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COMMUTATIVE IDEAL THEORY WITHOUT FINITENESS CONDITIONS: PRIMAL IDEALS

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ABSTRACT. Our goal is to establish an efficient decomposition of an ideal A of a commutative ring R as an intersection of primal ideals. We prove the existence of a canonical primal decomposition: $A = \bigcap_{P \in \mathcal{X}_A} A_{(P)}$, where the $A_{(P)}$ are isolated components of A that are primal ideals having distinct and incomparable adjoint primes P. For this purpose we define the set $\mathrm{Ass}(A)$ of associated primes of the ideal A to be those defined and studied by Krull. We determine conditions for the canonical primal decomposition to be irredundant, or residually maximal, or the unique representation of A as an irredundant intersection of isolated components of A. Using our canonical primal decomposition, we obtain an affirmative answer to a question raised by Fuchs in [5] and also prove for $P \in \operatorname{Spec} R$ that an ideal $A \subseteq P$ is an intersection of P-primal ideals if and only if the elements of $R \setminus P$ are prime to A. We prove that the following conditions are equivalent: (i) the ring R is arithmetical, (ii) every primal ideal of R is irreducible, (iii) each proper ideal of R is an intersection of its irreducible isolated components. We classify the rings for which the canonical primal decomposition of each proper ideal is an irredundant decomposition of irreducible ideals as precisely the arithmetical rings with Noetherian maximal spectrum. In particular, the integral domains having these equivalent properties are the Prüfer domains that satisfy (##).

INTRODUCTION

In her seminal paper [27], Emmy Noether proved that in a commutative ring satisfying the ascending chain condition on ideals every ideal is the intersection of a finite number of irreducible ideals, and the irreducible ideals are primary ideals. She went on to establish several intersection decompositions, one of which featured primary ideal components with distinct radicals.

Among rings without the ascending chain condition, the rings in which such a decomposition holds for all ideals (called Laskerian rings) are few and far between (see e.g. Heinzer-Lantz [13]). One reason for this is that irreducible ideals need not be primary. Thus, if we wish to obtain analogous decompositions in general rings, then we have to replace primary ideals by more general ideals. Natural candidates are the primal ideals introduced in Fuchs [5] where it is shown that every ideal is the intersection of (in general infinitely many) irreducible ideals and these ideals are primal ideals. This primal decomposition of an ideal gives components whose structure we know, but the components are not closely tied to the ideal and there is no indication of how many and what kind of primal ideals are needed for the

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intersection. It is also possible to represent an ideal as the intersection of its isolated components. In this case the components are closely tied to the ideal, but the structure of the components is not known. In general, the isolated components of an ideal are not primal ideals and there are almost always superfluous components.

Our main purpose here is to establish a more efficient decomposition of an ideal into the intersection of primal ideals. We show in Theorem 3.5 that if we take just those components that appear in both decompositions, then we can still represent the ideal even with the minimal isolated primal components. The existence of such decompositions in the Noetherian case was established in Fuchs [5]. In Theorem 3.5 we achieve a canonical primal decomposition (that is often irredundant) where the primal components belong to distinct, incomparable primes and where these distinct primes are the maximal primes in the set of associated primes of A; here we are using associated primes in the sense defined by Krull [19, page 742]. Also see [1, page 253]. With this definition of associated primes, maximal associated primes of A exist and every associated prime is contained in a maximal associated prime.

Krull [20, page 16] mentions that it was an undecided problem whether or not (phrased in our terminology) the principal component $A_{(P)}$ (for P a maximal prime divisor of A) is P-primal. Nagata in [24] and Gilmer in [8] gave a negative answer, but the full answer is given in Theorem 3.4: the isolated P-component of A is P-primal if and only if P is a Krull associated prime of A.

We establish in Corollary 3.6 an affirmative answer to the question raised in [5, page 3] of whether for a maximal prime divisor P of an ideal A the principal P-component $A_{(P)}$ is equal to the intersection of all primal ideals that contain A and have their associated adjoint primes contained in P. Indeed, Proposition 3.7 implies that $A_{(P)}$ is an intersection of P-primal ideals. In Example 3.8, we present an example where the ideal A has precisely two maximal prime divisors and where the principal component with respect to each of these maximal prime divisors is not a primal ideal.

We prove in Theorem 4.2 that under certain additional hypothesis the canonical primal decomposition of Theorem 3.5 is irredundant. We establish in Proposition 4.3 under certain conditions that a stronger form of irredundancy holds: namely, the canonical primal decomposition of Theorem 3.5 is residually maximal in the sense that no component can be replaced by any of its proper residuals. For P a maximal associated prime of the ideal A, we give in Theorem 4.6 necessary and sufficient conditions in order that the isolated component $A_{(P)}$ appear in every representation of A as an intersection of primal isolated components of A.

In Section 5 we consider irredundant decompositions into irreducible ideals. If an ideal A has a representation as an irredundant intersection of irreducible isolated components, we prove in Theorem 5.1 that this intersection is residually maximal and is the unique irredundant decomposition of A into irreducible isolated components of A. Example 5.4 shows there are limitations as to any sharpening of Theorem 5.1. In Theorem 5.8 we prove that a ring R is an arithmetical ring if and only if each proper ideal of R can be represented as an intersection of irreducible isolated components. Indeed, if R is an arithmetical ring and A can be represented as an irredundant intersection of irreducible ideals, then this is the unique representation of A as an irredundant intersection of irreducible ideals (Corollary 5.6).

We also consider in Section 5 the question of what rings R have the property that the canonical primal decomposition of each proper ideal A of R is an irredundant representation of A as an intersection of irreducible isolated components. We classify in Theorem 5.14 the rings R having this property as the arithmetical rings with Noetherian maximal spectrum. In particular, the integral domains having this property are precisely the Prüfer domains that satisfy condition (##).

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1. Preliminaries

Throughout R will denote a commutative ring with 1. For ideals I, J of R, the residual $I : J$ is defined as usual by

$$I : J = \{x \in R \ : \ xJ \subseteq I\}.$$

For an ideal A and for a prime ideal P of R, we use the notation

$$A_{(P)} = \{x \in R \ : \ sx \in A \text{ for some } s \in R \setminus P\} = \bigcup_{s \in R \setminus P} A : s$$

to denote the *isolated P-component* (isoliertes Komponentenideal) of A in the sense of Krull [20, page 16]. It will be useful to observe that $x \in A_{(P)}$ if and only if $A : x \not\subseteq P$. If R is a domain, then $A_{(P)} = AR_P \cap R$, where R_P denotes the localization of R at P.

Evidently, $A_{(P)} \subseteq P$ for every prime P with $A \subseteq P$. If P is a minimal prime of A, then $A_{(P)}$ is known to be a P-primary ideal: the *isolated P-primary component* of A.

Lemma 1.1. *For every ideal A, prime ideal P and element $x \in R$, we have*

$$(A : x)_{(P)} = A_{(P)} : x.$$

Proof. Evidently, $y \in (A : x)_{(P)} \Leftrightarrow \exists c \notin P$ such that $cy \in A : x \Leftrightarrow \exists c \notin P$ with $cxy \in A \Leftrightarrow xy \in A_{(P)} \Leftrightarrow y \in A_{(P)} : x$. □

An element $x \in R$ is called *non-prime to A* if $A \subset A : x$ (proper inclusion), i.e. there exists an element $c \in R \setminus A$ such that $cx \in A$. Evidently, the set $S(A)$ of non-prime elements to A corresponds to the set of zero-divisors in the factor ring R/A; it is the set union of prime ideals: the maximal prime divisors of A.

An ideal A is said to be *primal* (see [5]) if $S(A)$ itself is an ideal, i.e. if the elements of R that are not prime to A form an ideal P; this ideal is then a prime ideal, called the *adjoint ideal* of A. In this case, we also say that A is a *P-primal ideal.* An ideal A is primal if and only if the sum of any two ring elements non-prime to A is again non-prime to A.

Irreducible ideals are in general not primary (for a necessary and sufficient condition see Fuchs [4]). Primal ideals have the advantage over primary ideals that the following is true without any chain condition:

Lemma 1.2. (Fuchs [5])

(a) *Irreducible ideals are primal.*
(b) *Every ideal is the intersection of primal ideals.*

We use the following result.

Lemma 1.3. *If A is a P-primal ideal, then*

(i) $A_{(Q)} = A$ *if Q is a prime containing P;*
(ii) $A_{(Q)} \supset A$ *(proper inclusion) if Q is a prime not containing P;*
(iii) *if $A_{(Q)}$ is a Q-primal ideal for some prime Q containing A, then $Q \subseteq P$.*

Proof. (i) To show that $A_{(Q)} \subseteq A$, observe that $x \in A_{(Q)}$ means $cx \in A$ for some $c \in R \setminus Q$. But if A is P-primal, then c is prime to A, so $x \in A$.

(ii) There exists an $x \in P \setminus Q$. Such an x is not prime to A, so for some $y \notin A$ we have $xy \in A$. Then $y \in A_{(Q)} \setminus A$.

(iii) By assumption, $Q = \cup_{x \in R \setminus A_{(Q)}} A_{(Q)} : x$. If $Q \not\subseteq P$, then there exists $q \in Q \setminus P$ such that $qx \in A_{(Q)}$ for some $x \in R \setminus A_{(Q)}$. Thus there exists $c \in R \setminus Q$ such that $cqx \in A$. Now $q \notin P$, so since $A = A_{(P)}$, $cx \in A$. However, $c \notin Q$, so $x \in A_{(Q)}$, contrary to assumption. We conclude that $Q \subseteq P$. □

Primal ideals behave similarly to primary ideals, but there are notable differences. For instance, the intersection of two primal ideals with the same adjoint prime need not be primal again. A counterexample is the principal ideal xyR in the polynomial ring $R = F[x, y]$, where x and y are indeterminates over a field F. The ideal xyR is not primal (x, y are not prime to xyR, but $x - y$ is), though it is the intersection of the $(x, y)R$-primal ideals $(x^2, xy)R$ and $(xy, y^2)R$. However, if we have a *reduced* intersection (i.e. none of the components may be replaced by a larger ideal without changing the intersection), then we can claim:

Lemma 1.4. (Fuchs [5]) *A reduced intersection $A = A_1 \cap \cdots \cap A_n$ of primal ideals A_i with adjoint primes P_i is again primal if and only if there is a unique maximal member in the set $\{P_1, \ldots, P_n\}$. This prime is then the adjoint prime to A.*

Recall that the closed sets of the prime spectrum $\operatorname{Spec} R$ of the ring R in the *Zariski topology* are the sets $V(I) = \{P \in \operatorname{Spec} R : P \supseteq I\}$ with I ranging over the set of ideals of R. $\operatorname{Spec} R$ is called *Noetherian* if the closed subsets in the Zariski topology satisfy the descending chain condition, or equivalently, if the radical ideals of R satisfy the ascending chain condition. The *maximal spectrum* of R is the set $\operatorname{Max} R$ of maximal ideals of R with the subspace topology from $\operatorname{Spec} R$. We say that R has *Noetherian maximal spectrum* if the closed subsets of $\operatorname{Max} R$ satisfy the descending chain condition, or equivalently, if the J-radical ideals of R satisfy the ascending chain condition, where an ideal is a *J-radical ideal* if it is an intersection of maximal ideals.

Some of the following characterizations of rings with Noetherian prime spectra will be needed in our discussions.

Lemma 1.5. (Mori [23] or Ohm-Pendelton [28]) *For a commutative ring R the following conditions are equivalent:*

(i) $\operatorname{Spec} R$ *is Noetherian;*
(ii) *R has the maximum condition on radical ideals;*
(iii) *the ascending chain condition holds for prime ideals and each ideal of R has only finitely many minimal prime ideals.*

We will pay special attention to the class of arithmetical rings. In addition to providing an interesting contrast to the theory of primal decompositions in Noetherian rings, Theorem 1.8 and the results of Section 5 show that this class of rings arises naturally in our context.

Recall that a ring R is a *chained* ring or *valuation* ring if the ideals of R are linearly ordered with respect to inclusion; and R is an *arithmetical* ring if R_M is a valuation ring for each maximal ideal M of R. An integral domain is an arithmetical ring if and only if it is a Prüfer domain. Among several results that are relevant to our study is the theorem of Jensen that primary ideals of an arithmetical ring are irreducible and that a Prüfer domain R has the property that each ideal of R with prime radical is irreducible if and only if each nonzero prime ideal of R is contained in a unique maximal ideal [17, Theorem 6]. It is readily seen that a ring R is an arithmetical ring if and only if for each proper ideal A of R, A_M is an irreducible ideal of R_M for every maximal ideal M of R containing A.

