s, t]$ is also nonzero. Equations 22.28.1 and 22.28.2 together yield

$$s f_1(s, t) h(s^2, st) = f_2(s, t) g(s^2, st). \tag{22.28.3}$$

The term of lowest total degree in s and t on the left hand side of Equation 22.28.3 has odd degree, while the term of lowest total degree in s and t on the right hand side has even degree, a contradiction. Therefore the assumption that A is Noetherian leads to a contradiction. We conclude that A is not Noetherian. $\square$

Remarks 22.29. (i) Although A is not Noetherian, the proof of Theorem 22.28 does not rule out the possibility that A is a UFD. The proof does show that if A is a UFD, then $\operatorname{ht}(P_1 \cap A) > \operatorname{ht}(P_1)$. It would be interesting to know whether the non-flat map $A \to \widehat{A} = \widehat{R}$ has the property that $\operatorname{ht}(\widehat{Q} \cap A) \leq \operatorname{ht}(\widehat{Q})$, for each $\widehat{Q} \in \operatorname{Spec} \widehat{R}$. It would also be interesting to know the dimension of A.
(ii) We observe the close connection of the integral domains $A \subseteq D$ of Setting 22.26. The extension of fields $\mathcal{Q}(A) \subseteq \mathcal{Q}(D)$ has degree two and $A = \mathcal{Q}(A) \cap D$, yet A is non-Noetherian, while D is Noetherian.

Theorem 22.30. *With the notation of (22.26), C is a two-dimensional Noetherian local domain having completion $\widehat{R}/I$ and the generic formal fiber of C is not Cohen-Macaulay.*

Proof. It follows from Construction Properties Theorem 6.17, that the completion of C is $\widehat{R}/I$. Hence if C is Noetherian, then $\dim(C) = \dim(\widehat{R}/I) = 2$. To show that C is Noetherian, by the Noetherian Flatness Theorem 8.3, it suffices to show that the canonical map φ is flat, where:

$$\begin{aligned} R = k[x,y,z]_{(x,y,z)} \xrightarrow{\varphi} & (\widehat{R}/I)[1/x] = (k[[x,y,z]]/I)[1/x] \\ & = (k[[x,s,t]]/(s^2,st)k[[x,s,t]])[1/x]. \end{aligned}$$

Thus it suffices to show for every prime ideal $\widehat{Q}$ of $\widehat{R}$ with $x \notin \widehat{Q}$ that the map

$$\varphi_{\widehat{Q}} : R \longrightarrow \widehat{R}_{\widehat{Q}}/I\widehat{R}_{\widehat{Q}} = (\widehat{R}/I)_{\widehat{Q}}$$

is flat. We may assume $I = P_1P_2 \subseteq \widehat{Q}$.

If $\widehat{Q} = P_2 = (s,t)\widehat{R}$, then $\varphi_{\widehat{Q}}$ is flat since $P_2 \cap R = (0)$.

If $\widehat{Q} \neq P_2$, then $P_2\widehat{R}_{\widehat{Q}} = \widehat{R}_{\widehat{Q}}$, because $\operatorname{ht} P_2 = 2$. Hence $I\widehat{R}_{\widehat{Q}} = P_1\widehat{R}_{\widehat{Q}} = s\widehat{R}_{\widehat{Q}}$. Thus we need to show

$$\varphi_{\widehat{Q}} : R \longrightarrow \widehat{R}_{\widehat{Q}}/s\widehat{R}_{\widehat{Q}} = (\widehat{R}/s\widehat{R})_{\widehat{Q}}$$

is flat. To see that $\varphi_{\widehat{Q}}$ is flat, we observe that, since $R \subseteq D_{\widehat{Q}\cap D} \subseteq \widehat{R}_{\widehat{Q}}$ and $s\widehat{R} \cap R = (0)$, the map $\varphi_{\widehat{Q}}$ factors through a homomorphic image of $D = V[y,z]_{(x,y,z)}$. That is, $\varphi_{\widehat{Q}}$ is the composition of the following maps:

$$R \xrightarrow{\ \gamma\ } (D/sD)_{D\cap\widehat{Q}} \xrightarrow{\ \psi_{\widehat{Q}}\ } (\widehat{R}/s\widehat{R})_{\widehat{Q}}.$$

Since D is Noetherian, the map $\psi_{\widehat{Q}}$ is faithfully flat. Thus it remains to show that γ is flat. Since $x \notin \widehat{Q}$, the ring $(D/sD)_{D\cap\widehat{Q}}$ is a localization of $(D/sD)[1/x]$. Thus it is a localization of the polynomial ring:

$$k[x,y,z,\tau_1,\tau_2]/sk[x,y,z,\tau_1,\tau_2] = k[x,y,z,s,t]/sk[x,y,z,s,t],$$

which is clearly flat over R. Thus C is Noetherian.

Now $P_2/I = \mathbf{p}$ is an embedded associated prime of (0) of $\widehat{C}$ so $\widehat{C}_{\mathbf{p}}$ is not Cohen-Macaulay. Since $\mathbf{p} \cap C = (0)$ the generic formal fiber of C is not Cohen-Macaulay. $\square$

Proposition 22.31. *The canonical map $B \longrightarrow \widehat{R}/I$ factors through C. We have $B/Q \cong C$, where $Q = I \cap B = s\widehat{R} \cap B$. On the other hand, the canonical map $A \longrightarrow \widehat{R}/I$ fails to factor through C.*

Proof. We have canonical maps $B \to \widehat{R}/I$ and $C \to \widehat{R}/I$. We define a map $\phi : B \twoheadrightarrow C$ such that the following diagram commutes:

(22.4.6.1)
$$\begin{array}{ccc} B & \longrightarrow & \widehat{R} \\ {\scriptstyle\phi}\big\downarrow & & \big\downarrow \\ C & \longrightarrow & \widehat{R}/I. \end{array}$$

We write C as a nested union as is done in Chapter 8 (8.3):

$$C = \bigcup_{n=1}^{\infty} R[\bar{\rho_n}, \bar{\sigma_n}]_{(x,y,z,\bar{\rho_n},\bar{\sigma_n})}$$

where $\bar{\rho_n}, \bar{\sigma_n}$ are the n^{th} frontpieces of ρ and σ:

$$\bar{\rho_n} = \frac{1}{x^n}(y + \sum_{j=1}^{n} a_{1j}x^j)^2 \qquad \text{and} \qquad \bar{\sigma_n} = \frac{1}{x^n}(y + \sum_{j=1}^{n} a_{1j}x^j)(z + \sum_{j=1}^{n} a_{2j}x^j).$$

Then

$$B = \bigcup_{n=1}^{\infty} R[\rho_n, \sigma_n]_{(x,y,z,\rho_n,\sigma_n)} \qquad \text{and} \qquad C = \bigcup_{n=1}^{\infty} R[\bar{\rho_n}, \bar{\sigma_n}]_{(x,y,z,\bar{\rho_n},\bar{\sigma_n})}.$$

It is clear that for each $n \in \mathbb{N}$ there is a surjection

$$R[\rho_n, \sigma_n]_{(x,y,z,\rho_n,\sigma_n)} \longrightarrow R[\bar{\rho_n}, \bar{\sigma_n}]_{(x,y,z,\bar{\rho_n},\bar{\sigma_n})}$$

which maps $\rho_n \mapsto \bar{\rho_n}$ and $\sigma_n \mapsto \bar{\sigma_n}$ and which extends to a surjective homomorphism ϕ on the directed unions such that diagram (22.4.6.1) commutes. This shows that $C \cong B/Q$ is a homomorphic image of B.

In order to see the canonical map $\zeta : A \longrightarrow \widehat{R}/I$ fails to factor through C, we note that $I \cap D = (\rho, \sigma)D$ and so ζ factors through D:

$$A \xrightarrow{\quad\gamma\quad} D/(\rho,\sigma)D \xrightarrow{\quad\delta\quad} \widehat{R}/I$$

where δ is injective. The map γ sends the element $\sigma^2/\rho = t^2$ to the residue class of $t^2 = (z + \tau_2)^2$ in $D/(\rho,\sigma)D$. This element is algebraically independent over R, which shows that the ring $A/\ker(\gamma)$ is transcendental over R. Since C is a birational extension of R, the map $A \longrightarrow \widehat{R}/I$ fails to factor through C. $\square$

Exercise

(1) Let $(R, \mathbf{m})$ be a Noetherian local ring, let y be an element in $\mathbf{m}$ and let R^* be the (y)-adic completion of R. Let S be the localized polynomial ring $R[t]_{(\mathbf{m},t)}$ and let S^* denote the I-adic completion completion of S, where $I = (y,t)S$. Let a be an element in the ideal yR^*.

   (a) Prove that R^* is complete in the (a)-adic topology on R^*, and that S^* is complete in the $(t-a)$-adic topology on S^*.
   (b) Prove that S^* is the formal power series ring $R^*[[t]]$.
   (c) Prove that $R^*[[t]] = R^*[[t-a]]$. Thus S^* is the formal power series ring in $t-a$ over R^*, as is used in the proof of Theorem 22.12.

   Comment: Item a is a special case of Exercise 2 of [**96**, p. 63].

   Suggestion: For item c, prove that every element of S^* has a unique expression as a power series in t over R^* and also a unique expression as a power series in $t-a$ over R^*.

CHAPTER 23

Approximating discrete valuation rings by regular local rings

Let k be a field of characteristic zero and let $(V, \mathbf{n})$ be a discrete rank-one valuation domain (DVR) containing k and having residue field $V/\mathbf{n} \cong k$. If the field of fractions L of V has finite transcendence degree s over k, we prove that for every positive integer $d \leq s$, the ring V can be realized as a directed union of regular local rings each of which is a k-subalgebras of V of dimension d. We use a technique inspired by Nagata [**103**] and examined in Chapters 4, 5, 6, 7, 8 for the construction of Noetherian domains.

23.1. Local quadratic transforms and local uniformization

The concepts of local quadratic transformations and local uniformization are relevant for our work in this chapter.

DEFINITIONS 23.1. Let $(R, \mathbf{m})$ be a Noetherian local domain and let $(V, \mathbf{n})$ be a valuation domain that birationally dominates R.

(1) The *first local quadratic transform* of $(R, \mathbf{m})$ along $(V, \mathbf{n})$ is the ring
$$R_1 = R[\mathbf{m}/a]_{\mathbf{m}_1},$$
where $a \in \mathbf{m}$ is such that $\mathbf{m}V = aV$ and $\mathbf{m}_1 := \mathbf{n} \cap R[\mathbf{m}/a]$. The ring R_1 is also called the *dilatation of R by the ideal* $\mathbf{m}$ *along* V [**104**, page 141].

(2) More generally, if $I \subseteq \mathbf{m}$ is a nonzero ideal of R, the *dilatation of R by I along V* is the ring $R[I/a]_{\mathbf{m}_1}$, where $a \in I$ is such that $IV = aV$ and $\mathbf{m}_1 = \mathbf{n} \cap R[I/a]$; moreover, R_1 is uniquely determined by R, V and the ideal I [**104**, page 141].

(3) For each positive integer i, the $(i+1)^{\text{st}}$ *local quadratic transform* R_{i+1} *of R along V* is defined inductively: R_{i+1} is the first local quadratic transform of R_i along V.

REMARKS 23.2. Let $(R, \mathbf{m})$ be a regular local ring and let $(V, \mathbf{n})$ be a valuation domain that birationally dominates R.

(1) It is well known that the local quadratic transform R_1 of R along V is again a regular local ring [**104**, 38.1]; moreover, R_1 is uniquely determined by R and V [**104**, page 141].

(2) With the notation of Definition 23.1.3, we have the following relationship among iterated local quadratic transforms:
$$R_{i+j} = (R_i)_j \quad \text{for all} \quad i, j \geq 0.$$
Associated with the set $\{R_i\}_{i \in \mathbb{N}}$ of local quadratic transformations of R along V, it is natural to consider the subring $R_\infty := \bigcup_{i=1}^{\infty} R_i$ of V.

(3) If $(R, \mathbf{m})$ is a regular local ring of dimension 2 and V is a valuation domain that birationally dominates R, a classical result of Zariski and Abhyankar is that $R_\infty = \bigcup_{n=1}^\infty R_n = V$ [**2**, Lemma 12].

(4) In the case where R is a regular local ring of dimension $d \geq 3$, for certain valuation rings V that birationally dominate R, the union $\bigcup_{n=1}^\infty R_n$ of the local quadratic transforms of R along V is strictly smaller than V [**130**, 4.13]. In many cases Shannon proves in [**130**, (4.5), page 308] that V is a directed union of iterated monoidal transforms of R, where a *monoidal transform of* R is a dilatation of R by a prime ideal P for which the residue class ring R/P is regular.

(5) Assume that $R \subseteq S \subseteq V$, where S is regular local ring birationally dominating R and V is a valuation domain birationally dominating S. Using monoidal transforms, Cutkosky has shown in [**26**] and [**27**] that there exists an iterated local monoidal transform T of S along V such that T is an iterated local monoidal transform of R .

In the case where V is a DVR that birationally dominates a regular local ring, the following useful result is proved by Zariski [**147**, pages 27-28] and Abhyankar [**2**, page 336]. In this connection, for a related result, see Example 6.6.

Proposition 23.3. *Let $(V, \mathbf{n})$ be a DVR that birationally dominates a regular local ring $(R, \mathbf{m})$, and let R_n be the n^{th} local quadratic transform of R along V. Then $R_\infty = \bigcup_{n=1}^\infty R_n = V$. In the case where V is essentially finitely generated over R, we have $R_n = V$ for some positive integer n, and thus $R_{n+i} = R_n$ for all $i \geq 0$.*

Proof. A nonzero element η of V has the form $\eta = b/c$, where $b, c \in R$. If $(b, c)V = V$, then $b/c \in V$ implies $cV = V$. Since V dominates R, it follows that $cR = R$, so $b/c \in R$ in this case. If $\eta = b/c$, with $b, c \in R$ and $(b, c)V = \mathbf{n}^n$, we prove by induction on n that $\eta \in R_n$. We have already done the case where $n = 0$. Assume for every regular local domain $(S, \mathbf{p})$ birationally dominated by V, and every nonzero element $\beta/\gamma \in V$ with $\beta, \gamma \in S$, $(\beta, \gamma)V = \mathbf{n}^j$ and $0 \leq j < n$ we have $\beta/\gamma \in S_j$, where S_j is the j-th iterated local quadratic transform of S along V. Suppose $\beta, \gamma \in S$, $\beta/\gamma \in V$ with $(\beta, \gamma)V = \mathbf{n}^n$. Let $S_1 = S[\mathbf{p}/a]_{\mathbf{p}_1}$, where $a \in \mathbf{p}$ is such that $\mathbf{p}V = aV$ and $\mathbf{p}_1 := \mathbf{n} \cap S[\mathbf{p}/a]$; that is, S_1 is the first local quadratic transform of S along V. Then $\beta_1 := \beta/a$ and $\gamma_1 := \gamma/a$ are in S_1. Thus $a \in \mathbf{n}$ implies $(\beta_1, \gamma_1)V = \mathbf{n}^j$ where $0 \leq j < n$, so by induction

$$\beta/\gamma = \beta_1/\gamma_1 \in (S_1)_j = S_{j+1} \subseteq S_n.$$

This completes the proof of Proposition 23.3. □

Definition 23.4. Let $(R, \mathbf{m})$ be a Noetherian local domain that is essentially finitely generated over a field k and let $(V, \mathbf{n})$ be a valuation domain that birationally dominates R. In algebraic terms *local uniformization of R along V* asserts the existence of a regular local domain extension S of R such that S is essentially finitely generated over R and S is dominated by V.

If R is a regular local ring and P is a prime ideal of R, *embedded local uniformization of R along V* asserts the existence of a regular local domain extension S of R such that S is essentially finitely generated over R and is dominated by V, and has the property that there exists a prime ideal Q of S with $Q \cap R = P$ such that the residue class ring S/Q is a regular local ring.

Discussion 23.5. The classical approach for obtaining embedded local uniformization, introduced by Zariski in the 1940's [**146**], uses local quadratic transforms of R along V. Let $(R, \mathbf{m})$ be a s-dimensional regular local ring. If $(V, \mathbf{n})$ is a DVR that birationally dominates R and has the property that $V/\mathbf{n}$ is algebraic over $R/\mathbf{m}$, then the classical method of taking local quadratic transforms of R along V gives by Proposition 23.3 a representation for V as a directed union $R_\infty = \bigcup_{n\in\mathbb{N}} R_n = V$, where each R_n is an iterated local quadratic transforms of R. The dimension formula [**96**, page 119] implies that $\dim R_n = \dim R = s$ for each n. Thus the DVR V has been represented as a directed union of s-dimensional RLRs. We prove in Theorem 23.6 that certain rank-one discrete valuation rings (DVRs) can be represented as a directed union of regular local domains of dimension d for every positive integer d less than or equal to $s = \dim R$.

23.2. Expressing a DVR as a directed union of regular local rings

We prove the following theorem:

Theorem 23.6. *Let k be a field of characteristic zero and let $(V, \mathbf{n})$ be a DVR containing k with $V/\mathbf{n} = k$. Assume that the field of fractions L of V has finite transcendence degree s over k. Then for every integer d with $1 \le d \le s$, there exists a nested family $\{C_n^{(\alpha)} : n \in \mathbb{N}, \alpha \in \Gamma\}$ of d-dimensional regular local k-subalgebras of V such that V is the directed union of the $C_n^{(\alpha)}$ and V dominates each $C_n^{(\alpha)}$.*

Moreover, if the field L is finitely generated over k, then V is a countable union $\bigcup_{n=1}^\infty C_n$, where, for each $n \in \mathbb{N}$,

(1) *C_n is a d-dimensional regular local k-subalgebra of V,*
(2) *C_n has field of fractions L,*
(3) *C_{n+1} dominates C_n and*
(4) *V dominates C_n.*

We have the following corollary to Theorem 23.6.

Corollary 23.7. *Let k be a field of characteristic zero and let $(R, \mathbf{m})$ be a local domain essentially of finite type over k with coefficient field $k = R/\mathbf{m}$ and field of fractions L. Let $(V, \mathbf{n})$ be a DVR birationally dominating R with $V/\mathbf{n} = k$. For every integer d with $1 \le d \le s$, $s = trdeg_k(L)$, there exists a sequence of d-dimensional regular local k-subalgebras C_n of V such that $V = \cup_{n=1}^\infty C_n$, and for each $n \in \mathbb{N}$, C_{n+1} dominates C_n and V dominates C_n. Moreover C_n dominates R for all sufficiently large n.*

Discussion 23.8. (1) If L/k is finitely generated of transcendence degree s, then the fact that V is a directed union of s-dimensional regular local rings follows from classical theorems of Zariski. The local uniformization theorem of Zariski [**146**] implies the existence of a regular local domain $(R, \mathbf{m})$ containing the field k such that V birationally dominates R. Since k is a coefficient field for V, we have

- $k \hookrightarrow V \twoheadrightarrow V/\mathbf{n} \cong k$; thus k is relatively algebraically closed in L.
- $R/\mathbf{m} = k$ (because V dominates R).
- Every iterated local quadratic transform of R along V has dimension s.

Now by Proposition 23.3, V is a directed union of s-dimensional RLRs.

(2) If $d = 1$, the main theorem is trivially true by taking each $C_n = V$. Thus if L/k is finitely generated of transcendence degree $s = 2$, then the theorem is saying nothing new.

(3) If $s > 2$, then the classical local uniformization theorem says nothing about expressing V as a directed union of d-dimensional RLRs, where $2 \le d \le s-1$. If $(S, \mathbf{p})$ is a Noetherian local domain containing k and birationally dominated by V with $\dim(S) = d < s$, then S does not satisfy the dimension formula. It follows that S is not essentially finitely generated over k [**96**, page 119].

We use the following remark and notation in the proof of Theorem 23.6.

Remark 23.9. With the notation of Theorem 23.6, let $y \in \mathbf{n}$ be such that $yV = \mathbf{n}$. Then the $\mathbf{n}$-adic completion $\widehat{V}$ of V is $k[[y]]$, and we have

$$k[y]_{(y)} \subseteq V \subseteq k[[y]].$$

Then $V = L \cap k[[y]]$, for example since $V \hookrightarrow k[[y]]$ is flat. Since the transcendence degree of L over $k(y)$ is $s-1$, there are $s-1$ elements $\sigma_1, \ldots, \sigma_{s-d}, \tau_1, \ldots, \tau_{d-1} \in yV$ such that L is algebraic over $F := k(y, \sigma_1, \ldots, \sigma_{s-d}, \tau_1, \ldots, \tau_{d-1})$.

Notation 23.10. Continuing with the terminology of (23.9), we set

$$K := k(y, \sigma_1, \ldots, \sigma_{s-d}) \quad \text{and} \quad R := V \cap K.$$

Then R is a DVR and the (y)-adic completion of R is $R^* = k[[y]]$. We also have $B_0 := R[\tau_1, \ldots, \tau_{d-1}]_{(y, \tau_1, \ldots, \tau_{d-1})}$ is a d-dimensional regular local ring and $V_0 := V \cap F$ is a DVR that birationally dominates B_0 and has y-adic completion $\widehat{V_0} = k[[y]]$. The following diagram displays these domains:

$$\begin{tikzcd}
k \arrow[r, "\subseteq"] & K \arrow[r, "\subseteq"] & F \arrow[r, equal] & F \arrow[r, "\subseteq"] & L := \mathcal{Q}(V) \\
k \arrow[u, "\parallel"] \arrow[r, "\subseteq"] & R := V \cap K \arrow[u, "\cup|"] \arrow[r, "\subseteq"] & B_0 \arrow[u, "\cup|"] \arrow[r, "\subseteq"] & V_0 := V \cap F \arrow[u, "\cup|"] \arrow[r, "\subseteq"] & V \arrow[u, "\cup|"]
\end{tikzcd}$$

Let R^* denote the (y)-adic completion of R. Then $\tau_1, \ldots, \tau_s \in yR^*$ are regular elements of R^* that are algebraically independent over K. As in Notation 6.1 we represent each of the τ_i by a power series expansion in y; we use these representations to obtain for each positive integer n the n^{th}-endpieces τ_{in} and corresponding n^{th}-localized polynomial ring B_n. For $1 \le i \le s$, and $\tau_i := \sum_{j=1}^{\infty} r_{ij} y^j$, where the $r_{ij} \in R$, we set, for each $n \in \mathbb{N}$,

$$\tau_{in} := \sum_{j=n+1}^{\infty} r_{ij} y^{j-n}, \qquad B_n := R[\tau_{1n}, \ldots, \tau_{sn}]_{(\mathbf{m}, \tau_{1n}, \ldots, \tau_{sn})}$$
$$(23.10.1) \qquad B := \bigcup_{n=0}^{\infty} B_n = \varinjlim B_n \qquad \text{and} \qquad A := K(\tau_1, \ldots, \tau_s) \cap R^*.$$

Recall that A birationally dominates B. Also by Proposition 6.3, the definition of B_n is independent of the representations of the τ_i.

Since $V_0 = F \cap k[[y]]$, each $\tau_{in} \in V_0$ and $B_n = R[\tau_{1n}, \ldots, \tau_{d-1,n}]_{(y, \tau_{1n}, \ldots, \tau_{d-1,n})}$ is, for each $n \in \mathbb{N}$, the first quadratic transform of B_{n-1} along V_0.

In the proof of Theorem 23.6, we make use of Theorem 8.3 of Chapter 8. We also use Theorem 23.11 in the proof of Theorem 23.6:

Theorem 23.11. *With the notation of (23.10), for each positive integer n, let B_n^h denote the Henselization of B_n. Then $\bigcup_{n=1}^{\infty} B_n^h = V_0^h = V^h$.*

PROOF. Since R^*_y is a field, it is flat as an $R[\tau_1,\dots,\tau_{d-1}]$-module. By Theorem 8.3, $V_0 = \bigcup_{n=1}^\infty B_n$. An alternate way to justify this description of V_0 is to use Proposition 23.3, where the ring R is B_0, and V is V_0, and each $R_n = B_n$. We have

$$V_0 \underset{\text{alg}}{\longrightarrow} V \longrightarrow k[[y]],$$

where V_0 and V are DVRs of characteristic zero having completion $k[[y]]$. Since V_0 and V are excellent, their Henselizations V_0^h and V^h are the set of elements of $k[[y]]$ algebraic over V_0 or V [**104**, (44.3)]. Thus $V_0^h = V^h$ and V is a directed union of étale extensions of V_0, see Definition 14.15.

The ring $C := \bigcup B_n^h$ is Henselian and contains V_0, so $V_0^h = V^h \subseteq C$. Moreover, the inclusion map $V \to C = \bigcup B_n^h$ extends to a map $V^h \overset{\sigma}{\to} C = \bigcup B_n^h$. On the other hand, the maps $B_n \to V$ extend to maps: $B_n^h \to V^h$ yielding a map $\rho : C \to V^h$ with $\sigma\rho = 1_C$, and $\rho\sigma = 1_{V^h}$. Thus $\bigcup_{n=1}^\infty B_n^h = V^h$. $\square$

Proof of Theorem 23.6 if the field L is finitely generated over k.

PROOF. Since L is algebraic over F, it follows that L is finite algebraic over F. Since $\bigcup_{n=1}^\infty B_n^h = V^h$, we have $\bigcup_{n=1}^\infty \mathcal{Q}(B_n^h) = \mathcal{Q}(V^h)$ and $L \subseteq \mathcal{Q}(V^h)$. Since L/F is finite algebraic, $L \subseteq \mathcal{Q}(B_n^h)$ for all sufficiently large n. By relabeling, we may assume $L \subseteq \mathcal{Q}(B_n^h)$ for all n. Let $C_n := B_n^h \cap L$. Since B_n is a regular local ring, C_n is a regular local ring with $C_n^h = B_n^h$. [**122**, (1.3)].

We observe that for every n, C_{n+1} dominates C_n and V dominates C_n. Also $\bigcup_{n=1}^\infty C_n = V$. Indeed, since B_{n+1} dominates B_n, we have B_{n+1}^h dominates B_n^h and hence $C_{n+1} = B_{n+1}^h \cap L$ dominates $C_n = B_n^h \cap L$. Since $C_n = B_n^h \cap L \subseteq V^h \cap L = V$, it follows that V dominates C_n and $V_0 \subseteq \bigcup_{n=1}^\infty C_n \subseteq V$. Since V birationally dominates $\bigcup_{n=1}^\infty C_n$, it suffices to show that $\bigcup_{n=1}^\infty C_n$ is a DVR.

But by the same argument as before, $\bigcup_{n=1}^\infty C_n^h = (\bigcup_{n=1}^\infty C_n)^h = V^h$. This shows that $\bigcup_{n=1}^\infty C_n$ is a DVR, and therefore $\bigcup_{n=1}^\infty C_n = V$. Thus in the case where L/k is finitely generated we have completed the proof of Theorem 23.6 including the "moreover" statement. $\square$

REMARK 23.12. An alternate approach to the definition of C_n is as follows. Since V is a directed union of étale extensions of V_0 and $\mathbb{Q}(V) = L$ is finite algebraic over $\mathbb{Q}(V_0) = F$, V is étale over V_0 and therefore $V = V_0[\theta] = V_0[X]/(f(X))$, where $f(X)$ is a monic polynomial such that $f(\theta) = 0$ and $f'(\theta)$ is a unit of V. Let B_n' denote the integral closure of B_n in L and let $C_n = (B_n')_{(\mathbf{n}\cap B_n')}$. Since $\cup_{n=1}^\infty B_n = V_0$, it follows that $\cup_{n=1}^\infty C_n = V$. Moreover, for all sufficiently large n, $f(X) \in B_n[X]$ and $f'(\theta)$ is a unit of C_n. Therefore C_n is a regular local ring for all sufficiently large n [**104**, (38.6)]. As we note in (23.13) below, this allows us to deduce a version of Theorem 23.6 also in the case where k has characteristic $p > 0$ provided the field F can be chosen so that L/F is separable.

Proof of Theorem 23.11 if the field L is not finitely generated over k.

PROOF. If L is not finitely generated over k, we choose a nested family of fields L_α, with $\alpha \in \Gamma$, such that

(1) $F \subseteq L_\alpha$, for all α.
(2) L_α is finite algebraic over F.
(3) $\cup_{\alpha\in\Gamma} L_\alpha = L$.

The rings $V_\alpha = L_\alpha \cap V$ are DVRs with $\cup_{\alpha\in\Gamma} V_\alpha = V$ and $V_\alpha^h = V^h$, since $V_0 \subseteq V_\alpha$, for each $\alpha \in \Gamma$.

As above, $\cup_{n=1}^{\infty} B_n^h = V^h$, $\cup_{n=1}^{\infty} \mathcal{Q}(B_n^h) = \mathcal{Q}(V^h)$ and $L \subseteq \mathcal{Q}(V^h)$. Thus we see that for each $\alpha \in \Gamma$, there is an $n_\alpha \in \mathbb{N}$ such that $L_\alpha \subseteq \mathcal{Q}(B_n^h)$ for all $n \geq n_\alpha$.

Put $C_n^{(\alpha)} = L_\alpha \cap B_n^h$ for each $n \geq n_\alpha$. Then $V_\alpha = \cup_{n=n_\alpha}^{\infty} C_n^{(\alpha)}$ and V_α birationally dominates $C_n^{(\alpha)}$. Hence

$$V = \underset{\alpha\in\Gamma, n\geq n_\alpha}{\rightarrow} \bigcup C_n^{(\alpha)}.$$

This completes the proof of Theorem 23.6. □

Remark 23.13. If the characteristic of k is $p > 0$ then the Henselization V_0^h of $V_0 = F \cap k[[y]]$ may not equal the Henselization V^h of $V = L \cap k[[y]]$, because the algebraic field extension L/F may not be separable. But in the case where L/F is separable algebraic, the fact that the DVRs V and V_0 have the same completion implies that V is a directed union of étale extensions of V_0 (see, for example, [**4**, Theorem 2.7]). Therefore in the case where L/F is separable algebraic, V is a directed union of regular local rings of dimension d.

Thus for a local domain $(R, \mathbf{m})$ essentially of finite type over a field k of characteristic $p > 0$, a result analogous to Corollary 1.3 is true provided there exists a subfield F of L such that F is purely transcendental over k, L/F is separable algebraic, and F contains a generator for the maximal ideal of V.

In characteristic $p > 0$, with V excellent and the extension separable, the ring V_0 need not be excellent (see for example Proposition 9.3 or [**53**, (3.3) and (3.4)]).

23.3. More general valuation rings as directed unions of RLRs

A useful method for constructing rank-one valuation rings is to make use of generalized power series rings as in [**149**, page 101].

