COMMUTATIVE IDEAL THEORY WITHOUT FINITENESS CONDITIONS: IRREDUCIBILITY IN THE QUOTIENT FIELD

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ABSTRACT. Let R be an integral domain and let Q denote the quotient field of R. We investigate the structure of R-submodules of Q that are Q-irreducible, or completely Q-irreducible. One of our goals is to describe the integral domains that admit a completely Q-irreducible ideal, or a nonzero Q-irreducible ideal. If R has a nonzero finitely generated Q-irreducible ideal, then R is quasilocal. If R is integrally closed and admits a nonzero principal Q-irreducible ideal, then R is a valuation domain. If R has an m-canonical ideal and admits a completely Q-irreducible ideal, then R is quasilocal and all the completely Q-irreducible ideals of R are isomorphic. We consider the condition that every nonzero ideal of R is an irredundant intersection of completely Q-irreducible submodules of Q and present eleven conditions that are equivalent to this. We classify the domains for which every nonzero ideal can be represented uniquely as an irredundant intersection of completely Q-irreducible submodules of Q. The domains with this property are the Prüfer domains that are almost semi-artinian, that is, every proper homomorphic image has a nonzero socle. We characterize the Prüfer or Noetherian domains that possess a completely Q-irreducible ideal or a nonzero Q-irreducible ideal.

1. Introduction

This article continues a study of commutative ideal theory in rings without finiteness conditions begun in [15], [16], [17] and [26]. In [15] and [16] we examine irreducible and completely irreducible ideals of commutative rings. In the present article we investigate stronger versions of these two notions of irreducibility for ideals of integral domains. In particular, we consider irreducibility of an ideal of an integral domain when it is viewed as a submodule of the quotient field of the domain.

All rings in this paper are commutative and contain a multiplicative identity. Our notation is as in [18]. Let R be a ring and let C be an R-module. An R-submodule A of C is *C-irreducible* if $A = B_1 \cap B_2$, where B_1 and B_2 are R-submodules of C, implies that either $B_1 = A$ or $B_2 = A$. An R-submodule A of C is *completely C-irreducible* (or *completely irreducible* when the module C is clear from context)

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if $A = \bigcap_{i \in I} B_i$, where $\{B_i\}_{i \in I}$ is a family of R-submodules of C, implies $A = B_i$ for some $i \in I$.

In the case where the module C is the ring R, an ideal A of R is R-irreducible as a submodule of R precisely if A is *irreducible* as an ideal in the conventional sense that A is not the intersection of two strictly larger ideals. It is established by Fuchs in [14, Theorem 1] that a proper irreducible ideal A of the ring R is a *primal ideal* in the sense that the set of elements of R that are non-prime to A form an ideal P that is necessarily a prime ideal and is called the *adjoint prime ideal* of A. One then says that A is P-primal. For such an ideal A, it is the case that $A = A_{(P)}$, where $A_{(P)} = \bigcup_{x \in R \setminus P} (A :_R x)$.

In Remark 1.1 we record several general facts about completely C-irreducible submodules. The straightforward proofs are omitted.

Remark 1.1. For a proper submodule A of C the following are equivalent:

(1) A is completely C-irreducible.
(2) There exists an element $x \in C \setminus A$ such that $x \in B$ for every submodule B of C that properly contains A.
(3) C/A has a simple essential socle, that is, C/A is a cocyclic R-module.
(4) C/A is subdirectly irreducible in the sense that in any representation of C/A as a subdirect product of R-modules, one of the projections to a component is an isomorphism.

It is also straightforward to see that every submodule of a module C is an intersection of completely C-irreducible submodules of C. Thus a nonzero module C contains proper completely C-irreducible submodules.

The main focus of our present study is the case where R is an integral domain and $C = Q$ is the quotient field of R. (Throughout this paper Q is understood to be the quotient field of the integral domain R.) We are thus interested in Q-irreducible and completely Q-irreducible submodules of Q. We are particularly interested in determining conditions on an integral domain R in order that R admit a completely Q-irreducible ideal, or a nonzero Q-irreducible ideal. The zero ideal of R is always Q-irreducible, but if $R \neq Q$, the zero ideal of R is not completely Q-irreducible. In the case where R admits completely Q-irreducible ideals, or nonzero Q-irreducible ideals, we are interested in describing the structure of such ideals. Ideals with either of these properties are necessarily primal ideals.

It is frequently the case that an integral domain R may fail to have any fractional ideals that are completely Q-irreducible, or any nonzero ideals that are Q-irreducible. If $R = \mathbb{Z}$ is the ring of integers, then every nonzero proper Q-irreducible

R-submodule of Q is completely Q-irreducible and has the form $p^n\mathbb{Z}_{p\mathbb{Z}}$, where p is a prime integer and n is an integer. Thus for $R = \mathbb{Z}$ every nonzero proper Q-irreducible R-submodule of Q is a fractional ideal of a valuation overring of R. Moreover, every nonzero fractional R-ideal has a unique representation as an irredundant intersection of infinitely many completely Q-irreducible R-submodules of Q. It follows that R has no nonzero fractional ideal that is Q-irreducible.

In Section 2 we establish basic properties of irreducible submodules of an R-module C with special emphasis on the case where $C = Q$. We prove in Theorem 2.5 that if R admits a nonzero principal Q-irreducible fractional ideal, then R is quasilocal, and R is integrally closed if and only if R is a valuation domain. In Theorem 2.11 we give several necessary conditions for an integral domain to possess a nonzero Q-irreducible ideal. If A is a nonzero Q-irreducible ideal, we prove that $\mathrm{End}(A)$ is quasilocal, and that A is a primal ideal of $\mathrm{End}(A)$ with adjoint prime the maximal ideal of $\mathrm{End}(A)$. If the integral domain R admits a nonzero finitely generated Q-irreducible ideal, we prove that R is quasilocal. Moreover, every nonzero Q-irreducible ideal of a Noetherian domain is completely Q-irreducible.

In Section 3 we review some relevant results and examples regarding completely Q-irreducible fractional ideals. Over a quasilocal domain, an m-canonical ideal (if it exists) is an example of a completely Q-irreducible ideal. If R has an m-canonical ideal and admits a completely Q-irreducible ideal, we prove that R is quasilocal and all completely Q-irreducible ideals of R are isomorphic. We classify the Noetherian domains that admit a nonzero Q-irreducible ideal.

In Proposition 4.3 of Section 4 we show that a proper submodule A of the quotient field Q of a domain is an irredundant intersection of Q-irreducible submodules if and only if the injective hull of Q/A is an interdirect sum of indecomposable injectives.

In Section 5 we continue to examine irredundant intersections of Q-irreducible submodules in Q. We draw on the literature to give in Theorem 5.2 eleven different module- and ideal-theoretic conditions that are equivalent to the assertion that every nonzero ideal of a domain is an irredundant intersection of completely irreducible submodules of Q. We show in particular that such a domain is locally almost perfect, and from this observation we answer in the negative a question of Bazzoni and Salce of whether every locally almost perfect domain R has the property that Q/R is semi-artinian (Example 5.5). In Theorem 5.9 we classify the domains for which every nonzero ideal can be represented *uniquely* as an irredundant intersection of completely Q-irreducible submodules of Q. The domains having this property have Krull dimension at most one and are necessarily Prüfer, that is, every nonzero finitely generated ideal is invertible. They may be described precisely as the Prüfer domains R that are almost semi-artinian, that is, every proper homomorphic image of R has a nonzero socle.

In light of Theorem 5.9 it is useful to describe the completely irreducible submodules of the quotient field of a Prüfer domain. This is done in Theorem 6.2. Also in Section 6 we characterize the Prüfer domains that possess a completely Q-irreducible ideal, or a nonzero Q-irreducible ideal. We prove that a Prüfer domain R that admits a nonzero Q-irreducible ideal also admits a completely Q-irreducible ideal, and this holds if and only if every proper R-submodule of Q is a fractional R-ideal.

In Section 7 we discuss several open questions, and in an appendix we correct some errors in the article [17] that were pointed out to us by Jung-Chen Liu and her student Zhi-Wei Ying. We are grateful to them for showing us these mistakes.

2. The structure of Q-irreducible ideals

We begin with several general results.

Proposition 2.1. *Let R be a ring and C an R-module. The following statements are equivalent for a proper R-submodule A of C.*

(i) *A is a completely C-irreducible R-submodule of C.*
(ii) *There exists $x \in C \setminus A$ such that for all $y \in C \setminus A$ we have $x \in A + Ry$.*
(iii) *A is C-irreducible and there exists a maximal ideal M of R such that $A \subset (A :_C M)$, where $(A :_C M) = \{y \in C : yM \subseteq A\}$.*

Furthermore, if R is a domain, A is torsionfree and C is the divisible hull of A, then statements (i)-(iii) are equivalent to:

(iv) *There is a maximal ideal M of R such that $A = AR_M$ and A is completely C-irreducible as an R_M-submodule of C.*

Proof. (i) $\Rightarrow$ (ii) Let A^* be the intersection of all R-submodules of C properly containing A. Then $A \subset A^*$, and A^*/A is a simple R-module. Hence $A^* = Rx + A$ for some $x \in Q \setminus A$, and (ii) follows.