Remark 1.6. Let A be an ideal of a ring R and let $P \in \operatorname{Spec} R$. The ideal A_P is irreducible in R_P if and only if $A_{(P)}$ is irreducible in R. This statement is clear in view of the following facts: (i) finite intersections of ideals of R behave well with respect to the map $R \to R_P$, (ii) every ideal of R_P comes from an ideal of R, and (iii) $A_{(P)}$ is the preimage in R of the ideal A_P of R_P.

From this observation we deduce the characterization: *A ring R is arithmetical if and only if for all ideals A of R, $A_{(M)}$ is an irreducible ideal for all maximal ideals M that contain A.* For if R is arithmetical, then every ideal of R_M, M a maximal ideal of R, is irreducible, so if A is an ideal of R contained in M, A_M, and hence $A_{(M)}$, are irreducible ideals of R_M and R, respectively. Conversely, as noted in the paragraph before Remark 1.6, to prove R is arithmetical it is enough to show for every ideal A and maximal ideal M containing A that A_M is an irreducible ideal. This is indeed the case, since by assumption $A_{(M)}$ is irreducible and we have from the above observation that A_M is an irreducible ideal of R_M.

Lemma 1.7. *Let A be a finitely generated ideal contained in a prime ideal P of the ring R. If $A_P \neq 0$, then $(AP)_{(P)}$ is a P-primal ideal of R.*

Proof. Set $B = AP$. Clearly, the elements of R not prime to $B_{(P)}$ are contained in P, so to show $B_{(P)}$ is a P-primal ideal of R, it suffices to prove that the elements of P are not prime to $B_{(P)}$. Since A is finitely generated and $A_P \neq 0$, Nakayama's Lemma implies that $B_P \neq A_P$, and it follows that $B_{(P)} \neq A_{(P)}$. Thus $B_{(P)} \subset A_{(P)} \subseteq B_{(P)} : P$, so there exists $y \in R \setminus B_{(P)}$ such that $yP \subseteq B_{(P)}$. This proves that the elements of P are not prime to $B_{(P)}$ and $B_{(P)}$ is a P-primal ideal. □

Theorem 1.8. *The following statements are equivalent for a ring R.*

(i) *R is an arithmetical ring.*
(ii) *Every primal ideal of R is irreducible.*

Proof. (i) ⇒ (ii). The implication (i) ⇒ (ii) is proved in [7] for the case where R is a domain. Allowing zero divisors requires a simple modification of this argument. If A is a primal ideal, then $A = A_{(P)}$, where P is the adjoint prime of A. By (i), A_P, and hence $A_{(P)}$ (Remark 1.6), are irreducible ideals, and this proves (ii).

(ii) ⇒ (i) We use the following observation: (†) *Let $I = (a, b)R$ be a finitely generated ideal contained in a maximal ideal M of R. If $I_M \neq 0$ and IM_M is an irreducible ideal of R_M, then $I_M = R_M a$ or $I_M = R_M b$.* For the R/M-vector space I_M/IM_M is finite-dimensional. Since IM_M is irreducible, it follows that $I_M/IM_M \cong R/M$. In particular, $I_M = aR_M + IM_M$ or $I_M = bR_M + IM_M$. By Nakayama's Lemma, $I_M = aR_M$ or $I_M = bR_M$.

To prove (i), it suffices to show for each maximal ideal M of R and $a, b \in M$ that either $R_M a \subseteq R_M b$ or $R_M b \subseteq R_M a$, for this implies that R_M is a valuation ring. We may also assume that $I = (a, b)R$ is such that $I_M \neq 0$. By Lemma 1.7, $(IM)_{(M)}$ is an M-primal ideal, so by (ii) and Remark 1.6, IMR_M is an irreducible ideal of R_M. Thus by the observation (†), either $R_M a \subseteq R_M b$ or $R_M b \subseteq R_M a$. $\square$

2. Associated Primes

We start with the study of primes associated to an ideal. There are at least six inequivalent notions of associated primes in the literature, all of which are equivalent in the Noetherian case. (See Heinzer-Ohm [14] and Iroz-Rush [16], for example.)

For our purposes, the most useful concept of associated primes is one introduced by Krull [19, page 742]. Let A be a proper ideal of the ring R. Following [16], we define a prime ideal P of a ring R to be a *Krull associated prime* of the ideal A if for every element $x \in P$, there exists $y \in R$ such that $x \in A : y \subseteq P$. A prime P of R is called a *weak-Bourbaki associated prime* to A if it is a minimal prime divisor of $A : x$ for some $x \in R \backslash A$. Following [14, page 279], we call P a *Zariski-Samuel associated prime* of A if $P = \sqrt{A : x}$ for some $x \in R$. (This differs slightly from the terminology of [31], where P is said to be a Zariski-Samuel associated prime of A if $A : x$ is P-primary for some $x \in R$. Of course if $A : x$ is P-primary, then P is a Zariski-Samuel associated prime in our sense.) It is clear that a Zariski-Samuel associated prime is a weak-Bourbaki associated prime.

Lemma 2.1. *Let A be a proper ideal of the ring R. Every weak-Bourbaki associated prime of A is a Krull associated prime of A. A prime ideal Q of R is a Krull associated prime of A if and only if Q is a set-theoretic union of weak-Bourbaki associated primes of A.*

Proof. Let P be a weak-Bourbaki associated prime of A. Then P is a minimal prime of $A : x$ for some $x \notin A$. It follows that the ideal $(A : x)_{(P)}$ is P-primary. Thus given $u \in P$, there is a smallest integer $k \geq 1$ such that $u^k \in (A : x)_{(P)}$. Hence $u^k v \in A : x$ for some $v \notin P$. Evidently, $u \in A : xu^{k-1}v$. If $A : xu^{k-1}v \subseteq P$ were not true, then we could find a $w \notin P$ with $w \in A : xu^{k-1}v$. But then $u^{k-1}vw \in A : x$, and $u^{k-1} \in (A : x)_{(P)}$ is impossible. Thus $A : xu^{k-1}v \subseteq P$, indeed. (See also [21, Theorem 1] or [16, page 346].) It follows that a prime ideal Q of R is a Krull associated prime of A if it is a set-theoretic union of weak-Bourbaki associated primes of A. The converse is clear, since $x \in A : y \subseteq Q$ implies that x is contained in every minimal prime of $A : y$. $\square$

It is obvious that every proper ideal of the ring has weak-Bourbaki associated primes; hence every proper ideal possesses at least one Krull associated prime. However, this is not the case for Zariski-Samuel associated primes: Nakano [26] gives an example of a Prüfer domain in which finitely generated ideals do not admit any Zariski-Samuel associated prime.

We are convinced that the notion of a Krull associated prime is more useful (and perhaps more natural) than any of the other definitions proposed in the literature, because besides being equivalent to the classical notion of associated prime in Noetherian rings, it has the following pleasant properties: 1) the set union of the Krull associated primes of an ideal A is the set of non-prime elements to A (so there is a sufficient supply of them); 2) every Krull associated prime is contained in a maximal Krull associated prime; 3) Krull associated primes are preserved under localization; and 4) they can be recognized by a distinguished property of the corresponding isolated components of A (see Theorem 3.4).

For an ideal A of R we follow [2, page 289] and define $\mathrm{Ass}_f(A)$ to be the set of weak-Bourbaki associated primes of A. We reserve the notation $\mathrm{Ass}(A)$ for the set of Krull associated primes of A, that is,

$$\mathrm{Ass}(A) = \{P \ : \ P \text{ is a prime that is the set union of primes in } \mathrm{Ass}_f(A)\}.$$

Finally, let

$$\mathcal{X}_A = \{P \ : P \text{ is a maximal member of } \mathrm{Ass}(A)\}$$

That there is a sufficient supply of primes in $\mathcal{X}_A$ (and hence in $\mathrm{Ass}(A)$) is shown by the next lemma (cf. [2, Exercise 17(b), page 289]).

Lemma 2.2. *Let A be a proper ideal of the ring R and let $S(A)$ denote the set of elements of R that are non-prime to A. Then*

$$S(A) = \bigcup_{P \in \mathrm{Ass}_f(A)} P = \bigcup_{P \in \mathrm{Ass}(A)} P = \bigcup_{P \in \mathcal{X}_A} P.$$

Proof. The last two equalities being obvious, it suffices to verify the first. Let $u \in P$ for some $P \in \mathrm{Ass}_f(A)$. Since P is a minimal prime of $A : x$ for some $x \in R \setminus A$, and $(A : x)_{(P)} = A_{(P)} : x$ is a P-primary ideal, there is an integer $k > 0$ such that $u^k \in A_{(P)} : x$. Thus $su^k x \in A$ for some $s \notin P$. But $A : x \subseteq P$ implies $sx \notin A$, so $u^k \in S(A)$ and hence $u \in S(A)$ as well.

On the other hand, if $u \in S(A)$, then there is a $v \notin A$ such that $u \in A : v$. If P is a minimal prime divisor of $A : v$, then $u \in A : v \subseteq P \in \mathrm{Ass}_f(A)$. □

The maximal prime divisors of A (i.e. the ideals maximal in the set $S(A)$) need not belong to $\mathrm{Ass}(A)$. This situation is illustrated by examples given by Nagata [24], by Gilmer [8], and by Example 3.8 presented below. However, we have

Corollary 2.3. *If A is a primal ideal of R, then its adjoint prime ideal P is a Krull associated prime of A.*

Proof. By Lemma 2.2, $P = S(A)$ is as stated. □

If A is P-primal, then P is a Zariski-Samuel prime of A if and only if P is a weak-Bourbaki prime of A. For it is always true that Zariski-Samuel implies weak-Bourbaki. On the other hand, if $x \in R \setminus A$, then every minimal prime of $A : x$ consists of elements non-prime to A. Therefore if A is P-primal, then every minimal prime of $A : x$ is contained in P. Hence P is a minimal prime of $A : x$ implies $P = \sqrt{A : x}$ and P is a Zariski-Samuel prime of A.

As far as the behavior of associated primes under localization is concerned, we have the most satisfactory result (cf. [2, Exercise 17(d), page 289]):

Lemma 2.4. *Let A be an ideal of the ring R and let P be a prime ideal of R containing A.*

(i) *The following statements are equivalent for P.*
   (a) *$P \in \mathrm{Ass}_f(A)$.*
   (b) *$P_M \in \mathrm{Ass}_f(A_M)$ for every maximal ideal M of R containing P.*
   (c) *$P_M \in \mathrm{Ass}_f(A_M)$ for some maximal ideal M of R containing P.*

(ii) *The following statements are equivalent for P.*

(a) $P \in \mathrm{Ass}(A)$.
(b) $P_M \in \mathrm{Ass}(A_M)$ *for every maximal ideal* M *of* R *containing* P.
(c) $P_M \in \mathrm{Ass}(A_M)$ *for some maximal ideal* M *of* R *containing* P.

Proof. (i) To see that (a) implies (b), suppose P is a minimal prime over $A : r$ for some $r \in R \setminus A$, and let M be a maximal ideal of R containing P. There exists a prime ideal Q of R such that Q_M is minimal over $A_M : r$, so $A_M : r \subseteq Q_M \subseteq P_M$. In this case, $A : r \subseteq Q \subseteq P$, so $Q = P$. This proves (a) implies (b). That (b) implies (c) is clear, so to prove that (c) implies (a), suppose there is a maximal ideal M of R such that P_M is minimal over $A_M : s$ for some $s \in R \setminus A_M$. Then $A : s \subseteq P$, and there exists a minimal prime Q of $A : s$ such that $A : s \subseteq Q \subseteq P$. This implies $A_M : s \subseteq Q_M \subseteq P_M$, so $Q_M = P_M$ and $Q = P$.

(ii) If $P \in \mathrm{Ass}(A)$, then P is the union of weak-Bourbaki primes Q of A. Let M be a maximal ideal of R that contains P. By (i), Q_M is a weak-Bourbaki prime of A_M, so P_M is the union of weak-Bourbaki primes Q_M of A_M. This proves (a) implies (b). The assertion that (b) implies (c) is clear. To complete the proof, we verify (c) implies (a). Suppose $P_M \in \mathrm{Ass}(A_M)$ for some maximal ideal M containing P. Then P_M is the union of weak-Bourbaki primes Q_M of A_M. By (i), the preimage Q in R of each Q_M is a weak-Bourbaki prime of A, so P is the union of weak-Bourbaki primes of A. □

With a hypothesis of Noetherian spectrum we can say more about the primes in $\mathrm{Ass}(A)$ and about the set $\mathcal{X}_A$. However, as Example 3.8 below illustrates, it is possible even in a ring with Noetherian spectrum for there to exist an ideal A having a maximal prime divisor P such that $P \notin \mathrm{Ass}(A)$. (This cannot happen if the ring is Noetherian.)