Definition 23.14. Let k be a field and let $e_0 < e_1 < \cdots$ be real numbers such that $\lim_{n\to\infty} e_n = \infty$. For a variable t and elements $a_i \in k$, consider the *generalized power series expansion*

$$z(t) := a_0 t^{e_0} + a_1 t^{e_1} + \cdots + a_n t^{e_n} + \cdots$$

The *generalized power series ring* $k\{t\}$ is the set of all generalized power series expansions $z(t)$ with the usual addition and multiplication.

Remarks 23.15. With the notation of Definition 23.14, we have:

(1) The generalized power series ring $k\{t\}$ is a field.
(2) The field $k\{t\}$ admits a valuation v of rank one defined by setting $v(z(t))$ to be the order of the generalized power series $z(t)$. Thus $v(z(t)) = e_0$ if a_0 is a nonzero element of k.
(3) The valuation ring V of v is the set of generalized power series of nonnegative order together with zero. The value group of v is the additive group of real numbers.
(4) If $x_1, \ldots, x_r$ are variables over k, then every k-algebra isomorphism of the polynomial ring $k[x_1, \ldots, x_r]$ into $k\{t\}$ determines a valuation ring of rank one on the field $k(x_1, \ldots, x_r)$. Moreover, every such valuation ring has residue field k.

(5) Thus if $z_1(t), \dots, z_r(t) \in k\{t\}$ are algebraically independent over k, then the k-algebra isomorphism defined by mapping $x_i \mapsto z_i(t)$ determines a valuation on the field $k(x_1, \dots, x_r)$ of rank one. MacLane and Schilling prove in [**90**] a result that implies for a field k of characteristic zero the existence of a vaulation on $k(x_1, \dots, x_r)$ of rank one with any preassigned value group of rational rank less than r. In particular, if $r \geq 2$, then every additive subgroup of the group of rational numbers is the value group of a suitable valuation on the field of rational functions in r variables over k.

(6) As a specific example, let k be a field of characteristic zero and consider the k-algebra isomorphism of the polynomial ring $k[x, y]$ into $k\{t\}$ defined by mappping $x \mapsto t$ and $y \mapsto \sum_{n=1}^{\infty} t^{e_1+\cdots+e_n}$, where $e_i = 1/i$ for each positive integer i. The result of MacLane and Schilling [**90**] mentioned above implies that the value group of the valuation ring V defined by this embedding is the group of all rational numbers.

Exercises

(1) Let $(R, \mathbf{m})$ be a two-dimensional regular local ring with $\mathbf{m} = (x, y)R$ and let $a \in R \setminus \mathbf{m}$. Define:
$$S \ := \ R\big[\frac{y}{x}\big] \ = \ R\big[\frac{\mathbf{m}}{x}\big], \quad \mathbf{n} \ := \ (x, \frac{y}{x} - a)S \quad \text{and} \quad R_1 \ := \ S_{\mathbf{n}} \ = \ \big(R\big[\frac{y}{x}\big]\big)_{\mathbf{n}}.$$
Thus R_1 is a first local quadratic transform of R. Prove that there exists a maximal ideal $\mathbf{n}'$ of the ring $S' := R\big[\frac{x}{y}\big]$ such that $R_1 = S'_{\mathbf{n}'}$, and describe generators for $\mathbf{n}'$.

Suggestion: Notice that $\frac{y}{x}$ is a unit of R_1.

(2) Let $(R, \mathbf{m})$ be a two-dimensional regular local ring with $\mathbf{m} = (x, y)R$. Define:
$$S \ := \ R\big[\frac{x}{y}\big] \ = \ R\big[\frac{\mathbf{m}}{y}\big], \quad \mathbf{n} \ := \ (y, \frac{x}{y})S \quad \text{and} \quad R_1 \ := \ (S)_{\mathbf{n}} \ = \ \big(R\big[\frac{x}{y}\big]\big)_{\mathbf{n}},$$
and define $P = (x^2 - y^3)R$.

(a) Prove that R/P is a one-dimensional local domain that is not regular.

(b) Prove that there exists a prime ideal Q of R_1 such that $Q \cap R = P$ and R_1/Q is a DVR and hence is regular.

Comment: This is an example of embedded local uniformization.

CHAPTER 24

Weierstrass techniques for generic fiber rings

Let k be a field, let m and n be positive integers, and let $X = \{x_1, \dots, x_n\}$ and $Y = \{y_1, \dots, y_m\}$ be sets of independent variables over k. We define the rings A, B and C as follows:

$$(24.0) \qquad A := k[X]_{(X)}, \; B := k[[X]]\,[Y]_{(X,Y)} \quad \text{and} \quad C := k[Y]_{(Y)}[[X]].$$

That is, A is the usual localized polynomial ring in the variables of X. The rings B and C are "mixed polynomial-power series rings", formed from k using X, the power series variables, and Y, the polynomial variables, in two different ways: For the ring B we take polynomials in Y with coefficients in the power series ring $k[[X]]$ and for C we take power series in the X variables over the localized polynomial ring $k[Y]_{[Y]}$. We have the following local embeddings.

$$A := k[X]_{(X)} \hookrightarrow \widehat{A} := k[[X]], \quad \widehat{A} \hookrightarrow \widehat{B} = \widehat{C} = k[[X,Y]] \quad \text{and}$$
$$B := k[[X]]\,[Y]_{(X,Y)} \hookrightarrow C := k[Y]_{(Y)}[[X]] \hookrightarrow \widehat{B} = \widehat{C} = k[[X]]\,[[Y]].$$

There is a canonical inclusion map $B \hookrightarrow C$, and the ring C has infinite transcendence degree over B, even if $m = n = 1$. In Chapter 26 we consider this embedding further and we analyze the associated spectral map.

In this chapter, we develop techniques using the Weierstrass Preparation Theorem. We use these techniques in Chapter 25 to describe the prime ideals maximal in the generic fiber rings associated to the polynomial-power series rings A, B, and C. In particular, in Chapter 25, we prove every prime ideal P in $k[[X]]$ that is maximal with respect to $P \cap A = (0)$ has $\operatorname{ht} P = n - 1$. For every prime ideal P of $k[[X]][[Y]]$ such that P is maximal with respect to either $P \cap B = (0)$ or $P \cap C = (0)$, we prove $\operatorname{ht}(P) = n + m - 2$. In addition we prove each prime ideal P of $k[[X,Y]]$ that is maximal with respect to $P \cap k[[X]] = (0)$ has $\operatorname{ht} P = m$ or $n + m - 2$; see Theorem 24.3.

24.1. Terminology, Background and Results

We begin with definitions and notation for generic formal fiber rings.

NOTATION 24.1. Let $(R, \mathbf{m})$ be a Noetherian local domain and let $\widehat{R}$ be the $\mathbf{m}$-adic completion of R. The *generic formal fiber ring* of R is the localization $(R \setminus (0))^{-1}\widehat{R}$ of $\widehat{R}$ with respect to the multiplicatively closed set of nonzero elements of R. Let $\operatorname{Gff}(R)$ denote the generic formal fiber ring of R.

The *formal fibers* of R are the fibers of the map $\operatorname{Spec} \widehat{R} \to \operatorname{Spec} R$. For a prime ideal P of R, the formal fiber over P is $\operatorname{Spec}(\,(R_P/PR_P) \otimes_R \widehat{R}\,)$, or equivalently $\operatorname{Spec}((R \setminus P)^{-1}(\widehat{R}/P\widehat{R}))$; see Discussion 3.21 and Definition 3.28. Let $\operatorname{Gff}(R/P)$

denote the generic formal fiber ring of R/P. Since $\widehat{R}/P\widehat{R}$ is the completion of R/P, the formal fiber over P is $\operatorname{Spec}(\operatorname{Gff}(R/P))$.

Let $R \hookrightarrow S$ be an injective homomorphism of commutative rings. If R is an integral domain, the *generic fiber ring* of the map $R \hookrightarrow S$ is the localization $(R \setminus (0))^{-1}S$ of S.

The formal fibers encode important information about the structure of R. For example, R is excellent provided it is universally catenary and has geometrically regular formal fibers [**44**, (7.8.3), page 214]; see Definition 14.10.

The following historical remarks concern dimensions and heights of maximal ideals of generic formal fiber rings for Noetherian local domains:

REMARKS 24.2. (1) Let $(R, \mathbf{m})$ be a Noetherian local domain. In [**95**] Matsumura remarks that, as the ring R gets closer to its $\mathbf{m}$-adic completion $\widehat{R}$, it is natural to think that the dimension of the generic formal fiber ring $\operatorname{Gff}(R)$ gets smaller. Concerning the rings A, B and C of Equation 24.0, he proves that the generic formal fiber ring of A has dimension $\dim A - 1$, and the generic formal fiber rings of B and C have dimension $\dim B - 2 = \dim C - 2$ in [**95**]. Matsumura speculates as to whether $\dim R - 1, \dim R - 2$ and 0 are the only possible values for $\dim(\operatorname{Gff}(R))$ in [**95**, p. 261].

(2) In answer to Matsumura's question Rotthaus, establishes the following result in [**121**]: For every positive integer n and every integer t between 0 and $n-1$, there exists an excellent regular local ring R such that $\dim R = n$ and such that the generic formal fiber ring of R has dimension t.

(3) For $(R, \mathbf{m})$ an n-dimensional universally catenary Noetherian local domain, Loepp and Rotthaus in [**88**] compare the dimension of the generic formal fiber ring of R with that of the localized polynomial ring $R[x]_{(\mathbf{m},x)}$. Matsumura shows in [**95**] that the dimension of the generic formal fiber ring $\operatorname{Gff}(R[x]_{(\mathbf{m},x)})$ is either n or $n-1$. Loepp and Rotthaus in [**88**, Theorem 2] prove that $\dim(\operatorname{Gff}(R[x]_{(\mathbf{m},x)})) = n$ implies that $\dim(\operatorname{Gff} R) = n-1$. They show by example that in general the converse is not true, and they give sufficient conditions for the converse to hold.

(4) Let (T, M) be a complete Noetherian local domain that contains a field of characteristic zero. Assume that T/M has cardinality at least the cardinality of the real numbers. By adapting techniques developed by Heitmann in [**75**], in the articles [**86**] and [**87**], Loepp proves, among other things, for every prime ideal p of T with $p \neq M$, there exists an excellent regular local ring R that has completion T and has generic formal fiber ring $\operatorname{Gff}(R) = T_p$. By varying the height of p, this yields examples where the dimension of the generic formal fiber ring is any integer t with $0 \leq t < \dim T$. Loepp shows for these examples that there exists a unique prime q of T with $q \cap R = P$ and $q = PT$, for each nonzero prime P of R.

(5) In the case where R is a countable Noetherian local domain, Heinzer, Rotthaus and Sally show in [**53**, Proposition 4.10, page 36] that:

(a) The generic formal fiber ring $\operatorname{Gff}(R)$ is a Jacobson ring in the sense that each prime ideal of $\operatorname{Gff}(R)$ is an intersection of maximal ideals of $\operatorname{Gff}(R)$.

(b) $\dim(\widehat{R}/P) = 1$ for each prime ideal $P \in \operatorname{Spec} \widehat{R}$ that is maximal with respect to $P \cap R = (0)$.

(c) If $\widehat{R}$ is equidimensional, then $\operatorname{ht} P = n-1$ for each prime ideal $P \in \operatorname{Spec} \widehat{R}$ that is maximal with respect to $P \cap R = (0)$.

(d) If $Q \in \operatorname{Spec} \widehat{R}$ with $\operatorname{ht} Q \geq 1$, then there exists a prime ideal $P \subset Q$ such that $P \cap R = (0)$ and $\operatorname{ht}(Q/P) = 1$.

If the field k is countable, it follows from this result that all ideals maximal in the generic formal fiber ring of the ring A of Equation 24.0 have the same height.

(6) In Matsumura's article [**95**] from item 1 above, he does not address the question of whether all ideals maximal in the generic formal fiber rings for the rings A, B and C of Equation 24.0 have the same height. In general, for an excellent regular local ring R it can happen that $\operatorname{Gff}(R)$ contains maximal ideals of different heights; see the article [**121**, Corollary 3.2] of Rotthaus.

(7) Charters and Loepp in [**20**, Theorem 3.1] extend Rotthaus's result of item 6: Let (T, M) be a complete Noetherian local ring and let G be a nonempty subset of $\operatorname{Spec} T$ such that the number of maximal elements of G is finite. They prove there exists a Noetherian local domain A whose completion is T and whose generic formal fiber is exactly G if G satisfies the following conditions:

(a) $M \notin G$ and G contains the associated primes of T,
(b) If $P \subset Q$ are in $\operatorname{Spec} T$ and $Q \in G$, then $P \in G$, and
(c) Every $Q \in G$ meets the prime subring of T in (0).

If T contains the ring of integers and, in addition to items 1, 2, and 3, one also has

(d) T is equidimensional, and
(e) T_P is a regular local ring for each maximal element P of G,

then Charters and Loepp prove there exists an excellent local domain A whose completion is T and whose generic formal fiber is exactly G; see [**20**, Theorem 4.1]. Since the maximal elements of the set G may be chosen to have different heights, this result provides many examples where the generic formal fiber ring contains maximal ideals of different heights.

The Weierstrass techniques developed in this chapter enable us to prove the following theorem in Chapter 25:

Maximal Generic Fibers Theorem 24.3. *Let k be a field, let m and n be positive integers, and let $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_m\}$ be sets of independent variables over k. Then, for each of the rings $A := k[X]_{(X)}, B := k[[X]]\,[Y]_{(X,Y)}$ and $C := k[Y]_{(Y)}[[X]]$, every prime ideal maximal in the generic formal fiber ring has the same fixed height; more precisely*:

(1) *If P is a prime ideal of $\widehat{A}$ maximal with respect to $P \cap A = (0)$, then $\operatorname{ht}(P) = n - 1$.*
(2) *If P is a prime ideal of $\widehat{B}$ maximal with respect to $P \cap B = (0)$, then $\operatorname{ht}(P) = n + m - 2$.*
(3) *If P is a prime ideal of $\widehat{C}$ maximal with respect to $P \cap C = (0)$, then $\operatorname{ht}(P) = n + m - 2$.*
(4) *In addition, there are at most two possible values for the height of a maximal ideal of the generic fiber ring $(\widehat{A} \setminus (0))^{-1}\widehat{C}$ of the inclusion map $\widehat{A} \hookrightarrow \widehat{C}$.*
  (a) *If $n \geq 2$ and P is a prime ideal of $\widehat{C}$ maximal with respect to $P \cap \widehat{A} = (0)$, then either $\operatorname{ht} P = n + m - 2$ or $\operatorname{ht} P = m$.*
  (b) *If $n = 1$, then all ideals maximal in the generic fiber ring $(\widehat{A} \setminus (0))^{-1}\widehat{C}$ have height m.*

We were motivated to consider generic fiber rings for the embeddings displayed above because of questions related to Chapters 26 and **??** and ultimately because of the following question posed by Melvin Hochster and Yongwei Yao.

QUESTION 24.4. Let R be a complete Noetherian local domain. Can one describe or somehow classify the local maps of R to a complete Noetherian local domain S such that $U^{-1}S$ is a field, where $U = R \setminus (0)$, i.e., such that the generic fiber of $R \hookrightarrow S$ is trivial?

REMARK 24.5. By Cohen's structure theorems [**23**], [**104**, (31.6)], a complete Noetherian local domain R is a finite integral extension of a complete regular local domain R_0. If R has the same characteristic as its residue field, then R_0 is a formal power series ring over a field; see Remarks 3.13. The generic fiber $R \hookrightarrow S$ is trivial if and only if the generic fiber of $R_0 \hookrightarrow S$ is trivial.

A local ring R is called "equicharacteristic", if the ring and its residue field have the same characteristic; see Definition 3.12.1. If the equicharacteristic local ring has characteristic zero, then we say R is "equicharacteristic zero" or "of equal characteristic zero". Such a ring contains the field of rational numbers; see Exercise 24.1.

Thus, as Hochster and Yao remark, there is a natural way to construct such extensions in the case where the local ring R has characteristic zero and contains the rational numbers; consider

$$R = k[[x_1, ..., x_n]] \hookrightarrow T = L[[x_1, ..., x_n, y_1, ..., y_m]] \to T/P = S, \tag{24.4.0}$$

where k is a subfield of L, the x_i, y_j are formal indeterminates, and P is a prime ideal of T maximal with respect to being disjoint from the image of $R \setminus \{0\}$. Such prime ideals P correspond to the maximal ideals of the generic fiber $(R \setminus (0))^{-1}T$. The composite extension $T \hookrightarrow S$ satisfies the condition of Question 24.4.

In Theorem 25.6, we answer Question 24.4 in the special case where the extension arises from the embedding in Sequence 24.4.0 with the field $L = K$. We prove in this case that the dimension of the extension ring S must be either 2 or n.

We introduce the following terminology for the condition of Question 24.4 with a more general setting:

DEFINITION 24.6. For R and S integral domains with R a subring of S, we say that S is a *trivial generic fiber* extension of R, or a *TGF* extension of R, if every nonzero prime ideal of S has nonzero intersection with R. If $R \overset{\varphi}{\hookrightarrow} S$, then φ is also called a *trivial generic fiber extension* or *TGF extension*.

As in Remark 24.5, every extension $R \hookrightarrow T$ from an integral domain R to a commutative ring T yields a TGF extension by considering a composition

$$R \;\hookrightarrow\; T \;\to\; T/P = S, \tag{24.6.0}$$

where $P \in \operatorname{Spec} T$ is maximal with respect to $P \cap R = (0)$. Thus the generic fiber ring and so also Theorem 24.3 give information regarding TGF extensions in the case where the smaller ring is a mixed mixed polynomial-power series ring.

In addition, Theorem 24.3 is useful in the study of Sequence 24.4.0, because the map in Sequence 24.4.0 factors through:

$$R = k[[x_1, \dots, x_n]] \hookrightarrow k[[x_1, \dots, x_n]]\,[y_1, \dots, y_m] \hookrightarrow T = L[[x_1, \dots, x_n, y_1, \dots, y_n]].$$

The second extension of this sequence is TGF if $n = m = 1$ and $k = L$; see Exercise 1 of this chapter. We study TGF extensions in Chapters 26 and **??**.

Section 24.2 contains implications of Weierstrass' Preparation Theorem to the prime ideals of power series rings. We first prove a technical proposition regarding a change of variables that provides a "nice" generating set for a given prime ideal P of a power series ring; then in Theorem 24.11 we prove that, in certain circumstances, a larger prime ideal can be found with the same contraction as P to a certain subring. In Section 24.3 we use Valabrega's, Theorem 4.2, concerning subrings of a two-dimensional regular local domain.

In Sections 25.1 and 25.2, we prove parts 2 and 3 of Theorem 24.3 stated above. We apply Theorem 24.11 in Section 25.3 to prove part 1 of Theorem 24.3, and in Section 25.4 we prove part 4.

24.2. Variations on a theme of Weierstrass

We apply the Weierstrass Preparation Theorem 24.7 below to examine the structure of a given prime ideal P in the power series ring $\widehat{A} = k[[X]]$, where $X = \{x_1, \ldots, x_n\}$ is a set of n variables over the field k. Here $A = k[X]_{(X)}$ is the localized polynomial ring in these variables. Our procedure is to make a change of variables that yields a regular sequence in P of a nice form.

We recall the statement of the Weierstrass Preparation Theorem.

THEOREM 24.7. (Weierstrass) [**149**, Theorem 5, p. 139; Corollary 1, p. 145] *Let $(R, \mathbf{m})$ be a complete Noetherian local ring, let $f \in R[[x]]$ be a formal power series and let $\overline{f}$ denote the image of f in $(R/\mathbf{m})[[x]]$. Assume that $\overline{f} \neq 0$ and that* $\operatorname{ord} \overline{f} = s > 0$. *There exists a unique ordered pair (u, F) such that u is a unit in $R[[x]]$ and $F \in R[x]$ is a distinguished monic polynomial of degree s such that $f = uF$. Here $F = x^s + a_{s-1}x^{s-1} + \cdots + a_0 \in R[x]$ is* **distinguished** *if $a_i \in \mathbf{m}$ for $0 \le i \le s-1$.*

We often write "By Weierstrass", where we use Theorem 24.7.

COROLLARY 24.8. *The ideal $fR[[x]]$ is extended from $R[x]$ and $R[[x]]/(f)$ is a free R-module of rank s. Every $g \in R[[x]]$ is of the form $g = qf + r$, where $q \in R[[x]]$ and $r \in R[x]$ is a polynomial with* $\deg r \le s - 1$.

NOTATION 24.9. By a *change of variables*, we mean a finite sequence of 'polynomial' change of variables of the type described below, where $X = \{x_1, \ldots, x_n\}$ is a set of n variables over the field k. For example, with $e_i, f_i \in \mathbb{N}$, consider

$$x_1 \mapsto x_1 + x_n^{e_1} = z_1, \qquad x_2 \mapsto x_2 + x_n^{e_2} = z_2, \qquad \ldots,$$
$$x_{n-1} \mapsto x_{n-1} + x_n^{e_{n-1}} = z_{n-1}, \qquad x_n \mapsto x_n = z_n,$$

followed by:

$$z_1 \mapsto z_1 = t_1, \qquad z_2 \mapsto z_2 + z_1^{f_2} = t_2, \qquad \ldots,$$
$$z_{n-1} \mapsto z_{n-1} + z_1^{f_{n-1}} = t_{n-1}, \qquad x_n \mapsto z_n + z_1^{f_n} = t_n.$$

Thus a change of variables defines an automorphism of $\widehat{A}$ that restricts to an automorphism of A.

We also consider a change of variables for subrings of A and $\widehat{A}$. For example, if $A_1 = k[x_2, \dots, x_n] \subseteq A$ and $S = k[[x_2, \dots, x_n]] \subseteq \widehat{A}$, then by a *change of variables inside* A_1 and S, we mean a finite sequence of automorphisms of A and $\widehat{A}$ of the type described above on $x_2, \dots, x_n$ that leave the variable x_1 fixed. In this case we obtain an automorphism of $\widehat{A}$ that restricts to an automorphism on each of S, A and A_1.

PROPOSITION 24.10. *Let $\widehat{A} := k[[X]] = k[[x_1, \dots, x_n]]$ and let $P \in \operatorname{Spec} \widehat{A}$ with $x_1 \notin P$ and $\operatorname{ht} P = r$, where $1 \le r \le n-1$. There exists a change of variables $x_1 \mapsto z_1 := x_1$ (x_1 is fixed), $x_2 \mapsto z_2, \dots, x_n \mapsto z_n$ and a regular sequence $f_1, \dots, f_r \in P$ so that, upon setting $Z_1 = \{z_1, \dots, z_{n-r}\}$, $Z_2 = \{z_{n-r+1}, \dots, z_n\}$ and $Z = Z_1 \cup Z_2$, we have*

$$\begin{array}{ll} f_1 \in k[[Z_1]]\,[z_{n-r+1}, \dots, z_{n-1}]\,[z_n] & \textit{is monic as a polynomial in } z_n \\ f_2 \in k[[Z_1]]\,[z_{n-r+1}, \dots, z_{n-2}]\,[z_{n-1}] & \textit{is monic as a polynomial in } z_{n-1}, \textit{ etc} \\ \vdots & \\ f_r \in k[[Z_1]]\,[z_{n-r+1}] & \textit{is monic as a polynomial in } z_{n-r+1}. \end{array}$$

In addition:

(1) *P is a minimal prime of the ideal $(f_1, \dots, f_r)\widehat{A}$.*
(2) *The (Z_2)-adic completion of $k[[Z_1]]\,[Z_2]_{(Z)}$ is identical to the $(f_1, \dots, f_r)$-adic completion and both equal $\widehat{A} = k[[X]] = k[[Z]]$.*
(3) *If $P_1 := P \cap k[[Z_1]]\,[Z_2]_{(Z)}$, then $P_1\widehat{A} = P$, that is, P is extended from $k[[Z_1]]\,[Z_2]_{(Z)}$.*
(4) *The ring extension:*
$$k[[Z_1]] \hookrightarrow k[[Z_1]]\,[Z_2]_{(Z)}/P_1 \cong k[[Z]]/P$$
is finite (and integral).

PROOF. Since $\widehat{A}$ is a unique factorization domain, there exists a nonzero prime element f in P. The power series f is therefore not a multiple of x_1, and so f must contain a monomial term $x_2^{i_2} \dots x_n^{i_n}$ with a nonzero coefficient in k. This nonzero coefficient in k may be assumed to be 1. There exists an automorphism $\sigma : \widehat{A} \to \widehat{A}$ defined by the change of variables:

$$x_1 \mapsto x_1 \qquad x_2 \mapsto t_2 := x_2 + x_n^{e_2} \quad \dots \quad x_{n-1} \mapsto t_{n-1} := x_{n-1} + x_n^{e_{n-1}} \qquad x_n \mapsto x_n,$$

with $e_2, \dots, e_{n-1} \in \mathbb{N}$ chosen suitably so that f written as a power series in the variables $x_1, t_2, \dots, t_{n-1}, x_n$ contains a term $a_n x_n^{s_n}$, where s_n is a positive integer, and $a_n \in k$ is nonzero. We assume that the integer s_n is minimal among all integers i such that a term $a x_n^i$ occurs in f with a nonzero coefficient $a \in k$; we further assume that the coefficient $a_n = 1$. By Weierstrass, that is, Theorem 24.7, we have that:

$$f = m\epsilon,$$

where $m \in k[[x_1, t_2, \dots, t_{n-1}]]\,[x_n]$ is a monic polynomial in x_n of degree s_n and ϵ is a unit in $\widehat{A}$. Since $f \in P$ is a prime element, $m \in P$ is also a prime element. Using Weierstrass again, every element $g \in P$ can be written as:

$$g = mh + q,$$

where $h \in k[[x_1, t_2, \ldots, t_{n-1}, x_n]] = \widehat{A}$ and $q \in k[[x_1, t_2, \ldots, t_{n-1}]]\,[x_n]$ is a polynomial in x_n of degree less than s_n. Note that

$$k[[x_1, t_2, \ldots, t_{n-1}]] \hookrightarrow k[[x_1, t_2, \ldots, t_{n-1}]]\,[x_n]/(m)$$

is an integral (finite) extension. Thus the ring $k[[x_1, t_2, \ldots, t_{n-1}]]\,[x_n]/(m)$ is complete. Moreover, the two ideals $(x_1, t_2, \ldots, t_{n-1}, m) = (x_1, t_2, \ldots, t_{n-1}, x_n^{s_n})$ and $(x_1, t_2, \ldots, t_{n-1}, x_n)$ of $B_0 := k[[x_1, t_2, \ldots, t_{n-1}]]\,[x_n]$ have the same radical. Therefore $\widehat{A}$ is the (m)-adic and the (x_n)-adic completion of B_0 and P is extended from B_0.

This implies the statement for $r = 1$, with $f_1 = m$, $z_n = x_n$, $z_1 = x_1$, $z_2 = t_2$, $\ldots$, $z_{n-1} = t_{n-1}$, $Z_1 = \{x_1, t_2, \ldots, t_{n-1}\}$ and $Z_2 = \{z_n\} = \{x_n\}$. In particular, when $r = 1$, P is minimal over $m\widehat{A}$, so $P = m\widehat{A}$.

For $r > 1$ we continue by induction on r. Let $P_0 := P \cap k[[x_1, t_2, \ldots, t_{n-1}]]$. Since $m \notin k[[x_1, t_2, \ldots, t_{n-1}]]$ and P is extended from $B_0 := k[[x_1, t_2, \ldots, t_{n-1}]]\,[x_n]$, then $P \cap B_0$ has height r and $\operatorname{ht} P_0 = r - 1$. Since $x_1 \notin P$, we have $x_1 \notin P_0$, and by the induction hypothesis there is a change of variables $t_2 \mapsto z_2, \ldots, t_{n-1} \mapsto z_{n-1}$ of $k[[x_1, t_2, \ldots, t_{n-1}]]$ and elements $f_2, \ldots, f_r \in P_0$ so that:

$$\begin{array}{ll}
f_2 \in k[[x_1, z_2 \ldots, z_{n-r}]]\,[z_{n-r+1}, \ldots, z_{n-2}]\,[z_{n-1}] & \text{is monic in } z_{n-1} \\
f_3 \in k[[x_1, z_2 \ldots, z_{n-r}]]\,[z_{n-r+1}, \ldots, z_{n-3}]\,[z_{n-2}] & \text{is monic in } z_{n-2}, \text{ etc} \\
\quad \vdots & \\
f_r \in k[[x_1, z_2, \ldots, z_{n-r}]]\,[z_{n-r+1}] & \text{is monic in } z_{n-r+1},
\end{array}$$

and $f_2, \ldots, f_r$ satisfy the assertions of Proposition 24.10 for P_0.

It follows that $m, f_2, \ldots, f_r$ is a regular sequence of length r and that P is a minimal prime of the ideal $(m, f_2, \ldots, f_r)\widehat{A}$. Set $z_n = x_n$. We now prove that m may be replaced by a polynomial $f_1 \in k[[x_1, z_2, \ldots, z_{n-r}]]\,[z_{n-r+1}, \ldots, z_n]$. Write

$$m = \sum_{i=0}^{s_n} a_i z_n,$$

where the $a_i \in k[[x_1, z_2, \ldots, z_{n-1}]]$. For each $i < s_n$, apply Weierstrass to a_i and f_2 in order to obtain:

$$a_i = f_2 h_i + q_i,$$

where h_i is a power series in $k[[x_1, z_2, \ldots, z_{n-1}]]$ and $q_i \in k[[x_1, z_2, \ldots, z_{n-2}]]\,[z_{n-1}]$ is a polynomial in z_{n-1}. With $q_{s_n} = 1 = a_{s_n}$, we define

$$m_1 = \sum_{i=0}^{s_n} q_i z_n^i.$$

Now $(m_1, f_2, \ldots, f_r)\widehat{A} = (m, f_2, \ldots, f_r)\widehat{A}$ and we may replace m by m_1 which is a polynomial in z_{n-1} and z_n. To continue, for each $i < s_n$, write:

$$q_i = \sum_{j,k} b_{ij} z_{n-1}^j \quad \text{with} \quad b_{ij} \in k[[x_1, z_2, \ldots, z_{n-2}]].$$

For each b_{ij}, we apply Weierstrass to b_{ij} and f_3 to obtain:

$$b_{ij} = f_3 h_{ij} + q_{ij},$$

where $q_{ij} \in k[[x_1, z_2, \dots, z_{n-3}]]\,[z_{n-2}]$. Set

$$m_2 = \sum_{i,j} q_{ij} z_{n-1}^j z_n^i \in k[[x_1, z_2, \dots, z_{n-3}]]\,[z_{n-2}, z_{n-1}, z_n]$$

with $q_{s_n 0} = 1$. It follows that $(m_2, f_2, \dots, f_r)\widehat{A} = (m, f_2, \dots, f_r)\widehat{A}$. Continuing this process by applying Weierstrass to the coefficients of $z_{n-2}^k z_{n-1}^j z_n^i$ and f_4, we establish the existence of a polynomial $f_1 \in k[[Z_1]]\,[z_{n-r+1}, \dots, z_n]$ that is monic in z_n so that $(f_1, f_2, \dots, f_r)\widehat{A} = (m, f_2, \dots, f_r)\widehat{A}$. Therefore P is a minimal prime of $(f_1, \dots, f_r)\widehat{A}$.