(ii) $\Rightarrow$ (iii) By (ii) there exists $x \in C \setminus A$ such that $A^* := A + Rx$ is contained in every R-submodule of C properly containing A. Hence A^*/A is a simple R-module and $A^*/A \cong R/M$ for some maximal ideal M of R. Thus $A^* \subseteq (A :_C M)$ so that $(A :_C M) \neq A$.

(iii) $\Rightarrow$ (i) Since A is irreducible, $(A :_C M)/A \cong R/M$ and every proper submodule containing A contains $(A :_C M)$, proving (i).

(i) $\Rightarrow$ (iv) Since R is a domain and A is torsion-free, $A = \bigcap_{M \in \mathrm{Max}(R)} A_M$, where each A_M is identified with its image in $C = QA$. Because A is completely C-irreducible, $A = A_M$ for some maximal ideal M of R. The assumption that A is completely C-irreducible as an R-module clearly implies A is completely C-irreducible as an R_M-submodule of C.

(iv) $\Rightarrow$ (iii) Since we have established the equivalence of (i)-(iii), and since by assumption A is a completely irreducible R_M-submodule of C, we have by (iii) (applied to the R_M-module A) that there exists $x \in (A :_C MR_M) \setminus A$. Now since $A = A_M$, we have $A \neq (A :_C MR_M) = (A :_C M)$. Thus it remains to observe that A is C-irreducible. Suppose $A = B \cap D$ for some R-submodules B and D of C. Then $A = A_M = B_M \cap D_M$, so since by assumption A is irreducible as an R_M-submodule of C, it must be that $A = B_M$ or $A = D_M$. Thus $B \subseteq A$ or $D \subseteq A$, proving that A is irreducible. $\square$

Remark 2.2. Let R be an integral domain that is properly contained in its quotient field Q.

(i) By Remark 1.1, every R-submodule of Q is an intersection of completely irreducible submodules of Q. In particular, every ideal of R is an intersection of completely irreducible submodules of Q.

(ii) A fractional ideal A of R is completely Q-irreducible if and only if A is not the intersection of fractional R-ideals that properly contain A. If A is a fractional R-ideal and $A \neq Q$, then A is completely Q-irreducible if and only if there exists $x \in Q \setminus A$ such that x is in every fractional ideal that properly contains A.

(iii) A maximal ideal P of R is completely Q-irreducible if it is Q-irreducible. This is immediate from Proposition 2.1, since $P \subsetneq R \subseteq (P :_Q P)$.

In Lemma 2.3, we establish several general facts about Q-irreducible and completely Q-irreducible ideals.

Lemma 2.3. *Let A be a proper ideal of the integral domain R. Then*

(i) *A is Q-irreducible if and only if for each nonzero $r \in R$ the ideal rA is irreducible.*

(ii) *For a nonzero $q \in Q$, the fractional ideal qA is Q-irreducible if and only if A is Q-irreducible. Therefore the property of being Q-irreducible is an invariant of isomorphism classes of fractional R-ideals.*

(iii) *A is Q-irreducible if and only if there is a prime ideal P of R such that $A = AR_P$ and A is a Q-irreducible ideal of R_P. It then follows that P is uniquely determined by A and A is P-primal.*

(iv) *For a nonzero $q \in Q$, the fractional ideal qA is completely Q-irreducible if and only if A is completely Q-irreducible. Therefore the property of being completely Q-irreducible is an invariant of isomorphism classes of fractional R-ideals.*

(v) *If A is completely R-irreducible and if for each nonzero $r \in R$ the ideal rA is irreducible, then A is completely Q-irreducible.*

Proof. (i) Assume A is Q-irreducible and r is a nonzero element of R. If $rA = B \cap C$ for ideals B and C of R, then $A = r^{-1}B \cap r^{-1}C$. Since A is Q-irreducible, either $A = r^{-1}B$ or $A = r^{-1}C$. Hence either $rA = B$ or $rA = C$ and rA is irreducible. Conversely, assume A is not Q-irreducible. Then there exist R-submodules B and C of Q that properly contain A such that $A = B \cap C$. We may assume that B and C are fractional ideals of R. Then there exists a nonzero $r \in R$ such that rB and rC are integral ideals of R. Moreover, $A = B \cap C$ implies $rA = rB \cap rC$ and $A \subset B$ implies $rA \subset rB$ and similarly $A \subset C$ implies $rA \subset rC$. Therefore rA is reducible. This completes the proof of (i).

Statements (ii) and (iv) are clear since $A = \bigcap_{i \in I} B_i$ if and only if $qA = \bigcap_{i \in I} qB_i$ and multiplication by q (or by q^{-1}) preserves strict inclusion.

(iii) Assume A is Q-irreducible. Then A is P-primal for some prime ideal P of R, so that $A = A_{(P)} = AR_P \cap R$. Since A is Q-irreducible, this forces $A = AR_P$. Clearly then A is Q-irreducible as an R_P-module since it is Q-irreducible as an R-module. Conversely, suppose that $A = AR_P$ and A is Q-irreducible as an ideal of R_P. If $A = B \cap C$ for some R-submodules B and C of Q, then $A = AR_P = BR_P \cap CR_P$, and since A is a Q-irreducible R_P-submodule of Q, $A = BR_P$ or $A = CR_P$. Thus $B \subseteq A$ or $C \subseteq A$, which completes the proof.

(v) Since A is completely R-irreducible, there exists an element $x \in R \setminus A$ such that x is in every ideal of R that properly contains A. Let $A^* = A + xR$. If A is not completely Q-irreducible, then there exists an R-submodule B of Q that properly contains A but does not contain x. Since there are no ideals properly between A and A^*, $A = A^* \cap B$ and this intersection is irredundant. We may assume that B is a fractional ideal of A. Then there exists a nonzero $r \in R$ such that rB is an integral ideal of R. Therefore $rA = rA^* \cap rB$ is an irredundant intersection. It follows that rA is not irreducible. □

Remark 2.4. With regard to Lemma 2.3 we have:

(1) If A is a nonzero Q-irreducible ideal of R and P is as in Lemma 2.3(iii), then $R_P \subseteq \mathrm{End}(A)$ and $rA = A$ for each $r \in R \setminus P$. It follows that A is contained in every ideal of R not contained in P. Thus if P is a maximal ideal of R and A is P-primary with $A = AR_P$, then R is quasilocal.

(2) It is also true that if A and B are isomorphic R-submodules of Q, then A is (completely) Q-irreducible if and only if B is (completely) Q-irreducible. For A and B are R-isomorphic if and only if there exists $q \in Q$ such that $A = qB$.

Theorem 2.5. *If the integral domain R has a nonzero principal fractional ideal that is Q-irreducible, then R is quasilocal and every principal ideal of R is Q-irreducible. If R is integrally closed, then*

(i) *R is Q-irreducible if and only if R is a valuation domain, and*
(ii) *R is completely Q-irreducible if and only if R is a valuation domain with principal maximal ideal.*

Proof. (i) Lemma 2.3 implies that R has a nonzero principal fractional ideal that is (completely) Q-irreducible if and only if every nonzero principal fractional ideal of R is (completely) Q-irreducible. Suppose R has distinct maximal ideals M and N. Then there exist $x \in M$ and $y \in N$ such that $x + y = 1$. It follows that $xyR = xR \cap yR$ is an irredundant intersection. By Lemma 2.3(i), R is not Q-irreducible.

(ii) Suppose that R is integrally closed and Q-irreducible but is not a valuation domain. Then there exists $x \in Q$ such that neither x nor $1/x$ is in R. Let $\mathcal{F}$ be the set of valuation overrings of R that contain x and let $\mathcal{G}$ be the set of valuation overrings of R that contain $1/x$. Let $A = \bigcap_{V \in \mathcal{F}} V$ and $B = \bigcap_{W \in \mathcal{G}} W$. Then $x \in A$ implies $R \subsetneq A$ and $1/x \in B$ implies $R \subsetneq B$. Observe that every valuation overring of R is a member of at least one of the sets $\mathcal{F}$ or $\mathcal{G}$. Since R is integrally closed, we have $R = A \cap B$, a contradiction to the assumption that R is Q-irreducible. Conversely, it is clear that if R is a valuation domain, then R is integrally closed and Q-irreducible.

(iii) By (ii) we need only observe the well-known fact that a valuation domain R is completely Q-irreducible if and only if the maximal ideal of R is principal. (See for example [3].) □

Remark 2.6. There exist integral domains R that are completely Q-irreducible and are not integrally closed. If R is a one-dimensional Gorenstein local domain, then R, and every nonzero principal fractional ideal of R, is completely Q-irreducible. Thus, for example, if k is a field and a and b are relatively prime positive integers, then the subring $R := k[[t^a, t^b]]$ of the formal power series ring $k[[t]]$ is completely Q-irreducible.

Theorem 2.5(ii) characterizes among integrally closed domains R the ones that are valuation domains as precisely those R that are Q-irreducible. As a corollary to Proposition 2.1, we have the following additional characterizations of the valuation property in terms of Q-irreducibility.

Corollary 2.7. *The following are equivalent for a domain R with quotient field Q.*

(i) *R is a valuation domain.*
(ii) *Every irreducible ideal is Q-irreducible.*
(iii) *Every completely irreducible ideal is completely Q-irreducible.*
(iv) *There exists a maximal ideal of R that is Q-irreducible.*
(v) *There exists a maximal ideal of R that is completely Q-irreducible.*

Proof. (i) $\Rightarrow$ (ii) If R is a valuation domain, then it is easy to see that irreducible ideals are Q-irreducible since the R-submodules of Q are linearly ordered.