Lemma 2.5. *Suppose that* A *is a proper ideal of the ring* R *and the spectrum of* R/A *is Noetherian.*

(i) *All the ideals in* $\mathrm{Ass}_f(A)$ *are Zariski-Samuel associated primes.*
(ii) *The primes in* $\mathcal{X}_A$ *are either maximal members in the set*

$$\{\sqrt{A : x} \ : \ x \in R \setminus A\} \tag{1}$$

of radical ideals or unions of incomparable primes in $\mathrm{Ass}_f(A)$.
(iii) *If* $P = \sqrt{A : y}$ *is maximal in the set* (1), *then* $A : y$ *is* P*-primary.*

Proof. (i) By Heinzer-Ohm [14, Lemma 3.2], the members of $\mathrm{Ass}_f(A)$ are all Zariski-Samuel associated primes, if the radical ideals containing A satisfy the ascending chain condition (cf. Lemma 1.5). Then $\mathrm{Ass}(A)$ is the set union of $\mathrm{Ass}_f(A)$ and unions of incomparable primes in $\mathrm{Ass}_f(A)$. (That the second kind of ideals may exist is illustrated by Example 4.5 infra.)

(ii) By Lemma 1.5(ii), the set $\{\sqrt{A : x} \ : \ x \in R \setminus A\}$ of radical ideals contains maximal members. Let $A : y$ $(y \in R \setminus A)$ be an ideal with maximal radical $\sqrt{A : y}$. Lemma 1.5(iii) guarantees that $A : y$ has at most finitely many minimal primes, thus there are finitely many primes $P_1, \ldots, P_m$ with $\sqrt{A : y} = P_1 \cap \cdots \cap P_m$, where no P_i may be omitted. If $m > 1$, pick a $u \in (P_1 \cap \cdots \cap P_{m-1}) \setminus P_m$ and a $v \in P_m \setminus (P_1 \cap \cdots \cap P_{m-1})$. Then $u^k \notin A : y$ for any $k > 0$, so by the maximal choice of $\sqrt{A : y}$, we must have $\sqrt{A : yu^k} = P_1 \cap \cdots \cap P_m$ for every $k > 0$. There is certainly a $k > 0$ such that $(uv)^k \in A : y$. Hence we obtain $v^k \in A : yu^k$, so

$v \in \sqrt{A : yu^k} \subseteq P_1 \cap \cdots \cap P_m$, a contradiction. We have thus shown that the radical of $A : y$ is the prime ideal $P = P_m$.

(iii) To show that $A : y$ is P-primary, let $r, s \in R$ such that $rs \in A : y$, but $r \notin A : y$. Then by maximality again $\sqrt{A : ry} = P$, whence $s \in A : ry \subseteq P$. □

Note that from Lemma 2.5 it follows that if R is a Prüfer domain and $\mathrm{Spec}(R/A)$ is a Noetherian space, then every Krull associated prime of A is a Zariski-Samuel associated prime of A.

Example 2.6. *For an ideal A, the set $\mathcal{X}_A$ need not be finite even if R has Noetherian spectrum.* Let F be a field, and x, y indeterminates over F. Let V be the discrete valuation domain obtained by localizing the polynomial ring $F[x, y]$ at the prime ideal generated by y. Then $V = F(x) + P$, where P denotes the maximal ideal of V.

It is easily seen that $R = F[x]+P$ is a Prüfer domain with P as the unique prime ideal of height one. Evidently, the factor ring R/P is isomorphic to the polynomial ring $F[x]$, and as a result, R has infinitely many maximal ideals of height 2.

Consider the ideal $A = yR$. Manifestly, Spec R is Noetherian. It is straightforward to check that each of the height 2 maximal ideals of R consists of zero-divisors in R/A and is a Krull associated prime to A. Hence $\mathcal{X}_A$ is an infinite set.

For an ideal A of a Noetherian domain R, it may happen that there exist prime ideals $P \subset Q$ with P and Q in $\mathrm{Ass}(A)$ and yet there exist infinitely many prime ideals L with $P \subset L \subset Q$ and $L \notin \mathrm{Ass}(A)$. Consider, for example, $R = F[x, y, z]$, where x, y, z are indeterminates over a field F and $A = (x^2, xy, xz)R$. Then $\mathrm{Ass}(A) = \{P, Q\}$, where $P = xR$ and $Q = (x, y, z)R$ and there exist infinitely many prime ideals L such that $P \subset L \subset Q$ (see, for example [18, page 7]. Moreover, P is the radical of A and Q is the adjoint of the primal ideal A.

To illustrate the properties of $\mathrm{Ass}(A)$ in a non-Noetherian setting, we turn our attention to arithmetical rings. In this case we can assert very specific properties of the set $\mathrm{Ass}(A)$. It is interesting to compare Proposition 2.7 to the example of the preceding paragraph.

Proposition 2.7. *Let R be an arithmetical ring and let A be a proper ideal of R. If $P \in \mathrm{Ass}(A)$, then $Q \in \mathrm{Ass}(A)$ for every prime ideal Q of R such that $A \subseteq Q \subseteq P$.*

Proof. By Lemma 2.4, it is enough to prove the proposition in the case where R is a valuation ring. Furthermore, we may assume that $A = (0)$. If $P \in \mathrm{Ass}(A)$, then R_P is a valuation ring whose maximal ideal consists of zero divisors. Thus if $Q \subseteq P$, then by Lemma 1.1 of [11], QR_Q consists of zero divisors of R_Q, and it follows that $Q \in \mathrm{Ass}(A)$. □

We conclude from Proposition 2.7 that if R is a valuation ring and $A \neq 0$ is a proper ideal of R, then $\mathrm{Ass}(A)$ is exactly the set of all primes between the radical of A and the adjoint prime of A. In this case, all the primes in $\mathrm{Ass}(A)$ are Zariski-Samuel associated primes with the possible exception of those primes that are infinite unions of strictly smaller primes.

Let A be a nonzero proper ideal of a Prüfer domain R and let M be a maximal ideal of R with $A \subseteq M$. Since R_M is a valuation domain, $\mathrm{End}(AR_M) = R_P$ for some $P \in \mathrm{Spec}\, R$ with $P \subseteq M$. Let

$$\mathcal{E}_A = \{P \in \mathrm{Spec}\, R : R_P = \mathrm{End}(AR_Q) \text{ for some prime ideal } Q \text{ containing } A\}.$$

Proposition 2.8. *Let A be a nonzero proper ideal of a Prüfer domain R. Then* $\mathrm{Ass}(A) = \mathcal{E}_A$. *In particular, if $P \in \mathcal{X}_A$, then there exists a maximal ideal M of R such that* $\mathrm{End}(A_M) = R_P$.

Proof. We first show $\mathrm{Ass}(A) \subseteq \mathcal{E}_A$. By Proposition 2.7, it is enough to verify that $\mathcal{X}_A \subseteq \mathcal{E}_A$. Let $P \in \mathcal{X}_A$, and let M be some maximal ideal of R containing P. Since $P \in \mathrm{Ass}(A)$, $P_M \in \mathrm{Ass}(A_M)$ by Lemma 2.4. If Q_M is the adjoint prime of the irreducible (hence, by Lemma 1.2, primal) ideal A_M, then $P_M \subseteq Q_M$, and $Q_M \in \mathrm{Ass}(A_M)$ by Corollary 2.3. Thus, by Lemma 2.4, $Q \in \mathrm{Ass}(A)$. Since $P \subseteq Q$ and P is in $\mathcal{X}_A$, it must be the case that $P = Q$. Thus A_M has adjoint prime P_M, and by Lemma 2.4 in [7], $\mathrm{End}(A_M) = R_P$. Consequently, $P \in \mathcal{E}_A$.

To prove the converse, by Proposition 2.7, it suffices to show that every maximal element P (with respect to set inclusion) of $\mathcal{E}_A$ is a member of $\mathrm{Ass}(A)$. Note that every member of $\mathcal{E}_A$ is contained in at least one maximal element of $\mathcal{E}_A$ and that such a maximal element P must satisfy $\mathrm{End}(A_M) = R_P$ for some maximal ideal M of R. Let M be such a maximal ideal of R. By Lemma 2.4 in [7], A_M is a primal ideal with adjoint prime P_M. It follows that $P_M \in \mathrm{Ass}(A_M)$; hence $P \in \mathrm{Ass}(A)$ by Lemma 2.4. □

The following example shows that in the setting of Prüfer domains, even with additional hypothesis such as Noetherian spectrum, the maximal prime divisors of an ideal need not be Krull associated primes. (Compare this to the Noetherian case, in which maximal prime divisors are always associated primes.) In [6], we consider this problem in more depth.

Example 2.9. *There exists a 2-dimensional Prüfer domain R for which* $\mathrm{Spec}(R)$ *is Noetherian and for which there exist an ideal A and a finitely generated maximal ideal M such that M is a maximal prime divisor of A and yet $A : M = A$ and* $M \notin \mathrm{Ass}(A)$. Let D be a Dedekind domain with non-torsion class group and let P be a maximal ideal of D that is not the radical of a principal ideal. Let x be an indeterminate over the field of fractions K of D and let $V = K[x]_{(xK[x])} = K + Q$, where Q is the maximal ideal of V. Define $R = D + Q$. Then R is a 2-dimensional Prüfer domain with Q as the unique prime ideal of R of height one. The maximal ideals of R are all of height two and have the form $M' = P'R$, where P' is a maximal ideal of D. Thus $\mathrm{Spec}\, R$ is Noetherian. Let $E = \bigcup_{n=1}^{\infty} D :_K P^n$ be the P-transform of D. Consider the ideal $A = xER$ of R and the maximal ideal $M = PR$ of R. Since $ED_P = K$, it follows that $AR_M = QR_M$ and $M \notin \mathrm{Ass}(A)$. However, for P' a maximal ideal of D with $P' \neq P$ and $M' = P'R$, we have $AR_{M'} = xR_{M'}$. Therefore $M' \in \mathrm{Ass}(A)$. Since every element of P is contained in a maximal ideal of D distinct from P, every element of M is non-prime to A and M is a maximal prime divisor of A.

3. Isolated Components and Primal Ideals

In view of Lemma 1.2, every ideal of the ring is the intersection of primal ideals. Also, every ideal is the intersection of its isolated components. We would like to have an intersection representation of an ideal with components that are both isolated components and primal ideals with distinct adjoint primes; in this way we will have more information about the components and the decomposition.

In the representation of an ideal as the intersection of its isolated components we do not need all them to obtain the ideal: those belonging to maximal prime divisors, the *principal components* (Hauptkomponenten) in the sense of Krull [20, page 16], would perfectly suffice (see the comment after Lemma 3.1). However, an objection to using these components is that they need not be ideals with well-understood properties, for example, they may fail to be primal. Therefore, we focus our attention on the isolated components belonging to the maximal Krull associated primes; they will provide sufficiently many isolated components to represent the ideals, as is shown by the next result.

Lemma 3.1. *For every proper ideal A of the ring R we have*

$$A = \bigcap_{P \in \mathcal{X}_A} A_{(P)}.$$

Proof. Suppose $x \notin A$. If Q is a minimal prime of $A : x$, then Q is contained in some $P \in \mathcal{X}_A$. Hence $A : x \subseteq P$, and it follows that $x \notin A_{(P)}$. Consequently, $x \notin \bigcap_{P \in \mathcal{X}_A} A_{(P)}$, and we conclude that $A = \bigcap_{P \in \mathcal{X}_A} A_{(P)}$. □

Observe that every prime P in $\mathcal{X}_A$ is contained in a maximal prime divisor P' of A. The principal component of A with respect to P' is obviously contained in the isolated component of A with respect to P. Therefore, the decomposition of Lemma 3.1 continues to hold if the intersection is taken not over the primes in $\mathcal{X}_A$ but over the maximal prime divisors of A. In particular, the following result was proved by Krull [20, page 16]:

Corollary 3.2. *Every ideal A is the intersection of its principal components $A_{(P)}$, P a maximal prime divisor of A. In particular, if A has only finitely many maximal prime divisors, $P_1, \ldots, P_n$, then A is the intersection of finitely many principal components $A_{(P_i)}$.*

It is natural to wonder when the isolated components, and in particular those in the intersection presented in Lemma 3.1, are primal ideals. Examples given by Nagata [24] and by Gilmer [8] show that if P is a maximal prime divisor of the ideal A, then the ideal $A_{(P)}$ need not be P-primal. In these examples, A has infinitely many maximal prime divisors. In Example 3.8 below, we present an example where the ideal A has precisely two maximal prime divisors and where the principal component with respect to each of these maximal prime divisors is not a primal ideal.