The extension

$$k[[Z_1]] \longrightarrow k[[Z_1]]\,[Z_2]/(f_1, \dots, f_r)$$

is integral and finite. Thus the ring $k[[Z_1]]\,[Z_2]/(f_1, \dots, f_r)$ is complete. This implies $\widehat{A} = k[[x_1, z_2, \dots, z_n]]$ is the $(f_1, \dots, f_r)$-adic (and the (Z_2)-adic) completion of $k[[Z_1]]\,[Z_2]_{(Z)}$ and that P is extended from $k[[Z_1]]\,[Z_2]_{(Z)}$. This completes the proof of Proposition 24.10. $\square$

The following theorem is the technical heart of this section.

THEOREM 24.11. *Let k be a field and let y and $X = \{x_1, \dots, x_n\}$ be variables over k. Assume that V is a discrete valuation domain with completion $\widehat{V} = k[[y]]$ and that $k[y] \subseteq V \subseteq k[[y]]$. Also assume that the field $k((y)) = k[[y]]\,[1/y]$ has uncountable transcendence degree over the quotient field $\mathcal{Q}(V)$ of V. Set $R_0 := V[[X]]$ and $R = \widehat{R_0} = k[[y, X]]$. Let $P \in \operatorname{Spec} R$ be such that:*

(i) *$P \subseteq (X)R$ (so $y \notin P$), and*
(ii) *$\dim(R/P) > 2$.*

Then there is a prime ideal $Q \in \operatorname{Spec} R$ such that

(1) *$P \subset Q \subset XR$,*
(2) *$\dim(R/Q) = 2$, and*
(3) *$P \cap R_0 = Q \cap R_0$.*

In particular, $P \cap k[[X]] = Q \cap k[[X]]$.

PROOF. Assume that P has height r. Since $\dim(R/P) > 2$, we have $0 \le r < n-1$. If $r > 0$, then there exist a transformation $x_1 \mapsto z_1, \dots, x_n \mapsto z_n$ and elements $f_1, \dots, f_r \in P$, by Proposition 24.10, so that the variable y is fixed, and

$$\begin{aligned}
&f_1 \in k[[y, z_1, \dots, z_{n-r}]]\,[z_{n-r+1}, \dots, z_n] \text{ is monic in } z_n,\\
&f_2 \in k[[y, z_1, \dots, z_{n-r}]]\,[z_{n-r+1}, \dots, z_{n-1}] \text{ is monic in } z_{n-1} \text{ etc},\\
&\quad\vdots\\
&f_r \in k[[y, z_1, \dots, z_{n-r}]]\,[z_{n-r+1}] \text{ is monic in } z_{n-r+1},
\end{aligned}$$

and the assertions of Proposition 24.10 are satisfied. In particular, P is a minimal prime of $(f_1, \dots, f_r)R$. Let $Z_1 = \{z_1, \dots, z_{n-r}\}$ and $Z_2 = \{z_{n-r+1}, \dots, z_{n-1}, z_n\}$. By Proposition 24.10, if $D := k[[y, Z_1]]\,[Z_2]_{(Z)}$ and $P_1 := P \cap D$, then $P_1 R = P$.

The following diagram shows these rings and ideals.

$$R = k[[y, X]] = k[[y, Z_1, Z_2]]$$

$$(X)R$$

$$D = k[[y, Z_1]]\,[Z_2]_{(Z)}$$

$$P = P_1 R$$

$$P_1 = P \cap D$$

Note that $f_1, \dots, f_r \in P_1$. Let $g_1, \dots, g_s \in P_1$ be other generators such that $P_1 = (f_1, \dots, f_r, g_1, \dots, g_s)D$. Then $P = P_1 R = (f_1, \dots, f_r, g_1, \dots, g_s)R$. For each $(i) := (i_1, \dots, i_n) \in \mathbb{N}^n$ and j, k with $1 \le j \le r$, $1 \le k \le s$, let $a_{j,(i)}$, $b_{k,(i)}$ denote the coefficients in $k[[y]]$ of the f_j, g_k, so that

$$f_j = \sum_{(i)\in\mathbb{N}^n} a_{j,(i)} z_1^{i_1} \dots z_n^{i_n}\,, \qquad g_k = \sum_{(i)\in\mathbb{N}^n} b_{k,(i)} z_1^{i_1} \dots z_n^{i_n} \in k[[y]]\,[[Z]].$$

Define

$$\Delta := \begin{cases} \{a_{j,(i)}, b_{k,(i)}\} \subseteq k[[y]], & \text{for } r > 0 \\ \emptyset, & \text{for } r = 0. \end{cases}$$

A key observation here is that in either case the set Δ is countable.

To continue the proof, we consider $S := \mathcal{Q}(V(\Delta)) \cap k[[y]]$, a discrete valuation domain, and its field of quotients $L := \mathcal{Q}(V(\Delta))$. Since Δ is a countable set, the field $k((y))$ is (still) of uncountable transcendence degree over L. Let $\gamma_2, \dots, \gamma_{n-r}$ be elements of $k[[y]]$ that are algebraically independent over L. We define $T := L(\gamma_2, \dots, \gamma_{n-r}) \cap k[[y]]$ and $E := \mathcal{Q}(T) = L(\gamma_2, \dots, \gamma_{n-r})$.

The diagram below shows the prime ideals P and P_1 and the containments among the relevant rings.

$$R = k[[y, Z]]$$

$$P = (\{f_j, g_k\})R$$

$$D := k[[y, Z_1]]\,[Z_2]_{(Z)} \qquad \mathcal{Q}(k[[y]]) = k[[y]]\,[1/y] = k((y))$$

$$P_1 = (\{f_j, g_k\})D$$

$$k[[y]]$$

$$E := \mathcal{Q}(T) = L(\gamma_2, \ldots, \gamma_{n-r})$$

$$T := L(\gamma_2, \ldots, \gamma_{n-r}) \cap k[[y]]$$

$$L := \mathcal{Q}(S) = \mathcal{Q}(V(\Delta))$$

$$S := \mathcal{Q}(V(\Delta)) \cap k[[y]]$$

$$\mathcal{Q}(V)$$

$$V$$

$$k[y]$$

Let $P_2 := P \cap S[[Z_1]]\,[Z_2]_{(Z)}$. Since $f_1, \ldots, f_r, g_1, \ldots, g_s \in S[[Z_1]]\,[Z_2]_{(Z)}$, we have $P_2 R = P$. Since $P \subseteq (x_1, \ldots, x_n)R = (Z)R$, there is a prime ideal $\widetilde{P}$ in $L[[Z]]$ that is minimal over $P_2 L[[Z]]$. Since $L[[Z]]$ is flat over $S[[Z]]$, $\widetilde{P} \cap S[[Z]] = P_2 S[[Z]]$. Note that $L[[X]] = L[[Z]]$ is the $(f_1, \ldots, f_r)$-adic (and the (Z_2)-adic) completion of $L[[Z_1]][Z_2]_{(Z)}$. In particular,

$$L[[Z_1]]\,[Z_2]/(f_1, \ldots, f_r) = L[[Z_1]]\,[[Z_2]]/(f_1, \ldots, f_r)$$

and this also holds with the field L replaced by its extension field E.

Since $L[[Z]]/\widetilde{P}$ is a homomorphic image of $L[[Z]]/(f_1, \ldots, f_r)$, it follows that $L[[Z]]/\widetilde{P}$ is integral (and finite) over $L[[Z_1]]$. This yields the commutative diagram:

$$\begin{array}{ccc} E[[Z_1]] & \longrightarrow & E[[Z_1]]\,[[Z_2]]/\widetilde{P}E[[Z]] \\ \uparrow & & \uparrow \\ L[[Z_1]] & \longrightarrow & L[[Z_1]]\,[[Z_2]]/\widetilde{P} \end{array} \qquad (24.11.0)$$

with injective integral (finite) horizontal maps. Recall that E is the subfield of $k((y))$ obtained by adjoining $\gamma_2, \ldots, \gamma_{n-r}$ to the field L. Thus the vertical maps in Diagram 24.11.0 are faithfully flat.

Let $\mathbf{q} := (z_2 - \gamma_2 z_1, \dots, z_{n-r} - \gamma_{n-r} z_1)E[[Z_1]] \in \operatorname{Spec}(E[[Z_1]])$ and let $\widetilde{W}$ be a minimal prime of the ideal $(\widetilde{P}, \mathbf{q})E[[Z]]$. Since

$$f_1, \dots, f_r, z_2 - \gamma_2 z_1, \dots, z_{n-r} - \gamma_{n-r} z_1$$

is a regular sequence in $T[[Z]]$ the prime ideal $W := \widetilde{W} \cap T[[Z]]$ has height $n-1$. Let $\widetilde{Q}$ be a minimal prime of $\widetilde{W}k((y))[[Z]]$ and let $Q := \widetilde{Q} \cap R$. Then $W = Q \cap T[[Z]]$, $P \subset Q \subset ZR = XR$, and pictorially we have:

$$(\widetilde{W}) \subseteq \widetilde{Q} \subset k((y))[[Z]]$$

$$R := k[[y, Z]] \qquad\qquad (\widetilde{P}, \mathbf{q}) \subseteq \widetilde{W} \subset \quad E[[Z]]$$

$$P = (\{f_j, g_k\})R \subseteq Q \subset R \qquad\qquad W \subset \quad T[[Z]]$$

$$(P_2) \subseteq \widetilde{P} \subset L[[Z]]$$

$$D := k[[y, Z_1]]\,[Z_2]_{(Z)}$$

$$P_1 = (\{f_j, g_k\})D \subset D \qquad\qquad \mathbf{q} \subset E[[Z_1]]$$

$$S[[Z]] \qquad\qquad L[[Z_1]]\,[Z_2]_{(Z)}$$

$$L[[Z_1]]$$

$$P_2 = (\{f_j, g_k\}) \subset S[[Z_1]]\,[Z_2]_{(Z)}$$

Notice that $\mathbf{q}$ is a prime ideal of height $n - r - 1$. Also, since $k((y))[[Z]]$ is flat over $k[[y, Z]] = R$, we have $\operatorname{ht} Q = n - 1$ and $\dim(R/Q) = 2$. We clearly have $P_2 \subseteq W \cap S[[Z_1]]\,[Z_2]_{(Z)}$.

Claim 24.12. $\mathbf{q} \cap L[[Z_1]] = (0)$.

To show this we argue as in [**95**]: Suppose that

$$h = \sum_{m \in \mathbb{N}} H_m \in \mathbf{q} \cap L[[z_1, \dots, z_{n-r}]],$$

where $H_m \in L[z_1, \dots, z_{n-r}]$ is a homogeneous polynomial of degree m:

$$H_m = \sum_{|(i)|=m} c_{(i)} z_1^{i_1} \dots z_{n-r}^{i_{n-r}},$$

where $(i) := (i_1, \dots, i_{n-r}) \in \mathbb{N}^{n-r}$, $|(i)| := i_1 + \cdots + i_{n-r}$ and $c_{(i)} \in L$. Consider the E-algebra homomorphism $\pi : E[[Z_1]] \to E[[z_1]]$ defined by $\pi(z_1) = z_1$ and $\pi(z_i) = \gamma_i z_1$ for $2 \le i \le n - r$. Then $\ker \pi = \mathbf{q}$, and for each $m \in \mathbb{N}$:

$$\pi(H_m) = \pi(\sum_{|(i)|=m} c_{(i)} z_1^{i_1} \dots z_{n-r}^{i_{n-r}}) = \sum_{|(i)|=m} c_{(i)} \gamma_2^{i_2} \dots \gamma_{n-r}^{i_{n-r}} z_1^m$$

and

$$\pi(h) = \sum_{m \in \mathbb{N}} \pi(H_m) = \sum_{m \in \mathbb{N}} \sum_{|(i)|=m} c_{(i)} \gamma_2^{i_2} \dots \gamma_{n-r}^{i_{n-r}} z_1^m.$$

Since $h \in \mathbf{q}$, $\pi(h) = 0$. Since $\pi(h)$ is a power series in $E[[z_1]]$, each of its coefficients is zero, that is, for each $m \in \mathbb{N}$,

$$\sum_{|(i)|=m} c_{(i)} \gamma_2^{i_2} \dots \gamma_{n-r}^{i_{n-r}} = 0.$$

Since the γ_i are algebraically independent over L, each $c_{(i)} = 0$. Therefore $h = 0$, and so $\mathbf{q} \cap L[[Z_1]] = (0)$. This proves Claim 24.12.

Using the commutativity of Diagram 24.11.0 and that the horizonal maps of this diagram are integral extensions, we deduce that $(\widetilde{W} \cap E[[Z_1]]) = \mathbf{q}$, and $\mathbf{q} \cap L[[Z_1]] = (0)$ implies $\widetilde{W} \cap L[[Z_1]] = (0)$. We conclude that $Q \cap S[[Z]] = P \cap S[[Z]]$ and therefore $Q \cap R_0 = P \cap R_0$. $\square$

We record the following corollary.

Corollary 24.13. *Let k be a field, let $X = \{x_1, \dots, x_n\}$ and y be independent variables over k, and let $R = k[[y, X]]$. Assume $P \in \operatorname{Spec} R$ is such that:*

(i): $P \subseteq (x_1, \dots, x_n)R$ *and*
(ii): $\dim(R/P) > 2$.

Then there is a prime ideal $Q \in \operatorname{Spec} R$ so that

(1) $P \subset Q \subset (x_1, \dots, x_n)R$,
(2) $\dim(R/Q) = 2$, *and*
(3) $P \cap k[y]_{(y)}[[X]] = Q \cap k[y]_{(y)}[[X]]$.

In particular, $P \cap k[[x_1, \dots, x_n]] = Q \cap k[[x_1, \dots, x_n]]$.

Proof. With notation as in Theorem 24.11, let $V = k[y]_{(y)}$. $\square$

24.3. Subrings of the power series ring $k[[z,t]]$

In this section we establish properties of certain subrings of the power series ring $k[[z,t]]$ that will be useful in considering the generic formal fiber of localized polynomial rings over the field k.

Notation 24.14. Let k be a field and let z and t be independent variables over k. Consider countably many power series:

$$\alpha_i(z) = \sum_{j=0}^{\infty} a_{ij} z^j \in k[[z]]$$

with coefficients $a_{ik} \in k$. Let s be a positive integer and let $\omega_1, \dots, \omega_s \in k[[z,t]]$ be power series in z and t, say:

$$\omega_i = \sum_{j=0}^{\infty} \beta_{ij} t^j, \quad \text{where} \quad \beta_{ij}(z) = \sum_{k=0}^{\infty} b_{ijk} z^k \in k[[z]] \quad \text{and} \quad b_{ijk} \in k,$$

for each i with $1 \le i \le s$. Consider the subfield $k(z, \{\alpha_i\}, \{\beta_{ij}\})$ of $k((z))$ and the discrete rank-one valuation domain

$$V := k(z, \{\alpha_i\}, \{\beta_{ij}\}) \cap k[[z]]. \tag{24.14.0}$$

The completion of V is $\widehat{V} = k[[z]]$. Assume that $\omega_1, \dots, \omega_r$ are algebraically independent over $\mathcal{Q}(V)(t)$ and that the elements $\omega_{r+1}, \dots, \omega_s$ are algebraic over the field $\mathcal{Q}(V)(t, \{\omega_i\}_{i=1}^r)$. Notice that the set $\{\alpha_i\} \cup \{\beta_{ij}\}$ is countable, and that also the set of coefficients of the α_i and β_{ij}

$$\Delta := \{a_{ij}\} \cup \{b_{ijk}\}$$

is a countable subset of the field k. Let k_0 denote the prime subfield of k and let F denote the algebraic closure in k of the field $k_0(\Delta)$. The field F is countable and the power series $\alpha_i(z)$ and $\beta_{ij}(z)$ are in $F[[z]]$. Consider the subfield $F(z, \{\alpha_i\}, \{\beta_{ij}\})$ of $F((z))$ and the discrete rank-one valuation domain

$$V_0 := F(z, \{\alpha_i\}, \{\beta_{ij}\}) \cap F[[z]].$$

The completion of V_0 is $\widehat{V_0} = F[[z]]$. Since $\mathcal{Q}(V_0)(t) \subseteq \mathcal{Q}(V)(t)$, the elements $\omega_1, \ldots, \omega_r$ are algebraically independent over the field $\mathcal{Q}(V_0)(t)$.

Consider the subfield $E_0 := \mathcal{Q}(V_0)(t, \omega_1, \ldots, \omega_r)$ of $\mathcal{Q}(V_0[[t]])$ and the subfield $E := \mathcal{Q}(V)(t, \omega_1, \ldots, \omega_r)$ of $\mathcal{Q}(V[[t]])$. A result of Valabrega, Theorem 4.2, implies that the integral domains:

$$(24.14.1) \qquad D_0 := E_0 \cap V_0[[t]] \qquad \text{and} \qquad D := E \cap V[[t]]$$

are two-dimensional regular local rings with completions $\widehat{D}_0 = F[[z,t]]$ and $\widehat{D} = k[[z,t]]$, respectively. Moreover, $\mathcal{Q}(D_0) = E_0$ is a countable field.

PROPOSITION 24.15. *Let D_0 be as defined in Equation 24.14.1. Then there exists a power series $\gamma \in zF[[z]]$ such that the prime ideal $(t-\gamma)F[[z,t]] \cap D_0 = (0)$, that is, $(t-\gamma)F[[z,t]]$ is in the generic formal fiber of D_0.*

PROOF. Since D_0 is countable there are only countably many prime ideals in D_0 and since D_0 is Noetherian there are only countably many prime ideals in $\widehat{D}_0 = F[[z,t]]$ that lie over a nonzero prime of D_0. There are uncountably many primes in $F[[z,t]]$, which are generated by elements of the form $t - \sigma$ for some $\sigma \in zF[[z]]$. Thus there must exist an element $\gamma \in zF[[z]]$ with $(t-\gamma)F[[z,t]] \cap D_0 = (0)$. $\square$

For $\omega_i = \omega_i(t) = \sum_{j=0}^{\infty} \beta_{ij} t^j$ as in Notation 24.14 and γ an element of $zk[[z]]$, let $\omega_i(\gamma)$ denote the following power series in $k[[z]]$:

$$\omega_i(\gamma) := \sum_{j=0}^{\infty} \beta_{ij} \gamma^j \in k[[z]].$$

PROPOSITION 24.16. *Let V and D be as defined in Equations 24.14.0 and 24.14.1. For an element $\gamma \in zk[[z]]$ the following conditions are equivalent:*

(i): $(t-\gamma)k[[z,t]] \cap D = (0)$.

(ii): *The elements $\gamma, \omega_1(\gamma), \ldots, \omega_r(\gamma)$ are algebraically independent over $\mathcal{Q}(V)$.*

PROOF. (i) $\Rightarrow$ (ii): Assume by way of contradiction that $\{\gamma, \omega_1(\gamma), \ldots, \omega_r(\gamma)\}$ is an algebraically dependent set over $\mathcal{Q}(V)$ and let $d_{(k)} \in V$ be finitely many elements such that

$$\sum_{(k)} d_{(k)} \omega_1(\gamma)^{k_1} \ldots \omega_r(\gamma)^{k_r} \gamma^{k_{r+1}} = 0$$

is a nontrivial equation of algebraic dependence for $\gamma, \omega_1(\gamma), \ldots, \omega_r(\gamma)$, where each $(k) = (k_1 \ldots, k_r, k_{r+1})$ is an $(r+1)$-tuple of nonnegative integers. It follows that

$$\sum_{(k)} d_{(k)} \omega_1^{k_1} \ldots \omega_r^{k_r} t^{k_{r+1}} \in (t-\gamma)k[[z,t]] \cap D = (0).$$

Since $\omega_1, \dots, \omega_r$ are algebraically independent over $\mathcal{Q}(V)(t)$, we have $d_{(k)} = 0$ for all (k), a contradiction. This completes the proof that $(i) \Rightarrow (ii)$.

(ii) $\Rightarrow$ (i): If $(t-\gamma)k[[z,t]] \cap D \neq (0)$, then there exists a nonzero element

$$\tau = \sum_{(k)} d_{(k)} \omega_1^{k_1} \dots \omega_r^{k_r} t^{k_{r+1}} \in (t-\gamma)k[[z,t]] \cap V[t, \omega_1, \dots, \omega_r].$$

But this implies that

$$\tau(\gamma) = \sum_{(k)} d_{(k)} \omega_1(\gamma)^{k_1} \dots \omega_r(\gamma)^{k_r} \gamma^{k_{r+1}} = 0.$$

Since $\gamma, \omega_1(\gamma), \dots, \omega_r(\gamma)$ are algebraically independent over $\mathcal{Q}(V)$, it follows that all the coefficients $d_{(k)} = 0$, a contradiction to the assumption that τ is nonzero. $\square$

Let $\gamma \in zF[[z]]$ be as in Proposition 24.15 with $(t-\gamma)F[[z,t]] \cap D_0 = (0)$. Then:

Proposition 24.17. *With notation as above, we have $(t-\gamma)k[[z,t]] \cap D = (0)$, that is, $(t-\gamma)k[[z,t]]$ is in the generic formal fiber of D.*

Proof. Let $L := F(\{t_i\}_{i \in I})$, where $\{t_i\}_{i \in I}$ is a transcendence basis of k over F. Then k is algebraic over L. Let $\{\alpha_i\}, \{\beta_{ij}\} \subset F[[z]]$ be as in (5.1) and define

$$V_1 = L(z, \{\alpha_i\}, \{\beta_{ij}\}) \cap L[[z]] \quad \text{and} \quad D_1 = \mathcal{Q}(V_1)(t, \omega_1, \dots, \omega_r) \cap L[[z,t]].$$

Then V_1 is a discrete rank-one valuation domain with completion $L[[z]]$ and D_1 is a two-dimensional regular local domain with completion $\widehat{D}_1 = L[[z,t]]$. Note that $\mathcal{Q}(V)$ and $\mathcal{Q}(D)$ are algebraic over $\mathcal{Q}(V_1)$ and $\mathcal{Q}(D_1)$, respectively. Since $(t-\gamma)k[[z,t]] \cap L[[z,t]] = (t-\gamma)L[[z,t]]$, it suffices to prove that $(t-\gamma)L[[z,t]] \cap D_1 = (0)$. By Proposition 24.16, it suffices to show that $\gamma, \omega_1(\gamma), \dots, \omega_r(\gamma)$ are algebraically independent over $\mathcal{Q}(V_1)$. The commutative diagram

$$\begin{array}{ccc} F[[z]] & \xrightarrow{\{t_i\}\text{algebraically ind.}} & L[[z]] \\ \uparrow & & \uparrow \\ \mathcal{Q}(V_0) & \xrightarrow{\text{transcendence basis } \{t_i\}} & \mathcal{Q}(V_1) \end{array}$$

implies that the set $\{\gamma, \omega_1(\gamma), \dots, \omega_r(\gamma)\} \cup \{t_i\}$ is algebraically independent over $\mathcal{Q}(V_0)$. Therefore $\{\gamma, \omega_1(\gamma), \dots, \omega_r(\gamma)\}$ is algebraically independent over $\mathcal{Q}(V_1)$. This completes the proof of Proposition 24.17. $\square$

Remark 24.18. If $\omega_{r+1}, \dots, \omega_s$ is algebraic over $\mathcal{Q}(V)(\omega_1, \dots, \omega_r)$ as in (5.1) and we define

$$\widetilde{D} := \mathcal{Q}(V)(t, \omega_1, \dots, \omega_s) \cap V[[t]],$$

then again by Valabrega's Theorem 4.2, $\widetilde{D}$ is a two-dimensional regular local domain with completion $k[[z,t]]$. Moreover, $\mathcal{Q}(\widetilde{D})$ is algebraic over $\mathcal{Q}(D)$ and $(t-\gamma)k[[z,t]] \cap D = (0)$ implies that $(t-\gamma)k[[z,t]] \cap \widetilde{D} = (0)$.

Exercise

(1) Prove that a local ring that has residue field of characteristic zero contains the field of rational numbers.

CHAPTER 25

Generic fiber rings of mixed polynomial-power series rings

Our primary project in this chapter is to prove Theorem 24.3 concerning generic fiber rings for extensions of the polynomial-power series rings A, B and C defined in Chapter 24; see Equation 24.1.0 and Notation 24.1. By Theorem 24.3, all ideals maximal in each of the generic formal fiber rings for A, B and C have the same height. These results are proved using the techniques developed in Chapter 24. Matsumura proves in [**95**] that the generic formal fiber ring of A has dimension $n-1 = \dim A - 1$, and the generic formal fiber rings of B and C have dimension $n+m-2 = \dim B - 2 = \dim C - 2$. Matsumura does not consider in [**95**] the question of whether all the maximal ideals in these generic formal fiber ring have the same height.

For a local extension $R \hookrightarrow S$ of Noetherian local integral domains, Theorem 25.12 gives sufficient conditions in order that all maximal ideals in $\operatorname{Gff}(S)$ have height $h = \dim \operatorname{Gff}(R)$. Using Theorem 25.12, we show in Theorem 25.10 that all prime ideals maximal in the generic formal fiber of a local domain essentially finitely generated over a field have the same height. For certain Noetherian local extensions S of the rings B and C, we show in Theorem 25.16 that the maximal ideals of $\operatorname{Gff}(S)$ all have height $n+m-2$.

In Sections 25.1 and 25.2, we prove parts 2 and 3 of Theorem 24.3 stated in Chapter 24. In Section 25.3 we prove part 1 of Theorem 24.3, by using the results of Section 24.3, and in Section 25.4 we prove part 4. Theorems 25.12, 25.10 and 25.16 are in Section 25.5.

25.1. Weierstrass implications for the ring $B = k[[X]]\,[Y]_{(X,Y)}$

As before, k denotes a field, n and m are positive integers, and $X = \{x_1, \dots, x_n\}$ and $Y = \{y_1, \dots, y_m\}$ denote sets of variables over k. Let

$$B := k[[X]]\,[Y]_{(X,Y)} = k[[x_1, \dots, x_n]]\,[y_1, \dots, y_m]_{(x_1,\dots,x_n,y_1,\dots,y_m)}.$$

The completion of B is $\widehat{B} = k[[X, Y]]$.

Theorem 25.1. *With the notation as above, every ideal Q of $\widehat{B} = k[[X,Y]]$ maximal with the property that $Q \cap B = (0)$ is a prime ideal of height $n+m-2$.*

Proof. Suppose first that Q is such an ideal. Then clearly Q is prime. Matsumura shows in [**95**, Theorem 3] that the dimension of the generic formal fiber of B is at most $n+m-2$. Therefore $\operatorname{ht} Q \le n+m-2$.

Now suppose $P \in \operatorname{Spec} \widehat{B}$ is an arbitrary prime ideal of height $r < n+m-2$ with $P \cap B = (0)$. We construct a prime $Q \in \operatorname{Spec} \widehat{B}$ with $P \subset Q$, $Q \cap B = (0)$, and $\operatorname{ht} Q = n + m - 2$. This will show that all prime ideals maximal in the generic fiber have height $n + m - 2$.

For the construction of Q we consider first the case where $P \not\subseteq X\widehat{B}$. Then there exists a prime element $f \in P$ that contains a term $\theta := y_1^{i_1} \cdots y_m^{i_m}$, where the i_j's are nonnegative integers and at least one of the i_j is positive. Notice that $m \geq 2$ for otherwise with $y = y_1$ we have $f \in P$ contains a term y^i. By Weierstrass, that is, by Theorem 24.7, it follows that $f = g\epsilon$, where $g \in k[[X]]\,[y]$ is a nonzero monic polynomial in y and ϵ is a unit of $\widehat{B}$. But $g \in P$ and $g \in B$ implies $P \cap B \neq (0)$, a contradiction to our assumption that $P \cap B = (0)$.

For convenience we now assume that the last exponent i_m appearing in θ above is positive. We apply a change of variables: $y_m \to t_m := y_m$ and, for $1 \leq \ell < m$, let $y_\ell \to t_\ell := y_\ell + {t_m}^{e_\ell}$, where the e_ℓ are chosen so that f, expressed in the variables $t_1, \dots, t_m$, contains a term t_m^q, for some positive integer q. This change of variables induces an automorphism of B. By Weierstrass $f = g_1 h$, where h is a unit in $\widehat{B}$ and $g_1 \in k[[X, t_1, \dots, t_{m-1}]]\,[t_m]$ is monic in t_m. Set $P_1 = P \cap k[[X, t_1, \dots, t_{m-1}]]$. If $P_1 \subseteq Xk[[X, t_1, \dots, t_{m-1}]]$, we stop the procedure and take $s = m - 1$ in what follows. If $P_1 \not\subseteq Xk[[X, t_1, \dots, t_{m-1}]]$, then there exists a prime element $\tilde{f} \in P_1$ that contains a term ${t_1}^{j_1} \cdots {t_{m-1}}^{j_{m-1}}$, where the j_k's are nonnegative integers and at least one of the j_k is positive. We then repeat the procedure using the prime ideal P_1. That is, we replace $t_1, \dots t_{m-1}$ with a change of variables so that a prime element of P_1 contains a term monic in some one of the new variables. After a suitable finite iteration of changes of variables, we obtain an automorphism of $\widehat{B}$ that restricts to an automorphism of B and maps $y_1, \dots, y_m \mapsto z_1, \dots, z_m$. Moreover, there exist a positive integer $s \leq m - 1$ and elements $g_1, \dots g_{m-s} \in P$ such that

$$\begin{array}{ll} g_1 \in k[[X, z_1, \dots, z_{m-1}]]\,[z_m] & \text{is monic in } z_m \\ g_2 \in k[[X, z_1, \dots, z_{m-2}]]\,[z_{m-1}] & \text{is monic in } z_{m-1}, \text{ etc} \\ \vdots & \\ g_{m-s} \in k[[X, z_1, \dots, z_s]]\,[z_{s+1}] & \text{is monic in } z_{s+1}, \end{array}$$

and such that, for $R_s := k[[X, z_1, \dots, z_s]]$ and $P_s := P \cap R_s$, we have $P_s \subseteq XR_s$.