(ii) $\Rightarrow$ (iii) If A is a completely irreducible ideal of R, then there is a maximal ideal M of R such that $(A :_R M) \neq A$. Thus $(A :_Q M) \neq A$, and since A is by (ii) Q-irreducible, we have from Proposition 2.1 (iii) that A is Q-irreducible.

(iii) $\Rightarrow$ (iv) This is clear from the fact that maximal ideals are completely irreducible.

(iv) $\Rightarrow$ (v) This follows from Remark 2.2(iii).

(v) $\Rightarrow$ (i) Let M be a completely Q-irreducible maximal ideal of R. For every nonzero $r \in R$, rM is completely irreducible by Proposition 2.3. It is shown in Lemma 5.1 of [16] that this property characterizes valuation domains, so the proof is complete. □

Corollary 2.8. *Let P be a prime ideal of a domain R. Then P is Q-irreducible if and only if $P = PR_P$ and R_P is a valuation domain. Thus if P is Q-irreducible, then R_P/P is the quotient field of R/P, and R is a pullback of R/P and the valuation domain R_P. Moreover P is completely Q-irreducible as an ideal of R_P.*

Proof. Suppose that P is Q-irreducible. By Lemma 2.3, $P = PR_P$ and PR_P is a Q-irreducible ideal of R_P. Hence, by Corollary 2.7, R_P is a valuation domain.

Conversely, assume $P = PR_P$ and R_P is a valuation domain. By Corollary 2.7, P is a Q-irreducible ideal of R_P. Hence, by Lemma 2.3, P is Q-irreducible. It follows from Remark 2.2(iii) that $P = PR_P$ is a completely Q-irreducible ideal of R_P. □

Remark 2.9. With $P = PR_P$ as in Corollary 2.8, if $R \neq R_P$, then P as an ideal of R is not completely Q-irreducible. For Proposition 2.1 (iii) implies that a completely Q-irreducible prime ideal is a maximal ideal, and by Remark 2.4(i), if P is maximal and Q-irreducible, then $R = R_P$. It can happen however that P is Q-irreducible and nonmaximal. This is the case, for example, if P is a nonmaximal prime of a valuation domain R.

Remark 2.10. Pullbacks arising as in Corollary 2.8 have been well-studied; for a recent survey see [20]. For example, a consequence of our Corollary 2.8 and Theorem 4.8 in [19] is that if a domain R has a Q-irreducible prime ideal P, then R is coherent if and only if R/P is coherent.

Theorem 2.11. *Assume that A is a nonzero Q-irreducible ideal of the integral domain R. Then*

(i) *If A is not principal, then AA^{-1} is contained in the Jacobson radical of R.*
(ii) $\operatorname{End}(A)$ *is a quasilocal integral domain.*

Let M denote the maximal ideal of $\operatorname{End}(A)$.

(iii) *A is an M-primal ideal of* $\operatorname{End}(A)$.
(iv) *If M is finitely generated as an ideal of* $\operatorname{End}(A)$*, then A is completely Q-irreducible as an ideal both of R and of* $\operatorname{End}(A)$.
(v) *If A is a finitely generated ideal of R, then R is quasilocal and the maximal ideal of R is the adjoint prime of A.*
(vi) *If both A and its adjoint prime are finitely generated ideals, then A is completely Q-irreducible.*

Proof. (i) Let $x \in A^{-1}$ and suppose that there is a maximal ideal N of R not containing xA. Then there exists $y \in N$ such that $xA + yR = R$. It follows that $xyA = xA \cap yR$. By Lemma 2.3(ii), xyA is irreducible. Therefore either $xyA = xA$ or $xyA = yR$. If $xA = xyA$, then $xA \subseteq yR \subseteq N$, a contradiction, while if $xyA = yR$, then $xA = R$ and A is principal. We conclude that every maximal ideal of R contains xA. Therefore AA^{-1} is contained in the Jacobson radical of R.

(ii) and (iii) Since A is Q-irreducible as an ideal of R, it is also Q-irreducible as an ideal of $\operatorname{End}(A)$. By Lemma 2.3(iii), there is a prime ideal M of $\operatorname{End}(A)$ such that $A = A\operatorname{End}(A)_M$. Thus $\operatorname{End}(A)_M \subseteq \operatorname{End}(A)$, which implies that M is the unique maximal ideal of $\operatorname{End}(A)$. Also by Lemma 2.3(iii), A is M-primal.

(iv) Let $x_1, \ldots, x_n$ generate M. By Lemma 2.1(iii), to show that A is completely Q-irreducible it suffices to prove that $(A :_Q M) \neq A$. Now $(A :_Q M) = x_1^{-1}A \cap \cdots \cap x_n^{-1}A$, so if $(A :_Q M) = A$, then the Q-irreduciblity of A implies $x_i^{-1}A = A$ for some i. In this case, $x_i^{-1} \in \operatorname{End}(A)$, which is impossible since $x_i \in M$, the maximal ideal of $\operatorname{End}(A)$.

(v) By Lemma 2.3(ii), $A = AR_P$ for some prime ideal P of R. Thus $R_P \subseteq \mathrm{End}(A)$. But A is a finitely generated ideal of R implies that $\mathrm{End}(A)$ is an integral extension of R. This forces $R = R_P$, so that P is the unique maximal ideal of R.

(vi) By (v), R is quasilocal with maximal ideal M, and M is the adjoint prime of A. As in the proof of (iv), we have $A \subset (A :_Q M)$. Therefore Lemma 2.1(ii) implies that A is completely Q-irreducible. □

Corollary 2.12. *Every nonzero Q-irreducible ideal over a Noetherian domain is completely Q-irreducible. If the Noetherian domain R admits a completely Q-irreducible ideal, then R is local and* $\dim R \leq 1$.

Proof. Suppose that A is a nonzero Q-irreducible ideal of R. By Theorem 2.11(vi) A is a completely Q-irreducible ideal of R, and hence also of $\mathrm{End}(A)$. By Theorem 2.11(ii), $\mathrm{End}(A)$ is quasilocal. Since R is Noetherian, $\mathrm{End}(A)$ is a finitely generated integral extension of R. Therefore R is local.

If $\dim R > 1$, then there exists a nonzero nonmaximal prime ideal P of R. Let $x \in P$ with $x \neq 0$. Then xM is completely irreducible by Lemma 2.3(iv). However, by Corollary 1.4 in [16] a completely irreducible ideal of a Noetherian local domain is primary for the maximal ideal, contradicting $xM \subseteq P$. Therefore $\dim R \leq 1$. □

Corollary 2.13. *If the integral domain R admits an invertible Q-irreducible ideal, then every invertible ideal of R is principal and completely Q-irreducible.*

Proof. Suppose that A is an invertible Q-irreducible ideal of R. By Theorem 2.11(i) A is principal. Let B be an invertible ideal of R. Since A is invertible, $A = (B :_Q : (B :_Q A))$. Moreover, $(B :_Q A)$ is an invertible, hence finitely generated, fractional ideal of R. Hence there are elements $q_1, \ldots, q_k \in Q$ such that $A = (B :_Q (q_1, \ldots, q_k)R) = q_1^{-1}B \cap \cdots \cap q_k^{-1}B$. Since A is Q-irreducible, there exists $i \in \{1, \ldots, k\}$ such that $B = q_iA$. Hence B is principal and R-isomorphic to A. By Lemma 2.3, B is Q-irreducible. □

Remark 2.14. Statement (ii) of Theorem 2.11 is true also when A is a completely irreducible submodule of Q. For by Lemma 2.1(iv) (with A viewed as a completely irreducible $\mathrm{End}(A)$-submodule of Q) there is a maximal ideal M of $\mathrm{End}(A)$ such that $A = A_M$ This forces $\mathrm{End}(A)_M \subseteq \mathrm{End}(A)$, so $\mathrm{End}(A)$ is quasilocal.

3. Completely Q-irreducible and m-canonical ideals

As noted in Remark 2.2 every ideal of a domain is the intersection of completely irreducible submodules of the quotient field. Thus for a given domain there exists an abundance of completely irreducible submodules of Q. However, as we observe in Section 1, a domain need not possess a completely Q-irreducible ideal (see also Example 3.7).

In this section we examine the existence and structure of completely Q-irreducible ideals. We also consider the class of "m-canonical" ideals. A nonzero fractional ideal A of a domain R is an *m-canonical* fractional ideal if for all nonzero ideals B of R, $B = (A :_Q (A :_Q B))$. This terminology is from [1] and [25]. Different terminology is used in [3] and [18] to express the same concept. An ideal A is, in our terminology, m-canonical if and only if, in the terminology of [3] and [18], R is an "A-divisorial" domain and $\mathrm{End}(A) = R$. Notice that the property of being an m-canonical ideal is invariant with respect to R-isomorphism for fractional ideals of R.

It follows from [25, Lemma 4.1] that an m-canonical ideal of a quasilocal domain is completely Q-irreducible. A deeper result is due to S. Bazzoni [3]: *A fractional ideal A of a quasilocal domain R is m-canonical if and only if A is completely Q-irreducible,* $\mathrm{End}(A) = R$ *and for all nonzero $r \in R$, A/rA satisfies the dual AB-5^* of Grothendieck's AB-5.* (An R-module B satisfies AB-5^* if for any submodule C of B and inverse system of submodules $\{B_i\}_{i\in I}$ of B, it is the case that $\bigcap_{i\in I}(C+B_i) = C + \bigcap_{i\in I}(B_i)$.)