There are important special cases in which isolated components are always primal, viz. the arithmetical rings.

Lemma 3.3. *If A is an ideal of an arithmetical ring, then $A_{(P)}$ is an irreducible (hence primal) ideal for every prime ideal P containing A. Its adjoint prime is contained in P.*

Proof. By Remark 1.6, A_P is irreducible if and only if $A_{(P)}$ is irreducible. Since R_P is a valuation ring, it follows that $A_{(P)}$ is an irreducible ideal of R. By Lemma 1.3, the adjoint prime of $A_{(P)}$ is contained in P. □

Let P be a prime ideal containing the ideal A. The isolated P-component $A_{(P)}$ is a P-primary ideal if and only if P is a minimal prime of A. The most important criterion for the primality of the isolated components is given by the equivalence of (1) and (3) of [21, Proposition 1.1] as we record in Theorem 3.4. Since our definition of Krull associated prime is different from that given in [21], we give a short direct proof of the result.

Theorem 3.4. *Let P be a prime ideal containing the ideal A. The isolated P-component $A_{(P)}$ of A is P-primal if and only if P is a Krull associated prime of A.*

Proof. Since the elements of $R \setminus P$ are prime to $A_{(P)}$, $A_{(P)}$ is P-primal if and only if the elements of P are nonprime to $A_{(P)}$. Equivalently, for every $x \in P$, there is a $z \notin A_{(P)}$ such that $xz \in A_{(P)}$, i.e., $xzu \in A$ for some $u \in R \setminus P$. Thus $x \in A : zu$, where $zu \notin A_{(P)}$. The last relation says that $A : zu \subseteq P$, so P is a Krull associated prime of A. □

We summarize our findings in

Theorem 3.5. *Every proper ideal A of a ring R can be written as an intersection*

$$A = \bigcap\nolimits_{P \in \mathcal{X}_A} A_{(P)} \tag{2}$$

where the (isolated) components $A_{(P)}$ are primal ideals with distinct, incomparable adjoint primes P.

We call the decomposition of Theorem 3.5, the *canonical primal decomposition* of A. Example 2.6 shows that the canonical primal decomposition may be infinite even if Spec R is Noetherian.

For a maximal prime divisor P of an ideal A the question was raised in [5, page 3] whether or not the principal P-component $A_{(P)}$ is equal to the intersection of all primal ideals that contain A and have their associated adjoint prime contained in P. It follows from Theorem 3.4 and observations made in [5] that the answer to this question is affirmative if $P \in \mathrm{Ass}(A)$. We show in Corollary 3.6 that the answer is affirmative in general.

Corollary 3.6. *Let P be a maximal prime divisor of an ideal A of the ring R. Let $A_{(P)}$ denote the principal P-component of A and let $A^*_{(P)}$ be the intersection of all primal ideals that contain A and have their associated adjoint prime contained in P. Then $A_{(P)} = A^*_{(P)}$.*

Proof. It is shown in [5, page 3] that $A_{(P)} \subseteq A^*_{(P)}$ and it is easily seen that the elements of $R \setminus P$ are prime to $A_{(P)}$. Hence if $Q \in \mathrm{Ass}(A_{(P)})$, then $Q \subseteq P$. Therefore the reverse inclusion follows from the canonical primal decomposition of $A_{(P)}$ given by Theorem 3.5. Indeed, if $Q \in \mathrm{Ass}(A_{(P)})$, then by Lemma 2.4 $Q \in \mathrm{Ass}(A_P)$, so that (again by Lemma 2.4) $Q \in \mathrm{Ass}(A)$. Thus by Theorem 3.5

$$A_{(P)} = \bigcap \{A_{(Q)} \ : \ Q \in \mathrm{Ass}(A), Q \subseteq P\}.$$

□

It is also true that $A_{(P)}$ is an intersection of P-primal ideals as we show in Proposition 3.7.

Proposition 3.7. *Let P be a prime ideal of the ring R. An ideal A contained in P is an intersection of P-primal ideals if and only if the elements of $R \setminus P$ are prime to A. In particular, $A_{(P)}$ is an intersection of P-primal ideals.*

Proof. If $A = \bigcap Q_i$, where each Q_i is P-primal, then for $s \in R \setminus P$, we have $A : s = (\bigcap Q_i) : s = \bigcap (Q_i : s) = \bigcap Q_i = A$. Thus the elements of $R \setminus P$ are prime to A.

To prove the converse, we may pass from R to R/A and assume that $A = (0)$. Then the elements of $R \setminus P$ are prime to (0) and R is a subring of the localization $T = R_P$. We observe that if J is a PT-primal ideal of T, then $I = J \cap R$ is P-primal in R. For T is a localization of R implies every ideal of T is the extension of an ideal of R, so $IT = J$. Moreover for $r \in R$, we have $(I :_R r)T = IT :_T r$. It follows that I is P-primal in R. Therefore it suffices to show that if R is quasilocal with maximal ideal P, then (0) is an intersection of P-primal ideals of R. Let a be a nonzero element of P and let $Q = aP$. Then $a \notin Q$ and $P = Q : a$, so Q is P-primal. Thus (0) is an intersection of P-primal ideals. □

Example 3.8. *There exists a ring R having Noetherian prime spectrum and precisely two maximal ideals, each of which is a maximal prime divisor of* (0) *but not a Krull associated prime of* (0). Consider the polynomial ring $E = F[x, y, z]$ in three indeterminates over a field F and the maximal ideals $N_1 = (x, y, z)E$ and $N_2 = (x, y, z - 1)E$ of E. Let D denote the localization of E at the multiplicative set $E \setminus (N_1 \cup N_2)$. Then D is a unique factorization domain with precisely two maximal ideals, N_1D and N_2D. Consider the height-two prime ideals $P_1 = (x, z)D$ and $P_2 = (y, z - 1)D$, and let C denote the D-module $C = \bigoplus D/p$, where p varies over the height-one primes of D other than xD and yD. Let $B = C \oplus D/P_1 \oplus D/P_2$.

Define R to be the direct sum of D and B made into a ring via Nagata's idealization [25] (see Exercises 6,7 on pp. 62-63 in Kaplansky [18]). It is clear that Spec R is isomorphic to Spec D. Thus $\operatorname{Spec} R$ is Noetherian. Since D is an integral domain, B is the nilradical of R and R has precisely two maximal ideals, $M_1 = N_1D \oplus B$ and $M_2 = N_2D \oplus B$. Moreover, every element of $M_1 \cup M_2$ is a zero divisor, so R is equal to its total quotient ring. For $i = 1, 2$, let $A_i = \{r \in R : \text{there exists } s \in R \setminus M_i \text{ such that } rs = 0\}$ denote the principal component of the zero ideal of R with respect to the maximal ideal M_i. Then A_1 and A_2 are contained in B; A_1 is the direct sum of D/P_2 with $\bigoplus D/p$, where p varies over the height-one primes of D not contained in M_1 and A_2 is the direct sum of D/P_1 with $\bigoplus D/p$, where p varies over the height-one primes of D not contained in M_2. Since $(0) : x = D/P_1$ and $(0) : y = D/P_2$, we see that $A_1 : y = A_1$ and $A_2 : x = A_2$. Therefore the ideal A_i is not M_i-primal and M_i is not an associated prime of (0). Indeed, the ideals A_1 and A_2 are not even primal since the set of elements non-prime to A_i does not form an ideal. For example, the elements $y + z$ and z are non-prime to A_1, while their difference y is prime to A_1.

4. Irredundant Primal Decompositions

We wish to consider primal decompositions of an ideal A with distinct adjoint primes, and to examine when such decompositions are irredundant and unique in some sense. Recall that if an ideal A is represented as an intersection of ideals $A = \bigcap_j B_j$, then a component B_j is said to be *relevant* in this representation if omitting B_j increases the intersection. If each component B_j is relevant, then the intersection is said to be *irredundant.*

In this section we concentrate on the general case and on rings with Noetherian spectra, while in the next section we deal with arithmetical rings.

First, we verify the following lemma.

Lemma 4.1. *Suppose that A is a proper ideal of R, and $P \in \mathcal{X}_A$ has one of the following properties:*

(i) *P is not contained in the set union of the primes $Q \in \mathcal{X}_A$ with $Q \neq P$.*
(ii) *P is a Zariski-Samuel associated prime.*

Then in the canonical primal decomposition (2) *the isolated P-component $A_{(P)}$ is relevant.*

Proof. (i) Suppose that P is not contained in the set union of the primes $Q \in \mathcal{X}_A$ with $Q \neq P$. Then there exists an element $p \in P$ that fails to belong to any of these $Q \in \mathcal{X}_A$. By Theorem 3.4, $A_{(Q)}$ is a Q-primal ideal, thus $A_{(Q)} : p = A_{(Q)}$ for every $Q \neq P$. From (2) we obtain

$$A : p = (A_{(P)} : p) \cap \bigcap_{Q \in \mathcal{X}_A, Q \neq P} A_{(Q)}.$$

In view of Lemma 2.2, $A : p$ properly contains A, whence it is clear that (a) $A_{(P)} \subset A_{(P)} : p$; and (b) in (2) we cannot replace $A_{(P)}$ by the larger $A_{(P)} : p$. Consequently, the component $A_{(P)}$ is relevant in (2).

(ii) Suppose that P is a Zariski-Samuel associated prime to A. Then by definition there exists an element $y \notin A$ such that $P = \sqrt{A : y}$. If $P \neq Q \in \mathcal{X}_A$, then $A_{(Q)} : y \subseteq Q$ is impossible, as it would imply $P = \sqrt{A : y} \subseteq \sqrt{A_{(Q)} : y} \subseteq Q$. We conclude that $y \in A_{(Q)}$ holds for all $Q \neq P$. In other words, $y \in \bigcap_{Q \in \mathcal{X}_A, Q \neq P} A_{(Q)}$. But then $A_{(P)} : y = A : y \subseteq P$, which shows that $y \notin A_{(P)}$, establishing the claimed relevance of $A_{(P)}$. $\square$

In conclusion, from Theorem 3.5 and Lemma 4.1 we have established:

Theorem 4.2. *Let A be a proper ideal of the ring R. If each $P \in \mathcal{X}_A$ has the property that either* (i) *P is not contained in the set union of the other primes in $\mathcal{X}_A$, or* (ii) *P is a Zariski-Samuel associated prime of A, then the canonical primal decomposition $A = \bigcap_{P \in \mathcal{X}_A} A_{(P)}$ is irredundant.*

If each $P \in \mathcal{X}_A$ is a Zariski-Samuel prime of A, we prove that a stronger form of irredundancy holds. An intersection $A = \bigcap_j B_j$ is said to be *residually maximal at B_j* if the intersection no longer equals A whenever B_j is replaced by a residual $B_j : c$ that properly contains B_j. The intersection $A = \bigcap_j B_j$ is said to be *residually maximal* if it is residually maximal at each B_j. Residually maximal intersections are evidently irredundant.

Proposition 4.3. *If $P \in \mathcal{X}_A$ is a Zariski-Samuel associated prime of A, then the canonical primal decomposition $A = \bigcap_{Q \in \mathcal{X}_A} A_{(Q)}$ is residually maximal at $A_{(P)}$. If each $P \in \mathcal{X}_A$ is a Zariski-Samuel associated prime of A, then the canonical primal decomposition is residually maximal.*

Proof. Supposing that the component $A_{(P)}$ can be replaced by $A_{(P)} : c$ for some $c \in R$, we show that such a residual cannot be larger than $A_{(P)}$. We argue as in the proof of Lemma 4.1(ii) (keeping the same notation) that in this case

$$A : y = A_{(P)} : y = (A_{(P)} : c) : y = (A_{(P)} : y) : c.$$

Thus c is prime to the P-primary ideal $A : y$ (see Lemma 2.5(iii)), i.e. $c \notin P$. Therefore, $A_{(P)} : c = A_{(P)}$, since by Theorem 3.4 $A_{(P)}$ is P-primal. $\square$

We now give several examples to illustrate our theorems.