As in the proof of Proposition 24.10, we replace the regular sequence $g_1, \dots .g_{m-s}$ by a regular sequence $f_1, \dots, f_{m-s}$ so that:

$$\begin{array}{ll} f_1 \in R_s[z_{s+1}, \dots, z_m] & \text{is monic in } z_m \\ f_2 \in R_s[z_{s+1}, \dots, z_{m-1}] & \text{is monic in } z_{m-1}, \text{ etc} \\ \vdots & \\ f_{m-s} \in R_s[z_{s+1}] & \text{is monic in } z_{s+1}. \end{array}$$

and $(g_1, \dots .g_{m-s})\widehat{B} = (f_1, \dots, f_{m-s})\widehat{B}$.

Let $G := k[[X, z_1, \dots, z_s]]\,[z_{s+1}, \dots, z_m] = R_s[z_{s+1}, \dots, z_m]$. By Proposition 24.10, P is extended from G. Let $\mathbf{q} := P \cap G$ and extend $f_1, \dots, f_{m-s}$ to a generating system of $\mathbf{q}$, say, $\mathbf{q} = (f_1, \dots, f_{m-s}, h_1, \dots, h_t)G$. For integers k, ℓ with $1 \leq k \leq m - s$ and $1 \leq \ell \leq t$, express the f_k and h_ℓ in G as power series in

$\widehat{B} = k[[z_1]][[z_2,\dots,z_m]]\,[[X]]$ with coefficients in $k[[z_1]]$:

$$f_k = \sum a_{k(i)(j)} z_2^{i_2}\dots z_m^{i_m} x_1^{j_1}\dots x_n^{j_n} \quad \text{and} \quad h_\ell = \sum b_{\ell(i)(j)} z_2^{i_2}\dots z_m^{i_m} x_1^{j_1}\dots x_n^{j_n}\,,$$

where $a_{k(i)(j)}, b_{\ell(i)(j)} \in k[[z_1]]$, $(i) = (i_2,\dots,i_m)$ and $(j) = (j_1,\dots,j_n)$. The set $\Delta = \{a_{k(i)(j)}, b_{\ell(i)(j)}\}$ is countable. We define $V := k(z_1,\Delta) \cap k[[z_1]]$. Then V is a discrete valuation domain with completion $k[[z_1]]$ and $k((z_1))$ has uncountable transcendence degree over $\mathcal{Q}(V)$. Let $V_s := V[[X,z_2,\dots,z_s]] \subseteq R_s$. Notice that $R_s = \widehat{V_s}$, the completion of V_s. Also $f_1,\dots,f_{m-s} \in V_s[z_{s+1},\dots,z_m] \subseteq G$ and $(f_1,\dots,f_{m-s})G \cap R_s = (0)$. Furthermore the extension

$$V_s := V[[X,z_2,\dots,z_s]] \hookrightarrow V_s[z_{s+1},\dots,z_m]/(f_1,\dots,f_{m-s})$$

is finite. Set $P_0 := P \cap V_s$. Then $P_0 \subseteq XR_s \cap V_s = XV_s$.

Consider the commutative diagram:

$$\begin{array}{ccc} R_s := k[[X,z_1,\dots,z_s]] & \longrightarrow & R_s[[z_{s+1},\dots,z_m]]/(f_1,\dots,f_{m-s}) \\ \uparrow & & \uparrow \\ V_s := V[[X,z_2,\dots,z_s]] & \longrightarrow & V_s[z_{s+1},\dots,z_m]/(f_1,\dots,f_{m-s})\,. \end{array} \tag{25.1}$$

The horizontal maps are injective and finite and the vertical maps are completions.

The prime ideal $\bar{\mathbf{q}} := PR_s[[z_{s+1},\dots,z_m]]/(f_1,\dots,f_{m-s})$ lies over P_s in R_s. By assumption $P_s \subseteq (X)R_s$ and by Theorem 24.11 there is a prime ideal Q_s of R_s such that $P_s \subseteq Q_s \subseteq (X)R_s$, $Q_s \cap V_s = P_s \cap V_s = P_0$, and $\dim(R_s/Q_s) = 2$. There is a prime ideal $\bar{Q}$ in $R_s[[z_{s+1},\dots,z_m]]/(f_1,\dots,f_{m-s})$ lying over Q_s with $\bar{\mathbf{q}} \subseteq \bar{Q}$ by the "going-up theorem" [**96**, Theorem 9.4]. Let Q be the preimage in $\widehat{B} = k[[X,z_1,\dots,z_m]]$ of $\bar{Q}$. We show the rings and ideals of Theorem 25.1 below.

$$\widehat{B} = k[[X,Y]] = k[[X,z_1,\dots,z_m]] = R_s[[z_{s+1},\dots,z_m]]$$
$$(\mathbf{q}, Q_s)\widehat{B} \subseteq Q$$
$$P \nsubseteq X\widehat{B}$$

$$G := R_s[z_{s+1},\dots,z_m]$$
$$\mathbf{q} := P \cap G$$
$$\mathbf{q} = (\{f_i, h_j\})G$$

$$f_i \notin R_s := k[[X,z_1,\dots,z_s]]$$
$$P_s \subseteq Q_s \subset R_s$$
$$P_s := P \cap R_s \subseteq XR_s$$

$$V_s := V[[X,z_2,\dots,z_s]] \qquad\qquad \widehat{V} = k[[z_1]]$$
$$P_0 := P \cap V_s$$

$$V := k(z_1,\Delta) \cap k[[z_1]]$$

Then Q has height $n+s-2+m-s = n+m-2$. Moreover, it follows from Diagram 25.1 that Q and P have the same contraction to $V_s[z_{s+1},\dots,z_m]$. This implies that $Q \cap B = (0)$ and completes the proof in the case where $P \not\subseteq X\widehat{B}$.

In the case where $P \subseteq X\widehat{B}$, let $h_1,\dots,h_t \in \widehat{B}$ be a finite set of generators of P, and as above, let $b_{\ell(i)(j)} \in k[[z_1]]$ be the coefficients of the h_ℓ's. Consider the countable set $\Delta = \{b_{\ell(i)(j)}\}$ and the valuation domain $V := k(z_1,\Delta) \cap k[[z_1]]$. Set $P_0 := P \cap V[[X,z_2,\dots,z_m]]$. By Theorem 24.11, there exists a prime ideal Q of $\widehat{B} = k[[X,z_1,\dots,z_m]]$ of height $n+m-2$ such that $P \subset Q$ and $Q \cap V[[X,z_2,\dots,z_m]] = P \cap V[[X,z_2,\dots,z_m]] = P_0$. Therefore $Q \cap B = (0)$. This completes the proof of Theorem 25.1. $\square$

25.2. Weierstrass implications for the ring $C = k[Y]_{(Y)}[[X]]$

As before, k denotes a field, n and m are positive integers, and $X = \{x_1,\dots,x_n\}$ and $Y = \{y_1,\dots,y_m\}$ denote sets of variables over k. Consider the ring

$$C = k[y_1,\dots,y_m]_{(y_1,\dots,y_m)}[[x_1,\dots,x_n]] = k[Y]_{(Y)}[[X]].$$

The completion of C is $\widehat{C} = k[[Y,X]]$.

THEOREM 25.2. *With notation as above, let $Q \in \operatorname{Spec}\widehat{C}$ be maximal with the property that $Q \cap C = (0)$. Then* $\operatorname{ht} Q = n+m-2$.

PROOF. Let $B = k[[X]]\,[Y]_{(X,Y)} \subset C$. If $P \in \operatorname{Spec}\widehat{C} = \operatorname{Spec}\widehat{B}$ and $P \cap C = (0)$, then $P \cap B = (0)$, and so $\operatorname{ht} P \le n+m-2$ by Theorem 25.1. Consider a nonzero prime $P \in \operatorname{Spec}\widehat{C}$ with $P \cap C = (0)$ and $\operatorname{ht} P = r < n+m-2$. If $P \subseteq X\widehat{C}$ then Theorem 24.11 implies the existence of $Q \in \operatorname{Spec}\widehat{C}$ with $\operatorname{ht} Q = n+m-2$ such that $P \subset Q$ and $Q \cap C = (0)$.

Assume that P is not contained in $X\widehat{C}$ and consider the ideal $J := (P,X)\widehat{C}$. Since C is complete in the XC-adic topology, a lemma of Rotthaus implies that if J is primary for the maximal ideal of $\widehat{C}$, then P is extended from C; see [**120**, Lemma 2]. Since we are assuming $P \cap C = (0)$, J is not primary for the maximal ideal of $\widehat{C}$ and we have $\operatorname{ht} J = n+s < n+m$, where $0 < s < m$. Let $W \in \operatorname{Spec}\widehat{C}$ be a minimal prime of J such that $\operatorname{ht} W = n+s$. Let $W_0 = W \cap k[[Y]]$. Then $W = (W_0,X)\widehat{C}$ and W_0 is a prime ideal of $k[[Y]]$ with $\operatorname{ht} W_0 = s$. By Proposition 24.10 applied to $k[[Y]]$ and the prime ideal $W_0 \in \operatorname{Spec} k[[Y]]$, there exists a change of variables $Y \mapsto Z$ with $y_1 \mapsto z_1,\dots,y_m \mapsto z_m$ and elements $f_1,\dots,f_s \in W_0$ so that with $Z_1 = \{z_1,\dots,z_{m-s}\}$, we have

$$\begin{array}{ll} f_1 \in k[[Z_1]]\,[z_{m-s+1},\dots,z_m] & \text{is monic in } z_m \\ f_2 \in k[[Z_1]]\,[z_{m-s+1},\dots,z_{m-1}] & \text{is monic in } z_{m-1}, \text{ etc} \\ \quad\vdots & \\ f_s \in k[[Z_1]]\,[z_{m-s+1}] & \text{is monic in } z_{m-s+1}. \end{array}$$

Now $z_1,\dots,z_{m-s},f_1,\dots,f_s$ is a regular sequence in $k[[Z]] = k[[Y]]$. Let $T = \{t_{m-s+1},\dots,t_m\}$ be a set of additional variables and consider the map:

$$\varphi : k[[Z_1,T]] \longrightarrow k[[z_1,\dots,z_m]]$$

defined by $z_i \mapsto z_i$ for all $1 \le i \le m-s$ and $t_{m-i+1} \mapsto f_i$ for all $1 \le i \le s$. The embedding φ is finite (and free) and so is the extension to power series rings in X:

$$\rho : k[[Z_1, T]]\,[[X]] \longrightarrow k[[z_1, \dots, z_m]]\,[[X]] = \widehat{C}.$$

The contraction $\rho^{-1}(W) \in \operatorname{Spec} k[[Z_1, T, X]]$ of the prime ideal W of $\widehat{C}$ has height $n+s$, since $\operatorname{ht} W = n+s$. Moreover $\rho^{-1}(W)$ contains $(T, X)k[[Z_1, T, X]]$, a prime ideal of height $n+s$. Therefore $\rho^{-1}(W) = (T, X)k[[Z_1, T, X]]$. By construction, $P \subseteq W$ which yields that $\rho^{-1}(P) \subseteq (T, X)k[[Z_1, T, X]]$.

To complete the proof we construct a suitable base ring related to C. Consider the expressions for the f_i's as power series in $z_2, \dots, z_m$ with coefficients in $k[[z_1]]$:

$$f_j = \sum a_{j(i)} z_2^{i_2} \dots z_m^{i_m},$$

where $(i) := (i_2, \dots, i_m), 1 \le j \le s, a_{j(i)} \in k[[z_1]]$. Also consider a finite generating system $g_1, \dots, g_q$ for P and expressions for the g_k, where $1 \le k \le q$, as power series in $z_2, \dots, z_m, x_1, \dots, x_n$ with coefficients in $k[[z_1]]$:

$$g_k = \sum b_{k(i)(\ell)} z_2^{i_2} \dots z_m^{i_m} x_1^{\ell_1} \dots x_n^{\ell_n},$$

where $(i) := (i_2, \dots, i_m)$, $(\ell) := (\ell_1, \dots, \ell_n)$, and $b_{k(i)(\ell)} \in k[[z_1]]$. We take the subset $\Delta = \{a_{j(i)}, b_{k(i)(\ell)}\}$ of $k[[z_1]]$ and consider the discrete valuation domain:

$$V := k(z_1, \Delta) \cap k[[z_1]].$$

Since V is countably generated over $k(z_1)$, the field $k((z_1))$ has uncountable transcendence degree over $\mathcal{Q}(V) = k(z_1, \Delta)$. Moreover, by construction the ideal P is extended from $V[[z_2, \dots, z_m]]\,[[X]]$. Consider the embedding:

$$\psi : V[[z_2, \dots, z_{m-s}, T]] \longrightarrow V[[z_2, \dots, z_m]],$$

which is the restriction of φ above, so that $z_i \mapsto z_i$ for all $2 \le i \le m-s$ and $t_{m-i+1} \mapsto f_i$ for all i with $1 \le i \le s$.

Let σ be the extension of ψ to the power series rings:

$$\sigma : V[[z_2, \dots, z_{m-s}, T]]\,[[X]] \longrightarrow V[[z_2, \dots, z_m]]\,[[X]]$$

with $\sigma(x_i) = x_i$ for all i with $1 \le i \le n$.

Notice that ρ defined above is the completion $\widehat{\sigma}$ of the map σ, that is, the extension of σ to the completions. Consider the commutative diagram:

$$\begin{array}{ccc} k[[Z_1, T]]\,[[X]] & \xrightarrow{\widehat{\sigma}=\rho} & k[[Z]]\,[[X]] = \widehat{C} \\ \uparrow & & \uparrow \\ V[[z_2, \dots, z_{m-s}, T]]\,[[X]] & \xrightarrow{\sigma} & V[[z_2, \dots, z_m]]\,[[X]] \end{array} \tag{25.2.0}$$

where $\widehat{\sigma} = \rho$ is a finite map.

Recall that $\rho^{-1}(W) = (T, X)k[[Z_1, T, X]]$, and so $\rho^{-1}(P) \subseteq (T, X)k[[Z_1, T, X]]$ by Diagram 25.2.0. By Theorem 24.11, there exists a prime ideal Q_0 of the ring $k[[Z_1, T, X]]$ such that $\rho^{-1}(P) \subseteq Q_0$, $\operatorname{ht} Q_0 = n+m-2$, and

$$Q_0 \cap V[[z_2, \dots, z_{m-s}, T]]\,[[X]] = \rho^{-1}(P) \cap V[[z_2, \dots, z_{m-s}, T]]\,[[X]].$$

By the "going-up theorem" [**96**, Theorem 9.4] , there is a prime ideal $Q \in \operatorname{Spec} \widehat{C}$ that lies over Q_0 and contains P. Moreover, Q also has height $n+m-2$. The commutativity of Diagram 25.2.0 implies that

$$P_1 := P \cap V[[z_2, \dots, z_{m-s}, T]]\,[[X]] \;\subseteq\; Q_1 := Q \cap V[[z_2, \dots, z_{m-s}, T]]\,[[X]].$$

Consider the finite homomorphism:

$$\lambda : V[[z_2, \dots, z_{m-s}]]\,[T]_{(Z_1,T)}[[X]] \longrightarrow V[[z_2, \dots, z_{m-s}]]\,[z_{m-s+1}, \dots, z_m]_{(Z)}[[X]]$$

(determined by $t_i \mapsto f_i$ for $1 \le i \le m$) and the commutative diagram:

$$\begin{array}{ccc} V[[z_2, \dots, z_{m-s}]]\,[[T]]\,[[X]] & \xrightarrow{\sigma} & V[[z_2, \dots, z_m]]\,[[X]] \\ \uparrow & & \uparrow \\ V[[z_2, \dots, z_{m-s}]]\,[T]_{(Z_1,T)}[[X]] & \xrightarrow{\lambda} & V[[z_2, \dots, z_{m-s}]]\,[z_{m-s+1}, \dots, z_m]_{(Z)}[[X]]. \end{array}$$

Since $Q \cap V[[z_2, \dots, z_{m-s}, T]]\,[[X]] = P \cap V[[z_2, \dots, z_{m-s}, T]]\,[[X]]$ and since λ is a finite map we conclude that

$$\begin{aligned} Q_1 \cap\ & V[[z_2, \dots, z_{m-s}]]\,[z_{m-s+1}, \dots, z_m]_{(Z)}[[X]] \\ &= P_1 \cap V[[z_2, \dots, z_m]]\,[z_{m-s+1}, \dots, z_m]_{(Z)}[[X]]. \end{aligned}$$

Since $C \subseteq V[[z_2, \dots, z_{m-s}]]\,[z_{m-s+1}, \dots, z_m]_{(Z)}[[X]]$, we obtain that the intersection $Q \cap C = P \cap C = (0)$. This completes the proof of Theorem 25.2. $\square$

REMARK 25.3. With B and C as in Sections 25.1 and 25.2, we have

$$B = k[[X]]\,[Y]_{(X,Y)} \hookrightarrow k[Y]_{(Y)}[[X]] = C \quad \text{and} \quad \widehat{B} = k[[X,Y]] = \widehat{C}.$$

Thus for $P \in \operatorname{Spec} k[[X,Y]]$, if $P \cap C = (0)$, then $P \cap B = (0)$. By Theorems 25.1 and 25.2, each prime of $k[[X,Y]]$ maximal in the generic formal fiber of B or C has height $n+m-2$. Therefore each $P \in \operatorname{Spec} k[[X,Y]]$ maximal with respect to $P \cap C = (0)$ is also maximal with respect to $P \cap B = (0)$. However, if $n+m \ge 3$, the generic fiber of $B \hookrightarrow C$ is nonzero (see Propositions 26.24 and 26.25 of Chapter 26), and so there exist primes of $k[[X,Y]]$ maximal in the generic formal fiber of B that are not in the generic formal fiber of C.

25.3. Weierstrass implications for the localized polynomial ring A

Let n be a positive integer, let $X = \{x_1, \dots, x_n\}$ be a set of n variables over a field k, and let $A := k[x_1, \dots, x_n]_{(x_1, \dots, x_n)} = k[X]_{(X)}$ denote the localized polynomial ring in these n variables over k. Then the completion of A is $\widehat{A} = k[[X]]$.

THEOREM 25.4. *For the localized polynomial ring $A = k[X]_{(X)}$ defined above, if Q is an ideal of $\widehat{A}$ maximal with respect to $Q \cap A = (0)$, then Q is a prime ideal of height $n-1$.*

PROOF. It is clear that Q as described in the statement is a prime ideal. Also the assertion holds for $n = 1$. Thus we assume $n \ge 2$. By Proposition 24.17, there exists a nonzero prime $\mathbf{p}$ in $k[[x_1, x_2]]$ such that $\mathbf{p} \cap k[x_1, x_2]_{(x_1,x_2)} = (0)$. It follows that $\mathbf{p}\widehat{A} \cap A = (0)$. Thus the generic formal fiber of A is nonzero.

Let $P \in \operatorname{Spec} \widehat{A}$ be a nonzero prime ideal with $P \cap A = (0)$ and $\operatorname{ht} P = r < n-1$. We construct $Q \in \operatorname{Spec} \widehat{A}$ of height $n-1$ with $P \subseteq Q$ and $Q \cap A = (0)$. By

Proposition 24.10, there exists a change of variables $x_1 \mapsto z_1, \dots, x_n \mapsto z_n$ and polynomials

$$\begin{array}{ll} f_1 \in k[[z_1, \dots, z_{n-r}]]\,[z_{n-r+1}, \dots, z_n] & \text{monic in } z_n \\ f_2 \in k[[z_1, \dots, z_{n-r}]]\,[z_{n-r+1}, \dots, z_{n-1}] & \text{monic in } z_{n-1}, \text{ etc} \\ \quad\vdots & \\ f_r \in k[[z_1, \dots, z_{n-r}]]\,[z_{n-r+1}] & \text{monic in } z_{n-r+1}, \end{array}$$

so that P is a minimal prime of $(f_1, \dots, f_r)\widehat{A}$ and P is extended from

$$R := k[[z_1, \dots, z_{n-r}]]\,[z_{n-r+1}, \dots, z_n].$$

Let $P_0 := P \cap R$ and extend $f_1, \dots, f_r$ to a system of generators of P_0, say:

$$P_0 = (f_1, \dots, f_r, g_1, \dots, g_s)R.$$

Using an argument similar to that in the proof of Theorem 24.11, write

$$f_j = \sum_{(i) \in \mathbb{N}^{n-1}} a_{j,(i)} z_2^{i_2} \dots z_n^{i_n} \quad \text{and} \quad g_k = \sum_{(i) \in \mathbb{N}^{n-1}} b_{k,(i)} z_2^{i_2} \dots z_n^{i_n},$$

where $a_{j,(i)}, b_{k,(i)} \in k[[z_1]]$. Let

$$V_0 := k(z_1, a_{j,(i)}, b_{k,(i)}) \cap k[[z_1]].$$

Then V_0 is a discrete rank-one valuation domain with completion $k[[z_1]]$, and $k((z_1))$ has uncountable transcendence degree over the field of fractions $\mathcal{Q}(V_0)$ of V_0. Let $\gamma_3, \dots, \gamma_{n-r} \in k[[z_1]]$ be algebraically independent over $\mathcal{Q}(V_0)$ and define

$$\mathbf{q} := (z_3 - \gamma_3 z_2, z_4 - \gamma_4 z_2, \dots, z_{n-r} - \gamma_{n-r} z_2) k[[z_1, \dots, z_{n-r}]].$$

We see that $\mathbf{q} \cap V_0[[z_2, \dots, z_{n-r}]] = (0)$ by an argument similar to that in [**95**] and in Claim 24.12. Let $R_1 := V_0[[z_2, \dots, z_{n-r}]]\,[z_{n-r+1}, \dots, z_n]$, let $P_1 := P \cap R_1$ and consider the commutative diagram:

$$\begin{array}{ccc} k[[z_1, \dots, z_{n-r}]] & \longrightarrow & R/P_0 \\ \uparrow & & \uparrow \\ V_0[[z_2, \dots, z_{n-r}]] & \longrightarrow & R_1/P_1 \end{array}$$

The horizontal maps are injective finite integral extensions. Let W be a minimal prime of $(\mathbf{q}, P)\widehat{A}$. Then $\operatorname{ht} W = n - 2$ and $\mathbf{q} \cap V_0[[z_2, \dots, z_{n-r}]] = (0)$ implies that $W \cap R_1 = P_1$. Thus the prime ideal $W \in \operatorname{Spec} \widehat{A}$ satisfies $\operatorname{ht} W = n - 2$, $W \cap A = (0)$ and $P \subseteq W$. Since $f_1, \dots, f_r \in W$ and since $\widehat{A} = k[[z_1, \dots, z_n]]$ is the $(f_1, \dots, f_r)$-adic completion of $k[[z_1, \dots, z_{n-r}]]\,[z_{n-r+1}, \dots, z_n]$, the prime ideal W is extended from $k[[z_1, \dots, z_{n-r}]]\,[z_{n-r+1}, \dots, z_n]$.

We claim that W is actually extended from $k[[z_1, z_2]]\,[z_3, \dots, z_n]$. To see this, let $g \in W \cap k[[z_1, \dots, z_{n-r}]]\,[z_{n-r+1}, \dots, z_n]$ and write:

$$g = \sum_{(i)} a_{(i)} z_{n-r+1}^{i_{n-r+1}} \dots z_n^{i_n} \in k[[z_1, \dots, z_{n-r}]]\,[z_{n-r+1}, \dots, z_n],$$

where the sum is over all $(i) = (i_{n-r}, \dots, i_n)$ and $a_{(i)} \in k[[z_1, \dots, z_{n-r}]]$. For all $a_{(i)}$ by Weierstrass, that is, by Theorem 24.7, we can write

$$a_{(i)} = (z_{n-r} - \gamma_{n-r} z_2) h_{(i)} + q_{(i)},$$

where $h_{(i)} \in k[[z_1, \dots z_{n-r}]]$ and $q_{(i)} \in k[[z_1, \dots, z_{n-r-1}]]$. If $n - r > 3$, we write

$$q_{(i)} = (z_{n-r-1} - \gamma_{n-r-1} z_2) h'_{(i)} + q'_{(i)},$$

where $h'_{(i)} \in k[[z_1, \dots z_{n-r-1}]]$ and $q'_{(i)} \in k[[z_1, \dots, z_{n-r-2}]]$. In this way we replace a generating set for W in $k[[z_1, \dots, z_{n-r}]]\,[z_{n-r+1}, \dots, z_n]$ by a generating set for W in $k[[z_1, z_2]]\,[z_3, \dots, z_n]$.

In particular, we can replace the elements $f_1, \dots, f_r$ by elements:

$$\begin{array}{ll} h_1 \in k[[z_1, z_2]]\,[z_3, \dots, z_n] & \text{monic in } z_n \\ h_2 \in k[[z_1, z_2]]\,[z_3, \dots, z_{n-1}] & \text{monic in } z_{n-1}, \text{ etc} \\ \quad\vdots & \\ h_r \in k[[z_1, z_2]]\,[z_3 \dots, z_{n-r+1}] & \text{monic in } z_{n-r+1} \end{array}$$

and set $h_{r+1} = z_3 - \gamma_3 z_2, \dots, h_{n-2} = z_{n-r} - \gamma_{n-r} z_2$, and then extend to a generating set $h_1, \dots, h_{n+s-2}$ for

$$W_0 = W \cap k[[z_1, z_2]]\,[z_3, \dots, z_n]$$

such that $W_0 \widehat{A} = W$. Consider the coefficients in $k[[z_1]]$ of the h_j:

$$h_j = \sum_{(i)} c_{j(i)} z_2^{i_2} \dots z_n^{i_n}$$

with $c_{j(i)} \in k[[z_1]]$. The set $\{c_{j(i)}\}$ is countable. Define

$$V := \mathcal{Q}(V_0)(\{c_{j(i)}\}) \cap k[[z_1]]$$

Then V is a rank-one discrete valuation domain that is countably generated over $k[z_1]_{(z_1)}$ and W is extended from $V[[z_2]]\,[z_3, \dots, z_n]$.

We may also write each h_i as a polynomial in $z_3, \dots, z_n$ with coefficients in $V[[z_2]]$:

$$h = \sum \omega_{(i)} z_3^{i_3} \dots z_n^{i_n}$$

with $\omega_{(i)} \in V[[z_2]] \subseteq k[[z_1, z_2]]$. By Valabrega's Theorem 4.2, the integral domain

$$D := \mathcal{Q}(V)(z_2, \{\omega_{(i)}\}) \cap k[[z_1, z_2]]$$

is a two-dimensional regular local domain with completion $\widehat{D} = k[[z_1, z_2]]$. Let $W_1 := W \cap D[z_3, \dots, z_n]$. Then $W_1 \widehat{A} = W$. We have shown in Section 24.3 that there exists a prime element $q \in k[[z_1, z_2]]$ with $qk[[z_1, z_2]] \cap D = (0)$. Consider the finite extension

$$D \longrightarrow D[z_3, \dots, z_n]/W_1.$$

Let $Q \in \operatorname{Spec} \widehat{A}$ be a minimal prime of $(q, W)\widehat{A}$. Since $\operatorname{ht} W = n-2$ and $q \notin W$, $\operatorname{ht} Q = n-1$. Moreover, $P \subseteq W$ implies $P \subseteq Q$. We claim that

$$Q \cap D[z_3, \dots, z_n] = W_1 \quad \text{and therefore} \quad Q \cap A = (0).$$

To see this consider the commutative diagram:

$$\begin{array}{ccc} k[[z_1, z_2]] & \longrightarrow & k[[z_1, \dots, z_n]]/W \\ \uparrow & & \uparrow \\ D & \longrightarrow & D[z_3, \dots, z_n]/W_1\,, \end{array}$$

which has injective finite horizontal maps. Since $qk[[z_1, z_2]] \cap D = (0)$, it follows that $Q \cap D[z_3, \dots, z_n] = W_1$. This completes the proof of Theorem 25.4. $\square$

25.4. Generic fibers of power series ring extensions

In this section we apply the Weierstrass machinery from Section 24.2 to the generic fiber rings of power series extensions.

THEOREM 25.5. *Let $n \geq 2$ be an integer and let $y, x_1, \dots, x_n$ be variables over the field k. Let $X = \{x_1, \dots, x_n\}$ and let R_1 be the formal power series ring $k[[X]]$. Consider the extension $R_1 \hookrightarrow R_1[[y]] = R$. Let $U = R_1 \setminus (0)$. For $P \in \operatorname{Spec} R$ such that $P \cap U = \emptyset$, we have:*

1. *If $P \not\subseteq XR$, then $\dim R/P = n$ and P is maximal in the generic fiber $U^{-1}R$.*
2. *If $P \subseteq XR$, then there exists $Q \in \operatorname{Spec} R$ such that $P \subseteq Q$, $\dim R/Q = 2$ and Q is maximal in the generic fiber $U^{-1}R$.*

If $n > 2$ for each prime ideal Q maximal in the generic fiber $U^{-1}R$, we have

$$\dim R/Q = \begin{cases} n & \text{and } R_1 \hookrightarrow R/Q \text{ is finite, or} \\ 2 & \text{and } Q \subset XR. \end{cases}$$

PROOF. Let $P \in \operatorname{Spec} R$ be such that $P \cap U = \emptyset$ or equivalently $P \cap R_1 = (0)$. Then R_1 embeds in R/P. If $\dim(R/P) \leq 1$, then the maximal ideal of R_1 generates an ideal primary for the maximal ideal of R/P. By Theorem 3.10, R/P is finite over R_1, and so $\dim R_1 = \dim(R/P)$, a contradiction. Thus $\dim(R/P) \geq 2$.

If $P \not\subseteq XR$, then there exists a prime element $f \in P$ that contains a term y^s for some positive integer s. By Weierstrass, that is, by Theorem 24.7, it follows that $f = g\epsilon$, where $g \in k[[X]]\,[y]$ is a nonzero monic polynomial in y and ϵ is a unit of R. We have $fR = gR \subseteq P$ is a prime ideal and $R_1 \hookrightarrow R/gR$ is a finite integral extension. Since $P \cap R_1 = (0)$, we must have $gR = P$.