As examples later in this section show, a domain need not possess an m-canonical ideal. However if R admits an m-canonical ideal, then all completely Q-irreducible ideals of R are isomorphic:

Proposition 3.1. *Let R be a domain that is not a field. If R has an m-canonical ideal A, then every completely Q-irreducible ideal of R is isomorphic to A. Consider the following statements.*

(i) *R has an m-canonical ideal.*
(ii) *Any two completely Q-irreducible ideals of R are isomorphic.*

Then (i) $\Rightarrow$ (ii). If every completely irreducible proper submodule of Q is a fractional ideal of R, then (ii) $\Rightarrow$ (i).

Proof. Suppose that R has an m-canonical ideal A. If B is a nonzero ideal of R, then $B = \bigcap_q q^{-1}A$, where q ranges over all nonzero elements of $(A :_Q B)$. Thus if B is completely Q-irreducible, then $B = q^{-1}A$ for some $0 \neq q \in (A :_Q B)$. Thus every proper completely Q-irreducible ideal is isomorphic to A, and (i) $\Rightarrow$ (ii).

Assume that any two completely Q-irreducible ideals are isomorphic and every completely Q-irreducible proper submodule of Q is a fractional ideal of R. Let A be a completely irreducible R-ideal. By Remark 1.1 every ideal of R is an intersection of completely Q-irreducible submodules of Q and therefore of completely Q-irreducible fractional ideals of R. Thus every ideal of R is an intersection of ideals isomorphic to A; that is, for any ideal B, there exists a set $X \subseteq Q$ such that $B = \bigcap_{q \in X} qA$. It follows that $B = (A :_Q (A :_Q B))$. Hence A is an m-canonical ideal. □

Remark 3.2. An integral domain may have an m-canonical ideal, but not admit a completely Q-irreducible fractional ideal. For example, if R is a Dedekind domain having more than one maximal ideal, then R admits an m-canonical ideal, but does not have any completely Q-irreducible fractional ideals. Indeed, as we observe in Proposition 3.3, if R has an m-canonical ideal and admits a completely Q-irreducible ideal, then R is quasilocal.

Proposition 3.3. *If R has an m-canonical ideal and a completely Q-irreducible ideal, then R is quasilocal.*

Proof. Let A be a completely Q-irreducible ideal of R. By Proposition 3.1, A is an m-canonical ideal. Therefore $R = \mathrm{End}(A)$. By Theorem 2.11, $\mathrm{End}(A)$ is quasilocal. Therefore R is quasilocal. □

Remark 3.4. If A is a proper R-submodule of Q, then A is contained in a completely irreducible proper submodule of R. Thus if every completely irreducible proper submodule of Q is a fractional ideal of R, then every proper submodule of Q is a fractional ideal of R. The latter property holds for R if and only if there exists a valuation overring of R which is a fractional ideal of R [31, Theorem 79].

Routine arguments show that a nonzero fractional ideal of a valuation domain is m-canonical if and only if it is completely Q-irreducible. Also in the Noetherian case, the condition AB-5* is redundant, as we note next. The following proposition is essentially due in the case of Krull dimension 1 to Matlis [32] and in the general case with the assumption that $\mathrm{End}(A) = R$ to Bazzoni [3]. Bazzoni's proof shows that you can omit in our context the assumption that $\mathrm{End}(A) = R$. We outline how to do this in the proof. We also include a different proof of the step (iv) $\Rightarrow$ (iii).

Proposition 3.5. (Bazzoni [3, Theorem 3.2], Matlis [32, Theorem 15.5]) *The following statements are equivalent for a nonzero fractional ideal A of a Noetherian local domain (R, M) that is not a field.*

(i) *Q/A is an injective R-module.*
(ii) *R has Krull dimension 1 and $(A : M)/A$ is a simple R-module.*

(iii) *A is an m-canonical ideal.*

(iv) *A is Q-irreducible.*

Proof. (i) $\Rightarrow$ (ii) By Proposition 4.4 in [33] a Noetherian domain that admits an ideal of injective dimension 1 necessarily has Krull dimension 1. Thus $\dim(R) = 1$, so we may apply Theorem 15.5 in [32] to obtain (ii).

(ii) $\Rightarrow$ (iii) This is contained in Theorem 15.5 of [32].

(iii) $\Rightarrow$ (i) If A is an m-canonical ideal, then necessarily $\mathrm{End}(A) = R$, so Theorem 3.2 of [3] applies.

(iii) $\Rightarrow$ (iv) An m-canonical ideal of a quasilocal domain is completely Q-irreducible [25, Lemma 4.1].

(iv) $\Rightarrow$ (iii) Suppose that A is Q-irreducible. By Corollary 2.12 $\dim R = 1$ and A is completely Q-irreducible. By Theorem 2.11 $\mathrm{End}(A)$ is a quasilocal domain. Since R is Noetherian, $\mathrm{End}(A)$ is Noetherian. Thus by Theorem 3.2 in [3] A is an m-canonical ideal of $\mathrm{End}(A)$.

By [32, Theorem 15.7] a Noetherian local domain of Krull dimension 1 has an m-canonical ideal if and only if the total quotient ring of the completion of the domain is Gorenstein. Therefore the total quotient ring of the completion of $\mathrm{End}(A)$ is Gorenstein. Now $\mathrm{End}(A)$ is an overring of R that is finitely generated as a module over R. Hence there exists a nonzero $x \in R$ such that $x\,\mathrm{End}(A) \subseteq R$. It follows that the total quotient ring T of the completion of R coincides with the completion of $\mathrm{End}(A)$. Thus T is a Gorenstein ring, and by the result cited above, R has an m-canonical ideal, say B. By Proposition 3.1 B is isomorphic to A, so A is an m-canonical ideal of R. $\square$

Remark 3.6. Let R be a Noetherian domain of positive dimension. If R admits a nonzero Q-irreducible ideal, then R is local and $\dim R = 1$. Every proper R-submodule of Q is a fractional R-ideal if and only if the integral closure $\overline{R}$ of R is local (so a DVR) and is a finitely generated R-module. In this case every proper R-submodule of Q that is completely Q-irreducible is a fractional R-ideal. There exist, however, other one-dimensional Noetherian local domains R that admit completely Q-irreducible ideals. By Proposition 3.5, R admits a completely Q-irreducible ideal if and only if the total quotient ring of the completion of R is Gorenstein. In particular, this is true if R is Gorenstein. There exist examples where R is Gorenstein and $\overline{R}$ is not local, or not a finitely generated R-module, or both. For such an R, nonzero principal fractional ideals of R are completely Q-irreducible, and there also exist completely Q-irreducible proper R-submodules of Q that are not fractional R-ideals.

Example 3.7. *A one-dimensional Noetherian local domain need not possess a nonzero Q-irreducible ideal.* As noted in the proof of Proposition 3.5 it suffices to exhibit a Noetherian local domain R of Krull dimension 1 such that the total quotient ring of R is not Gorenstein. Such examples can be found in Proposition 3.1 of [12] and Theorem 1.26 and Corollary 1.27 of [27]. A specific example, based on [27] is obtained as follows. Let x, y, z be algebraically independent over the field k and let $R = k[x, y, z]_{(x,y,z)}$. Let $f, g \in xk[[x]]$ be such that x, f, g are algebraically independent over k. Let $u = y - f$ and $v = z - g$. Then $P := (u, v)k[[x, y, z]]$ is a prime ideal of height 2 of the completion $\widehat{R} = k[[x, y, z]]$ of R having the property that $P \cap R = (0)$. If $\mathbf{q}$ is a P-primary ideal of $\widehat{R}$, it follows from [27, Theorem 1.26] that $(\widehat{R}/\mathbf{q}) \cap k(x, y, z)$ is a one-dimensional Noetherian local domain having $\widehat{R}/\mathbf{q}$ as its completion. If we take $\mathbf{q} = P^2 = (u^2, uv, v^2)\widehat{R}$, then the total quotient ring of $\widehat{R}/\mathbf{q}$ is not Gorenstein.

Remark 3.8. (i) It is an open question whether a completely Q-irreducible ideal of a quasilocal integrally closed domain R is an m-canonical ideal if $\mathrm{End}(A) = R$ [3, Question 5.5]. The answer is affirmative when $A = R$: this is Theorem 2.3 of [3].

(ii) In [3] Bazzoni relates the question in (i) to a 1968 question of Heinzer [24]: If R is a domain for which every nonzero ideal is divisorial, is the integral closure of R a Prüfer domain? To obtain that R has a Prüfer integral closure the weaker requirement that R be completely Q-irreducible is not sufficient, as we note below in Example 3.10.

(iii) Bazzoni constructs in Example 2.11 of [3] an example of a quasilocal domain R such that R is completely Q-irreducible but not m-canonical. By Lemma 3.5 and (i) such a domain is neither Noetherian nor integrally closed.

The $D + M$ construction provides a source of interesting examples of completely Q-irreducible ideals. The following example is from [25, Remark 5.3], as strengthened in [1]. We recall it here, since it is relevant to Example 3.10.