Example 4.4. *An irredundant primal decomposition of an ideal A of a Noetherian ring R need not consist of isolated components.* Consider the polynomial ring $R = F[x, y, z]$ in three indeterminates with coefficients in a field F. The ideal $A = (x^2y, xy^2z^2)R$ has a primary decomposition

$$A = xR \cap yR \cap (x^2, y^2)R \cap (x^2, z^2)R$$

with radicals $xR, yR, (x, y)R, (x, z)R$, respectively. The set $\mathcal{X}_A$ consists of two prime ideals (they are Zariski-Samuel associated primes):

$$\sqrt{A : xyz^2} = (x, y)R \quad \text{and} \quad \sqrt{A : xy^2z} = (x, z)R.$$

The canonical primal decomposition of A given by (2) of Theorem 3.5 is

$$A = A_{(x,y)R} \cap A_{(x,z)R},$$

where

$$A_{(x,y)R} = (x^2y, xy^2)R \quad \text{and} \quad A_{(x,z)R} = (x^2, xz^2)R.$$

Note that $A = (x^2y, xy^2)R \cap (x^2, z^2)R$ and $A = (x^2y, y^2)R \cap (x^2, xz^2)R$ are representations of A as intersections of primal ideals, however, the primal ideals $(x^2, z^2)R$ and $(x^2y, y^2)R$ are not isolated components of A.

Example 4.5. *There exists a ring R with Noetherian prime spectrum such that R has a P-primal ideal A that can be represented as an irredundant intersection of primal isolated components whose associated adjoint primes are different from the adjoint prime P of A.* Consider the localized polynomial ring $D = F[x, y]_{(x,y)F[x,y]}$, where x, y are two indeterminates over a field F. Visibly, D is a 2-dimensional local UFD. Let B be the D-module which is the direct sum of the D/pD as p varies over the non-associate irreducible elements of D.

As in Example 3.8, define R to be the direct sum of D and B made into a ring via Nagata's idealization. It is clear that Spec R is isomorphic to Spec D. Hence Spec R is Noetherian. Let $M = (x, y)D + B$ denote the maximal ideal of R and let $A = (0)$ in R. Since M consists of zero-divisors, A is a primal ideal of R whose adjoint prime is M.

The set $\mathcal{X}_A$ consists of the single element M. Moreover, the height-one primes P of R have the form $P = pD+B$, where p is an irreducible element of D. Hence M is the union of the height one primes of R, each of which is a Zariski-Samuel associated prime of A. For $P = pD+B$ a height-one prime of R, the isolated component $A_{(P)}$ is the submodule of B that is the direct sum of D/qD as q varies over the non-associate irreducible elements of D distinct from p. Thus the primal ideal A of R can be represented with primal components both as A (one component), and also as an irredundant intersection $A = \cap A_{(P)}$ of P-primal ideals, where P varies over the height-one primes of R (infinitely many components).

As Examples 4.4 and 4.5 show, an ideal may have various decompositions into the intersection of primal ideals. For primes $P \in \text{Ass}(A)$, the isolated P-component $A_{(P)}$ of an ideal A is certainly the minimal P-primal ideal containing A, so the representation established in Lemma 3.1 is the minimal one in the obvious sense. Since there might exist other decompositions into primal components, the usual question arises: to what extent is the representation of an ideal as an intersection of primal isolated components unique? Needless to say, we cannot expect complete uniqueness, not even for irredundant representations with distinct adjoint primes, since this fails already in the Noetherian case. However, the uniqueness of the set of adjoint primes can be ascertained if we assume some sort of maximality (in the Noetherian case, it was assumed that the intersection was reduced; see [5]).

By a *primal isolated component* of an ideal A we mean a primal ideal B such that $B = A_{(Q)}$ for some prime ideal Q. If P is the adjoint prime of B, then necessarily $P \subseteq Q$. It is possible that P is properly contained in Q. However, regardless of whether this containment is proper, it follows that $A_{(P)} = A_{(Q)}$. For $P \subseteq Q$ implies $A_{(Q)} \subseteq A_{(P)}$ and by Lemma 1.3(i), $B = B_{(P)}$. Since $A \subseteq B$, we have $B = A_{(Q)} \subseteq A_{(P)} \subseteq B$. Thus if B is a primal isolated component of A, then $B = A_{(P)}$ for some $P \in \text{Ass}(A)$. In Theorem 4.6 we pinpoint a necessary and sufficient condition for $P \in \mathcal{X}_A$ to appear in every representation of A as an intersection of a subset of its primal isolated components.

Theorem 4.6. *If A is an ideal of the ring R and $P \in \mathcal{X}_A$, then the following statements are equivalent for P:*

(i) *P is a Zariski-Samuel associated prime of A.*
(ii) *The isolated component $A_{(P)}$ must appear in every representation of A as an intersection of primal isolated components of A.*

Proof. (i) $\Rightarrow$ (ii) The proof of Lemma 4.1(ii) applies, since it requires only that $Q \in \text{Ass}(A)$.

(ii) $\Rightarrow$ (i) Suppose that P is not a Zariski-Samuel associated prime of A. We claim that $A = \cap_{Q \neq P} A_{(Q)}$, where Q ranges over the members of Ass(A) distinct from P, thus contradicting (ii). To prove this, it suffices to show that for every $y \in R \setminus A$, there exists $Q \in \text{Ass}(A)$ with $Q \neq P$ and $A : y \subseteq Q$. Let $y \in R \setminus A$. By assumption $\sqrt{A : y} \neq P$, so there exists a minimal prime Q of $A : y$ with $Q \neq P$. Necessarily, $Q \in \text{Ass}(A)$. $\square$

Corollary 4.7. *For an ideal A of the ring R the following statements are equivalent.*

(i) *For each set of prime ideals $\mathcal{C} \subseteq \text{Ass}(A)$ such that $A = \bigcap_{P \in \mathcal{C}} A_{(P)}$, it is the case that $\mathcal{X}_A \subseteq \mathcal{C}$.*
(ii) *Each $P \in \mathcal{X}_A$ is a Zariski-Samuel associated prime of A.*

In particular, if each $P \in \mathcal{X}_A$ is a Zariski-Samuel associated prime of A, then the canonical primal decomposition $A = \bigcap_{P \in \mathcal{X}_A} A_{(P)}$ is irredundant.

Proof. That (i) implies (ii) follows from Theorems 3.4 and 4.6. To see that (ii) implies (i), suppose $A = \bigcap_{P \in \mathcal{C}} A_{(P)}$ for some $\mathcal{C} \subseteq \text{Ass}(A)$. By Theorem 4.6, for each $P \in \mathcal{X}_A$, we have $A_{(P)} = A_{(Q)}$, for some $Q \in \mathcal{C}$. Since $P, Q \in \text{Ass}(A)$, we have by Theorem 3.4 that $A_{(P)}$ is P-primal and $A_{(Q)}$ is Q-primal. Since $A_{(P)} = A_{(Q)}$ we conclude $P = Q$. $\square$

Remark 4.8. If $P \in \mathcal{X}_A$ is of the form $P = \sqrt{A : x}$, then $A : x$ is P-primary. For if $r, s \in R$ are such that $rs \in A : x$ and $r \notin A : x$, then $A : rx$ is a proper ideal and $P \in \mathcal{X}_A$ implies that $P = \sqrt{A : rx}$. Hence $s \in A : rx \subseteq P$, so $A : x$ is P-primary.

Example 4.5 shows that the canonical primal decomposition of A given in (2) of Theorem 3.5 may be irredundant, but not the unique representation of A as an irredundant intersection of pairwise incomparable isolated primal components of A. For the ideal $A = yR$ of Example 2.6, the canonical primal decomposition of A given in (2) of Theorem 3.5 is irredundant and yet the set $\mathcal{X}_A$ is infinite. There are also examples where A is an ideal of a one-dimensional Prüfer domain and the canonical primal decomposition given in (2) of Theorem 3.5 is redundant, but a proper subset of $\mathcal{X}_A$ gives an irredundant representation of A. Indeed, it can happen that all but one of the elements of $\mathcal{X}_A$ is a Zariski-Samuel prime of A and that A is the intersection of its primal components with respect to this proper subset of $\mathcal{X}_A$ (cf. Example 2.2 in [14]).

We would like to have conditions that are both necessary and sufficient for irredundance of the canonical primal decomposition $A = \cap_{P\in\mathcal{X}_A} A_{(P)}$.

The following interesting example shown to us by Alan Loper and included here with his permission shows that the sufficient conditions of Theorem 4.2 are not in general necessary conditions for irredundance. The example is a Prüfer domain R having property (#), that is, the representation $R = \cap\{R_P : P \in \mathrm{Max}(R)\}$ is irredundant. The integral domain R is constructed so that it has a nonzero principal ideal $A = yR$ that is contained in every maximal ideal of R. Since R is Prüfer and A is principal, it follows from Proposition 2.8 that $\mathcal{X}_A = \mathrm{Max}(R)$. Moreover the representation $A = \cap_{P\in\mathcal{X}_A} A_{(P)}$ is irredundant and yet there exists $P \in \mathcal{X}_A$ such that P is contained in the union of the prime ideals of $\mathcal{X}_A$ distinct from P and P is not a Zariski-Samuel associated prime of A.

Example 4.9. (Alan Loper) Let V be a valuation domain on a field K such that the maximal ideal M of V is the union of the prime ideals properly contained in M. Let $E = \{a_1, a_2, a_3, a_4\}$ where $\{a_i\}_{i=1}^4$ are nonzero elements of M having the property that

$$a_4V = a_3V \supset (a_3 - a_4)V \supset a_2V \supseteq a_1V.$$

Thus if v is a valuation on K associated to the valuation domain V, then

$$v(a_4) = v(a_3) < v(a_3 - a_4) < v(a_2) \leq v(a_1).$$

Let x be an indeterminate and let $D = \mathrm{Int}(E, V) = \{f(x) \in K[x] : f(E) \subseteq V\}$ be the ring of integer-valued polynomials on the set E. It is well-known that D is a Prüfer domain [22, Corollary 7]. Moreover, D has precisely four maximal ideals of infinite height. These are the maximal ideals containing MD. They are M_i, $1 \leq i \leq 4$, where $M_i = \{f(x) \in K[x] : f(a_i) \in M\}$. The other maximal ideals of D are of height one and are the contraction to D of maximal ideals of $K[x]$. Each of the maximal ideals of $K[x]$ other than $(x - a_i)K[x]$ intersects D in a maximal ideal of D, while $(x - a_i)K[x] \cap D$, for $1 \leq i \leq 4$, is a height-one prime of D that is contained in M_i. Each of the valuation domains D_{M_i} has the property that its maximal ideal is the union of the prime ideals it properly contains. Therefore M_i in D is the union of the prime ideals it properly contains.

The following argument shows that M_i, for $1 \leq i \leq 4$, is contained in the union of the other maximal ideals of D. We have $D \subseteq K[x]$ and $M_i \cap K = M$ for each i with $1 \leq i \leq 4$. Therefore every element of K that is in one of the M_i is in all four of the M_i. A nonconstant polynomial $f(x) \in D \subseteq K[x]$ that is in one of that M_i and is not of the form $f(x) = d(x - a_j)^n$, where $d \in K$, n is a positive integer and $1 \leq j \leq 4$ is easily seen to be also in another maximal ideal of D. Thus assume $f(x) = d(x - a_j)^n \in D$. By definition of D, $f(a_i) \in V$ for $1 \leq i \leq 4$. Also $f(a_j) = 0 \in M$, so $f(x) \in M_j$. Using the fact that

$$v(a_4) = v(a_3) < v(a_3 - a_4) < v(a_2) \leq v(a_1).$$

it follows by examining cases that at most two of the values $v(f(a_i))$ can be 0. Therefore $f(x)$ is in at least two of the M_j.

Let y be an indeterminate over $K(x)$, let $Q = yK(x)[y]_{(y)}$ and let $R = D + Q$. For each maximal ideal P of R, we have $P = (P \cap D)R$ and $P \cap D$ is a maximal ideal of D. It follows that M_iR, for $1 \leq i \leq 4$, is a maximal ideal of R that is contained in the union of the other maximal ideals of R.

To show that R satisfies property (#), it suffices to show that for each maximal ideal P of R there exists a finitely generated ideal contained in P but not contained in any other maximal ideal of R. Since Q is a height-one prime ideal of R that is contained in every nonzero prime ideal of R and since $D = R/Q$, it suffices to show this property for D. Each maximal ideal of D is either the contraction of a maximal ideal of $K[x]$ or else in the set $\{M_i\}_{i=1}^4$. For each of the maximal ideals P contracted from $K[x]$ there exists an element f contained in P and contained in no other maximal ideal of D that is contracted from $K[x]$. By choosing an element of $P \setminus \cup_{i=1}^4 M_i$, we see that P is the radical of a finitely generated ideal of D. Thus each of the maximal ideals of D contracted from $K[x]$ is the radical of a finitely generated ideal. Since a nonzero element of D is contained in only finitely many maximal ideals of $K[x]$, we also easily get, for $1 \leq i \leq 4$, the existence of a finitely generated ideal A_i contained in M_i such that A_i is contained in no other maximal ideal of D. Therefore D and $R = D + Q$ satisfy property (#).