If $P \subseteq XR$ and $\dim(R/P) > 2$, then Theorem 24.11 implies there exists $Q \in \operatorname{Spec} R$ such that $\dim(R/Q) = 2$, $P \subset Q \subset XR$ and $P \cap R_1 = (0) = Q \cap R_1$, and so P is not maximal in the generic fiber. Thus $Q \in \operatorname{Spec} R$ maximal in the generic fiber of $R_1 \hookrightarrow R$ implies that the dimension of $\dim(R/Q)$ is 2, or equivalently that $\operatorname{ht} Q = n - 1$. $\square$

THEOREM 25.6. *Let n and m be positive integers, and let $X = \{x_1, \dots, x_n\}$ and $Y = \{y_1, \dots, y_m\}$ be sets of independent variables over the field k. Consider the formal power series rings $R_1 = k[[X]]$ and $R = k[[X, Y]]$ and the extension $R_1 \hookrightarrow R_1[[Y]] = R$. Let $U = R_1 \setminus (0)$. Let $Q \in \operatorname{Spec} R$ be maximal with respect to $Q \cap U = \emptyset$. If $n = 1$, then $\dim R/Q = 1$ and $R_1 \hookrightarrow R/Q$ is finite.*

If $n \geq 2$, there are two possibilities:

1. *$R_1 \hookrightarrow R/Q$ is finite, in which case $\dim R/Q = \dim R_1 = n$, or*
2. *$\dim R/Q = 2$.*

PROOF. First assume $n = 1$, and let $x = x_1$. Since Q is maximal with respect to $Q \cap U = \emptyset$, for each $P \in \operatorname{Spec} R$ with $Q \subsetneq P$ we have $P \cap U$ is nonempty and therefore $x \in P$. It follows that $\dim R/Q = 1$, for otherwise,

$$Q = \bigcap \{P \mid P \in \operatorname{Spec} R \text{ and } Q \subsetneq P\},$$

which implies $x \in Q$. By Theorem 3.10, $R_1 \hookrightarrow R/Q$ is finite.

It remains to consider the case where $n \geq 2$. We proceed by induction on m. Theorem 25.5 yields the assertion for $m = 1$. Suppose $Q \in \operatorname{Spec} R$ is maximal with respect to $Q \cap U = \emptyset$. As in the proof of Theorem 25.5, we have $\dim R/Q \geq 2$. If $Q \subseteq (X, y_1, \ldots, y_{m-1})R$, then by Theorem 24.11 with $R_0 = k[y_m]_{(y_m)}[[X, y_1, \ldots, y_{m-1}]]$, there exists $Q' \in \operatorname{Spec} R$ with $Q \subseteq Q'$, $\dim R/Q' = 2$, and $Q \cap R_0 = Q' \cap R_0$. Since $R_1 \subseteq R_0$, we have $Q' \cap U = \emptyset$. Since Q is maximal with respect to $Q \cap U = \emptyset$, we have $Q = Q'$, and so $\dim R/Q = 2$.

Otherwise, if $Q \not\subseteq (X, y_1, \ldots, y_{m-1})R$, then there exists a prime element $f \in Q$ that contains a term y_m^s for some positive integer s. Let $R_2 = k[[X, y_1, \ldots, y_{m-1}]]$. By Weierstrass, it follows that $f = g\epsilon$, where $g \in R_2[y_m]$ is a nonzero monic polynomial in y_m and ϵ is a unit of R. We have $fR = gR \subseteq Q$ is a prime ideal and $R_2 \hookrightarrow R/gR$ is a finite integral extension. Thus $R_2/(Q \cap R_2) \hookrightarrow R/Q$ is an integral extension. It follows that $Q \cap R_2$ is maximal in R_2 with respect to being disjoint from U. By induction $\dim R_2/(Q \cap R_2)$ is either n or 2. Since R/Q is integral over $R_2/(Q \cap R_2)$, $\dim R/Q$ is either n or 2. $\square$

Remark 25.7. In the notation of Theorem 24.3, Theorem 25.6 proves the second part of the theorem, since $\dim R = n + m$. Thus if $n = 1$, $\operatorname{ht} Q = m$. If $n \geq 2$, the two cases are (i) $\operatorname{ht} Q = m$ and (ii) $\operatorname{ht} Q = n + m - 2$, as in (a) and (b) of Theorem 24.3, part 4.

Using the TGF terminology of Definition 24.6, we have the following corollary to Theorem 25.6.

Corollary 25.8. *With the notation of Theorem 25.6, assume $P \in \operatorname{Spec} R$ is such that $R_1 \hookrightarrow R/P =: S$ is a TGF extension. Then* $\dim S = \dim R_1 = n$ *or* $\dim S = 2$.

25.5. Formal fibers of prime ideals in polynomial rings

In this section, we present a generalization of Theorem 25.4 above. We also discuss in this section related results concerning generic formal fibers of certain extensions of mixed polynomial-power series rings.

We were inspired to revisit and generalize Theorem 25.4 by the following two questions asked by Youngsu Kim, regarding formal fibers.

Questions 25.9. For $n \in \mathbb{N}$, let $x_1, \ldots, x_n$ be indeterminates over the field $\mathbb{C}$ of complex numbers, and let $R = \mathbb{C}[x_1, \ldots, x_n]_{(x_1, \ldots, x_n)}$ denote the localized polynomial ring with maximal ideal $\mathbf{m} = (x_1, \ldots, x_n)R$. Let $\widehat{R}$ be the $\mathbf{m}$-adic completion of R.

(1) What is the dimension of the generic formal fiber ring $\operatorname{Gff}(R)$?
(2) For $P \in \operatorname{Spec} R$, what is the dimension of the generic formal fiber ring $\operatorname{Gff}(R/P)$?

Matsumura in [**95**, Theorem 2 and Corollary, p. 263] answers both of these questions over an arbitrary field k in place of $\mathbb{C}$. The answer to Question 25.9.1 is that $\dim \operatorname{Gff}(R) = n - 1$. In connection with Question 25.9.2, for $P \in \operatorname{Spec} R$, the ring R/P is essentially finitely generated over a field, and Matumura's result states that $\dim \operatorname{Gff}(R/P) = n - 1 - \operatorname{ht} P$. We discuss related work of Matsumura and others in Remarks 24.2.

As a sharpening of Matsumura's result and of Theorem 25.4, we show in Theorem 25.10 that the height of *every* maximal ideal of $\operatorname{Gff}(R/P)$ is $n - 1 - \operatorname{ht} P$.

THEOREM 25.10. *Let S be a local domain essentially finitely generated over a field; thus $S = k[s_1, \dots, s_r]_{\mathbf{p}}$, where k is a field, $r \in \mathbb{N}$, the elements s_i are in S and $\mathbf{p}$ is a prime ideal of the finitely generated k-algebra $k[s_1, \dots, s_r]$. Let $\mathbf{n} := \mathbf{p}S$ and let $\widehat{S}$ denote the $\mathbf{n}$-adic completion of S. Then every maximal ideal of $\operatorname{Gff}(S)$ has height $\dim S - 1$. Equivalently, if $Q \in \operatorname{Spec} \widehat{S}$ is maximal with respect to $Q \cap S = (0)$, then $\operatorname{ht} Q = \dim S - 1$.*

For the proof of Theorem 25.10, see Remark 25.14 and Corollary 25.15.

We use the following observations related to ideal-adic topologies and subspace topologies:

DISCUSSION 25.11. Let $(R, \mathbf{m})$ be a Noetherian local domain and let Q be a prime ideal of $\widehat{R}$ such that $Q \cap R = (0)$. Then there exists an injective local map $\varphi : R \hookrightarrow \widehat{R}/Q$. The map φ extends to a map $\widehat{\varphi} : \widehat{R} \to \widehat{R}/Q$; the map $\widehat{\varphi}$ is the canonical map of $\widehat{R}$ onto $\widehat{R}/Q$. Thus the map $\widehat{\varphi}$ is injective if and only if $Q = (0)$. In the case where $Q \neq (0)$, we explain below why the topology defined on R as a subspace of $\widehat{R}/Q$ is not the same as the $\mathbf{m}$-adic topology on R.

To see these topologies are different if $Q \neq (0)$, we consider more generally an injective local map $\phi : (R, \mathbf{m}) \hookrightarrow (S, \mathbf{n})$ of the Noetherian local ring $(R, \mathbf{m})$ into a Noetherian local ring $(S, \mathbf{n})$. Let $\widehat{R} = \varprojlim_n R/\mathbf{m}^n$ denote the $\mathbf{m}$-adic completion of R and let $\widehat{S} = \varprojlim_n S/\mathbf{n}^n$ denote the $\mathbf{n}$-adic completion of S. For each $n \in \mathbb{N}$, we have $\mathbf{m}^n \subseteq \mathbf{n}^n \cap R$. Hence there exists a map

$$\phi_n : R/\mathbf{m}^n \;\to\; R/(\mathbf{n}^n \cap R) \;\hookrightarrow\; S/\mathbf{n}^n, \qquad \text{for each } \; n \in \mathbb{N}.$$

The family of maps $\{\phi_n\}_{n \in \mathbb{N}}$ determines a unique map $\widehat{\phi} : \widehat{R} \to \widehat{S}$.

Since $\mathbf{m}^n \subseteq \mathbf{n}^n \cap R$, the $\mathbf{m}$-adic topology on R is the subspace topology from S if and only if for each positive integer n there exists a positive integer s_n such that $\mathbf{n}^{s_n} \cap R \subseteq \mathbf{m}^n$. Since $R/\mathbf{m}^n$ is Artinian, the descending chain of ideals $\{\mathbf{m}^n + (\mathbf{n}^s \cap R)\}_{s \in \mathbb{N}}$ stabilizes. The ideal $\mathbf{m}^n$ is closed in the $\mathbf{m}$-adic topology, and it is closed in the subspace topology if and only if $\bigcap_{s \in \mathbb{N}} (\mathbf{m}^n + (\mathbf{n}^s \cap R)) = \mathbf{m}^n$. Hence $\mathbf{m}^n$ is closed in the subspace topology if and only if there exists a positive integer s_n such that $\mathbf{n}^{s_n} \cap R \subseteq \mathbf{m}^n$.

Assume $Q \neq (0)$. To justify the assertion that the topology defined on R as a subspace of $\widehat{R}/Q$ is not the same as the $\mathbf{m}$-adic topology on R, let $\mathbf{n} := \mathbf{m}(\widehat{R}/Q)$ denote the maximal ideal of $\widehat{R}/Q$. Notice that the preimage in $\widehat{R}$ of $\mathbf{n}^s$ is $(\mathbf{m}^s, Q)\widehat{R}$ for each $s \in \mathbb{N}$. Hence $\mathbf{n}^s \cap R = (\mathbf{m}^s, Q)\widehat{R} \cap R$. Fix $n \in \mathbb{N}$ such that $Q \nsubseteq \mathbf{m}^n \widehat{R}$, and choose $s \in \mathbb{N}$ such that $\{\mathbf{m}^n + (\mathbf{n}^s \cap R)\}_{s \in \mathbb{N}}$ stabilizes at s. Since $(\mathbf{m}^t, Q)\widehat{R}$ is primary for the maximal ideal of $\widehat{R}$, for each $t \geq s$, we have

$$(\mathbf{n}^t \;\cap\; R)\widehat{R} \;=\; \big((\mathbf{m}^t, Q)\widehat{R} \;\cap R\big)\widehat{R} \;=\; (\mathbf{m}^t, Q)\widehat{R} \;\nsubseteq\; \mathbf{m}^n \;\widehat{R}.$$

Since these ideals are primary for the maximal ideal of $\widehat{R}$, they are the extensions to $\widehat{R}$ of their contractions to R, and we have

$$(\mathbf{n}^t \;\cap\; R) \;=\; \big((\mathbf{m}^t, Q)\widehat{R} \;\cap R\big) \;\nsubseteq\; \mathbf{m}^n \;,$$

for every $t \in \mathbb{N}$. Thus $\mathbf{m}^n$ is not closed in the subspace topology.

$\mathrm{Gff}(R)$ and $\mathrm{Gff}(S)$ for S an extension domain of R

Theorem 25.12 is useful in considering properties of generic formal fiber rings.

THEOREM 25.12. *Let $\phi : (R, \mathbf{m}) \hookrightarrow (S, \mathbf{n})$ be an injective local map of Noetherian local integral domains. Consider the following properties:*

(1) *$\mathbf{m}S$ is $\mathbf{n}$-primary, and $S/\mathbf{n}$ is finite algebraic over $R/\mathbf{m}$.*
(2) *$R \hookrightarrow S$ is a TGF-extension and $\dim R = \dim S$; see Definition 24.6.*
(3) *R is analytically irreducible.*
(4) *R is analytically normal and S is universally catenary.*
(5) *All maximal ideals of $\mathrm{Gff}(R)$ have the same height.*

If items 1, 2 and 3 hold, then $\dim \mathrm{Gff}(R) = \dim \mathrm{Gff}(S)$. If, in addition, items 4 and 5 hold, then the maximal ideals of $\mathrm{Gff}(S)$ all have height $h = \dim \mathrm{Gff}(R)$.

PROOF. Let $\widehat{R}$ and $\widehat{S}$ denote the $\mathbf{m}$-adic completion of R and $\mathbf{n}$-adic completion of S, and let $\widehat{\phi} : \widehat{R} \to \widehat{S}$ be the natural extension of φ as defined above. Consider the commutative diagram

$$\text{(25.12.a)} \qquad \begin{array}{ccc} \widehat{R} & \xrightarrow{\widehat{\phi}} & \widehat{S} \\ \uparrow & & \uparrow \\ R & \xrightarrow{\phi} & S \end{array},$$

where the vertical maps are the natural inclusion maps to the completion. Assume items 1, 2 and 3 hold. Item 1 implies that $\widehat{S}$ is a finite $\widehat{R}$-module with respect to the map $\widehat{\phi}$, [**96**, Theorem 8.4]. By item 2, we have $\dim \widehat{R} = \dim R = \dim S = \dim \widehat{S}$. Item 3 says that $\widehat{R}$ is an integral domain. It follows that the map $\widehat{\phi} : \widehat{R} \hookrightarrow \widehat{S}$ is injective. Let $Q \in \operatorname{Spec} \widehat{S}$ and let $P = Q \cap \widehat{R}$. Since $R \hookrightarrow S$ is a TGF-extension, by item 2, commutativity of Diagram 25.12.a implies that

$$Q \ \cap \ S \ = \ (0) \quad \Longleftrightarrow \quad P \ \cap \ R \ = \ (0).$$

Therefore $\widehat{\phi}$ induces an injective finite map $\mathrm{Gff}(R) \hookrightarrow \mathrm{Gff}(S)$. We conclude that $\dim \mathrm{Gff}(R) = \dim \mathrm{Gff}(S)$.

Assume in addition that items 4 and 5 hold, and let $h = \dim \mathrm{Gff}(R)$. The assumption that S is universally catenary implies that $\dim(\widehat{S}/\mathbf{q}) = \dim S$ for each minimal prime $\mathbf{q}$ of $\widehat{S}$; see [**96**, Theorem 31.7]. Since $\frac{\widehat{R}}{\mathbf{q} \cap \widehat{R}} \hookrightarrow \frac{\widehat{S}}{\mathbf{q}}$ is an integral extension, we have $\mathbf{q} \cap \widehat{R} = (0)$. The assumption that $\widehat{R}$ is a normal domain implies that the going-down theorem holds for $\widehat{R} \hookrightarrow \widehat{S}/\mathbf{q}$ [**96**, Theorem 9.4(ii)]. Therefore for each $Q \in \operatorname{Spec} \widehat{S}$ we have $\operatorname{ht} Q = \operatorname{ht} P$, where $P = Q \cap \widehat{R}$. Hence if $\operatorname{ht} P = h$ for each $P \in \operatorname{Spec} \widehat{R}$ that is maximal with respect to $P \cap R = (0)$, then $\operatorname{ht} Q = h$ for each $Q \in \operatorname{Spec} \widehat{S}$ that is maximal with respect to $Q \cap S = (0)$. This completes the proof of Theorem 25.12. □

DISCUSSION 25.13. As in the statement of Theorem 25.10, let $S = k[z_1, \ldots, z_r]_{\mathbf{p}}$ be a local domain essentially finitely generated over a field k. We observe that by enlarging the ground field if necessary, we may reduce to the case where S is a localization at a maximal ideal of an integral domain that is a finitely generated algebra over a field.

To see this, let $A = k[x_1, \ldots, x_r]$ be a polynomial ring in r variables over k, and let Q denote the kernel of the k-algebra homomorphism of A onto $k[z_1, \ldots, z_r]$

defined by mapping $x_i \mapsto z_i$ for each i with $1 \le i \le r$. Using permutability of localization and residue class formation, there exists a prime ideal $N \supset Q$ of A such that $S = A_N/QA_N$. A version of Noether normalization as in [**94**, Theorem 24 (14.F) page 89] states that if $\operatorname{ht} N = s$, then there exist elements $y_1, \dots, y_r$ in A such that A is integral over $B = k[y_1, \dots, y_r]$ and $N \cap B = (y_1, \dots, y_s)B$. It follows that $y_1, \dots, y_r$ are algebraically independent over k and A is a finitely generated B-module. Let U denote the multiplicatively closed set $k[y_{s+1}, \dots, y_r] \setminus (0)$, and let F denote the field $k(y_{s+1}, \dots, y_r)$. Then $U^{-1}B$ is the polynomial ring $F[y_1, \dots, y_s]$, and $U^{-1}A := C$ is a finitely generated $U^{-1}B$-module, and NC is a prime ideal of C such that $NC \cap U^{-1}B$ is the maximal ideal $(y_1, \dots, y_s)U^{-1}B = (y_1, \dots, y_s)F[y_1, \dots, y_s]$l. Hence NC is a maximal ideal of C and $S = C_{NC}/QC_{NC}$ is a localization of a finitely generated F-algebra.

Therefore we are reduced to the case where S is a localization of an integral domain C at a maximal ideal of C, and C is a finitely generated algebra over a field.

REMARK 25.14. By Discussion 25.13, in order to prove Theorem 25.10 it suffices to consider the case where S is a localization at a maximal ideal of a finitely generated algebra over a field. Hence to complete the proof of Theorem 25.10, it suffices to prove the following corollary to Theorem 25.12:

COROLLARY 25.15. *Let $A = k[s_1, \dots, s_r]$ be an integral domain that is a finitely generated algebra over a field k, let N be a maximal ideal of A, and let $Q \subset N$ be a prime ideal of A. Set $S = A_N/QA_N$ and $\mathbf{n} = NS$. If $\dim S = d$, then every maximal ideal of the generic formal fiber ring $\operatorname{Gff}(S)$ has height $d-1$.*

PROOF. Choose $x_1, \dots, x_d$ in $\mathbf{n}$ such that $x_1, \dots, x_d$ are algebraically independent over k and $(x_1, \dots, x_d)S$ is $\mathbf{n}$-primary. Set $R = k[x_1, \dots, x_d]_{(x_1, \dots, x_d)}$, a localized polynomial ring over k, and let $\mathbf{m} = (x_1, \dots, x_d)R$. To prove Corollary 25.15, it suffices to show that the inclusion map $\phi : R \hookrightarrow S$ satisfies items 1 - 5 of Theorem 25.12. By construction ϕ is an injective local homomorphism and $\mathbf{m}S$ is $\mathbf{n}$-primary. Also $R/\mathbf{m} = k$ and $S/\mathbf{n} = A/N$ is a field that is a finitely generated k-algebra and hence a finite algebraic extension field of k; see [**96**, Theorem 5.2]. Therefore item 1 holds. Since $\dim S = d = \dim A/Q$, the field of fractions of S has transcendence degree d over the field k. Therefore S is algebraic over R. It follows that $R \hookrightarrow S$ is a TGF extension. Thus item 2 holds. Since R is a regular local ring, R is analytically irreducible and analytically normal. Since S is essentially finitely generated over a field, S is universally catenary. Therefore items 3 and 4 hold. Since R is a localized polynomial ring in d variables, Theorem 25.4 implies that every maximal ideal of $\operatorname{Gff}(R)$ has height $d-1$. By Theorem 25.12, every maximal ideal of $\operatorname{Gff}(S)$ has height $d-1$. □

Other results on generic formal fibers

Theorems 25.1 and 25.3 give descriptions of the generic formal fiber ring of mixed polynomial-power series rings. We use Theorems 25.12, 25.1 and 25.3 to deduce Theorem 25.16.

THEOREM 25.16. *Let R be either $k[[X]]\,[Y]_{(X,Y)}$ or $k[Y]_{(Y)}[[X]]$, where m and n are positive integers and $X = \{x_1, \dots, x_n\}$ and $Y = \{y_1, \dots, y_m\}$ are sets of independent variables over a field k. Let $\mathbf{m}$ denote the maximal ideal $(X,Y)R$ of R. Let $(S, \mathbf{n})$ be a Noetherian local integral domain containing R such that:*

(1) *The injection* $\varphi : (R, \mathbf{m}) \hookrightarrow (S, \mathbf{n})$ *is a local map.*
(2) $\mathbf{m}S$ *is* $\mathbf{n}$*-primary, and* $S/\mathbf{n}$ *is finite algebraic over* $R/\mathbf{m}$.
(3) $R \hookrightarrow S$ *is a TGF-extension and* $\dim R = \dim S$.
(4) S *is universally catenary.*

Then every maximal ideal of the generic formal fiber ring $\mathrm{Gff}(S)$ *has height* $n+m-2$. *Equivalently, if* P *is a prime ideal of* $\widehat{S}$ *maximal with respect to* $P \cap S = (0)$, *then* $\mathrm{ht}(P) = n + m - 2$.

Proof. We check that the conditions 1–5 of Theorem 25.12 are satisfied for R and S and the injection φ. Since the completion of R is $k[[X, Y]]$, R is analytically normal, and so also analytically irreducible. Items 1–4 of Theorem 25.16 ensure that the rest of conditions 1–4 of Theorem 25.12 hold. By Theorems 25.1 and 25.3, every maximal ideal of $\mathrm{Gff}(R)$ has height $n+m-2$, and so condition 5 of Theorem 25.12 holds. Thus by Theorem 25.12, every maximal ideal of $\mathrm{Gff}(S)$ has height $n + m - 2$. □

Remark 25.17. Let k, X, Y, and R be as in Theorem 25.16. Let A be a finite integral extension domain of R and let S be the localization of A at a maximal ideal. As observed in the proof of Theorem 25.16, R is a local analytically normal integral domain. Since S is a localization of a finitely generated R-algebra and R is universally catenary, it follows that S is universally catenary. We also have that conditions 1–3 of Theorem 25.16 hold. Thus the extension $R \hookrightarrow S$ satisfies the hypotheses of Theorem 25.16. Hence every maximal ideal of $\mathrm{Gff}(S)$ has height $n + m - 2$.

Example 25.18 is an application of Theorem 25.16 and Remark 25.17.

Example 25.18. Let k, X, Y, and R be as in Theorem 25.16. Let K denote the field of fractions of R, and let L be a finite algebraic extension field of K. Let A be the integral closure of R in L, and let S be a localization of A at a maximal ideal. The ring R is a Nagata ring by a result of Marot; see [**91**, Prop.3.5]. Therefore A is a finite integral extension of R and the conditions of Remark 25.17 apply to show that every maximal ideal of $\mathrm{Gff}(S)$ has height $n + m - 2$.

Exercise

(1) Let x and y be indeterminates over a field k. Let $R = k[[x]][y]$ and let $\tau \in k[[y]]$ be such that y and τ are algebraically independent over k. Then we have the embedding $R = k[[x]][y] \hookrightarrow k[[x, y]]$. For $P := (x - \tau)k[[x, y]]$, prove that
(a) $P \cap k[x, y] = (0)$, but
(b) $P \cap R \neq (0)$.

Suggestion: For item b, apply Theorem 3.10.

CHAPTER 26

Mixed polynomial-power series rings and relations among their spectra (ppssec), Oct. 29, 2013

We are interested in the following sequence of two-dimensional nested mixed polynomial-power series rings:

$$(26.0.1) \qquad A := k[x,y] \hookrightarrow B := k[[y]]\,[x] \hookrightarrow C := k[x]\,[[y]] \hookrightarrow E := k[x,1/x]\,[[y]],$$

where k is a field and x and y are indeterminates over k. [1] That is, A is the usual polynomial ring in the two variables x and y over k, the ring B is all polynomials in the variable x with coefficients in the power series ring $k[[y]]$, the ring C is all power series in the variable y over the polynomial ring $k[x]$, and E is power series in the variable y over the polynomial ring $k[x,1/x]$. In Sequence 26.0.1 all the maps are flat; see Propositions 2.21.4 and 3.2.2. We also consider Sequence 26.0.2 consisting of embeddings between the rings C and E of Sequence 26.0.1:

$$(26.0.2) \qquad C \hookrightarrow D_1 := k[x]\,[[y/x]] \hookrightarrow \cdots \hookrightarrow D_n := k[x]\,[[y/x^n]] \hookrightarrow \cdots \hookrightarrow E.$$

With regard to Sequence 26.0.2, for n a positive integer, the map $C \hookrightarrow D_n$ is not flat, since $\mathrm{ht}(xD_n \cap C) = 2$ but $\mathrm{ht}(xD_n) = 1$; see Proposition 2.21.10. The map $D_n \hookrightarrow E$ is a localization followed by an ideal-adic completion of a Noetherian ring and therefore is flat. We discuss the spectra of the rings in Sequences 26.0.1 and 26.0.1, and we consider the maps induced on the spectra by the inclusion maps on the rings. For example, we determine whether there exist nonzero primes of one of the larger rings that intersect a smaller ring in zero.

26.1. Two motivations

We were led to consider these rings by questions that came up in two contexts.

The first motivation is a question about formal schemes that is discussed in the introduction to the paper [**8**] by Alonzo-Tarrio, Jeremias-Lopez and Lipman:

QUESTION 26.1. If a map between Noetherian formal schemes can be factored as a closed immersion followed by an open immersion, can this map also be factored as an open immersion followed by a closed immersion? [2]

Brian Conrad observed that an example to show the answer to Question 26.1 is "No" can be constructed for every triple $(R, x, \mathbf{p})$ that satisfies the following three conditions; see [**8**]:

[1] The material in this chapter is adapted from our article [**69**] dedicated to Robert Gilmer, an outstanding algebraist, scholar and teacher.

[2] See Scheme Terminology 26.3 for a brief explanation of this terminology.

(26.1.1) R is an ideal-adic domain, that is, R is a Noetherian domain that is separated and complete with respect to the powers of a proper ideal I.

(26.1.2) x is a nonzero element of R such that the completion of $R[1/x]$ with respect to the powers of $IR[1/x]$, denoted $S := R_{\{x\}}$, is an integral domain.

(26.1.3) $\mathbf{p}$ is a nonzero prime ideal of S that intersects R in (0).

The following example of such a triple $(R, x, \mathbf{p})$ is described in [**8**]:

EXAMPLE 26.2. Let w, x, y, z be indeterminates over a field k. Let

$$R := k[w, x, z]\,[[y]] \quad \text{and} \quad S := k[w, x, 1/x, z]\,[[y]].$$

Notice that R is complete with respect to yR and S is complete with respect to yS. An indirect proof that there exist nonzero primes p of S for which $p \cap R = (0)$ is given in the paper [**8**] of Lipman, Alonzo-Tarrio and Jeremias-Lopez, using a result of Heinzer and Rotthaus [**52**, Theorem 1.12, p. 364]. A direct proof is given in [**69**, Proposition 4.9]. In Proposition 26.30 below we give a direct proof of a more general result due to Dumitrescu [**29**, Corollary 4].

In Scheme Terminology 26.3 we explain some of the terminology of formal schemes necessary for understanding Question 26.1; more details may be found in [**45**]. In Remark 26.4 we explain why a triple satisfying (26.1.1) to (26.1.3) yields examples that answer Question 26.1.

SCHEME TERMINOLOGY 26.3. Let R be a Noetherian integral domain and let K be its field of fractions. Let X denote the topological space $\operatorname{Spec} R$ with the Zariski topology defined in Section 2.1. We form a *sheaf*, denoted $\mathcal{O}$, on X by associating, to each open set U of X, the ring

$$\mathcal{O}(U) \;=\; \bigcap_{x \,\in\, U} R_{\mathfrak{p}_x},$$

where $\mathfrak{p}_x$ is the prime associated to the point $x \in U$; see [**128**, p. 235 and Theorem 1, p. 238]. For each pair $U \subseteq V$ of open subsets of X, there exists a natural inclusion map $\rho_U^V : \mathcal{O}(V) \hookrightarrow \mathcal{O}(U)$. The "ringed space" $(X, \mathcal{O})$ is identified with $\operatorname{Spec} R$ and is called an *affine scheme*; see [**128**, p. 242-3], [**45**, Definition I.10.1.2, p. 402]. Assume that $R = R^*$ is complete with respect to the I-adic topology, where I is a nonzero proper ideal of R (see Definition 3.1). Then the ringed space $(X, \mathcal{O})$ is denoted Spf (R) and is called the *formal spectrum* of R. It is also called a *Noetherian formal adic affine scheme*; see [**45**, I.10.1.7, p. 403]. An *immersion* is a morphism $f : Y \to X$ of schemes that factors as an isomorphism to a subscheme Z of X followed by a canonical injection $Z \to X$; see [**45**, (I.4.2.1)].

REMARK 26.4. Assume, in addition to R being a Noetherian integral domain complete with respect to the I-adic topology, that x is a nonzero element of R, that S is the completion of $R[1/x]$ with respect to the powers of $IR[1/x]$, and that $\mathbf{p}$ is a prime ideal of S such that the triple $(R, x, \mathbf{p})$ satisfies the three conditions 26.1.1 to 26.1.3.

The composition of the maps $R \to S \to S/\mathbf{p}$ determines a map on formal spectra Spf $(S/\mathbf{p}) \to$ Spf $(S) \to$ Spf (R) that is a closed immersion followed by an open immersion. This is because a surjection such as $S \to S/\mathbf{p}$ of adic rings gives rise to a closed immersion Spf $(S/\mathbf{p}) \to$ Spf (S) while a localization, such as that of R with respect to the powers of x, followed by the completion of $R[1/x]$

with respect to the powers of $IR[1/x]$ to obtain S gives rise to an open immersion Spf $(S) \to$ Spf (R) [**45**, I.10.14.4].