Example 3.9. Let $k \subset F$ be a proper extension of fields and V be a valuation domain (that is not a field) of the form $V = F + M$, where M is the maximal ideal of V. Define $R = k + M$. Then R is a quasilocal domain with maximal ideal M. If U is any k-subspace of F of codimension 1, then the fractional ideal $A = U + M$ is a completely Q-irreducible fractional ideal of R since every R-submodule of the quotient field Q of R that properly contains A contains also V.

It is proved in Theorem 3.2 of [1] that if F is an algebraic extension of k with $[F:k]$ infinite, then there exist codimension 1 subspaces U and W of F such that $U+M$ and $W+M$ are non-isomorphic completely Q-irreducible fractional ideals of R. Thus by Proposition 3.1 R does not possess an m-canonical ideal. Indeed, it is shown in Theorem 3.1 of [1] that R has an m-canonical ideal if and only if $[F:k]$ is finite.

We shall see in Theorem 6.3 that it is possible for a domain R to possess a completely Q-irreducible ideal A and not be quasilocal. It follows from this result that $\mathrm{End}(A)$ need not equal R. However, in this situation, R is not quasilocal. The next example shows that even when R is quasilocal, it is possible for a completely Q-irreducible ideal to have an endomorphism ring not equal to R.

Gilmer and Hoffmann in [21] establish the existence of an integral domain R that admits a unique minimal overring, but has the property that the integral closure of R is not Prüfer. In Example 3.10 we modify this example to establish the existence of an integral domain R that has infinitely many distinct fractional overrings R_t, $t \in \mathbb{N}$, such that each R_t is completely Q-irreducible as a fractional ideal of R. Since R_t is a fractional overring of R, $\mathrm{End}(R_t) = R_t$. We remark that Bazzoni in [3, Section 4] has abstracted and greatly generalized the example of [21].

Example 3.10. Let K be a field and let $L = K((X))$ be the quotient field of the formal power series ring $K[[X]]$. Every nonzero element of L has a unique expression as a Laurent series $\sum_{n\geq k} a_n X^n$, where k is an integer, the $a_n \in K$ and $a_k \neq 0$. Let Y be an indeterminate over L and let $V = L[[Y]]$ denote the formal power series ring in Y over the field L. Thus V is a rank-one discrete valuation domain (DVR) of the form $L+M$, where $M = YL[[Y]]$ is the maximal ideal of V. Let $R = K + M^2$. It is well-known and readily established that R is a one-dimensional quasilocal domain with maximal ideal M^2. For t a positive integer, let W_t be the set of all elements $f \in K((X))$ such that $f = 0$ or the coefficient of X^{-t} in the Laurent expansion of f is 0. Notice that W_t is a K-subspace of L and $L = W_t \oplus KX^{-t}$ as K vector spaces. Let $R_t = K + W_tY + M^2$. Then R_t is an overring of R and $Y^2R_t \subseteq M^2$, so R_t is a fractional ideal of R.

We show that R_t is completely Q irreducible as a fractional R-ideal by proving that $X^{-t}Y$ is in every fractional ideal of R that properly contains R_t. Let $f \in Q \backslash R_t$. Since $Q = L((Y))$, there exists an integer j such that $f = \sum_{n \geq j} b_n Y^n$, where each $b_n \in L$ and $b_j \neq 0$. Notice that $f \notin R_t$ implies $j \leq 1$. Since $L = K((X))$, there exists an integer k such that $b_j = \sum_{n\geq k} a_n X^n$, where each $a_n \in K$ and $a_k \neq 0$. Since a_k is a unit of R, the fractional ideal $R_t + Rf = R_t + a_k^{-1} f$, so we may assume that $a_k = 1$. If $j < 0$, then $X^{-k-t}Y^{1-j} \in M^2 \subset R$ and $X^{-k-t}Y^{1-j}f = X^{-t}Y + \alpha Y + \beta Y^2$, where $\alpha \in K[[X]]$ and $\beta \in V = L[[Y]]$. Since $\alpha \in W_t$, $\alpha Y + \beta Y^2 \in R_t$. Hence $X^{-t}Y \in R_t + Rf$ if $j < 0$. If $j = 0$ and $k \neq 0$, then $X^{-k-t}Y^{1-j} \in W_tY \subset R_t$ and $X^{-k-t}Yf = X^{-t}Y + \alpha Y + \beta Y^2$, where $\alpha Y + \beta Y^2 \in R_t$, so $X^{-t}Y \in R_t + Rf$ in this case. If $j = 0$ and $k = 0$, replace f by $f - 1$ to obtain a situation where $k > 0$ and $j \geq 0$. If $j = 1$, then $f \notin R_t$ implies $b_1 \notin W_t$. Hence $b_1 = c + dX^{-t}$, where $c \in W_t$ and $0 \neq d \in K$. Hence $f - cY = dX^{-t}Y + \alpha Y^2$, where $\alpha \in L[[Y]]$. Therefore also in this case $X^{-t}Y \in R_t + Rf$. We conclude that R_t is completely Q-irreducible.

In Example 3.10 the completely Q-irreducible fractional ideals that are constructed have endomorphism rings integral over the base ring. In Example 3.13 we exhibit a Noetherian local domain R and a completely Q-irreducible R-submodule A of Q such that $\mathrm{End}(A)$ is not integral over R. We first give a partial characterization of when valuation overrings are (completely) Q-irreducible.

Theorem 3.11. *Let V be a valuation overring of the domain R. Then the following two statements hold for V.*

(i) *If V/R is a divisible R-module, then V is a Q-irreducible R-submodule of Q. Moreover, V has a principal maximal ideal if and only if V is a completely Q-irreducible R-submodule of Q.*

(ii) *Suppose that V is a DVR. Then V is a completely Q-irreducible R-submodule of Q if and only if V/R is a divisible R-module.*

Proof. (i) The assumption that V/R is divisible implies that every R-submodule of Q containing V is also a V-submodule of Q. For if $x \notin V$, then $1/x \in V$. Since V/R is divisible, $V = (1/x)V + R$. Thus $V + xR = xV$. Hence $V + xR$ is a V-submodule of Q. This implies that any R-submodule of Q containing V is a V-module. Since V is Q-irreducible as a V-submodule of Q, it follows that V is Q-irreducible as an R-submodule of Q.

If the valuation domain V has principal maximal ideal, then, by Theorem 2.5, V is a completely Q-irreducible V-submodule of Q. Therefore V is a completely Q-irreducible R-submodule of Q.

Conversely, if V is a completely Q-irreducible R-submodule of Q, then necessarily V is a completely Q-irreducible V-submodule of Q. By Corollary 2.7 every principal ideal of V is Q-irreducible. Hence by Theorem 2.5 V has a principal maximal ideal.

(ii) Suppose that V is a completely Q-irreducible R-submodule of Q. Let $0 \neq x \in R$. We claim that $V = R + xV$. Consider the ideal $C = (R + xV :_Q V)$ of V.

Since V is a DVR, C is isomorphic to V. Also, $C = \cap_{y \in V} y^{-1}(R + xV)$, so since C is completely Q-irreducible, C is isomorphic to $R + xV$. Thus V and $R + xV$ are isomorphic as R-modules, and since these two modules are rings, this forces $R + xV = V$, proving that V/R is divisible. The converse follows from (i). □

Remark 3.12. Let V be a DVR overring of the integral domain R and let P be the center of V on R. Necessary and sufficient conditions in order that V/R be a divisible R-module are that (i) PV is the maximal ideal of V, and (ii) the canonical inclusion map of $R/P \hookrightarrow V/PV$ is an isomorphism. By Theorem 3.11(ii), these conditions are also necessary and sufficient in order that V be completely Q-irreducible as an R-submodule of Q.

Example 3.13. Let K be a field, and let X and Y be indeterminates for K. Define $R = K[X,Y]_{(X,Y)}$. We construct a valuation overring V of R such that V is a completely Q-irreducible R-submodule of Q. Let $g(X) \in XK[[X]]$ be such that X and $g(X)$ are algebraically independent over K. Define a mapping v on $K[X,Y] \backslash \{0\}$ by $v(f(X,Y)) =$ smallest exponent of X appearing in the power series $f(X, g(X))$. Then v extends to a rank-one discrete valuation on $K(X,Y)$ centered on $(X,Y)R$ and having residue field K. (More details regarding this construction can be found in Chapter VI, Section 15, of [37].) Since the valuation ring V of v has maximal ideal $(X,Y)V$ and residue field $V/(X,Y)V = K$, it follows that $V = R + (X,Y)^k V$ for all $k > 0$. Since V is a DVR, $V = R + fV$ for every nonzero $f \in R$. Hence V/R is a divisible R-module. By Theorem 3.11, V is a completely Q-irreducible R-submodule of Q.

4. Q-irreducibility and injective modules

Let N be a submodule of the torsion-free R-module M. N is said to be an *RD-submodule* (relatively divisible) if $rN = N \cap rM$ for all $r \in R$. An R-module X is called *RD-injective* if every homomorphism from an RD-submodule N of any R-module M can be extended to a homomorphism $M \to X$. Every R-module M can be embedded as an RD-submodule in an RD-injective module, and among such RD-injectives there is a minimal one, unique up to isomorphisms over M, called the *RD-injective hull* $\widehat{M}$ of M. If M is torsion-free, then so are both $\widehat{M}$ and $\widehat{M}/M$.