Let $A = yR$. Then A is a nonzero principal ideal of R that is contained in each maximal ideal of R. Hence by Proposition 2.8, $\mathcal{X}_A = \mathrm{Max}(R)$. Since R satisfies property (#), the representation of the principal ideal A as $A = \cap_{P \in \mathcal{X}_A} A_{(P)}$ is irredundant, but M_iR, for $1 \leq i \leq 4$, is not a Zariski-Samuel associated prime of A and M_iR is contained in the union of the prime ideals of $\mathcal{X}_A$ distinct from M_iR. This completes the presentation of Example 4.9.

5. Irredundant Irreducible Decompositions

In this section our focus is on irredundant decompositions of ideals A into irreducible isolated components $A_{(P)}$. We prove a uniqueness statement about such decompositions, and then characterize those rings for which the canonical primal decomposition of every proper ideal A is an irredundant representation of A as an intersection of irreducible isolated components. In the statement of Theorem 5.1 we use the concept of a residually maximal intersection as defined in Section 4.

Theorem 5.1. *Let A be a proper ideal of a ring R. If $\mathcal{C}$ is a collection of prime ideals of R and $A = \bigcap_{P \in \mathcal{C}} A_{(P)}$ is an irredundant intersection of irreducible isolated components of A, then this intersection is residually maximal and is the unique irredundant decomposition of A into irreducible isolated components of A.*

Proof. Since the isolated components $A_{(P)}$ are irreducible, they are primal. Without loss of generality, we assume P is the adjoint prime of $A_{(P)}$ and $\mathcal{C}$ consists of the adjoint primes P of the $A_{(P)}$. Thus $\mathcal{C} \subseteq \mathrm{Ass}(A)$, and $A = \bigcap_{P \in \mathcal{C}} A_{(P)}$ is an irredundant decomposition of A, where each $P \in \mathcal{C}$ is the adjoint prime of the irreducible ideal $A_{(P)}$.

If P is a prime ideal of R and $A = B \cap C$ for ideals B and C of R, then $A_{(P)} = B_{(P)} \cap C_{(P)}$. We will use this fact frequently in the proof.

Suppose that for some $P \in \mathcal{C}$ there exists an $x \in R$ such that

$$A = (A_{(P)} : x) \cap (\bigcap \{A_{(Q)} : Q \in \mathcal{C}, Q \neq P\}.$$

Let $C = \bigcap_{Q \neq P} A_{(Q)}$. By Lemma 1.1, we have

$$A_{(P)} = (A_{(P)} : x) \cap C_{(P)}.$$

Since $A_{(P)}$ is irreducible, we have either $A_{(P)} = A_{(P)} : x$ or $A_{(P)} = C_{(P)}$. The latter equality, however, contradicts the irredundancy of the decomposition of A, so we have that $A_{(P)} = A_{(P)} : x$. We conclude that any irredundant decomposition of A into irreducible isolated components is residually maximal.

It remains to show that the representation $A = \bigcap_{P \in \mathcal{C}} A_{(P)}$ is unique among irredundant intersections of A into irreducible isolated components of A. Suppose $\mathcal{C}' = \{Q_i\}_{i \in I} \subseteq \mathrm{Ass}(A)$ is such that $A = \bigcap_{i \in I} A_{(Q_i)}$ is also an irredundant decomposition of A into irreducible components $B_i = A_{(Q_i)}$ of A with adjoint primes Q_i. We show that for each $i' \in I$, there exists $P_{i'} \in \mathcal{C}$ such that $P_{i'} \subseteq Q_{i'}$. Fix $j \in I$, and define $C_j = \bigcap_{i \neq j} B_i$. We exhibit $P \in \mathcal{C}$ such that $P \subseteq Q_j$. Observe that for each $P \in \mathcal{C}$,

$$A_{(P)} = (B_j)_{(P)} \cap (C_j)_{(P)}.$$

Since $A_{(P)}$ is irreducible, it must be the case that $A_{(P)} = (B_j)_{(P)}$ or $A_{(P)} = (C_j)_{(P)}$. If the latter case holds for every $P \in \mathcal{C}$, then

$$C_j \subseteq \bigcap_{P \in \mathcal{C}} A_{(P)} = A,$$

and it follows that $\bigcap_{i \neq j} B_i = C_j \subseteq A$. However, this contradicts the assumption that the intersection $A = \bigcap_{i \in I} B_i$ is irredundant, so it must be the case that for some $P \in \mathcal{C}$, $A_{(P)} = (B_j)_{(P)}$. Since $A_{(P)}$ is P-primal, we have that $(B_j)_{(P)}$ is P-primal. However, by assumption, B_j is Q_j-primal, so statement (iii) of Lemma 1.3 implies $P \subseteq Q_j$. We conclude that for each $i' \in I$, there exists $P_{i'} \in \mathcal{C}$ such that $P_{i'} \subseteq Q_{i'}$.

Applying a symmetrical argument to the decomposition $A = \bigcap_{i \in I} B_i$, we have that for each $P \in \mathcal{C}$, there exists $i \in I$ such that $Q_i \subseteq P$. Thus $P_i \subseteq Q_i \subseteq P$. Since the decomposition $A = \bigcap_{P \in \mathcal{C}} A_{(P)}$ is irredundant, it follows that $\mathcal{C}$ consists of incomparable primes. Thus $P_i = Q_i = P$, and $\mathcal{C} \subseteq \{Q_i : i \in I\}$. Also, the symmetry of our assumptions yields $\{Q_i : i \in I\} \subseteq \mathcal{C}$. We conclude $\mathcal{C} = \{Q_i : i \in I\}$, and it follows that

$$\{A_{(P)} : P \in \mathcal{C}\} = \{A_{(Q_i)} : i \in I\}.$$

This proves the theorem. □

Discussion 5.2. Assume that $\mathcal{C}$ is a subset of $\mathrm{Ass}(A)$ containing at least two elements. If $A = \bigcap_{P \in \mathcal{C}} A_{(P)}$ and if the intersection is irredundant, then $\mathcal{C}$ consists of incomparable primes. Also, for each $P \in \mathcal{C}$ there exists $Q \in \mathcal{X}_A$ such that $P \subseteq Q$, and it follows that $A_{(Q)} \subseteq A_{(P)}$. Thus if $\mathcal{C}' = \mathcal{C} \setminus \{P\}$ and $B = \bigcap_{P' \in \mathcal{C}'} A_{(P')}$, then $A_{(Q)}$ is relevant in the intersection $A = A_{(Q)} \cap B$. Since $A_{(P)}$ is P-primal and $A_{(Q)}$ is Q-primal, we have $A_{(P)} = A_{(Q)}$ if and only if $P = Q$. If $A_{(Q)}$ is irreducible, we observe that $P = Q$. For $A = A_{(P)} \cap B$ implies $A_{(Q)} = A_{(P)} \cap B_{(Q)}$, and $P \neq Q$ and $A_{(Q)}$ irreducible imply $A_{(Q)} = B_{(Q)}$. But this implies $B \subseteq A_{(P)}$ which contradicts the irredundance of the intersection $A = A_{(P)} \cap B$. Therefore, if $A_{(Q)}$ is irreducible for each $Q \in \mathcal{X}_A$, then $\mathcal{C} \subseteq \mathcal{X}_A$.

From Discussion 5.2, we deduce the following corollary to Theorem 5.1.

Corollary 5.3. *Let A be an ideal of a ring R, and suppose $\mathcal{C} \subseteq \mathrm{Ass}(A)$. If $A_{(Q)}$ is irreducible for each $Q \in \mathcal{X}_A$ and $A = \bigcap_{P \in \mathcal{C}} A_{(P)}$ is an irredundant intersection, then $\mathcal{C} \subseteq \mathcal{X}_A$ and this is the unique decomposition of A into an intersection of irreducible isolated components of A.* □

The irreducibility and isolation of the components in Theorem 5.1 are crucial to the assertion of uniqueness in the theorem. For A as in Example 4.5 we have two distinct irredundant representations of A as an intersection of primal isolated component ideals of A. Example 5.4 exhibits a situation where an ideal has two distinct irredundant representations as an intersection of primal isolated components, one of which consists of irreducible isolated components. It is interesting to contrast Theorem 5.1 with the situation for infinite irredundant primary decompositions of ideals in general. In this case, neither the primary ideals in the representation nor even the primes associated to these ideals need be unique as is shown for example by Underwood in [31, Example 4.18]. On the other hand, it is readily seen that in a Noetherian ring there cannot exist an infinite irredundant intersection of ideals.

Example 5.4. *An ideal may have two distinct decompositions as an irredundant intersection of primal isolated components, one of which consists of irreducible isolated components.* Let $\{x_n : n \in \mathbb{N}\}$ be indeterminates over a field F and let R be the localization of the polynomial ring $F[x_1, x_2, \dots]$ at the maximal ideal generated by $\{x_n : n \in \mathbb{N}\}$. Consider the ideal A of R generated by $\{x_i x_j : i \neq j\}$. Then if M is the unique maximal ideal of R, R/A is one-dimensional with a unique maximal ideal M/A and infinitely many minimal primes P_n/A, $n \in \mathbb{N}$, where $P_n = (\{x_i : i \in \mathbb{N}, i \neq n\})R$. Moreover, x_n is not in P_n but is in every prime of R containing A other than P_n. The ideal A of R is M-primal and is the irredundant intersection of irreducible isolated components $A = \bigcap_{P \in \mathcal{C}} A_{(P)}$, where $\mathcal{C} = \{P_n : n \in \mathbb{N}\} \subset \mathrm{Ass}(A)$ consists of precisely the elements in $\mathrm{Ass}(A)$ other than M. Thus for this example, the canonical primal decomposition of A given in (2) of Theorem 3.5 is trivial (that is, $A = A_{(M)}$) and yet there exists a representation $A = \bigcap_{P \in \mathcal{C}} A_{(P)}$ that is irredundant and consists of irreducible isolated components of A and that is, by Theorem 5.1, unique among irredundant intersections of irreducible isolated components of A. However, the representation $A = \bigcap_{P \in \mathcal{C}} A_{(P)}$ is not unique among irredundant intersections of primal isolated components of A.

In light of Theorem 5.1, it is natural to consider rings R for which every ideal A is locally irreducible in the sense that its extension to R_M is irreducible for each maximal ideal M containing A. As noted in Section 1, these are precisely the arithmetical rings.

Returning to Theorem 5.1, we obtain a stronger result for arithmetical rings because of the following lemma which asserts that irredundant primal decompositions in arithmetical rings must involve isolated components.

Lemma 5.5. *Let R be an arithmetical ring, and let A be an ideal of R. Suppose $A = \bigcap_{i \in I} A_i$ is an intersection of ideals A_i of R. For each $i \in I$, if A_i is a primal ideal of R that is relevant to this decomposition, then $A_i = A_{(P_i)}$, where P_i is the adjoint prime of A_i.*

Proof. Let $i \in I$, and suppose A_i is relevant to the given decomposition. If P_i is the adjoint prime of A_i, then $A_i = (A_i)_{(P_i)}$ in view of Lemma 1.3. Also,

$$A_{(P_i)} = A_i \cap (\bigcap_{j \neq i} A_j)_{(P_i)}.$$

By Remark 1.6, $A_{(P_i)}$ is an irreducible ideal of R. Thus if $A_{(P_i)} \neq A_i$, then $A_{(P_i)} = (\bigcap_{j \neq i} A_j)_{(P_i)}$. But this implies that $\bigcap_{j \neq i} A_j \subseteq A_i$, contradicting the relevancy of A_i to the decomposition. We conclude that $A_{(P_i)} = A_i$. □

Corollary 5.6. *Let R be an arithmetical ring, and A an ideal of R. If A has an irredundant decomposition $A = \bigcap_{i \in I} A_i$ into irreducible ideals A_i, then this intersection is residually maximal and it is the unique irredundant decomposition of A into irreducible ideals. Moreover, if P_i is the adjoint prime of A_i, then $\{P_i\}_{i \in I} \subseteq \mathcal{X}_A$.*

Proof. Apply Theorem 5.1, Corollary 5.3 and Lemma 5.5. □

Remark 5.7. Corollary 5.6 need not be true for rings that are not arithmetical. In particular, the uniqueness assertion of the corollary fails in the Noetherian case. For example, if x is an indeterminate over a field F and (R, M) is the local domain $F[x^2, x^3]_{(x^2, x^3)}$, then $M^2 = (x^4, x^5)R$ is M-primary and has an irredundant decomposition $M^2 = x^2R \cap x^3R$, where x^2R and x^3R are irreducible. However, neither x^2R nor x^3R is an isolated component of M^2 since the only isolated component of M^2 is the ideal M^2 itself.