The map Spf $(S/\mathbf{p}) \to$ Spf (R) cannot be factored as an open immersion followed by a closed one. This is because a closed immersion into Spf (R) corresponds to a surjective map of adic rings $R \to R/J$, where J is an ideal of R [**45**, page 441]. Thus if the map Spf $(S/\mathbf{p}) \to$ Spf (R) factored as an open immersion followed by a closed one, we would have R-algebra homomorphisms from $R \to R/J \to S/\mathbf{p}$, where Spf $(S/\mathbf{p}) \to$ Spf (R/J) is an open immersion. Since $\mathbf{p} \cap R = (0)$, we must have $J = (0)$. This implies Spf $(S/\mathbf{p}) \to$ Spf (R) is an open immersion, that is, the composite map Spf $(S/\mathbf{p}) \to$ Spf $(S) \to$ Spf (R), is an open immersion. But also Spf $(S) \to$ Spf (R) is an open immersion. It follows that Spf $(S/\mathbf{p}) \to$ Spf (S) is both open and closed. Since S is an integral domain this implies Spf $(S/\mathbf{p}) \cong$ Spf (S). Since $\mathbf{p}$ is nonzero, this is a contradiction. Thus Example 26.2 shows that the answer to Question 26.1 is "No".

The second motivation for the material in this chapter comes from Question 24.4 of Melvin Hochster and Yongwei Yao "Can one describe or somehow classify the local maps $R \hookrightarrow S$ of complete local domains R and S such that every nonzero prime ideal of S has nonzero intersection with R?" The following example is a local map of the type described in the Hochster-Yao question.

Example 26.5. Let x and y be indeterminates over a field k and consider the extension $R := k[[x,y]] \hookrightarrow S := k[[x]]\,[[y/x]]$.

To see this extension is TGF—the "trivial generic fiber" condition of Definition 24.6, it suffices to show $P \cap R \neq (0)$ for each $P \in \operatorname{Spec} S$ with $\operatorname{ht} P = 1$. This is clear if $x \in P$, while if $x \notin P$, then $k[[x]] \cap P = (0)$, and so $k[[x]] \hookrightarrow R/(P \cap R) \hookrightarrow S/P$ and S/P is finite over $k[[x]]$. Therefore $\dim R/(P \cap R) = 1$, and so $P \cap R \neq (0)$.

Definition 24.6 is related to Question 26.1. If a ring R and a nonzero element x of R satisfies conditions 26.1.1 and 26.1.2, then condition 26.1.3 simply says that the extension $R \hookrightarrow R_{\{x\}}$ is *not* TGF.

In some correspondence to Lipman regarding Question 26.1, Conrad asked: "Is there a nonzero prime ideal of $E := k[x, 1/x]\,[[y]]$ that intersects $C = k[x]\,[[y]]$ in zero?" If there were such a prime ideal $\mathbf{p}$, then the triple $(C, x, \mathbf{p})$ would satisfy conditions 26.1.1 to 26.1.3. This would yield a two-dimensional example to show the answer to Question 26.1 is "No". Thus one can ask:

Question 26.6. Let x and y be indeterminates over a field k. Is the extension $C := k[x]\,[[y]] \hookrightarrow E := k[x, 1/x]\,[[y]]$ TGF?

We show in Proposition 26.12.2 below that the answer to Question 26.6 is "Yes"; thus the triple $(C, x, \mathbf{p})$ does not satisfy condition 26.1.3, although it does satisfy conditions 26.1.1 and 26.1.2. This is part of our analysis of the prime spectra of A, B, C, D_n and E, and the maps induced on these spectra by the inclusion maps on the rings.

Remarks 26.7. (1) The extension $k[[x,y]] \hookrightarrow k[[x, y/x]]$ is, up to isomorphism, the same as the extension $k[[x, xy]] \hookrightarrow k[[x,y]]$.

(2) We show in Chapter **??** that the extension $R := k[[x, y, xz]] \hookrightarrow S := k[[x, y, z]]$ is not TGF. We also give more information about TGF extensions of local rings there.

(3) Takehiko Yasuda gives additional information on the TGF property in [**145**]. In particular, he shows that

$$\mathbb{C}[x,y][[z]] \hookrightarrow \mathbb{C}[x,x^{-1},y][[z]]$$

is not TGF, where $\mathbb{C}$ is the field of complex numbers [**145**, Theorem 2.7].

26.2. Trivial generic fiber (TGF) extensions and prime spectra

We record in Proposition 26.8 several basic facts about TGF extensions. We omit the proofs since they are straightforward.

Proposition 26.8. *Let $R \hookrightarrow S$ and $S \hookrightarrow T$ be injective maps where R, S and T are integral domains.*

1. *If $R \hookrightarrow S$ and $S \hookrightarrow T$ are TGF extensions, then so is the composite map $R \hookrightarrow T$. Equivalently if the composite map $R \hookrightarrow T$ is not TGF, then at least one of the extensions $R \hookrightarrow S$ or $S \hookrightarrow T$ is not TGF.*
2. *If $R \hookrightarrow T$ is TGF, then $S \hookrightarrow T$ is TGF.*
3. *If the map $\operatorname{Spec} T \to \operatorname{Spec} S$ is surjective and $R \hookrightarrow T$ is TGF, then $R \hookrightarrow S$ is TGF.*

We use the following remark about prime ideals in a formal power series ring.

Remarks 26.9. Let R be a commutative ring and let $R[[y]]$ denote the formal power series ring in the variable y over R. Then

1. Each maximal ideal of $R[[y]]$ is of the form $(\mathbf{m}, y)R[[y]]$ where $\mathbf{m}$ is a maximal ideal of R. Thus y is in every maximal ideal of $R[[y]]$.
2. If R is Noetherian with $\dim R[[y]] = n$ and $x_1, \dots, x_m$ are independent indeterminates over $R[[y]]$, then y is in every height $n+m$ maximal ideal of the polynomial ring $R[[y]]\,[x_1, \dots, x_m]$.

Proof. Item 1 follows from [**104**, Theorem 15.1]. For item 2, let $\mathbf{m}$ be a maximal ideal of $R[[y]]\,[x_1, \dots, x_m]$ with $\operatorname{ht}(\mathbf{m}) = n+m$. By [**78**, Theorem 39], $\operatorname{ht}(\mathbf{m} \cap R[[y]]) = n$; thus $\mathbf{m} \cap R[[y]]$ is maximal in $R[[y]]$, and so, by item 1, $y \in \mathbf{m}$. □

Proposition 26.10. *Let n be a positive integer, let R be an n-dimensional Noetherian domain, let y be an indeterminate over R, and let $\mathbf{q}$ be a prime ideal of height n in the power series ring $R[[y]]$. If $y \notin \mathbf{q}$, then $\mathbf{q}$ is contained in a unique maximal ideal of $R[[y]]$.*

Proof. The assertion is clear if $\mathbf{q}$ is maximal. Otherwise $S := R[[y]]/\mathbf{q}$ has dimension one. Moreover, S is complete with respect to the yS-adic topology [**96**, Theorem 8.7] and every maximal ideal of S is a minimal prime of the principal ideal yS. Hence S is a complete semilocal ring. Since S is also an integral domain, it must be local by [**96**, Theorem 8.15]. Therefore $\mathbf{q}$ is contained in a unique maximal ideal of $R[[y]]$. □

In Section 26.3 we use the following corollary to Proposition 26.10.

Corollary 26.11. *Let R be a one-dimensional Noetherian domain and let $\mathbf{q}$ be a height-one prime ideal of the power series ring $R[[y]]$. If $\mathbf{q} \neq yR[[y]]$, then $\mathbf{q}$ is contained in a unique maximal ideal of $R[[y]]$.*

PROPOSITION 26.12. *Consider the nested mixed polynomial-power series rings:*

$$\begin{aligned} A := k[x,y] \hookrightarrow B := k[[y]]\,[x] \hookrightarrow C := k[x]\,[[y]] \\ \hookrightarrow D_1 := k[x]\,[[y/x]] \hookrightarrow D_2 := k[x]\,[[y/x^2]] \hookrightarrow \cdots \\ \hookrightarrow D_n := k[x]\,[[y/x^n]] \hookrightarrow \cdots \hookrightarrow E := k[x,1/x]\,[[y]], \end{aligned}$$

where k is a field and x and y are indeterminates over k. Then

(1) *If $S \in \{B, C, D_1, D_2, \cdots, D_n, \cdots, E\}$, then $A \hookrightarrow S$ is not TGF.*
(2) *If $\{R, S\} \subset \{B, C, D_1, D_2, \cdots, D_n \cdots, E\}$ are such that $R \subseteq S$, then $R \hookrightarrow S$ is TGF.*
(3) *Each of the proper associated maps on spectra fails to be surjective.*

PROOF. For item 1, let $\sigma(y) \in yk[[y]]$ be such that $\sigma(y)$ and y are algebraically independent over k. Then $(x - \sigma(y))S \cap A = (0)$, and so $A \hookrightarrow S$ is not TGF.

For item 2, observe that every maximal ideal of C, D_n or E is of height two with residue field finite algebraic over k. To show $R \hookrightarrow S$ is TGF, it suffices to show $\mathbf{q} \cap R \neq (0)$ for each height-one prime ideal $\mathbf{q}$ of S. This is clear if $y \in \mathbf{q}$. If $y \notin \mathbf{q}$, then $k[[y]] \cap \mathbf{q} = (0)$, and so $k[[y]] \hookrightarrow R/(\mathbf{q} \cap R) \hookrightarrow S/q$ are injections. By Corollary 26.11, $S/\mathbf{q}$ is a one-dimensional local domain. Since the residue field of $S/\mathbf{q}$ is finite algebraic over k, it follows that $S/\mathbf{q}$ is finite over $k[[y]]$. Therefore $S/\mathbf{q}$ is integral over $R/(\mathbf{q} \cap R)$. Hence $\dim(R/(\mathbf{q} \cap R) = 1$ and so $\mathbf{q} \cap R \neq (0)$.

For item 3, observe that xD_n is a prime ideal of D_n and x is a unit of E. Thus $\operatorname{Spec} E \to \operatorname{Spec} D_n$ is not surjective. Now, considering $C = D_0$ and $n > 0$, we have $xD_n \cap D_{n-1} = (x, y/x^{n-1})D_{n-1}$. Therefore xD_{n-1} is not in the image of the map $\operatorname{Spec} D_n \to \operatorname{Spec} D_{n-1}$. The map from $\operatorname{Spec} C \to \operatorname{Spec} B$ is not onto, because $(1+xy)$ is a prime ideal of B, but $1+xy$ is a unit in C. Similarly $\operatorname{Spec} B \to \operatorname{Spec} A$ is not onto, because $(1+y)$ is a prime ideal of A, but $1+y$ is a unit in B. This completes the proof. ☐

QUESTION AND REMARKS 26.13. Which of the Spec maps of Proposition 26.12 are one-to-one and which are finite-to-one?

(1) For $S \in \{B, C, D_1, D_2, \cdots, D_n, \cdots, E\}$, the generic fiber ring of the map $A \hookrightarrow S$ has infinitely many prime ideals and has dimension one. Every height-two maximal ideal of S contracts in A to a maximal ideal. Every maximal ideal of S containing y has height two. Also $yS \cap A = yA$ and the map $\operatorname{Spec} S/yS \to \operatorname{Spec} A/yA$ is one-to-one.
(2) Suppose $R \hookrightarrow S$ is as in Proposition 26.12.2. Each height-two prime of S contracts in R to a height-two maximal ideal of R. Each height-one prime of R is the contraction of at most finitely many prime ideals of S and all of these prime ideals have height one. If $R \hookrightarrow S$ is flat, which is true if $S \in \{B, C, E\}$, then "going-down" holds for $R \hookrightarrow S$, and so, for P a height-one prime of S, we have $\operatorname{ht}(P \cap R) \leq 1$.
(3) As mentioned in [**73**, Remark 1.5], C/P is Henselian for every nonzero prime ideal P of C other than yC.

26.3. Spectra for two-dimensional mixed polynomial-power series rings

Let x and y be indeterminates over a field k. We consider the prime spectra, as partially ordered sets, of the mixed polynomial-power series rings A, B, C,

$D_1, D_2, \cdots, D_n, \cdots$ and E as given in Sequences 26.0.1 and 26.0.2 at the beginning of this chapter.

Even for k a countable field there are at least two non-order-isomorphic partially ordered sets that can be the prime spectrum of the polynomial ring $A := k[x, y]$. Let $\mathbb{Q}$ be the field of rational numbers, let F be a field contained in the algebraic closure of a finite field and let $\mathbb{Z}$ denote the ring of integers. Then, by [**141**] and [**142**], $\operatorname{Spec} \mathbb{Q}[x, y] \ncong \operatorname{Spec} F[x, y] \cong \operatorname{Spec} \mathbb{Z}[y]$.

The prime spectra of the rings B, C, $D_1, \cdots, D_n, \cdots$, and E of Sequences 26.0.1 and 26.0.2 are simpler since they involve power series in y. Remark 26.9.2 implies that y is in every maximal ideal of height two of each of these rings.

The partially ordered set $\operatorname{Spec} B = \operatorname{Spec}(k[[y]]\,[x])$ is similar to a prime ideal space studied by Heinzer and S. Wiegand in the countable case in [**73**] and then generalized by Shah to other cardinalities in [**129**]. The ring $k[[y]]$ is uncountable, even if k is countable. It follows that $\operatorname{Spec} B$ is also uncountable. The partially ordered set $\operatorname{Spec} B$ can be described uniquely up to isomorphism by the axioms of [**129**] (similar to the CHP axioms of [**73**]), since $k[[y]]$ is Henselian and has cardinality at least equal to c, the cardinality of the real numbers $\mathbb{R}$. [3]

Theorem 26.14 characterizes $U := \operatorname{Spec} B$, for the ring B of Sequence 26.01, as a *Henselian affine* partially ordered set (where the "$\leq$" relation is "set containment").

THEOREM 26.14. [**73**, Theorem 2.7] [**129**, Theorem 2.4] *Let $B = k[[y]]\,[x]$ be as in Sequence 26.0.1, where k is a field, the cardinality of the set of maximal ideals of $k[x]$ is α and the cardinality of $k[[y]]$ is β. Then the partially ordered set $U := \operatorname{Spec} B$ under containment is called Henselian affine of type (β, α) and is characterized as a partially ordered set by the following axioms:*

(1) $|U| = \beta$.
(2) *U has a unique minimal element.*
(3) $\dim(U) = 2$ *and* $|\{$ *height-two elements of* $U\}| = \alpha$.
(4) *There exists a unique special height-one element $u \in U$ such that u is less than every height-two element of U.*
(5) *Every nonspecial height-one element of U is less than at most one height-two element.*
(6) *Every height-two element $t \in U$ is greater than exactly β height-one elements such that t is the unique height-two element above each. If $t_1, t_2 \in U$ are distinct height-two elements, then the special element from (4) is the unique height-one element less than both.*
(7) *There are exactly β height-one elements that are maximal.*

REMARK 26.15. (1) The axioms of Theorem 26.14 are redundant. We feel this redundancy helps in understanding the relationships among the prime ideals.

(2) The theorem applies to the spectrum of B by defining the unique minimal element to be the ideal (0) of B and the special height-one element to be the prime ideal yB. Every height-two maximal ideal $\mathbf{m}$ of B has nonzero intersection with $k[[y]]$. Thus $\mathbf{m}/yB$ is principal and so $\mathbf{m} = (y, f(x))$, for some monic irreducible polynomial $f(x)$ of $k[x]$. Consider $\{f(x) + ay \mid a \in k[[y]]\}$. This set has cardinality

[3] Kearnes and Oman observe in [**80**] that some cardinality arguments are incomplete in the paper [**129**]. R. Wiegand and S. Wiegand show that Shah's results are correct in [**144**]. In Remarks 26.15.2 we give proofs of some items of Theorem 26.14.

β and each $f(x) + ay$ is contained in a nonempty finite set of height-one primes contained in $\mathbf{m}$. If $\mathbf{p}$ is a height-one prime contained in $\mathbf{m}$ with $\mathbf{p} \neq yB$, then $\mathbf{p} \cap k[[y]] = (0)$, and so $\mathbf{p}k((y))[x]$ is generated by a monic polynomial in $k((y))[x]$. But for $a, b \in k[[y]]$ with $a \neq b$, we have $(f(x) + ay, f(x) + by)k((y))[x] = k((y))[x]$. Therefore no height-one prime contained in $\mathbf{m}$ contains both $f(x)+ay$ and $f(x)+by$. Since B is Noetherian and $|B| = \beta$ is an infinite cardinal, we conclude that the cardinality of the set of height-one prime ideals contained in $\mathbf{m}$ is β. Examples of height-one maximals are $(1 + xyf(x, y)\,)$, for various $f(x, y) \in k[[y]]\,[x]$. The set of height-one maximal ideals of B also has cardinality β.

(3) We observe that $\alpha = |k| \cdot \aleph_0$ and $\beta = |k|^{\aleph_0}$ in Theorem 26.14, where $\aleph_0 = |\mathbb{N}|$. The proof of this is Exercise 26.1.

(4) The axioms given in Theorem 26.14 *characterize* $\operatorname{Spec} B$ in the sense that every two partially ordered sets satisfying these axioms are order-isomorphic.

The picture of $\operatorname{Spec} B$ is shown below:

$\bullet \qquad \bullet \qquad \bullet \qquad \cdots$

$(\#\{\text{ bullets}\} = \alpha)$

$\boxed{\beta} \qquad (y) \qquad \boxed{\beta} \qquad \boxed{\beta} \qquad \cdots$

(0)

$\operatorname{Spec} k[[y]]\,[x]$

In the diagram, β is the cardinality of $k[[y]]$, and α is the cardinality of the set of maximal ideals of $k[x]$ (and also the cardinality of the set of maximal ideals of $k[[y]]\,[x]\,$); the boxed β means there are cardinality β height-one primes in that position with respect to the partial ordering.

Next we consider $\operatorname{Spec} R[[y]]$, for R a Noetherian one-dimensional domain. Then $\operatorname{Spec} R[[y]]$ has the following picture by Theorem 26.18 below:

$\bullet \qquad \bullet \qquad \bullet \qquad \cdots$

$(\#\{\text{ bullets}\} = \alpha)$

$(y) \qquad \boxed{\beta} \qquad \boxed{\beta} \qquad \cdots$

(0)

$$\mathrm{Spec}(R[[y]])$$

Here α is the cardinality of the set of maximal ideals of R (and also the cardinality of the set of maximal ideals of $R[[y]]$ by Remark 26.9.1) and β is the cardinality (uncountable) of $R[[y]]$; the boxed β (one for each maximal ideal of R) means that there are exactly β prime ideals in that position.

We give the following lemma and add some more arguments in order to justify the cardinalities that occur in the spectra of power series rings more precisely.

LEMMA 26.16. [**144**, Lemma 4.2] *Let T be a Noetherian domain, y an indeterminate and I a proper ideal of T. Let $\delta = |T|$ and $\gamma = |T/I|$. Then $\delta \leq \gamma^{\aleph_0}$, and $|T[[y]]| = \delta^{\aleph_0} = \gamma^{\aleph_0}$.*

PROOF. The first equality holds by Exercise 26.1. Since $\gamma \leq \delta$, we have $\delta^{\aleph_0} \geq \gamma^{\aleph_0}$. For the reverse inequality, the Krull Intersection Theorem [**96**, Theorem 8.10 (ii)] implies that $\bigcap_{n\geq 1} I^n = 0$. Therefore there is a monomorphism

$$R \hookrightarrow \prod_{n\geq 1} R/I^n. \tag{26.1}$$

Now R/I^n has a finite filtration with factors I^{r-1}/I^r for each r with $1 \leq r \leq n$. Since I^{r-1}/I^r is a finitely generated (R/I)-module, $|I^{r-1}/I^r| \leq \gamma^{\aleph_0}$. Therefore $|R/I^n| \leq (\gamma^{\aleph_0})^n = \gamma^{\aleph_0}$, for each n. Thus (26.1) implies $\delta \leq (\gamma^{\aleph_0})^{\aleph_0} = \gamma^{(\aleph_0^2)} = \gamma^{\aleph_0}$. Finally, $\delta^{\aleph_0} \leq (\gamma^{\aleph_0})^{\aleph_0} = \gamma^{\aleph_0}$, and so $\delta^{\aleph_0} = \gamma^{\aleph_0}$. $\square$

The following remarks, observed in the article [**144**] of R. Wiegand and S. Wiegand, are helpful for establishing the cardinaliies in Theorem 26.18.

REMARKS 26.17. Let $\aleph_0$ denote the cardinality of the set of natural numbers. Suppose that T is a commutative ring of cardinality δ, that $\mathbf{m}$ is a maximal ideal of T and that γ is the cardinality of $T/\mathbf{m}$. Then:

(1) The cardinality of $T[[y]]$ is $\delta^{\aleph_0}$, by Lemma 26.16 and Exercise 26.1. If T is Noetherian, then $T[[y]]$ is Noetherian, and so every prime ideal of $T[[y]]$ is finitely generated. Since the cardinality of the finite subsets of $T[[y]]$ is $\delta^{\aleph_0}$, it follows that $T[[y]]$ has at most $\delta^{\aleph_0}$ prime ideals.

(2) If T is Noetherian, then there are at least $\gamma^{\aleph_0}$ distinct height-one prime ideals (other than $(y)T[[y]]$) of $T[[y]]$ contained in $(\mathbf{m}, y)T[[y]]$. To see this, choose a set $C = \{c_i \,|\, i \in I\}$ of elements of T so that $\{c_i + \mathbf{m} \,|\, i \in I\}$ gives the distinct coset representatives for $T/\mathbf{m}$. Thus there are γ elements of C, and for $c_i, c_j \in C$ with $c_i \neq c_j$, we have $c_i - c_j \notin \mathbf{m}$. Now also let $a \in \mathbf{m}, a \neq 0$. Consider the set

$$G = \{a + \sum_{n\in\mathbb{N}} d_n y^n \,|\, d_n \in C \,\forall n \in \mathbb{N}\}.$$

Each of the elements of G is in $(\mathbf{m}, y)T[[y]] \setminus yT[[y]]$ and hence is contained in a height-one prime contained in $(\mathbf{m}, y)T[[y]]$ distinct from $yT[[y]]$.

Moreover, $|G| = |C|^{\aleph_0} = \gamma^{\aleph_0}$. Let P be a height-one prime ideal of $T[[y]]$ contained in $(\mathbf{m}, y)T[[y]]$ but such that $y \notin P$. If two distinct elements of G, say $f = a + \sum_{n\in\mathbb{N}} d_n y^n$ and $g = a + \sum_{n\in\mathbb{N}} e_n y^n$, with the $d_n, e_n \in C$, are both in P, then so is their difference; that is

$$f - g = \sum_{n\in\mathbb{N}} d_n y^n - \sum_{n\in\mathbb{N}} e_n y^n = \sum_{n\in\mathbb{N}} (d_n - e_n) y^n \in P.$$

Now let t be the smallest power of y so that $d_t \neq e_t$. Then $(f-g)/y^t \in P$, since P is prime and $y \notin P$, but the constant term, $d_t - e_t \notin \mathbf{m}$, which contradicts the fact that $P \subseteq (\mathbf{m}, y)T[[y]]$. Thus there must be at least $|C|^{\aleph_0} = \gamma^{\aleph_0}$ distinct height-one primes contained in $(\mathbf{m}, y)T[[y]]$.

(3) If T is Noetherian, then there are exactly $\gamma^{\aleph_0} = \delta^{\aleph_0}$ distinct height-one prime ideals (other than $yT[[y]]$) of $T[[y]]$ contained in $(\mathbf{m}, y)T[[y]]$. This follows from (1) and (2) and Lemma 26.16.

THEOREM 26.18. [**69**] [**144**] *Let R be a one-dimensional Noetherian domain with cardinality δ, let $\beta = \delta^{\aleph_0}$ and let α be the cardinality of the set of maximal ideals of R, where α may be finite. Let $U = \operatorname{Spec} R[[y]]$, where y is an indeterminate over R. Then U as a partially ordered set (where the "$\leq$" relation is "set containment") satisfies the following axioms:*

(1) $|U| = \beta$.
(2) *U has a unique minimal element, namely* (0).
(3) $\dim(U) = 2$ *and* $|\{$ *height-two elements of* $U\}| = \alpha$.
(4) *There exists a unique special height-one element $u \in U$ (namely $u = (y)$) such that u is less than every height-two element of U.*
(5) *Every nonspecial height-one element of U is less than exactly one height-two element.*
(6) *Every height-two element $t \in U$ is greater than exactly β height-one elements that are less than only t. If $t_1, t_2 \in U$ are distinct height-two elements, then the special element from (4) is the unique height-one element less than both.*
(7) *There are no height-one maximal elements in U. Every maximal element has height two.*

The set U is characterized as a partially ordered set by the axioms 1-7. Every partially ordered set satisfying the axioms 1-7 is isomorphic to every other such partially ordered set.

PROOF. Item 1 follows from Remarks 26.17.1 and 26.17.3. Item 2 and the first part of item 3 are clear. The second part of item 3 follows immediately from Remark 26.9.1.

For items 4 and 5, suppose that P is a height-one prime of $R[[y]]$. If $P = yR[[y]]$, then P is contained in each maximal ideal of $R[[y]]$ by Remark 26.9.1, and so $yR[[y]]$ is the special element. If $y \notin P$, then, by Corollary 26.11, P is contained in a unique maximal ideal of $R[[y]]$.

For item 6, use Remarks 26.17.2 and 26.17.3.

All partially ordered sets satisfying the axioms of Theorem 26.14 are order-isomorphic, and the partially ordered set U of the present theorem satisfies the same axioms as in Theorem 26.14 except axiom (7) that involves height-one maximals. Since U has no height-one maximals, an order-isomorphism between two partially ordered sets as in Theorem 26.18 can be deduced by adding on height-one maximals and then deleting them. □

COROLLARY 26.19. *In the terminology of Sequences 26.0.1 and 26.0.2 at the beginning of this chapter, we have $\operatorname{Spec} C \cong \operatorname{Spec} D_n \cong \operatorname{Spec} E$, but $\operatorname{Spec} B \ncong \operatorname{Spec} C$.*

PROOF. The rings C, D_n, and E are all formal power series rings in one variable over a one-dimensional Noetherian domain R, where R is either $k[x]$ or $k[x, 1/x]$.

Thus the domain R satisfies the hypotheses of Theorem 26.18. Also the number of maximal ideals is the same for C, D_n, and E, because in each case, it is the same as the number of maximal ideals of R which is $|k[x]| = |k| \cdot \aleph_0$.

Thus in the picture of $R[[y]]$ shown above, for $R[[y]] = C, D_n$ or E, we have $\alpha = |k| \cdot \aleph_0$ and $\beta = |R[[y]]| = |R|^{\aleph_0}$, and so the spectra are isomorphic. The spectrum of B is not isomorphic to that of C, however, because B contains height-one maximal ideals, such as that generated by $1 + xy$, whereas C has no height-one maximal ideals. $\square$

Remark 26.20. As mentioned at the beginning of this section, it is shown in [**141**] and [**142**] that $\operatorname{Spec} \mathbb{Q}[x,y] \not\cong \operatorname{Spec} F[x,y] \cong \operatorname{Spec} \mathbb{Z}[y]$, where F is a field contained in the algebraic closure of a finite field. Corollary 26.21 shows that the spectra of power series extensions in y behave differently in that $\operatorname{Spec} \mathbb{Z}[[y]] \cong \operatorname{Spec} \mathbb{Q}[x]\,[[y]] \cong \operatorname{Spec} F[x]\,[[y]]$.

Corollary 26.21. *If $\mathbb{Z}$ is the ring of integers, $\mathbb{Q}$ is the rational numbers, F is a field contained in the algebraic closure of a finite field, and $\mathbb{R}$ is the real numbers, then*

$$\operatorname{Spec} \mathbb{Z}[[y]] \cong \operatorname{Spec} \mathbb{Q}[x]\,[[y]] \cong \operatorname{Spec} F[x]\,[[y]] \not\cong \operatorname{Spec} \mathbb{R}[x]\,[[y]].$$

Proof. The rings $\mathbb{Z}, \mathbb{Q}[x]$ and $F[x]$ are all countable with countably infinitely many maximal ideals. Thus if $R = \mathbb{Z}, \mathbb{Q}[x]$ or $F[x]$, then R satisfies the hypotheses of Theorem 26.18 with the cardinality conditions of parts (b) and (c). On the other hand, $\mathbb{R}[x]$ has uncountably many maximal ideals; thus $\mathbb{R}[x]\,[[y]]$ also has uncountably many maximal ideals. $\square$

26.4. Higher dimensional mixed polynomial-power series rings

In analogy to Sequence 26.0.1, we display several embeddings involving three variables.

(26.4.0.1)

$$k[x,y,z] \xhookrightarrow{\alpha} k[[z]]\,[x,y] \xhookrightarrow{\beta} k[x]\,[[z]]\,[y] \xhookrightarrow{\gamma} k[x,y]\,[[z]] \xhookrightarrow{\delta} k[x]\,[[y,z]],$$

$$k[[z]]\,[x,y] \xhookrightarrow{\epsilon} k[[y,z]]\,[x] \xhookrightarrow{\zeta} k[x]\,[[y,z]] \xhookrightarrow{\eta} k[[x,y,z]],$$

where k is a field and x, y and z are indeterminates over k.

Remarks 26.22. (1) By Proposition 26.12.2 every nonzero prime ideal of $C = k[x]\,[[y]]$ has nonzero intersection with $B = k[[y]]\,[x]$. In three or more variables, however, the analogous statements fail. We show below that the maps $\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta$ in Sequence 26.4.0.1 fail to be TGF. Thus, by Proposition 26.8.2, no proper inclusion in Sequence 26.4.0.1 is TGF. The dimensions of the generic fiber rings of the maps in the diagram are either one or two.

(2) For those rings in Sequence 26.4.0.1 of form $R = S[[z]]$ (ending in a power series variable) where S is a ring, such as $R = k[x,y][[z]]$, we have some information concerning the prime spectra. By Proposition 26.10 every height-two prime ideal not containing z is contained in a unique maximal ideal. By [**104**, Theorem 15.1] the maximal ideals of $S[[z]]$ are of the form $(\mathbf{m}, z)S[[z]]$, where $\mathbf{m}$ is a maximal ideal of S, and thus the maximal ideals of $S[[z]]$ are in one-to-one correspondence with the maximal ideals

of S. As in section 26.3, using Remarks 26.9, we see that maximal ideals of $\operatorname{Spec} k[[z]]\,[x,y]$ can have height two or three, that (z) is contained in every height-three prime ideal, and that every height-two prime ideal not containing (z) is contained in a unique maximal ideal.

(3) It follows by arguments analogous to that in Proposition 26.12.1, that α, δ, ϵ are not TGF. For α, let $\sigma(z) \in zk[[z]]$ be transcendental over $k(z)$; then $(x-\sigma)k[[z]]\,[x,y] \cap k[x,y,z] = (0)$. For δ and ϵ: let $\sigma(y) \in k[[y]]$ be transcendental over $k(y)$; then $(x-\sigma)k[x]\,[[z,y]] \cap k[x]\,[[z]]\,[y] = (0)$, and $(x-\sigma)k[[y,z]]\,[x] \cap k[[z]]\,[x,y] = (0)$.