The R-topology of an R-module M is defined by declaring the submodules rM for all $0 \neq r \in R$ as a subbase of open neighborhoods of 0. If M is torsion-free, then it is Hausdorff in the R-topology if and only if it is reduced (i.e. it has no divisible submodules $\neq 0$). M is R-complete if it is complete (Hausdorff) in the R-topology. If M is reduced torsion-free, then it is an RD-submodule of its R-completion $\widetilde{M}$.

Observe that for a prime ideal P the R-completion and R_P-completion of R_P are identical. The R-completion $\widetilde{M}$ of a torsion-free R-module M is an RD-submodule of the RD-injective hull $\widehat{M}$ such that $\widehat{M}/\widetilde{M}$ is reduced torsion-free.

Lemma 4.1. *For a proper R-submodule A of Q the following conditions are equivalent:*

(i) *A is Q-irreducible;*

(ii) *the injective hull $E(Q/A)$ of the R-module Q/A is indecomposable;*

(iii) *the RD-injective hull $\widehat{A}$ of A is indecomposable.*

Proof. (i) $\Leftrightarrow$ (ii) An injective module is indecomposable exactly if it is uniform.

(ii) $\Leftrightarrow$ (iii) This equivalence is a consequence of Matlis' category equivalence between the category of h-divisible torsion R-modules T and the category of reduced R-complete torsion-free R-modules M, given by the correspondences

$$T \mapsto \mathrm{Hom}_R(Q/R, T) \quad \text{and} \quad M \mapsto Q/R \otimes_R M$$

which are inverse to each other. Under the category equivalence, Q/A and the R-completion $\widetilde{A}$ of A correspond to each other, and so do the injective hull of Q/A and the RD-injective hull $\widehat{A}$ of A. As equivalence preserves direct decompositions, the claim is evident. $\square$

Let I be an ideal of the ring R. It is well known that if $E(R/I)$ is indecomposable, then I is irreducible. Note that $E(R/I)$ can also be written as $E(Q/A)$ for a Q-irreducible R-submodule A of Q. In fact, $E(R/I)$ is a summand of $E(Q/I)$, so we can write: $E(Q/I) = E(R/I) \oplus E$ for an injective R-module E. The kernel of the projection of Q/I into the first summand is of the form A/I for a Q-irreducible submodule A of Q, and then $E(R/I) = E(Q/A)$.

Conversely, if A is a Q-irreducible proper submodule of Q, and $x \in Q \setminus A$, then the set $I = \{r \in R \mid rx \in A\}$ is a primal ideal of R such that $E(Q/A) = E(R/I)$. The adjoint prime P of the primal ideal I may be called the *prime associated to* A: this is uniquely determined by A, though I depends on the choice of x.

Lemma 4.2. *Every indecomposable injective R-module E can be written as $E(Q/A)$ for a Q-irreducible R-submodule of A of Q. Moreover, there is a unique prime ideal P of R such that $E(Q/A) \cong E(R/I)$ for a P-primal ideal I of R, and P is a maximal ideal whenever A is completely Q-irreducible.* $\square$

We can add that I can be replaced by P if and only if P is a strong Bourbaki associated prime for I. Indeed, $E(R/I) = E(R/P)$ if and only if there are elements

$r \in R \setminus I$ and $s \in R \setminus P$ such that $(I :_R r) = (P :_R s)$. Since $(P :_R s) = P$, this is equivalent to $P = (I :_R r)$, that is, P is a strong Bourbaki associated prime of I.

It is clear that every proper submodule of Q is the intersection of Q-irreducible submodules. This intersection is in general redundant. A criterion for irredundancy is as follows.

Proposition 4.3. *A proper submodule A of Q admits an irredundant representation as an intersection of Q-irreducible submodules if and only if $E(Q/A)$ is an interdirect sum of indecomposable injectives.*

Proof. Suppose $A = \bigcap_{i \in I} A_i$ is an irredundant intersection with Q-irreducible submodules A_i of Q. Setting $B_i = \bigcap_{j \in I, j \neq i} A_j$, it is clear that the submodule generated by B_i/A $(i \in I)$ in Q/A is their direct sum. Hence $E(Q/A)$ contains the direct sum of the injective hulls $E(Q/A_i) \cong E(B_i/A)$. As Q/A embeds in the direct product of the Q/A_i, $E(Q/A)$ embeds in the direct product of the $E(Q/A_i)$. Thus $E(Q/A)$ is an interdirect sum of the $E(Q/A_i)$ (these are evidently indecomposable).

Conversely, suppose $E(Q/A)$ is an interdirect sum of indecomposable injectives E_i $(i \in I)$. Since E_i is a uniform module, we have $(Q/A) \cap E_i \neq 0$ for each $i \in I$. Clearly, A_i (defined by $A_i/A = (Q/A) \cap \prod_{j \in I, j \neq i} E_j$) is a submodule of Q, which is maximal disjoint from E_i, so Q-irreducible. The intersection $A = \bigcap_{i \in I} A_i$ is evidently irredundant. □

5. Irredundant decompositions and semi-artinian modules

In this section we examine domains for which every nonzero submodule of Q is an irredundant intersection of completely irreducible submodules of Q. Such domains are closely related to the class of almost perfect rings.

A ring R is *perfect* if every R-module has a projective cover; equivalently, (since our rings are assumed to be commutative) R satisfies the descending chain condition on principal ideals [2]. In their study [6] of strongly flat covers of modules, Bazzoni and Salce introduced the class of *almost perfect domains*, consisting of those domains R for which every proper homomorphic image of R is perfect. Every Noetherian domain of Krull dimension 1 is almost perfect, but the class of almost perfect domains includes also non-Noetherian non-integrally closed domains– see for example Section 3 of [5].

There are a number of applications of perfect and almost perfect domains in the literature, most of which are motivated by the rich module theory for these classes of rings [5, 6, 10]. In this section we emphasize different features of the module and ideal theory of almost perfect domains, namely, the close connection with irredundant decompositions into completely irreducible submodules.

If R is a ring, then an R-module A is (almost) *semi-artinian* if every (proper) homomorphic image of A has a nonzero socle. In a semi-artinian module every irreducible submodule is completely irreducible (see for example [9, Lemma 2.4]), but this property does not characterize semi-artinian modules [16, Example 1.7].

As indicated by Lemma 5.1 below, the semi-artinian property is both necessary and sufficient for irredundant decompositions into completely irreducible submodules. Bazzoni and Salce note in [5] that:

$$R \text{ almost perfect} \Rightarrow Q/R \text{ semi-artinian} \Rightarrow R \text{ locally almost perfect.}$$

They show also that R is almost perfect if and only if R is h-local and every localization of R at a maximal ideal is almost perfect. In general, the first implication cannot be reversed [5, Example 2.1]. Smith asserts in [36] that the converse of the second implication is always true, but as noted in [5, p. 288] the proof is incorrect. Thus Bazzoni and Salce raise the question in [5, p. 288] of whether the converse is always true; namely, if R is locally almost perfect, is Q/R necessarily semi-artinian?

We give an example in this section to show that the answer is negative, and we characterize in Theorem 5.2(vi) and (vii) precisely when a locally almost perfect domain R has Q/R semi-artinian. We collect also in this theorem a number of different characterizations of domains R for which Q/R is semi-artinian.

The following lemma is a special case of a lattice theoretic result [9, Theorem 4.1]. A number of other properties of irredundant intersections of completely irreducible submodules of semi-artinian modules can be deduced from this same article.

Lemma 5.1. (Dilworth-Crawley [9]) *Let R be a ring and A be an R-module. Then A is (almost) semi-artinian if and only if every (nonzero) submodule of A is an irredundant intersection of completely irreducible submodules of A.* □

In order to formulate (vii) of the next theorem, we recall that a topological space X is *scattered* if every nonempty subspace of X contains an isolated point.

Theorem 5.2. *The following statements are equivalent for a domain R with quotient field Q.*

(i) *Q/R is semi-artinian.*
(ii) *Every nonzero torsion module is semi-artinian.*
(iii) *R is almost semi-artinian.*
(iv) *Q is almost semi-artinian.*

(v) *For each nonzero proper ideal A of R, there is a maximal ideal that is a strong Bourbaki associated prime of A.*

(vi) *R is locally almost perfect and for each nonzero radical ideal J of R, there is a maximal ideal of R/J that is principal.*

(vii) *R is locally almost perfect and for each nonzero radical ideal J of R,* $\mathrm{Spec}(R/J)$ *is scattered.*

(viii) *For each torsion R-module T, every submodule of T is an irredundant intersection of completely irreducible submodules of T.*

(ix) *For each torsion-free module A, every nonzero submodule of A is an irredundant intersection of completely irreducible submodules of QA.*

(x) *Each nonzero submodule of Q is an irredundant intersection of completely irreducible submodules of Q.*

(xi) *Each nonzero ideal of R is an irredundant intersection of completely irreducible submodules of Q.*

(xii) *Each nonzero ideal of R is an irredundant intersection of completely irreducible ideals.*

Proof. The equivalence of (i)-(iv) can be found in [10, Theorem 4.4.1]. It follows then from Lemma 5.1 that (i) - (iv) are equivalent to (viii), (ix), (x) and (xii). The equivalence of (vi) and (vii) is a consequence of Corollary 2.10 in [26]. To complete the proof it is enough to show that (v) and (vi) are equivalent to (i) and that (xi) is equivalent to (iii).

(i) $\Rightarrow$ (vi) Since Q/R is semi-artinian, R is locally almost perfect. We have already established that (i) is equivalent to (xii). That (xii) implies (vi) is a consequence of Corollary 2.10 of [26].