The uniqueness result of Theorem 5.1 suggests the question: What rings R have the property that for every proper ideal A of R the canonical primal decomposition of A given in (2) of Theorem 3.5 is an irredundant representation of A as an intersection of irreducible isolated components? The rest of this section is devoted to a resolution of this question.

Theorem 5.8. *The following statements are equivalent for a ring R.*

(i) *R is an arithmetical ring.*
(ii) *For each proper ideal A of R, $A = \bigcap_{P \in \mathcal{X}_A} A_{(P)}$ is a decomposition of A into irreducible ideals $A_{(P)}$.*
(iii) *Each proper ideal A of R can be represented as an intersection of irreducible isolated components of A.*

Proof. The implication (i) ⇒ (ii) follows from Theorems 1.8 and 3.5; the implication (ii) ⇒ (iii) is clear. To complete the proof it suffices to show (iii) ⇒ (i). To show R is arithmetical, we establish that R_M is a valuation ring for each maximal ideal M of R. Let M be a maximal ideal of R, and let $a, b \in M$. Define $I = (a, b)R$. Then by (iii), there is a collection $\mathcal{C}$ of prime ideals of R such that $IM = \bigcap_{P \in \mathcal{C}} (IM)_{(P)}$, and each $(IM)_{(P)}$ is irreducible. If $M \notin \mathcal{C}$, then $(IM)_{(P)} = I_{(P)}$ for all $P \in \mathcal{C}$. But this implies

$$I \subseteq \bigcap_{P \in \mathcal{C}} I_{(P)} = IM.$$

Since I is finitely generated, $I \not\subseteq IM$. It follows that $M \in \mathcal{C}$ and $(IM)_{(M)}$ is an irreducible ideal of R. As in Remark 1.6, IMR_M is an irreducible ideal of R_M. Thus by the observation (†) made in the course of proof of Theorem 1.8, either $R_Ma \subseteq R_Mb$ or $R_Mb \subseteq R_Ma$. This completes the proof. □

Lemma 5.9. *Let R be a ring, and A be an ideal of R that has only finitely many minimal primes $P_1, P_2, \dots, P_n$. If these minimal primes are comaximal, then each P_i is a Zariski-Samuel associated prime of A.*

Proof. We may assume $n > 1$. By assumption, there exist $a \in P_1$ and $b \in P_2 \cdots P_n$ such that $a + b = 1$. Moreover ab has some power $(ab)^n$ in A. Then $(a^n, b^n)R = R$ and $a^n \in A : b^n$. Hence $A : b^n$ has radical P_1 and P_1 is a Zariski-Samuel associated prime of A. Similarly, $P_2, P_3, \dots, P_n$ are Zariski-Samuel associated primes of A. $\square$

Lemma 5.10. *Let R be a ring in which incomparable prime ideals are comaximal, so, for example an arithmetical ring. If A is a radical ideal of R, then each $P \in \mathcal{X}_A$ is a minimal prime of A and $A_{(P)} = P$. Thus the following are equivalent for a radical ideal A of R:*

(i) *The representation of A as the intersection of its minimal primes is irredundant.*
(ii) *The canonical primal decomposition of A given in (2) of Theorem 3.5 is irredundant.*

Moreover, these conditions hold for every radical ideal of R if and only if every ideal of R has only finitely many minimal primes.

Proof. Let A be a radical ideal of R. Since incomparable prime ideals of R are comaximal each $P \in \mathcal{X}_A$ contains a unique minimal prime Q of A. Since A is a radical ideal, $A_{(P)} \subseteq A_{(Q)} = Q$ are radical ideals with Q the unique minimal prime of $A_{(P)}$. Thus $A_{(P)} = Q$ and $Q = P$. Therefore conditions (i) and (ii) are equivalent.

It is clear that if every ideal of R has only finitely many minimal primes, then conditions (i) and (ii) hold for each radical ideal of R. To prove the converse, let A be a radical ideal of R and let $\{P_i\}_{i \in I}$ be the set of minimal primes of A. By hypothesis, the representation $A = \cap_{i \in I} P_i$ is irredundant.

To prove Lemma 5.10, it suffices to prove the set I is finite. We may assume that I has cardinality at least two. For each $i \in I$, let $A_i = \cap P_j$, where $j \in I$ and $j \neq i$. The representation $A_i = \cap_{j \neq i} P_j$ is irredundant and each $P_j, j \neq i$, is a minimal prime of A_i. Since by hypothesis the representation of each radical ideal of R as the intersection of its minimal primes is irredundant, it follows that $\{P_j : j \in I, j \neq i\}$ is the set of minimal primes of A_i.

We observe that A_i and P_i are comaximal. For if there exists a prime ideal Q of R with $A_i + P_i \subseteq Q$, then Q must contain a minimal prime of A_i. But for $j \neq i$, P_j and P_i are distinct minimal primes of A and thus are incomparable and hence by hypothesis comaximal. It follows that $\sum_{i \in I} A_i = R$. For if $\sum_{i \in I} A_i \subseteq Q$, where Q is prime in R, then $A \subseteq Q$ implies Q contains some minimal prime of A, so Q contains P_i for some $i \in I$. But A_i and P_i are comaximal, a contradiction. Thus $\sum_{i \in I} A_i = R$. Hence there exists a finite subset J of I such that $\sum_{i \in J} A_i = R$. We must have $J = I$, for by construction $\sum_{j \in J} A_j \subseteq P_i$ for $i \in I \setminus J$. $\square$

Remark 5.11. Without the hypothesis that incomparable prime ideals of R are comaximal, it can happen that every radical ideal of R is the irredundant intersection of its minimal primes and yet there exist radical ideals of R having infinitely many minimal primes. This is illustrated by the ring R/A of Example 5.4. This ring is one-dimensional reduced with a unique maximal ideal and infinitely many minimal primes P_n/A. Every radical ideal of R/A is the irredundant intersection of its minimal primes. Moreover, the zero ideal of R/A is a primal radical ideal that has infinitely many minimal primes.

We remark that even in a 2-dimensional Prüfer domain it can happen that there exists a radical ideal A with infinitely many minimal primes such that A is the irredundant intersection of its minimal primes. Thus the hypothesis in Lemma 5.10 about every radical ideal is necessary. This is illustrated by the following example.

Example 5.12. Let F be a field of characteristic different from 2 and let x, y be indeterminates over F. Let V_{11} be the discrete rank-one valuation domain obtained by localizing the polynomial ring $F[x, y]$ at the prime ideal generated by y. Then $V_{11} = F(x) + P_{11}$, where P_{11} denotes the maximal ideal of V_{11}. Define $R_1 = W_{11}$ to be the rank-2 valuation domain $F[x]_{xF[x]} + P_{11}$.

Let $K_1 = F(x, y)$ and let K_2 be a finite algebraic extension field of K_1 in which the valuation domain V_{11} has at least two extensions V_{21} and V_{22}. For example, one may take $K_2 = K_1[z]$, where z is a root of the polynomial $Z^2 - y - 1$. Then $V_{11}[z]$ is the intersection of two valuation domains of the field K_2 that extend V_{11} which we may label as V_{21} and V_{22}, where $z - 1$ generates the maximal ideal of V_{21} and $z + 1$ generates the maximal ideal of V_{22}.

The valuation domains V_{21} and V_{22} are each localizations of the integral closure $\overline{W_{11}}$ of W_{11} in the field K_2, say at prime ideals Q_{21} and Q_{22} of $\overline{W_{11}}$. By the Going-Up Theorem [25, (10.9)], there exist prime ideals $N_{21} \supset Q_{21}$ and $N_{22} \supset Q_{22}$ of $\overline{W_{11}}$ with N_{21} and N_{22} each lying over the maximal ideal of W_{11}. Hence by [32, Theorem 13, page 31], there exist extensions W_{21} and W_{22} of the rank-2 valuation domain W_{11} to the field K_2 such that $W_{2i} \subset V_{2i}$, $i = 1, 2$. Define $R_2 = W_{21} \cap W_{22}$. By the Theorem of Independence of Valuations [25, (11.11)], we see that R_2 is a Prüfer domain with precisely 4 nonzero prime ideals, these prime ideals being $P_{21} \subset M_{21}$ contracted from W_{21} and $P_{22} \subset M_{22}$ contracted from W_{22}.

Let A_1 be the ideal P_{11} of R_1 and define the ideal A_2 of R_2 as $A_2 = M_{21} \cap P_{22}$. Observe that $A_2 \cap R_1 = A_1$ and $\text{Ass}(A_2) = \{M_{21}, P_{22}\}$, with M_{21} and P_{22} the minimal primes of A_2.

There exists a finite algebraic extension field K_3 of K_2 for which the valuation domain V_{22} has at least two distinct extensions V_{32} and V_{33}, see, for example [15, Lemma 1.3]. Let V_{31} be an extension of V_{21} to the field K_3. There exists an extension W_{31} to K_3 of the valuation domain W_{21} such that $W_{31} \subset V_{31}$. Also there exist extensions W_{32} and W_{33} of W_{22} to K_3 such that $W_{3i} \subset V_{3i}$, $i = 2, 3$.

Define $R_3 = W_{31} \cap W_{32} \cap W_{33}$. Again by [25, (11.11)], we see that R_3 is a Prüfer domain with precisely 6 nonzero prime ideals, these prime ideals being $P_{3i} \subset M_{3i}$ contracted from W_{3i}, $i = 1, 2, 3$. Define $A_3 = M_{31} \cap M_{32} \cap P_{33}$. This representation of A_3 is irredundant, so $\text{Ass}(A_3) = \{M_{31}, M_{32}, P_{33}\}$. Moreover, $A_3 \cap R_2 = A_2$.

Continuing in this way, we obtain by induction for each positive integer n a finite algebraic extension field K_n of $F(x, y)$ and a Prüfer domain $R_n = \cap_{i=1}^{n} W_{ni}$, where the W_{ni} are rank-2 valuation domains on the field K_n and where R_n has precisely $2n$ nonzero prime ideals, these prime ideals being $P_{ni} \subset M_{ni}$ contracted from W_{ni}, $i = 1, \dots, n$. Define $A_n = (\cap_{i=1}^{n-1} M_{ni}) \cap P_{nn}$. This representation of A_n is irredundant, so $\text{Ass}(A_n) = \{M_{n1}, M_{n2}, \dots, M_{nn-1}, P_{nn}\}$. Moreover, $A_n \cap R_{n-1} = A_{n-1}$.

Let $R = \cup_{n=1}^{\infty} R_n$ and $A = \cup_{n=1}^{\infty} A_n$. Also define $M = \cup_{n=1}^{\infty} M_{nn}$ and $P = \cup_{n=1}^{\infty} P_{nn}$. Then R is a 2-dimensional Prüfer domain. The maximal ideals of R are M and $M_r = \cup_{n=r}^{\infty} M_{nr}$, $r = 1, 2, \dots$, while the nonzero nonmaximal prime ideals of R are P and $P_r = \cup_{n=r}^{\infty} P_{nr}$, $r = 1, 2, \ldots$. We have $A = (\cap_{r=1}^{\infty} M_r) \cap P$ with this intersection irredundant. Moreover every element of M is in some M_r. Therefore every element of M is non-prime to A. Thus M is a maximal prime divisor of A. However, the M-principal component $A_{(M)}$ of A is P, so $M \notin \mathrm{Ass}(A)$. This completes the presentation of the example.

In the proof of Theorem 5.14, it will be convenient to use the following terminology from [9] or [3, page 62]. An integral domain R satisfies (#) if the intersection $R = \bigcap_M R_M$, where M ranges over the set of maximal ideals of R, is irredundant in the sense that if M is any maximal ideal of R, then $R \neq \bigcap_{N \neq M} R_N$ when N ranges over the set of all maximal ideals that are distinct from M. If every overring of R satisfies (#), then R satisfies (##). The relevance of condition (##) in the present context is that a Prüfer domain satisfies (##) if and only if for each prime ideal P of R, there exists a finitely generated ideal J of R such that J is contained in P and the only maximal ideals of R that contain J are those that contain P (Theorem 3 in [9]]). See also [3] for a recent analysis of the conditions (#) and (##) in Prüfer domains.