(4) By the main theorem of Chapter 24, η is not TGF and the dimension of the generic fiber ring of η is one.

In order to show in Proposition 26.24 below that the map β is not TGF, we first observe:

PROPOSITION 26.23. *The element* $\sigma = \sum_{n=1}^{\infty}(xz)^{n!} \in k[x]\,[[z]]$ *is transcendental over* $k[[z]]\,[x]$.

PROOF. Consider an expression

$$Z := a_\ell\sigma^\ell + a_{\ell-1}\sigma^{\ell-1} + \cdots + a_1\sigma + a_0,$$

where the $a_i \in k[[z]]\,[x]$ and $a_\ell \neq 0$. Let m be an integer greater than $\ell+1$ and greater than $\deg_x a_i$ for each i such that $0 \le i \le \ell$ and $a_i \neq 0$. Regard each $a_i\sigma^i$ as a power series in x with coefficients in $k[[z]]$.

For each i with $0 \le i \le \ell$, we have $i(m!) < (m+1)!$. It follows that the coefficient of $x^{i(m!)}$ in σ^i is nonzero, and the coefficient of x^j in σ^i is zero for every j with $i(m!) < j < (m+1)!$. Thus if $a_i \neq 0$ and $j = i(m!) + \deg_x a_i$, then the coefficient of x^j in $a_i\sigma^i$ is nonzero, while for j such that $i(m!) + \deg_x a_i < j < (m+1)!$, the coefficient of x^j in $a_i\sigma^i$ is zero. By our choice of m, for each i such that $0 \le i < \ell$ and $a_i \neq 0$, we have

$$(m+1)! > \ell(m!) + \deg_x a_\ell \ge i(m!) + m! > i(m!) + \deg_x a_i.$$

Thus in Z, regarded as a power series in x with coefficients in $k[[z]]$, the coefficient of x^j is nonzero for $j = \ell(m!) + \deg_x a_\ell$. Therefore $Z \neq 0$. We conclude that σ is transcendental over $k[[z]]\,[x]$. $\square$

PROPOSITION 26.24. *In Sequence 26.4.0.1,* $k[[z]]\,[x,y] \overset{\beta}{\hookrightarrow} k[x]\,[[z]]\,[y]$ *is not a TGF-extension.*

PROOF. Fix an element $\sigma \in k[x]\,[[z]]$ that is transcendental over $k[[z]]\,[x]$. We define $\pi : k[x]\,[[z]]\,[y] \to k[x]\,[[z]]$ to be the identity map on $k[x]\,[[z]]$ and $\pi(y) = \sigma z$. Let $\mathbf{q} = \ker \pi$. Then $y - \sigma z \in \mathbf{q}$. If $h \in \mathbf{q} \cap (k[[z]]\,[x,y])$, then

$$h = \sum_{j=0}^{s}\sum_{i=0}^{t}(\sum_{\ell\in\mathbb{N}} a_{ij\ell}z^\ell)x^iy^j, \text{ for some } \ s,t \in \mathbb{N} \text{ and } a_{ij\ell} \in k, \text{ and so}$$

$$0 = \pi(h) = \sum_{j=0}^{s}\sum_{i=0}^{t}(\sum_{\ell\in\mathbb{N}} a_{ij\ell}z^\ell)x^i(\sigma z)^j = \sum_{j=0}^{s}\sum_{i=0}^{t}(\sum_{\ell\in\mathbb{N}} a_{ij\ell}z^{\ell+j})x^i\sigma^j.$$

Since σ is transcendental over $k[[z]]\,[x]$, we have that x and σ are algebraically independent over $k((z))$. Thus each of the $a_{ij\ell} = 0$. Therefore $\mathbf{q} \cap (k[[z]][x,y]) = (0)$, and so the embedding β is not TGF. $\square$

PROPOSITION 26.25. *In Sequence 26.4.0.1, the extensions*

$$k[[y,z]]\,[x] \overset{\zeta}{\hookrightarrow} k[x]\,[[y,z]] \quad \textit{and} \quad k[x]\,[[z]]\,[y] \overset{\gamma}{\hookrightarrow} k[x,y]\,[[z]]$$

are not TGF.

PROOF. For ζ, let $t = xy$ and let $\sigma \in k[[t]]$ be algebraically independent over $k(t)$. Define $\pi : k[x]\,[[y,z]] \to k[x]\,[[y]]$ as follows. For

$$f := \sum_{\ell=0}^{\infty} \sum_{m+n=\ell} f_{mn}(x) y^m z^n \in k[x]\,[[y,z]],$$

where $f_{mn}(x) \in k[x]$, define

$$\pi(f) := \sum_{\ell=0}^{\infty} \sum_{m+n=\ell} f_{mn}(x) y^m (\sigma y)^n \in k[x]\,[[y]].$$

In particular, $\pi(z) = \sigma y$. Let $\mathbf{p} := \ker \pi$. Then $z - \sigma y \in \mathbf{p}$, and so $\mathbf{p} \neq (0)$. Let $h \in \mathbf{p} \cap k[[y,z]]\,[x]$. We show $h = 0$. Now h is a polynomial with coefficients in $k[[y,z]]$, and we define $g \in k[[y,z]]\,[t]$, by, if $a_i(y,z) \in k[[y,z]]$ and

$$h := \sum_{i=0}^{r} a_i(y,z) x^i, \text{ then set } g := y^r h = \sum_{i=0}^{r} (\sum_{\ell=0}^{\infty} \sum_{m+n=\ell} b_{imn} y^m z^n) t^i.$$

The coefficients of g are in $k[[y,z]]$, since $y^r x^i = y^{r-i} t^i$. Thus

$$\begin{aligned} 0 = \pi(g) = \sum_{i=0}^{r} (\sum_{\ell=0}^{\infty} \sum_{m+n=\ell} b_{imn} y^m (\sigma y)^n) t^i &= \sum_{i=0}^{r} (\sum_{\ell=0}^{\infty} \sum_{m+n=\ell} b_{imn} \sigma^n y^\ell) t^i \\ &= \sum_{\ell=0}^{\infty} (\sum_{m+n=\ell} (\sum_{i=0}^{r} b_{imn} t^i) \sigma^n) y^\ell. \end{aligned}$$

Now t and y are analytically independent over k, and so the coefficient of each y^ℓ (in $k[[t]]$) is 0; since σ and t are algebraically independent over k, the coefficient of each σ^n is 0. It follows that each $b_{imn} = 0$, that $g = 0$ and hence that $h = 0$. Thus the extension ζ is not TGF.

To see that γ is not TGF, we switch variables in the proof for ζ, so that $t = yz$. Again choose $\sigma \in k[[t]]$ to be algebraically independent over $k(t)$. Define $\psi : k[x,y]\,[[z]] \to k[y]\,[[z]]$ by $\psi(x) = \sigma z$ and ψ is the identity on $k[y]\,[[z]]$. Then ψ can be extended to $\pi : k[y]\,[[x,z]] \to k[y]\,[[z]]$, which is similar to the π in the proof above. As above, set $\mathbf{p} := \ker \pi$; then $\mathbf{p} \cap k[[x,z]]\,[y] = (0)$. Thus $\mathbf{p} \cap k[x]\,[[z]]\,[y] = (0)$ and γ is not TGF. $\square$

PROPOSITION 26.26. *Let D be an integral domain and let x and t be indeterminates over D. Then $\sigma = \sum_{n=1}^{\infty} t^{n!} \in D[[x,t]]$ is algebraically independent over $D[[x,xt]]$.*

PROOF. Let ℓ be a positive integer and consider an expression

$$\gamma := \gamma_\ell \sigma^\ell + \cdots + \gamma_i \sigma^i + \cdots + \gamma_1 \sigma, \text{ where } \gamma_i := \sum_{j=0}^{\infty} f_{ij}(x)(xt)^j \in D[[x,xt]],$$

that is, each $f_{ij}(x) \in D[[x]]$ and $1 \le i \le \ell$. Assume that $\gamma_\ell \neq 0$. Let a_ℓ be the smallest j such that $f_{\ell j}(x) \neq 0$, and let m_ℓ be the order of $f_{\ell a_\ell}(x)$, that is, $f_{\ell a_\ell}(x) = x^{m_\ell} g_\ell(x)$, where $g_\ell(0) \neq 0$. Let n be a positive integer such that

$$n \ge 2 + \max\{\ell, m_\ell, a_\ell\}.$$

Since $\ell < n$, for each i with $1 \le i \le \ell$, we have

$$\sigma^i = \sigma_{i1}(t) + c_i t^{i(n!)} + t^{(n+1)!}\tau_i(t), \tag{26.2}$$

where c_i is a nonzero element of D, $\sigma_{i1}(t)$ is a polynomial in $D[t]$ of degree at most $(i-1)n! + (n-1)!$ and $\tau_i(t) \in D[[t]]$. We use the following two claims to complete the proof. □

Claim 26.27. *The coefficient of $t^{\ell(n!)+a_\ell}$ in $\sigma^\ell\gamma_\ell = \sigma^\ell(\sum_{j=a_\ell}^\infty f_{\ell j}(x)(xt)^j)$ as a power series in $D[[x]]$ has order $m_\ell + a_\ell$, and hence, in particular, is nonzero.*

Proof. By the choice of n, $(n+1)! > \ell(n!) + a_\ell$. Hence by the expression for σ^ℓ given in Equation 26.2, we see that all of the terms in $\sigma^\ell\gamma_\ell$ of the form $bt^{\ell(n!)+a_\ell}$, for some $b \in D[[x]]$, appear in the product

$$(\sigma_{\ell 1}(t) + c_\ell t^{\ell(n!)})(\sum_{j=a_\ell}^{\ell(n!)+a_\ell} f_{\ell j}(x)(xt)^j).$$

One of the terms of the form $bt^{\ell(n!)+a_\ell}$ in this product is

$$c_\ell t^{\ell(n!)} f_{\ell a_\ell}(x)(xt)^{a_\ell} = (c_\ell x^{m_\ell+a_\ell} g_\ell(x))t^{\ell(n!)+a_\ell} = (c_\ell x^{m_\ell+a_\ell} g_\ell(0) + \dots)t^{\ell(n!)+a_\ell}.$$

Since $c_\ell g_\ell(0)$ is a nonzero element of k, $c_\ell x^{m_\ell+a_\ell} g_\ell(x) \in k[[x]]$ has order $m_\ell + a_\ell$. The other terms in the product $\sigma^\ell\gamma_\ell$ that have the form $bt^{\ell(n!)+a_\ell}$, for some $b \in D[[x]]$, are in the product

$$(\sigma_{\ell 1}(t))(\sum_{j=a_\ell}^{\ell(n!)+a_\ell} f_{\ell j}(x)(xt)^j) = \sum_{j=a_\ell}^{\ell(n!)+a_\ell} f_{\ell j}(x)(xt)^j \sigma_{\ell 1}(t).$$

Since $\deg_t \sigma_{\ell 1} \le (\ell-1)n! + (n-1)!$ and since, for each j with $f_{\ell j}(x) \neq 0$, we have $\deg_t f_{\ell j}(x)(xt)^j = j$, we see that each term in $f_{\ell j}(xt)^j \sigma_{\ell 1}(t)$ has degree in t less than or equal to $j + (\ell-1)n! + (n-1)!$. Thus each nonzero term in this product of the form $bt^{\ell(n!)+a_\ell}$ has

$$j \ge \ell(n!) + a_\ell - (\ell-1)(n!) - (n-1)! = a_\ell + (n-1)!(n-1) > m_\ell + a_\ell,$$

by choice of n. Moreover, for j such that $f_{\ell j}(x) \neq 0$, the order in x of $f_{\ell j}(x)(xt)^j$ is bigger than or equal to j. This completes the proof of Claim 26.27. □

Claim 26.28. *For $i < \ell$, the coefficient of $t^{\ell(n!)+a_\ell}$ in $\sigma^i\gamma_i$ as a power series in $D[[x]]$ is either zero or has order greater than $m_\ell + a_\ell$.*

Proof. As in the proof of Claim 26.27, all of the terms in $\sigma^i\gamma_i$ of the form $bt^{\ell(n!)+a_\ell}$, for some $b \in D[[x]]$, appear in the product

$$(\sigma_{i1} + c_i t^{i(n!)})(\sum_{j=0}^{\ell(n!)+a_\ell} f_{ij}(x)(xt)^j) = \sum_{j=0}^{\ell(n!)+a_\ell} f_{ij}(x)(xt)^j(\sigma_{i1} + c_i t^{i(n!)}).$$

Since $\deg_t(\sigma_{i1} + c_i t^{i(n!)}) = i(n!)$, each term in $f_{ij}(x)(xt)^j(\sigma_{i1} + c_i t^{i(n!)})$ has degree in t at most $j + i(n!)$. Thus each term in this product of the form $bt^{\ell(n!)+a_\ell}$, for some nonzero $b \in D[[x]]$, has

$$j \geq \ell(n!) + a_\ell - i(n!) \geq n! + a_\ell > m_\ell + a_\ell.$$

Thus $\operatorname{ord}_x b \geq j > m_\ell + a_\ell$. This completes the proof of Claim 26.28. Hence $\gamma \notin D[[x, xz]]$. This completes the proof of Proposition 26.26. □

Question and Remarks 26.29. (1) As we show in Proposition 26.12, the embeddings from Equation 1 involving two mixed polynomial-power series rings of dimension two over a field k with inverted elements are TGF. In the article [**69**] we ask whether this is true in the three-dimensional case. For example, is the embedding θ below TGF?

$$k[x, y]\,[[z]] \overset{\theta}{\hookrightarrow} k[x, y, 1/x]\,[[z]]\text{''}$$

Yasuda shows the answer for this example is “No” in [**145**]. Dumitrescu establishes the answer is “No” in more generality; see Theorem 26.30.

(2) For the four-dimensional case, as observed in the discussion of Question 26.1, it follows from a result of Heinzer and Rotthaus [**52**, Theorem 1.12, p. 364] that the extension $k[x, y, u]\,[[z]] \hookrightarrow k[x, y, u, 1/x]\,[[z]]$ is not TGF. Theorem 26.30 yields a direct proof of this fact.

We close this chapter with a result of Dumitresco that shows many extensions involving only one power series variable are not TGF.

Theorem 26.30. [**29**, Corollary 4] *Let D be an integral domain and let x, y, z be indeterminates over D. For every subring B of $D[[x, y]]$ that contains $D[x, y]$, the extension $B[[z]] \hookrightarrow B[1/x][[z]]$ is not TGF.*

Proof. Let $t = z/x$. Proposition 26.26 implies there exists $\sigma \in D[[t]]$, such as $\sigma = \sum_{r=1}^{\infty} t^{r!}$, that is algebraically independent over $D[[x, z]]$.

Let u be another variable and consider the D-algebra homomorphism

$$\pi : D[[x, y, u]]\,[\frac{1}{x}]\,[[z]] \to D[[x, u]]\,[\frac{1}{x}]\,[[z]]$$

defined by mapping

$$\sum_{i=0}^{\infty} a_i(x, y, u, 1/x) z^i \mapsto \sum_{i=0}^{\infty} a_i(x, \sigma u, u, 1/x) z^i,$$

where $a_i(x, y, u, 1/x) \in D[[x, y, u]][1/x]$. Let $\mathbf{p} = \ker \pi$. Then $y - \sigma u \in \mathbf{p}$. We show that $\mathbf{p} \cap D[[x, y, u, z]] = (0)$, and so also $\mathbf{p} \cap D[x, y, u]\,[[z]] = (0)$. Let

$$f := \sum_{\ell=0}^{\infty} (\sum_{i+j=\ell} d_{ij} u^i y^j) \in D[[x, y, u, z]],$$

where $d_{ij} \in D[[x, z]]$. If $f \in \mathbf{p}$, then

$$0 = \pi(f) = \sum_{\ell=0}^{\infty} (\sum_{i+j=\ell} d_{ij} u^i \sigma^j u^j) = \sum_{\ell=0}^{\infty} (\sum_{i+j=\ell} d_{ij} \sigma^j) u^\ell.$$

This is a power series in u, and so, for each ℓ, $\sum_{i+j=\ell} d_{ij}\sigma^j = 0$. Since σ is algebraically independent over $D[[x, z]]$, each $d_{ij} = 0$. Thus $f = 0$. This completes the proof of Theorem 26.30. □

Exercises

(1) Let k be a field and let $\aleph_0 = |\mathbb{N}|$. Prove that $\alpha = |k| \cdot \aleph_0$ and $\beta = |k|^{\aleph_0}$ in Theorem 26.14.
Suggestion: Notice that every polynomial of the form $x - a$, for $a \in k$, generates a maximal ideal of $k[x]$ and also that $|k[x]| = |k| \cdot \aleph_0$, since $k[x]$ can be considered as an infinite union of polynomials of each finite degree.

(2) Let y denote an indeterminate over the ring of integers $\mathbb{Z}$, and let $A = \mathbb{Z}[[y]]$.
(a) Prove that every maximal ideal of A has height two.
(b) Describe and make a diagram of the partially ordered set $\operatorname{Spec} A$.
(c) Let $B = A[\frac{1}{y+2}]$. Describe the partially ordered set $\operatorname{Spec} B$. Prove that B has maximal ideals of height one, and deduce that $\operatorname{Spec} B$ is not order-isomorphic to $\operatorname{Spec} A$.
(d) Let $C = A[\frac{y}{2}]$. Describe the partially ordered set $\operatorname{Spec} C$. Prove that C has precisely two nonmaximal prime ideal of height one that are an intersection of maximal ideals, while each of A and B has precisely one nonmaximal prime ideal of height one that is an intersection of maximal ideals. Deduce that $\operatorname{Spec} C$ is not order-isomorphic to either $\operatorname{Spec} A$ or $\operatorname{Spec} B$.

CHAPTER 27

Multi-ideal-adic completions of Noetherian rings

In this chapter we consider a variation of the usual I-adic completion of a Noetherian ring R.[1] Instead of successive powers of a fixed ideal I, we use a *multi-adic* filtration formed from a more general descending sequence $\{I_n\}_{n=0}^{\infty}$ of ideals. We develop the mechanics of a *multi-adic* completion R^* of R. With additional hypotheses on the ideals of the filtration, we show that R^* is Noetherian. In the case where R is local, we prove that R^* is excellent, or Henselian or universally catenary if R has the stated property.

27.1. Ideal filtrations and completions

Let R be a commutative ring with identity. A *filtration* on R is a decreasing sequence $\{I_n\}_{n=0}^{\infty}$ of ideals of R. Associated to a filtration there is a well-defined completion

$$R^* = \varprojlim_n \ R/I_n,$$

and a canonical homomorphism $\psi : R \to R^*$, [**108**, Chapter 9]. If $\bigcap_{n=0}^{\infty} I_n = (0)$, then ψ is injective and R may be regarded as a subring of R^*, [**108**, page 401]. In the terminology of Northcott, a filtration $\{I_n\}_{n=0}^{\infty}$ is said to be *multiplicative* if $I_0 = R$ and $I_n I_m \subseteq I_{n+m}$, for all $m \geq 0$, $n \geq 0$, [**108**, page 408]. A well-known example of a multiplicative filtration on R is the I-adic filtration $\{I^n\}_{n=0}^{\infty}$, where I is a fixed ideal of R.

In this chapter we consider filtrations of ideals of R that are *not* multiplicative, and examine the completions associated to these filtrations. We assume the ring R is Noetherian. Instead of successive powers of a fixed ideal I, we use a filtration formed from a more general descending sequence $\{I_n\}_{n=0}^{\infty}$ of ideals. We require that, for each $n > 0$, the n^{th} ideal I_n is contained in the n^{th} power of the Jacobson radical of R, and that $I_{nk} \subseteq I_n^k$ for all $k, n \geq 0$. We call the associated completion a *multi-adic* completion, and denote it by R^*. The basics of the multi-adic construction and the relationship between this completion and certain ideal-adic completions are considered in Section 27.2. In Sections 27.3 and 27.4, we prove that the multi-adic completion R^* with respect to such ideals $\{I_n\}$ has the properties stated above

The process of passing to completion gives an analytic flavor to algebra. Often we view completions in terms of power series, or in terms of coherent sequences as in [**9**, pages 103-104]. Sometimes results are established by demonstrating for each n that they hold at the n^{th} stage in the inverse limit.

Multi-adic completions are interesting from another point of view. Many examples in commutative algebra can be considered as subrings of R^*/J, where R^* is

[1]The material in this chapter is adapted from our paper [**71**] dedicated to Melvin Hochster on the occasion of his 65^{th} birthday. Hochster's brilliant work has had a tremendous impact on commutative algebra.

a multi-adic completion of a localized polynomial ring R over a countable ground field and J is an ideal of R^*. In particular, certain counterexamples of Brodmann and Rotthaus, Heitmann, Nishimura, Ogoma, Rotthaus and Weston can be interpreted in this way, see [**15**], [**16**], [**77**], [**105**], [**107**], [**110**], [**111**], [**119**], [**120**], [**140**]. For many of these examples, a particular enumeration, $\{p_1, p_2, \dots\}$, of countably many non-associate prime elements is chosen and the ideals I_n are defined to be $I_n := (p_1 p_2 \dots p_n)^n$. The Noetherian property in these examples is a trivial consequence of the fact that every ideal of R^* that contains one of the ideals I_n, or a power of I_n, is extended from R. In general, an advantage of R^* over the I_n-adic completion $\widehat{R}_n$ is that an ideal of R^* is more likely to be extended from R than is an ideal of $\widehat{R}_n$.

27.2. Basic mechanics for the multi-adic completion

SETTING 27.1. Let R be a Noetherian ring with Jacobson radical $\mathcal{J}$, and let $\mathbb{N}$ denote the set of positive integers. For each $n \in \mathbb{N}$, let Q_n be an ideal of R. Assume that the sequence $\{Q_n\}$ is descending, that is $Q_{n+1} \subseteq Q_n$, and that $Q_n \subseteq \mathcal{J}^n$, for each $n \in \mathbb{N}$. Also assume, for each pair of integers $k, n \in \mathbb{N}$, that $Q_{nk} \subseteq Q_n^k$.

Let $\mathcal{F} = \{Q_k\}_{k \geq 0}$ be the filtration

$$R = Q_0 \supseteq Q_1 \supseteq \cdots \supseteq Q_k \supseteq Q_{k+1} \supseteq \cdots$$

of R and define

$$R^* := \varprojlim_k \ R/Q_k \tag{27.1.1}$$

to be the completion of R with respect to $\mathcal{F}$.

Let $\widehat{R} := \varprojlim_k \ R/\mathcal{J}^k$ denote the completion of R with respect to the powers of the Jacobson radical $\mathcal{J}$ of R, and for each $n \in \mathbb{N}$, let

$$\widehat{R_n} := \varprojlim_k \ R/Q_n^k \tag{27.1.2}$$

denote the completion of R with respect to the powers of Q_n.

REMARK 27.2. Assume notation as in Setting 27.1. For each fixed $n \in \mathbb{N}$, we have

$$R^* = \varprojlim_k \ R/Q_k = \varprojlim_k \ R/Q_{nk},$$

where $k \in \mathbb{N}$ varies. This holds because the limit of a subsequence is the same as the limit of the original sequence.

We establish in Proposition 27.3 canonical inclusion relations among $\widehat{R}$ and the completions defined in (27.1.1) and (27.1.2).

PROPOSITION 27.3. *Let the notation be as in Setting 27.1. For each $n \in \mathbb{N}$, we have canonical inclusions*

$$R \subseteq R^* \subseteq \widehat{R_n} \subseteq \widehat{R_{n-1}} \subseteq \cdots \subseteq \widehat{R_1} \subseteq \widehat{R}.$$

PROOF. The inclusion $R \subseteq R^*$ is clear since the intersection of the ideals Q_k is zero. For the inclusion $R^* \subseteq \widehat{R_n}$, by Remark 27.2, $R^* = \varprojlim_k \ R/Q_{nk}$. Notice that

$$Q_{nk} \subseteq Q_n^k \subseteq Q_{n-1}^k \subseteq \cdots \subseteq \mathcal{J}^k.$$

$\square$

To complete the proof of Proposition 27.3, we state and prove a general result about completions with respect to ideal filtrations (see also [**108**, Section 9.5]). We define the respective completions using coherent sequences as in [**9**, pages 103-104].

LEMMA 27.4. *Let R be a Noetherian ring with Jacobson radical $\mathcal{J}$ and let $\{H_k\}_{k\in\mathbb{N}}$, $\{I_k\}_{k\in\mathbb{N}}$ and $\{L_k\}_{k\in\mathbb{N}}$ be descending sequences of ideals of R such that, for each $k \in \mathbb{N}$, we have inclusions*

$$L_k \subseteq I_k \subseteq H_k \subseteq \mathcal{J}^k.$$

We denote the families of natural surjections arising from these inclusions as:

$$\delta_k : R/L_k \to R/I_k, \quad \lambda_k : R/I_k \to R/H_k \quad \text{ and } \quad \theta_k : R/H_k \to R/\mathcal{J}^k,$$

and the completions with respect to these families as:

$$\widehat{R_L} := \varprojlim_k R/L_k, \quad \widehat{R_I} := \varprojlim_k R/I_k \quad \widehat{R_H} := \varprojlim_k R/H_k \quad \text{and} \quad \widehat{R} := \varprojlim_k R/\mathcal{J}^k.$$

Then

(1) *These families of surjections induce canonical injective maps Δ, Λ and Θ among the completions as shown in the diagram below.*
(2) *For each positive integer k we have a commutative diagram as displayed below, where the vertical maps are the natural surjections.*

$$\begin{array}{ccccccc} R/L_k & \xrightarrow{\delta_k} & R/I_k & \xrightarrow{\lambda_k} & R/H_k & \xrightarrow{\theta_k} & R/\mathcal{J}^k \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \widehat{R_L} & \xrightarrow{\Delta} & \widehat{R_I} & \xrightarrow{\Lambda} & \widehat{R_H} & \xrightarrow{\Theta} & \widehat{R}\ . \end{array}$$

(3) *The composition $\Lambda \cdot \Delta$ is the canonical map induced by the natural surjections $\lambda_k \cdot \delta_k : R/L_k \to R/H_k$. Similarly, the other compositions in the bottom row are the canonical maps induced by the appropriate natural surjections.*

PROOF. In each case there is a unique homomorphism of the completions. For example, the family of homomorphisms $\{\delta_k\}_{k\in\mathbb{N}}$ induces a unique homomorphism

$$\widehat{R_L} \xrightarrow{\Delta} \widehat{R_I}. \tag{27.1}$$

To define Δ, let $x = (x_k)_{k\in\mathbb{N}} \in \widehat{R_L}$, where each $x_k \in R/L_k$. Then $\delta_k(x_k) \in R/I_k$ and we define $\Delta(x) := (\delta_k(x_k))_{k\in\mathbb{N}} \in \widehat{R_I}$.

To show the maps on the completions are injective, consider for example the map Δ. Suppose $x = (x_k)_{k\in\mathbb{N}} \in \varprojlim_k R/L_k$ with $\Delta(x) = 0$. Then $\delta_k(x_k) = 0$ in R/I_k, that is, $x_k \in I_kR/L_k$, for every $k \in \mathbb{N}$. For $v \in \mathbb{N}$, consider the following commutative diagram:

$$\begin{array}{ccc} R/L_k & \xrightarrow{\delta_k} & R/I_k \\ {\scriptstyle \beta_{k,kv}}\uparrow & & {\scriptstyle \alpha_{k,kv}}\uparrow \\ R/L_{kv} & \xrightarrow{\delta_{kv}} & R/I_{kv} \end{array} \tag{27.2}$$

where $\beta_{k,kv}$ and $\alpha_{k,kv}$ are the canonical surjections associated with the inverse limits. We have $x_{kv} \in I_{kv}R/L_{kv}$. Therefore

$$x_k = \beta_{k,kv}(x_{kv}) \in I_{kv}(R/L_k) \subseteq \mathcal{J}^{kv}(R/L_k),$$

for every $v \in \mathbb{N}$. Since $\mathcal{J}(R/L_k)$ is contained in the Jacobson radical of R/L_k and R/L_k is Noetherian, we have

$$\bigcap_{v\in\mathbb{N}} \mathcal{J}^{kv}(R/L_k) = (0).$$

Therefore $x_k = 0$ for each $k \in \mathbb{N}$, and so Δ is injective. The remaining assertions are clear. $\square$

LEMMA 27.5. *With R^* and $\widehat{R_n}$ as in Setting 27.1, we have*

$$R^* = \bigcap_{n\in\mathbb{N}} \widehat{R_n}.$$

PROOF. The inclusion "$\subseteq$" is shown in Proposition 27.3. For the reverse inclusion, fix positive integers n and k, and let $L_\ell = Q_{nk\ell}$, $I_\ell = Q_{nk}^\ell$ and $H_\ell = Q_n^\ell$ for each $\ell \in \mathbb{N}$. Then $L_\ell \subseteq I_\ell \subseteq H_\ell \subseteq \mathcal{J}^\ell$, as in Lemma 27.4 and

$$\widehat{R_L} := \varprojlim_\ell R/Q_{nk\ell} = R^*, \quad \widehat{R_I} := \varprojlim_\ell R/Q_{nk}^\ell = \widehat{R_{nk}}, \quad \widehat{R_H} := \varprojlim_\ell R/Q_n^\ell = \widehat{R_n}.$$

(Also, as before, $\widehat{R} := \varprojlim_\ell R/\mathcal{J}^\ell$.) We define φ_n, φ_{nk}, $\varphi_{nk,n}$, θ and φ to be the canonical injective homomorphisms given by Lemma 27.4 among the rings displayed in the following diagram.

(27.5.1)
$$\begin{tikzcd}
\widehat{R} & & \widehat{R_n} \arrow[ll, "\theta"'] \\
R^* \arrow[u, "\varphi"] \arrow[urr, "\varphi_n"] \arrow[rr, "\varphi_{nk}"'] & & \widehat{R_{nk}} \arrow[u, "\varphi_{nk,n}"']
\end{tikzcd}$$

By Lemma 27.4, Diagram 27.5.1 is commutative.

Let $\widehat{y} \in \bigcap_{n\in\mathbb{N}} \widehat{R_n}$. We show that there is an element $\xi \in R^*$ such that $\varphi(\xi) = \widehat{y}$. This is sufficient to ensure that $\widehat{y} \in R^*$, since the maps θ_t are injective and Diagram 27.5.1 is commutative.