(vi) $\Rightarrow$ (v) Suppose that A is a proper nonzero ideal of R. Since for every nonzero radical ideal J of R, R/J has a maximal ideal of R that is principal, every nonzero ideal of R has a Zariski-Samuel associated prime M [26, Theorem 2.8]; that is, $M = \sqrt{A :_R x}$ for some $x \in R \setminus A$. Since R has Krull dimension 1, M is a maximal ideal of R. By (vi) $R_M/(A_M :_{R_M} x)$ contains a simple R_M-module. Thus there exists $y \in R \setminus (A_M :_{R_M} x)$ such that $MR_M = (A_M :_{R_M} x) :_{R_M} y = A_M :_{R_M} xy$. Since $A :_R x \subseteq A :_R xy \subseteq M$ and $\sqrt{A :_R x} = M$, it follows that M is the only maximal ideal of R containing $A :_R xy$. Thus since $A_M :_{R_M} xy = MR_M$, it is the case that $A :_R xy = M$.

(v) $\Rightarrow$ (iii) If A is a proper nonzero ideal of R and M is a strong Bourbaki associated prime of A, then $A :_R M \neq A$, so R/A contains a simple R-module.

(iii) $\Rightarrow$ (xi) Since (iii) is equivalent to (x), it is sufficient to note that (x) implies (xi).

(xi) $\Rightarrow$ (iii) Let A be a proper nonzero ideal of R. Then there exists a completely irreducible submodule C of Q such that $A = C \cap D$ is an irredundant intersection for some submodule D of Q. Let $x \in D \setminus C$. Now $(C :_Q M)/C$ is the essential socle of Q/C, so if $y \in (C :_Q M) \setminus C$, then $y \in xR + C$. Thus $rx \in yR + C$ for some $r \in R$ such that $rx \notin C$. Consequently, $rxM \subseteq C$, and since $x \in D$, it is the case that $rxM \subseteq A$ with $rx \notin A$. Thus $rx + A$ is a nonzero member of the socle of R/A. Statement (iii) now follows. $\square$

An integral domain R is *almost Dedekind* if for each maximal ideal M of R, R_M is a DVR. In [35, Theorem 3.2] it is shown that if X is a Boolean (i.e. compact Hausdorff totally disconnected) topological space, then there exists an almost Dedekind domain R with nonzero Jacobson radical such that $\mathrm{Max}(R)$ is homeomorphic to X. Thus we obtain the following corollary to Theorem 5.2(vii).

Corollary 5.3. *The following statements are equivalent for a Boolean topological space X.*

(i) *X is a scattered space.*
(ii) *There exists a domain R with nonzero Jacobson radical such that Q/R is semi-artinian and* $\mathrm{Max}(R)$ *is homeomorphic to X.* $\square$

Remark 5.4. In Example 2.1 of [5] an example is given of a domain R for which Q/R is semi-artinian but R is not almost perfect. Using the corollary, we may obtain many such examples. Indeed, let X be an infinite Boolean scattered space. Then there exists an almost Dedekind domain R such that $\mathrm{Max}(R)$ is homeomorphic to X and R is not a Dedekind domain. In particular, R is not h-local, since an h-local almost Dedekind domain is Dedekind. Thus Q/R is semi-artinian but R is not almost perfect.

It is not difficult to exhibit infinite Boolean scattered spaces. For example, let X be a well-ordered set such that not every element has an immediate successor. Then X is a scattered space with respect to the order topology on X, and the isolated points of X are precisely the smallest element of X and the immediate successors of elements in X (see [28, Example 17.3, p. 272]).

In [5] Bazzoni and Salce raise the question of whether every locally almost perfect domain R has the property that Q/R is semi-artinian. Using Theorem 5.2 we give an example to show that this is not the case.

Example 5.5. Let X be a Boolean space that is not scattered (e.g. let X be the Stone-Ĉech compactification of the set of natural numbers with the discrete topology). As noted above, there exists an almost Dedekind domain R such that $\mathrm{Max}(R)$ is homeomorphic to X and R has nonzero Jacobson radical. Then R is locally almost perfect but by Theorem 5.2(vii) Q/R is not semi-artinian.

In [15] it is shown that every irreducible ideal of an almost perfect domain is primary. A similar argument yields:

Lemma 5.6. *If R is a locally almost perfect domain, then every proper irreducible ideal is primary.*

Proof. Let A be a nonzero irreducible ideal. Then A is primary if and only if any strictly ascending chain of the form $A \subset A :_R b_1 \subset A :_R b_1b_2 \subset \cdots \subset A :_R b_1b_2\cdots b_n \subseteq \cdots$ for $b_1, b_2, \ldots, b_n, \ldots \in R$ terminates [14]. Suppose there is an infinite such strictly ascending chain, and let M be a maximal ideal containing every residual $A :_R b_1b_2\cdots b_n$. Since R_M is an almost perfect domain, R_M/A_M has the descending chain condition for principal ideals. Thus there exists $n > 0$ such that $A_M :_R b_1b_2\cdots b_n = A_M :_R b_1b_2\cdots b_{n+1}$. If $r \in A :_R b_1b_2\cdots b_{n+1}$, then there exists $x \in R \setminus M$ such that $xr \in A :_R b_1b_2 \ldots b_n$. An irreducible ideal of a domain of Krull dimension 1 is contained in a unique maximal ideal (see for example [26, Lemma 2.7]), so necessarily A is M-primal. Thus x is prime to A and it follows that $r \in A :_R b_1b_2\cdots b_n$. However, this forces $A :_R b_1b_2\cdots b_n = A :_R b_1b_2\cdots b_{n+1}$, contrary to assumption. Thus A is primary. □

Theorem 5.7. *If R is an almost semi-artinian domain, then every ideal of R is an irredundant intersection of primary completely irreducible ideals.*

Proof. The theorem follows from Lemma 5.6 and Theorem 5.2(xii). □

We characterize next the domains R for which every nonzero submodule of Q can be represented uniquely as an irredundant intersection of completely Q-irreducible R-submodules.

An R-module B is *distributive* if for all submodules A_1, A_2 and A_3 of B, $(A_1 \cap A_2) + A_3 = (A_1 + A_3) \cap (A_2 \cap A_3)$. The module B is *uniserial* if its submodules are linearly ordered by inclusion. An R-module is distributive if and only if for all maximal ideals M of R, B_M is a uniserial R_M-module [29].

Lemma 5.8. *Let R be a ring and B be an R-module. Let $\mathcal{A}$ be the set of all R-submodules of B that are finite intersections of completely irreducible submodules* *of B. Then the module B is distributive if and only if for each $A \in \mathcal{A}$, the representation of A as an irredundant intersection of completely irreducible submodules of B is unique.*

Furthermore, if a submodule B of a distributive R-module can be represented as a (possibly infinite) irredundant intersection of irreducible submodules, then this representation is unique.

Proof. Suppose that each representation of $A \in \mathcal{A}$ as an irredundant intersection of completely Q-irreducible submodules of B is unique. Then this property holds also for the R_M-submodules of B_M for each maximal ideal M of R. Thus by the remark preceding the theorem, to prove that B is distributive it suffices to show that B_M is a uniserial R_M-module. Thus we may reduce to the case where R is a quasilocal domain with maximal ideal M and show that B is a uniserial R-module. If B is not uniserial, there exist incomparable completely B-irreducible submodules C_1 and C_2 of B. Define $A = C_1 \cap C_2$, $C_1^* = C_1 :_B M$ and $C_2^* = C_2 :_B M$. By Lemma 2.1, $C_1 \subset C_1^*$ and $C_2 \subset C_2^*$. Now there exist $x \in (C_1^* \cap C_2) \setminus A$ and $y \in (C_1 \cap C_2^*) \setminus A$. (This follows from the irreduciblity of the C_i and the modularity of the lattice of submodules of Q; see for example Noether [34, Hilfssatz II].) We have Soc $B/A = (A + xR + yR)/A$ is a 2-dimensional vector space over R/M and $x + y \notin C_1 \cup C_3$. Let C_3 be an R-submodule of B containing $A + (x + y)R$ that is maximal with respect to $x \notin C_3$. Then C_3 is completely B-irreducible, distinct from C_1 and C_2 and $A = C_1 \cap C_3$. Yet $A \in \mathcal{A}$, so this contradiction means that the submodules of B are comparable. The converse and the last assertion follow from the fact that in a complete distributive lattice, an irredundant meet decomposition into meet-irreducible elements is unique [8, pp. 5-6] . □

Theorem 5.9. *The following are equivalent for a domain R with quotient field Q.*

(i) *Every nonzero submodule of Q can be represented uniquely as an irredundant intersection of completely irreducible submodules of Q.*
(ii) *Every nonzero ideal of R can be represented uniquely as an irredundant intersection of completely irreducible submodules of Q.*
(iii) *Every nonzero proper ideal of R can be represented uniquely as an irredundant intersection of completely irreducible ideals of R.*
(iv) *R is an almost Dedekind domain such that for each radical ideal J of R, R/J has a finitely generated maximal ideal.*
(v) *R is an almost semi-artinian Prüfer domain.*

Proof. (i) ⇒ (ii) This is clear.

(ii) $\Rightarrow$ (iii) This follows from Theorem 5.2 and Lemma 5.8.

(iii) $\Leftrightarrow$ (iv) This is proved in [26, Corollaries 2.10 and 3.9].