Lemma 5.13. *Let R be an arithmetical ring such that for every prime ideal P of R, there exists a finitely generated ideal J contained in P such that the only maximal ideals of R containing J are those that contain P. Then every ideal of R has only finitely many minimal primes. In particular, if R is a Prüfer domain satisfying (##), then every ideal of R has only finitely many minimal primes.*

Proof. Let A be a proper ideal of R. If P is a minimal prime ideal of A, then by assumption there exists a finitely generated ideal J of R such that $J \subseteq P$ and the only maximal ideals of R containing A are those that contain P. Hence the ideal P/A of the ring R/A is the radical of a finitely generated ideal. By Theorem 1.6 in [10], it follows that R/A has finitely many minimal prime ideals. Therefore A has only finitely many minimal primes in R. The statement about Prüfer domains satisfying (##) now follows from Theorem 3 of [9]. □

Theorem 5.14. *The following statements are equivalent for a ring R.*

(i) *For each proper ideal A of R, the decomposition $A = \bigcap_{P \in \mathcal{X}_A} A_{(P)}$ is irredundant and the components $A_{(P)}$ are irreducible ideals.*
(ii) *R is an arithmetical ring such that for every prime ideal P of R, there exists a finitely generated ideal J contained in P such that the only maximal ideals of R containing J are those that contain P.*
(iii) *R is an arithmetical ring with Noetherian maximal spectrum.*

In particular, the integral domains having these equivalent properties are precisely the Prüfer domains that satisfy (##).

Proof. (i) ⇒ (ii) First we observe that if B is a proper ideal of R, then R/B also satisfies (i). For if A is a proper ideal of R containing B, then by (i) the decomposition $A = \bigcap_{P \in \mathcal{X}_A} A_{(P)}$ is irredundant and each component $A_{(P)}$ is irreducible. It is straightforward to verify that for each $x \in R$, $(A : x)/B = A/B :_{R/B} x + B$, and from this it follows that $\mathrm{Ass}(A/B) = \{P/B : P \in \mathrm{Ass}(A)\}$; hence $\mathcal{X}_{A/B} = \{P/B : P \in \mathcal{X}_A\}$. Moreover for each $P \in \mathcal{X}_A$, $(A/B)_{(P/B)} = A_{(P)}/B$, so that each $(A/B)_{(P/B)}$ is irreducible. Now by Theorem 3.5,

$$A/B = \bigcap_{P \in \mathcal{X}_A} (A/B)_{(P/B)} = \bigcap_{P \in \mathcal{X}_A} A_{(P)}/B.$$

Hence the irredundancy of the canonical primal decomposition of A/B follows from the irredundancy of the canonical primal decomposition of A. Thus (i) holds for R/B.

We now prove that (i) implies (ii). By Theorem 5.8, R is an arithmetical ring and by Lemma 5.10 each ideal of R has only finitely many minimal primes. Since a domain homomorphic image of an arithmetical ring is a Prüfer domain and since in an arithmetical ring with finitely many minimal primes each of the minimal primes is the radical of a finitely generated ideal, it suffices by our above observation that (i) passes to homomorphic images to prove that the condition in (ii) holds in the case where R is a Prüfer domain. In this case, as noted above, the condition in (ii) is equivalent to showing R satisfies (##). Furthermore, to verify (##), it is enough to prove that every fractional overring S of R (i.e. S is an overring of R such that $rS \subseteq R$ for some nonzero $r \in R$) satisfies (#) (see, for example, Proposition 2.5 in [29]). Let S be a fractional overring of R. Since R is a Prüfer domain, each maximal ideal of S is of the form SP for some prime ideal P of R. Let $\{SP_i\}$ denote the collection of maximal ideals of S, where each P_i is a prime ideal of R. Using the fact that R_{P_i} is a valuation domain, it is easy to see that $S_{SP_i} = R_{P_i}$ since P_i survives in S_{SP_i}. We show that S satisfies (#) by verifying that for each j, the quasilocal ring S_{SP_j} is relevant to the decomposition $S = \bigcap_i S_{SP_i}$.

Let SP_j be a maximal ideal of S and let r be a nonzero element of P_j such that $rS \subset P_j$. We use Proposition 2.8 to calculate $\mathcal{X}_A$, where $A = rS$. We show that $\mathcal{X}_A$ consists of the prime ideals in $\{P_i\}$ that contain A. We first observe:

Claim 1. If $P \in \{P_i\}$ and $A \subseteq P$, then $P \in \mathrm{Ass}(A)$.

We have $\mathrm{End}(A_P) = \mathrm{End}(S_P) = R_P$. The first equality follows from the fact that A is a principal ideal of S, and the second equality from the fact that $S \subseteq R_P$. By Proposition 2.8, $P \in \mathrm{Ass}(A)$. We next observe:

Claim 2. $\mathcal{X}_A \subseteq \{P_i\}$.

Let $Q \in \mathcal{X}_A$. By Proposition 2.8, $R_Q = \mathrm{End}(A_M) = S_M$ for some maximal ideal M of R. Thus $QS \neq S$, so $QS \subseteq PS$ for some $P \in \{P_i\}$. Since S is flat over R, we have $Q \subseteq P$. By Claim 1, $P \in \mathrm{Ass}(A)$. Since $Q \in \mathcal{X}_A$ is maximal in $\mathrm{Ass}(A)$, we have $Q = P$.

Since there are no inclusion relations among the prime ideals in $\{P_i\}$, it follows from Claims 1 and 2 that $\mathcal{X}_A = \{P \in \{P_i\} : A \subseteq P\}$. By (i), the decomposition $A = \bigcap_{P \in \mathcal{X}_A} A_{(P)}$ is irredundant, and by an application of the pertinent definitions, this implies there exists $x \in R \setminus A$ such that $A : x = Ax^{-1} \cap R \subseteq P_j$ but $Ax^{-1} \cap R \not\subseteq P_i$ for all $i \neq j$ such that $A \subseteq P_i$. Since R is a Prüfer domain and $A = Sr$, we have $Ax^{-1} \cap S = S(Ax^{-1} \cap R) \subseteq SP_j$, and if $Ax^{-1} \cap S \subseteq SP_i$ for some $i \neq j$, then $Ax^{-1} \cap R \subseteq SP_i \cap R = P_i$. If $A \subseteq P_i$, this is a contradiction to the choice of x. Also, since $A \subseteq A : x$, it cannot be the case that $A : x \subseteq P_i$ for any i for which $A \not\subseteq P_i$. We conclude that the only maximal ideal of S that contains $A :_S x = rS :_S x$ is SP_j. It follows that $rx^{-1} \in \bigcap_{i \neq j} S_{SP_i}$ but $rx^{-1} \notin S_{SP_j}$. Thus S_{SP_j} is relevant to the decomposition $S = \bigcap_i S_{SP_i}$, and this proves that S satisfies (#). Consequently, R satisfies (##). This completes the proof that (i) implies (ii).

(ii) $\iff$ (iii) The equivalence of (ii) and (iii) follows from [30, Corollaries 1.3 and 1.5].

(ii) $\Rightarrow$ (i) Let A be a proper ideal of R. By Lemma 3.1, $A = \bigcap_{P \in \mathcal{X}_A} A_{(P)}$. We claim that this is an irredundant decomposition of A. Let $P \in \mathcal{X}_A$. By (ii), there exists a finitely generated ideal J of R that is contained in P but in no maximal ideal of R not containing P. Since R is arithmetical, $P = \bigcup_i P_i$ for some chain of weak-Bourbaki associated primes of A. Since J is finitely generated, there exists a weak-Bourbaki associated prime P_i of A such that $J \subseteq P_i \subseteq P$. It follows that $P_i \not\subseteq Q$ for any $Q \in \mathcal{X}_A$ with $Q \neq P$. For Q and P distinct in $\mathcal{X}_A$ are comaximal and $J \subseteq P_i$ is contained in no maximal ideal of R not containing P. By definition, P_i is a minimal prime ideal of $A : x$ for some $x \in R \setminus A$. By Lemma 5.13, $A : x$ has only finitely many minimal primes, so Lemma 5.9 implies that there exists $y \in R \setminus A$ such that $P_i = \sqrt{(A : x) : y} = \sqrt{A : xy}$. It follows that $A : xy \subseteq P$ but $A : xy \not\subseteq Q$ for any $Q \in \mathcal{X}_A$ with $Q \neq P$. Hence $xy \in \bigcap_{Q \neq P} A_{(Q)}$. Since $A : xy \subseteq P$, it is case that $xy \notin A$, so that $xy \notin A_{(P)}$. Therefore $\bigcap_{Q \neq P} A_{(Q)} \not\subseteq A_{(P)}$, and we conclude that the decomposition $A = \bigcap_{P \in \mathcal{X}_A} A_{(P)}$ is irredundant. □

Remark 5.15. *A ring R is reduced and satisfies (i) and (ii) of Theorem 5.8 if and only if R is a finite direct product of Prüfer domains that satisfy (##).* If R satisfies (i) and (ii) of Theorem 5.14, then by Lemma 5.13, R has finitely many minimal prime ideals. Since R is reduced, (0) is the intersection of finitely many prime ideals, and since R is arithmetical, these prime ideals must be comaximal. Consequently, R is a finite direct product of Prüfer domains, each of which satisfies (##). Conversely, it is not hard to verify that if R is a finite direct product of Prüfer domains satisfying (##), then R is a reduced ring that satisfies (ii) of Theorem 5.14.

Remark 5.16. *An arithmetical ring R has Noetherian prime spectrum if and only if R satisfies condition (ii) (or (i)) of Theorem 5.14 and R has the ascending chain condition on prime ideals.* This follows, for example, from Theorem 1.6 in [10] and the equivalence of (i) and (iii) of Lemma 1.5. In particular, an arithmetical ring R of finite Krull dimension satisfies (ii) of Theorem 5.14 if and only if R has Noetherian prime spectrum. On the other hand, it is easy to exhibit infinite-dimensional Prüfer domains satisfying (ii) of Theorem 5.14 that do not have Noetherian prime spectrum. For example, if V is a valuation domain with value group $\oplus_{i=1}^{\infty}\mathbb{Z}$ given the reverse lexicographic order, then the maximal ideal of V is the union of all the prime ideals properly contained in it.

Questions 5.17. We close with the following questions.

(1) In Theorem 5.14, we have characterized the rings R such that for each proper ideal A of R the canonical primal decomposition of A is irredundant with irreducible components. It would be interesting to classify the rings R having the property that for every proper ideal A of R there exists a subset C of $\mathrm{Ass}(A)$ such that $A = \bigcap_{P \in C} A_{(P)}$ is irredundant and each $A_{(P)}$ is irreducible. In view of Theorem 5.8, a ring with this property is an arithmetical ring, so the question is asking to classify the arithmetical rings R for which each proper ideal A can be written as an irredundant intersection $A = \bigcap_{P \in C} A_{(P)}$ for some subset C of $\mathrm{Ass}(A)$. There exist Prüfer domains with this property that do not satisfy the equivalent conditions of Theorem 5.14. For example, if R is the almost Dedekind domain that is not Dedekind of Example 2.2 of [14, pages 277-278], then R does not satisfy the equivalent conditions of Theorem 5.14, but every proper ideal A of R is an irredundant intersection $A = \bigcap_{P \in C} A_{(P)}$ for some subset C of $\mathrm{Ass}(A)$.

(2) What Prüfer domains R have the property that for each proper Zariski closed subset V of $\operatorname{Spec} R$ there exists an ideal A of R such that $\mathrm{Ass}(A) =$

V? If there exists a finitely generated ideal A such that $V = \{P \in \operatorname{Spec} R : A \subseteq P\}$, then Proposition 2.8 implies that $\operatorname{Ass}(A) = V$.

(3) Let R be an arithmetical ring. Is it possible to characterize the posets $\operatorname{Ass}(A)$ of $\operatorname{Spec} R$ as A varies over the ideals of R? By Proposition 2.7, if $P \in \operatorname{Ass}(A)$ and $Q \in \operatorname{Spec} R$ with $A \subseteq Q \subseteq P$, then $Q \in \operatorname{Ass}(A)$.

(4) In [6] we investigate for an ideal A of an arithmetical ring the relationship between the set $\operatorname{Max}(A)$ of maximal prime divisors of A and the set $\mathcal{X}_A$ of maximal members of the set of Krull associated primes of A. In this connection, it would be interesting to characterize the subsets C of the set of maximal ideals of an arithmetical ring (or Prüfer domain) R such that C is the set of maximal prime divisors of an ideal A of R.

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