First, we define ξ: For each $t \in \mathbb{N}$, we have

$$\widehat{y} = (y_{1,t}, y_{2,t}, \dots,) \in \varprojlim_\ell R/Q_t^\ell = \widehat{R_t},$$

where $y_{1,t} \in R/Q_t$, $y_{2,t} \in R/Q_t^2$ and $y_{2,t} + Q_t/Q_t^2 = y_{1,t}$ in R/Q_t, $\cdots$ and so forth, is a coherent sequence as in [**9**, pp. 103-104]. Now take $z_t \in R$ so that $z_t + Q_t = y_{1,t}$. Thus $\widehat{y} - z_t \in Q_t\widehat{R_t}$. For positive integers s and t with $s \geq t$, we have $Q_s \subseteq Q_t$. Therefore $z_t - z_s \in Q_t\widehat{R_t} \cap R = Q_tR$. Thus $\xi := (z_t)_{t\in\mathbb{N}} \in R^*$. We have $\widehat{y} - z_t \in Q_t\widehat{R_t} \subseteq \mathcal{J}^t\widehat{R}$, for all $t \in \mathbb{N}$. Hence $\varphi(\xi) = \widehat{y}$. This completes the proof of Lemma 27.5. $\square$

The following special case of Setting 27.1 is used by Brodmann, Heitmann, Nishimura, Ogoma, Rotthaus, and Weston for the construction of numerous examples.

SETTING 27.6. Let R be a Noetherian ring with Jacobson radical $\mathcal{J}$. For each $i \in \mathbb{N}$, let $p_i \in \mathcal{J}$ be a non-zero-divisor (that is, a regular element) on R.

For each $n \in \mathbb{N}$, let $q_n = (p_1 \cdots p_n)^n$. Let $\mathcal{F}_0 = \{(q_k)\}_{k \geq 0}$ be the filtration

$$R \supseteq (q_1) \supseteq \cdots \supseteq (q_k) \supseteq (q_{k+1}) \supseteq \cdots$$

of R and define $R^* := \varprojlim_k R/(q_k)$ to be the completion of R with respect to $\mathcal{F}_0$.

REMARK 27.7. In Setting 27.6, assume further that $R = K[x_1, \ldots, x_n]_{(x_1,\ldots,x_n)}$, a localized polynomial ring over a countable field K, and that $\{p_1, p_2, \ldots\}$ is an enumeration of all the prime elements (up to associates) in R. As in 27.6, let $R^* := \varprojlim_n R/(q_n)$, where each $q_n = (p_1 \cdots p_n)^n$.

The ring R^* is often useful for the construction of Noetherian local rings with a bad locus (regular, Cohen-Macaulay, normal). In particular, Brodmann, Heitmann, Nishimura, Ogoma, Rotthaus, and Weston make use of special subrings of this multi-adic completion R^* for their examples. The first such example was constructed by Rotthaus in [**119**]. In this paper, Rotthaus obtains a regular local Nagata ring A that contains a prime element ω so that the singular locus of the quotient ring $A/(\omega)$ is not closed. This ring A is situated between the localized polynomial ring R and its $*$-completion R^*; thus, in general R^* is bigger than R. In the Rotthaus example, the singular locus of $(A/(\omega))^*$ is defined by a height one prime ideal Q that intersects $A/(\omega)$ in (0). Since all ideals $Q + (p_n)$ are extended from $A/(\omega)$, the singular locus of $A/(\omega)$ is not closed.

REMARK 27.8. For R and R^* as in Remark 27.7, the ring R^* is also the "ideal-completion", or "R-completion of R. This completion is defined and used in the paper of Zelinsky [**150**], the work of Matlis [**92**] and [**93**], and the book of Fuchs and Salce[**36**]. The *ideal-topology*, or *R-topology* on an integral domain R is the linear topology defined by letting the nonzero ideals of R be a subbase for the open neighborhoods of 0. The nonzero principal ideals of R also define a subbase for the open neighborhoods of 0. Recent work on ideal completions has been done by Tchamna in [**137**]. In particular, Tchamna observes in [**137**, Theorem 4.1] that the ideal-completion of a countable Noetherian local domain is also a multi-ideal-adic completion.

27.3. Preserving Noetherian under multi-adic completion

THEOREM 27.9. *Let the notation be as in Setting 27.1. Then the ring R^* defined in (27.1.1) is Noetherian.*

PROOF. It suffices to show each ideal I of R^* is finitely generated. Since $\widehat{R}$ is Noetherian, there exist $f_1, \ldots, f_s \in I$ such that $I\widehat{R} = (f_1, \ldots, f_s)\widehat{R}$. Since $\widehat{R_n} \hookrightarrow \widehat{R}$ is faithfully flat, $I\widehat{R_n} = I\widehat{R} \cap \widehat{R_n} = (f_1, \ldots, f_s)\widehat{R_n}$, for each $n \in \mathbb{N}$.

Let $f \in I \subseteq R^*$. Then $f \in I\widehat{R_1}$, and so

$$f \ = \ \sum_{i=1}^{s} \widehat{b}_{i0} f_i,$$

where $\widehat{b}_{i0} \in \widehat{R_1}$. Consider R as "Q_0", and so $\widehat{b}_{i0} \in Q_0\widehat{R}_1$. Since $\widehat{R_1}/Q_1\widehat{R_1} \cong R/Q_1$, for all i with $1 \leq i \leq s$, we have $\widehat{b}_{i0} = a_{i0} + \widehat{c}_{i1}$, where $a_{i0} \in R = Q_0R$ and

$\widehat{c}_{i1} \in Q_1\widehat{R_1}$. Then

$$f \ = \ \sum_{i=1}^{s} a_{i0}f_i \ + \ \sum_{i=1}^{s} \widehat{c}_{i1}f_i.$$

Notice that

$$\widehat{d_1} \ := \ \sum_{i=1}^{s} \widehat{c}_{i1}f_i \in (Q_1I)\widehat{R_1} \cap R^* \subseteq \widehat{R_2}.$$

By the faithful flatness of the extension $\widehat{R_2} \hookrightarrow \widehat{R_1}$, we see $\widehat{d_1} \in (Q_1I)\widehat{R_2}$, and therefore there exist $\widehat{b}_{i1} \in Q_1\widehat{R_2}$ with

$$\widehat{d_1} = \sum_{i=1}^{s} \widehat{b}_{i1}f_i.$$

As before, using that $\widehat{R_2}/Q_2\widehat{R_2} \cong R/Q_2$, we can write $\widehat{b}_{i1} = a_{i1} + \widehat{c}_{i2}$, where $a_{i1} \in R$ and $\widehat{c}_{i2} \in Q_2\widehat{R}_2$. This implies that $a_{i1} \in Q_1\widehat{R_2} \cap R = Q_1$. We have:

$$f \ = \ \sum_{i=1}^{s}(a_{i0} + a_{i1})f_i + \sum_{i=1}^{s} \widehat{c}_{i2}f_i.$$

Now set

$$\widehat{d_2} \ := \ \sum_{i=1}^{s} \widehat{c}_{i2}f_i.$$

Then $\widehat{d_2} \in (Q_2I)\widehat{R_2} \cap R^* \subseteq \widehat{R_3}$ and, since the extension $\widehat{R_3} \hookrightarrow \widehat{R_2}$ is faithfully flat, we have $\widehat{d_2} \in (Q_2I)\widehat{R_3}$. We repeat the process. By a simple induction argument,

$$f = \sum_{i=1}^{s}(a_{i0} + a_{i1} + a_{i2} + \ldots)f_i,$$

where $a_{ij} \in Q_j$ and $a_{i0} + a_{i1} + a_{i2} + \ldots \in R^*$. Thus $f \in (f_1, \ldots, f_s)R^*$. Hence I is finitely generated and R^* is Noetherian. $\square$

COROLLARY 27.10. *With notation as in Setting 27.1, the maps $R \hookrightarrow R^*$, $R^* \hookrightarrow \widehat{R_n}$ and $R^* \hookrightarrow \widehat{R}$ are faithfully flat.*

We use Proposition 27.11 in the next section on preserving excellence.

PROPOSITION 27.11. *Assume notation as in Setting 27.1, and let the ring R^* be defined as in (27.1.1). If M is a finitely generated R^*-module, then*

$$M \cong \varprojlim_{k} \ (M/Q_kM),$$

that is, M is $*$-complete.

PROOF. If $F = (R^*)^n$ is a finitely generated free R^*-module, then one can see directly that

$$F \cong \varprojlim_{k} \ F/Q_kF,$$

and so F is $*$-complete.

Let M be a finitely generated R^*-module. Consider an exact sequence:

$$0 \longrightarrow N \longrightarrow F \longrightarrow M \longrightarrow 0,$$

where F is a finitely generated free R^*-module. This induces an exact sequence:

$$0 \longrightarrow \tilde{N} \longrightarrow F^* \longrightarrow M^* \longrightarrow 0,$$

where $\tilde{N}$ is the completion of N with respect to the induced filtration $\{Q_kF \cap N\}_{k\geq 0}$; see [**9**, (10.3)].

This gives a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & F & \longrightarrow & M & \longrightarrow & 0 \\ & & \big\downarrow & & \cong\big\downarrow & & \gamma\big\downarrow & & \\ 0 & \longrightarrow & \tilde{N} & \longrightarrow & F^* & \longrightarrow & M^* & \longrightarrow & 0 \end{array}$$

where γ is the canonical map $\gamma : M \longrightarrow M^*$. The diagram shows that γ is surjective. We have

$$\bigcap_{k=1}^{\infty}(Q_kM) \subseteq \bigcap_{k=1}^{\infty} J^kM = (0),$$

where the last equality is by [**9**, (10.19)]. Therefore γ is also injective. $\square$

REMARK 27.12. Let the notation be as in Setting 27.1, and let B be a finite R^*-algebra. Let $\widehat{B_n} \cong B \otimes_{R^*} \widehat{R_n}$ denote the Q_n-adic completion of B. By Proposition 27.3, and Corollary 27.10, we have a sequence of inclusions:

$$B \hookrightarrow \cdots \hookrightarrow \widehat{B_{n+1}} \hookrightarrow \widehat{B_n} \hookrightarrow \ldots \hookrightarrow \widehat{B_1} \hookrightarrow \widehat{B},$$

where $\widehat{B}$ denotes the completion of B with respect to $\mathcal{J}B$. Let $\mathcal{J}_0$ denote the Jacobson radical of B. Since every maximal ideal of B lies over a maximal ideal of R^*, we have $\mathcal{J}B \subseteq \mathcal{J}_0$.

THEOREM 27.13. *With the notation of Setting 27.1, let B be a finite R^*-algebra and let $\widehat{B_n} \cong B \otimes_{R^*} \widehat{R}_n$ denote the Q_n-adic completion of B. Let $\widehat{I}$ be an ideal of $\widehat{B}$, let $I := \widehat{I} \cap B$, and let $I_n := \widehat{I} \cap \widehat{B_n}$, for each $n \in \mathbb{N}$. If $\widehat{I} = I_n\widehat{B}$, for all n, then $\widehat{I} = I\widehat{B}$.*

PROOF. By replacing B by B/I, we may assume that $(0) = I = \widehat{I} \cap B$. To prove the theorem, it suffices to show that $\widehat{I} = 0$.

For each $n \in \mathbb{N}$, we define ideals $\mathfrak{c}_n$ of $\widehat{B_n}$ and $\mathfrak{a}_n$ of B:

$$\mathfrak{c}_n := I_n + Q_n\widehat{B_n}, \qquad \mathfrak{a}_n := \mathfrak{c}_n \cap B.$$

Since $B/Q_nB = \widehat{B_n}/Q_n\widehat{B_n}$, the ideals of B containing Q_n are in one-to-one inclusion-preserving correspondence with the ideals of $\widehat{B_n}$ containing $Q_n\widehat{B_n}$, and so

$$\mathfrak{a}_n\widehat{B_n} = \mathfrak{c}_n, \quad \mathfrak{a}_{n+1}\widehat{B_n} = \mathfrak{a}_{n+1}\widehat{B_{n+1}}\widehat{B_n} = \mathfrak{c}_{n+1}\widehat{B_n}. \tag{27.13.1}$$

Since $\widehat{B}$ is faithfully flat over $\widehat{B_n}$ and $\widehat{I}$ is extended,

$$I_{n+1}\widehat{B_n} = (I_{n+1}\widehat{B}) \cap \widehat{B_n} = \widehat{I} \cap \widehat{B_n} = I_n. \tag{27.13.2}$$

Thus, for all $n \in \mathbb{N}$, we have, using (27.13.1), (27.13.2) and $Q_{n+1}\widehat{B_n} \subseteq Q_n\widehat{B_n}$:

$$\mathfrak{a}_n\widehat{B_n} = \mathfrak{c}_n = I_n + Q_n\widehat{B_n} = I_{n+1}\widehat{B_n} + Q_n\widehat{B_n} = \mathfrak{c}_{n+1}\widehat{B_n} + Q_n\widehat{B_n} = \mathfrak{a}_{n+1}\widehat{B_n} + Q_n\widehat{B_n}.$$

Since $\widehat{B_n}$ is faithfully flat over B, the equation above implies that

$$\mathfrak{a}_{n+1} + Q_nB = (\mathfrak{a}_{n+1}\widehat{B_n} + Q_n\widehat{B_n}) \cap B = \mathfrak{a}_n\widehat{B_n} \cap B = \mathfrak{a}_n. \tag{27.13.3}$$

Thus also

$$\mathfrak{a}_n\widehat{B} \subseteq \mathfrak{a}_{n+1}\widehat{B} + Q_n\widehat{B} \subseteq I_{n+1}\widehat{B} + Q_n\widehat{B} = \widehat{I} + Q_n\widehat{B}. \tag{27.13.4}$$

Now $Q_n \subseteq \mathcal{J}^n\widehat{B}$ and $\mathcal{J} \subseteq \mathcal{J}_0$, and so using (27.13.4)

$$\bigcap_{n\in\mathbb{N}}(\mathfrak{a}_n\widehat{B}) \subseteq \bigcap_{n\in\mathbb{N}}(\widehat{I}+Q_n\widehat{B}) \subseteq \bigcap_{n\in\mathbb{N}}(\widehat{I}+\mathcal{J}^n\widehat{B}) = \widehat{I}.$$

Since $\widehat{I} \cap B = (0)$, we have

$$0 = \widehat{I}\cap B \supseteq (\bigcap_{n\mathbf{n}\mathbb{N}}(\mathfrak{a}_n\widehat{B}))\cap B \supseteq \bigcap_{n\in\mathbb{N}}((\mathfrak{a}_n\widehat{B})\cap B) = \bigcap_{n\in\mathbb{N}}\mathfrak{a}_n,$$

where the last equality is because $\widehat{B}$ is faithfully flat over B. Thus $\bigcap_{n\in\mathbb{N}}\mathfrak{a}_n = (0)$.

Claim. $\widehat{I} = (0)$.

Proof of Claim. Suppose $\widehat{I} \neq 0$. Then there exists $d \in \mathbb{N}$ so that $\widehat{I} \nsubseteq \mathcal{J}_0^d\widehat{B}$. By hypothesis, $\widehat{I} = I_d\widehat{B}$, and so $I_d\widehat{B} \nsubseteq \mathcal{J}_0^d\widehat{B}$. Since $\widehat{B}$ is faithfully flat over $\widehat{B_d}$, we have $I_d \nsubseteq \mathcal{J}_0^d\widehat{B_d}$. By (27.13.1),

$$\mathfrak{a}_d\widehat{B_d} = \mathfrak{c}_d = I_d + Q_d\widehat{B_d} \nsubseteq \mathcal{J}_0^d\widehat{B_d},$$

and so there exists an element $y_d \in \mathfrak{a}_d$ with $y_d \notin \mathcal{J}_0^d$.

By (27.13.3), $\mathfrak{a}_{d+1} + Q_dB = \mathfrak{a}_d$. Hence there exists $y_{d+1} \in \mathfrak{a}_{d+1}$ and $q_d \in Q_dB$ so that $y_{d+1}+q_d = y_d$. Recursively we construct sequences of elements $y_n \in \mathfrak{a}_n$ and $q_n \in Q_nB$ such that $y_{n+1} + q_n = y_n$, for each $n \geq d$.

The sequence $\xi = (y_n + Q_nB) \in \varprojlim_n B/Q_nB = B$ corresponds to a nonzero element $y \in B$ such that, for every $n \geq d$, we have $y = y_n + g_n$, for some element $g_n \in Q_nB$. This shows that $y \in \mathfrak{a}_n$, for all $n \geq d$, and therefore $\bigcap_{n\in\mathbb{N}}\mathfrak{a}_n \neq 0$, a contradiction. Thus $\widehat{I} = (0)$. □

27.4. Preserving excellence or Henselian under multi-adic completion

The first four results of this section concern preservation of excellence.

THEOREM 27.14. *Assume notation as in Setting 27.1, and let the ring R^* be defined as in (27.1.1). If $(R, \mathbf{m})$ is an excellent local ring, then R^* is excellent.*

The following result is critical to the proof of Theorem 27.14.

LEMMA 27.15. [**96**, Theorem 32.5, page 259] *Let A be a semilocal Noetherian ring. Assume that $(\widehat{B})_Q$ is a regular local ring, for every local domain $(B, \mathbf{n})$ that is a localization of a finite A-algebra and for every prime ideal Q of the $\mathbf{n}$-adic completion $\widehat{B}$ such that $Q \cap B = (0)$. Then A is a G-ring, that is, $A \hookrightarrow \widehat{A_p}$ is regular for every prime ideal p of A; thus all of the formal fibers of all the local rings of A are geometrically regular.*

indexIdeal-adic!completion
We use Proposition 27.16 in the proof of Theorem 27.14.

PROPOSITION 27.16. *Let $(R, \mathbf{m})$ be a Noetherian local ring with geometrically regular formal fibers. Then R^* has geometrically regular formal fibers.*

PROOF. Let B be a domain that is a finite R^*-algebra and let $P \in \mathrm{Sing}(\widehat{B})$, that is, $\widehat{B}_P$ is not a regular local ring. To prove that R^* has geometrically regular formal fibers, by Lemma 27.15, it suffices to prove that $P \cap B \neq (0)$.

The Noetherian complete local ring $\widehat{R}$ has the property J-2 in the sense of Matsumura, that is, for every finite $\widehat{R}$-algebra, such as $\widehat{B}$, the subset $\operatorname{Reg}(\operatorname{Spec}(\widehat{B}))$, of primes where the localization of $\widehat{B}$ is regular, is an open subset in the Zariski topology; see [**94**, pp. 246–249]. Thus there is a reduced ideal $\widehat{I}$ in $\widehat{B}$ so that

$$\operatorname{Sing}(\widehat{B}) \;=\; \mathcal{V}(\widehat{I}).$$

If $\widehat{I} = (0)$, then $\widehat{B}$ is a reduced ring and, for all minimal primes Q of $\widehat{B}$, the localization $\widehat{B}_Q$ is a field, contradicting $Q \in \operatorname{Sing}(\widehat{B})$. Thus $\widehat{I} \neq (0)$. For all $n \in \mathbb{N}$:

$$\widehat{B_n} \cong \widehat{R_n} \otimes_{R^*} B$$

is a finite $\widehat{R}_n$-algebra. Since by [**118**] $\widehat{R_n}$ has geometrically regular formal fibers so has $\widehat{B_n}$. This implies that $\widehat{I}$ is extended from $\widehat{B_n}$ for all $n \in \mathbb{N}$. By Theorem 27.13, $\widehat{I}$ is extended from B, and so $\widehat{I} = I\widehat{B}$, where $0 \neq I := \widehat{I} \cap B$. Since $\widehat{I} \subseteq P$, we have $(0) \neq I \subseteq P \cap B$. □

Proof of Theorem 27.14 It remains to show that R^* is universally catenary. We have injective local homomorphisms $R \hookrightarrow R^* \hookrightarrow \widehat{R}$, and R^* is Noetherian with $\widehat{R^*} = \widehat{R}$. Proposition 27.17 below implies that R^* is universally catenary. □

PROPOSITION 27.17. *Let $(A, \mathbf{m})$ be a Noetherian local universally catenary ring and let $(B, \mathbf{n})$ be a Noetherian local subring of the $\mathbf{m}$-adic completion $\widehat{A}$ of A with $A \subseteq B \subseteq \widehat{A}$ and $\widehat{B} = \widehat{A}$, where $\widehat{B}$ is the $\mathbf{n}$-adic completion of B. Then B is universally catenary.*

PROOF. By [**96**, Theorem 31.7], it suffices to show for $P \in \operatorname{Spec}(B)$ that $\widehat{A}/P\widehat{A}$ is equidimensional. We may assume that $P \cap A = (0)$, and hence that A is a domain.

Let Q and W in $\operatorname{Spec}(\widehat{A})$ be minimal primes over $P\widehat{A}$.

Claim: $\dim(\widehat{A}/Q) = \dim(\widehat{A}/W)$.

Proof of Claim: Since B is Noetherian, the canonical morphisms $B_P \longrightarrow \widehat{A}_Q$ and $B_P \longrightarrow \widehat{A}_W$ are flat. By [**96**, Theorem 15.1],

$$\dim(\widehat{A}_Q) = \dim(B_P) + \dim(\widehat{A}_Q/P\widehat{A}_Q), \quad \dim(\widehat{A}_W) = \dim(B_P) + \dim(\widehat{A}_W/P\widehat{A_W}).$$

Since Q and W are minimal over $P\widehat{A}$, it follows that:

$$\dim(\widehat{A}_Q) = \dim(\widehat{A}_W) = \dim(B_P).$$

Let $q \subseteq Q$ and $w \subseteq W$ be minimal primes of $\widehat{A}$ so that:

$$\dim(\widehat{A}_Q) = \dim(\widehat{A}_Q/q\widehat{A}_Q) \quad \text{and} \quad \dim(\widehat{A}_W) = \dim(\widehat{A}_W/w\widehat{A}_W).$$

Since we have reduced to the case where A is a universally catenary domain, its completion $\widehat{A}$ is equidimensional and therefore:

$$\dim(\widehat{A}/q) = \dim(\widehat{A}/w).$$

Since a complete local ring is catenary [**96**, Theorem 29.4], we have:

$$\dim(\widehat{A}/q) = \dim(\widehat{A}_Q/q\widehat{A}_Q) + \dim(\widehat{A}/Q),$$
$$\dim(\widehat{A}/w) = \dim(\widehat{A}_W/w\widehat{A}_W) + \dim(\widehat{A}/W).$$

Since $\dim(\widehat{A}/q) = \dim(\widehat{A}/w)$ and $\dim(\widehat{A}_Q) = \dim(\widehat{A}_W)$, it follows that

$$\dim(\widehat{A}/Q) = \dim(\widehat{A}/W).$$

This completes the proof of Proposition 27.17. □

REMARK 27.18. Let R be a universally catenary Noetherian local ring. Proposition 27.17 implies that *every* Noetherian local subring B of $\widehat{R}$ with $R \subseteq B$ and $\widehat{B} = \widehat{R}$ is universally catenary. Hence, for each ideal I of R, the I-adic completion of R is universally catenary. Also R^* as in Setting 27.1 is universally catenary. Proposition 27.17 also implies that the Henselization of R is universally catenary. Seydi shows that the I-adic completions of universally catenary rings are universally catenary in [**127**]. Proposition 27.17 establishes this result for a larger class of rings.

PROPOSITION 27.19. *With notation as in Setting 27.1, let $(R, \mathbf{m}, k)$ be a Noetherian local ring. If R is Henselian, then R^* is Henselian.*

PROOF. Assume that R is Henselian. It is well known that every ideal-adic completion of R is Henselian, see [**119**, p.6]. Thus $\widehat{R_n}$ is Henselian for all $n \in \mathbb{N}$. Let $\mathbf{n}$ denote the nilradical of $\widehat{R}$. Then $\mathbf{n} \cap R^*$ is the nilradical of R^*, and to prove R^* is Henselian, it suffices to prove that $R' := R^*/(\mathbf{n} \cap R^*)$ is Henselian [**104**, (43.15)]. To prove R' is Henselian, by [**119**, Prop. 3, page 76], it suffices to show:

If $f \in R'[x]$ is a monic polynomial and its image $\bar{f} \in k[x]$ has a simple root, then f has a root in R'.

Let $f \in R'[x]$ be a monic polynomial such that $\bar{f} \in k[x]$ has a simple root. Since $\widehat{R_n}/(\mathbf{n} \cap \widehat{R_n})$ is Henselian, for each $n \in \mathbb{N}$, there exists $\widehat{\alpha}_n \in \widehat{R_n}/(\mathbf{n} \cap \widehat{R_n})$ with $f(\widehat{\alpha}_n) = 0$. Since f is monic and $\widehat{R}/(\mathbf{n} \cap \widehat{R})$ is reduced, f has only finitely many roots in $\widehat{R}/(\mathbf{n} \cap \widehat{R})$. Thus there is an α so that $\alpha = \widehat{\alpha}_n$, for infinitely many $n \in \mathbb{N}$. By Lemma 27.13, $R^* = \bigcap_{n \in \mathbb{N}} \widehat{R_n}$. Hence

$$R' = R^*/(\mathbf{n} \cap R^*) = \bigcap_{n \in \mathbb{N}} \widehat{R_n}/(\mathbf{n} \cap \widehat{R_n},$$

and so there exists $\alpha \in R'$ such that $f(\alpha) = 0$. □

Exercise

(1) Let R denote the ring and $\{q_n\}$ the family of ideals given in Remark 27.7. Consider the linear topology obtained by letting the ideals q_n be a subbase for the open neighborhoods of 0. Prove the q_n are also a subbase for the ideal-topology on R.

CHAPTER 28

Examples discussed in this book

Here is a list of examples presented in this book, with a brief description of each.

(1) The "simplest" example of a Noetherian local domain A on an algebraic function field L/k of at least two variables that is not essentially finitely generated over its ground field k, i.e., A is not the localization of a finitely generated k-algebra; see Example 4.1.

(2) A two-dimensional regular local domain A that is a nested union of three-dimensional regular local domains that A birationally dominates; see Example 4.3.

(3) A two-dimensional regular local domain A that is a nested union of four-dimensional regular local domains that A birationally dominates; see Example 4.4.

(4) A one-dimensional Noetherian local domain A that is the local coordinate ring of a nodal plane curve singularity; see Example 4.6. The integral closure of A is a homomorphic image of a regular Noetherian domain of dimension two with precisely two maximal ideals.

(5) A two-dimensional regular local domain A that is not Nagata and thus not excellent. The ring A contains a prime element f that factors as a square in the completion $\widehat{A}$ of A, that is, $f = g^2$ for some element $g \in \widehat{A}$; see Example 4.8, Remarks 4.9.2, Proposition 10.2 and Remark 10.3, [**104**, Example 7, pp. 209-211].

(6) A two-dimensional normal Noetherian local domain D that is analytically reducible; see Example 4.8 and Remarks 4.9.1, [**104**, Example 7, pp. 209-211].

(7) A three-dimensional regular local domain A that is Nagata but not excellent. The formal fibers of A are reduced but not regular; see Examples 4.10 and 10.7 and Remark 4.11, [**118**].

(8) A non-Noetherian three-dimensional local Krull domain $(B, \mathbf{n})$ such that $\mathbf{n}$ is two-generated, the $\mathbf{n}$-adic completion of B is a two-dimensional regular local domain, and B birationally dominates a four-dimensional regular local domain; see Theorem 7.2 and Examples 7.3 and 8.11.

(9) Every Noetherian local domain $(A, \mathbf{n})$ having a coefficient field k, and having the property that the field of fractions L of A is finitely generated over k is realizable as an intersection $L \cap \widehat{R}/I$, where R is a Noetherian local domain essentially finitely generated over k with $\mathcal{Q}(R) = L$, and I is an ideal in the completion $\widehat{R}$ of R such that $P \cap R = (0)$ for each associated prime P of I; see Corollary 5.10.

(10) An example of Inclusion Construction 5.3 where the approximation domain B is equal to the intersection domain A; see Examples 6.6, 6.7 and 7.15.
(11) A strictly descending chain of one-dimensional analytically ramified Noetherian local domains that birationally dominate a polynomial ring in two variables over a field; see Example 8.7.
(12) A non-excellent DVR obtained by Localized Polynomial Example Theorem 9.7; see Proposition 9.3.
(13) A two-dimensional non-excellent regular local domain obtained by Localized Polynomial Example Theorem 9.7; see Remark 9.4.
(14) For each pair of positive integers r, n, a Noetherian local domain A with $\dim A = r$ and a principal ideal-adic completion A^* of A such that A^* has nilradical with nilpotency index n; see Example 9.8.
(15) A non-universally catenary two-dimensional Noetherian local domain B that birationally dominates a three-dimensional regular local domain. The completion of B has two minimal primes, one of dimension one and one of dimension two. The ring B is not a homomorphic image of a regular local ring; see Example 9.11.
(16) An example of Insider Construction 13.1 where the approximation domain B is equal to the intersection domain A. The domain B is a non-catenary non-Noetherian four-dimensional local UFD that is very close to being Noetherian. The ring B has exactly one prime ideal Q of height three; the ideal Q is not finitely generated; see Examples 10.8 and 18.1.
(17) For every $m, n \in \mathbb{N}$ with $n \geq 4$, an example of Insider Construction 13.1 where the approximation domain B is equal to the intersection domain A, B has dimension n, and B has exactly m prime ideals of height $n-1$. The domain B is a non-catenary non-Noetherian UFD, and every prime ideal of B of height $n-1$ is not finitely generated; see Examples 10.8 and 13.8.
(18) An example of Insider Construction 13.1, where the approximation domain B is properly contained in the intersection domain A, and neither A nor B is Noetherian. The local domain B is a UFD that fails to have Cohen-Macaulay formal fibers; see Example 10.10 and Section 22.4.

From Chapter **??**

(19) A general example of a nonfinite TGF-complete embedding of a power series ring $R = k[[x_1, \ldots, x_n]]$ in n variables over a field k into a power series ring in two variables over k; see Example **??** and Section 22.4. A particular case is given in Example **??**¿
(20) An example where $\sigma : R \hookrightarrow S$ is an inclusion map, $\tau : S \hookrightarrow T$ is a TGF-embedding, and $\tau \cdot \sigma = \varphi$ is TGF, but $\sigma : R \hookrightarrow S$ is not TGF.
(21) An example where A is a 3-dimensional normal local domain, B is a 2-dimensional regular local domain, the residue field of B is transcendental over that of A and $(A, \mathbf{m}) \hookrightarrow (B, \mathbf{n})$ is a TGF extension, but $\widehat{A} \hookrightarrow \widehat{B}$ is not TGF-complete.

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