(iv) $\Rightarrow$ (v) This follows from Theorem 5.2.

(v) $\Rightarrow$ (i) Since R is a Prüfer domain, Q is a distributive R-module. Thus (i) is a consequence of Theorem 5.2 and Lemma 5.8. $\square$

6. Prüfer domains

In light of Theorem 5.9 it is of interest to describe the completely irreducible submodules of the quotient field of a Prüfer domain. We do this in Theorem 6.2. We need for the proof of this theorem a description of the completely irreducible ideals of a Prüfer domain. This is a special case of Theorem 5.3 in [16]: *A proper ideal A of a Prüfer domain is completely irreducible if and only if $A = MB_{(M)}$ for some maximal ideal M and nonzero principal ideal B of R.*

Lemma 6.1. *Let R be an integral domain and let A be a flat R-submodule of Q. If A is Q-irreducible, then* $\mathrm{End}(A)$ *is quasilocal and is Q-irreducible as an R-submodule of Q.*

Proof. Since A is a flat R-submodule of Q, it is the case that $A(B \cap C) = AB \cap AC$ for all R-submodules B and C of Q [7, I.2, Proposition 6]. Suppose now that $\mathrm{End}(A) = B \cap C$ for R-submodules B and C of Q. Then $A = A\,\mathrm{End}(A) = A(B \cap C) = AB \cap AC$, and since A is Q-irreducible, $A = AB$ or $A = AC$. Thus $B \subseteq \mathrm{End}(A)$ or $C \subseteq \mathrm{End}(A)$, so that $\mathrm{End}(A)$ is Q-irreducible. Finally, if $\mathrm{End}(A)$ is not quasilocal, then there exist two nonzero non-units $x, y \in \mathrm{End}(A)$ such that $x\,\mathrm{End}(A) + y\,\mathrm{End}(A) = \mathrm{End}(A)$. Thus $xy\,\mathrm{End}(A) = x\,\mathrm{End}(A) \cap y\,\mathrm{End}(A)$, so $\mathrm{End}(A) = y^{-1}\,\mathrm{End}(A) \cap x^{-1}\,\mathrm{End}(A)$. Since $\mathrm{End}(A)$ is Q-irreducible, this forces x or y to be a unit, a contradiction. $\square$

Theorem 6.2. *Let R be a Prüfer domain. Then*

(i) *the Q-irreducible R-submodules of Q are precisely the R-submodules of Q that are also R_P-submodules for some prime ideal P, and*

(ii) *the completely Q-irreducible proper R-submodules of Q are precisely the R-submodules of Q that are isomorphic to MR_M for some maximal ideal M of R.*

Conversely, either of statements (i) and (ii) characterizes among the class of domains those that are Prüfer.

Proof. (i) If A is Q-irreducible submodule of Q, then by Lemma 6.1 $\mathrm{End}(A)$ is quasilocal. Since R is a Prüfer domain, there is a prime ideal P of R such that

$R_P = \mathrm{End}(A)$ and A is an R_P-submodule of Q. Conversely, if P is a prime ideal of R, A is an R_P-submodule of Q and $A = B \cap C$ for some R-submodules B and C of Q, then $A = BR_P \cap CR_P$. Since R_P is a valuation domain $A = BR_P$ or $A = CR_P$. Thus $A = B$ or $A = C$ and A is Q-irreducible.

(ii) Suppose that R is a Prüfer domain and let A be a completely Q-irreducible proper R-submodule of Q. Then by Proposition 2.1, $A = AR_M$ for some maximal ideal M of R and A is a completely Q-irreducible submodule of R_M. Since R_M is a valuation domain, there exists $q \in Q$ such that $qA \subseteq R_M$. Moreover, qA is a completely irreducible ideal of R_M, so by Lemma 5.1 of [16], $qA = xMR_M$ for some $x \in R_M$. Hence A is isomorphic to MR_M.

On the other hand, if A is an R-submodule of the form xMR_M for some $x \in Q$, then A is a completely irreducible fractional ideal of the valuation domain R_M [16, Lemma 5.1]. Since R_M is a valuation domain, A is a completely Q-irreducible of R_M. Thus by Proposition 2.1, A is a completely Q-irreducible R-submodule of Q.

It is easy to see that statement (i) characterizes Prüfer domains. For let M is a maximal ideal of R, and observe that since by (i) the ideals of R_M are irreducible, they are linearly ordered.

Finally, suppose that each completely Q-irreducible proper R-submodule of Q is isomorphic for some maximal ideal M to the maximal ideal of R_M. Let M be a maximal ideal of R. Then by assumption rMR_M is an irreducible ideal of R_M for all $r \in R$. By Lemma 5.1 of [16], R_M must be a valuation domain. Thus R is a Prüfer domain since every localization of R at a maximal ideal is a valuation domain. □

In Theorem 6.3, we describe the Prüfer domains that have a completely Q-irreducible ideal.

Theorem 6.3. *The following statements are equivalent for a Prüfer domain R.*

(i) *There exists a completely Q-irreducible ideal of R.*
(ii) *There exists a nonzero Q-irreducible ideal of R.*
(iii) *There is a nonzero prime ideal contained in the Jacobson radical of R.*
(iv) *Every proper R-submodule of Q is a fractional ideal of R.*

Proof. (i) $\Rightarrow$ (ii) This is clear.

(ii) $\Rightarrow$ (iii) Suppose that A is a Q-irreducible ideal of R. By Lemma 2.3, $A = AR_P$ for some prime ideal of R. If A is an invertible ideal of R, then by Theorem 2.11 P is the unique maximal ideal of R, so that statement (iii) is clearly true. It remains to consider the case where A is not invertible. By Theorem 2.11, if x is a nonzero element in A^{-1}, then xA is contained in the Jacobson radical of R. Since by Lemma 2.3(ii), xA is Q-irreducible we may assume without loss of generality that A itself is contained in the Jacobson radical of R.

Now let $\{N_i\}$ be the set of maximal ideals of R. Since AR_P is an ideal of R and A is contained in each N_i, it follows that for each i, $AR_PR_{N_i} = AR_{N_i} \subset R_{N_i}$. Thus there is prime ideal P_i contained in P and N_i that contains A (the ideal P_i can be chosen to be the contraction of the maximal ideal of the ring $R_PR_{N_i}$ that contains A). Because R is a Prüfer domain, the prime ideals contained in P are linearly ordered by inclusion. Thus if $Q = \bigcap_i P_i$, then Q is a nonzero prime ideal of R (for it contains A) and Q is contained in every maximal ideal of R.

(iii) $\Rightarrow$ (i) Let P be a nonzero prime ideal of R contained in the Jacobson radical of R. Since R is a Prüfer domain, $P = PR_P$, so if $0 \neq x$ is in P it follows that xR_M is contained in P. Thus xMR_M is contained in $P = PR_M$. Moreover by Proposition 6.2 xMR_M is a completely Q-irreducible R-submodule of Q.

(iii) $\Rightarrow$ (iv) Statement (iv) is equivalent to the assertion that there exists a valuation overring $V \subset Q$ of R such that $(R :_Q V) \neq 0$ [31, Theorem 79]. If R satisfies (iii), then a nonzero prime ideal P contained in the Jacobson radical of R has the property that $PR_P = P$. Thus V can be chosen to be R_P.

(iv) $\Rightarrow$ (ii) By the theorem of Matlis cited in (iii) $\Rightarrow$ (iv), there exists a valuation ring V with $(R :_Q V) \neq 0$. Thus since R is a Prüfer domain there is a prime ideal P with $V = R_P$ and $rR_P \subseteq R$ for some nonzero $r \in R$. By Proposition 6.2, rR_P is a Q-irreducible ideal of R. □

Remark 6.4. If R is a Prüfer domain with nonzero Jacobson radical ideal J, then there exists a unique largest prime ideal P contained in J. If M is a maximal ideal of R, then $PR_M = PR_P$ since R_M is a valuation domain. Thus $P = \bigcap_{M \in \mathrm{Max}(R)} PR_M = PR_P$. It follows that R_P/P is the quotient field of R/P. Using this observation it is not hard to see that a Prüfer domain R satisfies the equivalent conditions of Theorem 6.3 if and only if R occurs in a pullback diagram of the form

$$\begin{array}{ccc} R & \longrightarrow & D \\ \big\downarrow & & \big\downarrow \alpha \\ V & \xrightarrow{\beta} & K \end{array}$$

where

- α is injective and D is a Prüfer domain such that the Jacobson radical of D does not contain a nonzero prime ideal,
- K is isomorphic to the quotient field of D, and

- β is surjective with V a valuation domain.

Thus if D is any Prüfer domain with quotient field Q and X is an indeterminate for Q, then $D + XQ[[X]]$ is a Prüfer domain satisfying the equivalent conditions of Theorem 6.3.

7. Questions

We conclude with several questions that we have not been able to resolve. Other questions touching on similar issues can be found in [1], [3] and [25].

Questions 7.1. What conditions on a domain R guarantee that any two completely Q-irreducible fractional ideals are necessarily isomorphic?

Proposition 3.1 gives an answer to this question in the case where every proper submodule of Q is a fractional R-ideal. By Theorem 6.2 if R is a valuation domain, then all completely Q-irreducible ideals of R are isomorphic. If R is a Noetherian local domain, then by Propositions 3.1 and 3.5 any two Q-irreducible ideals are isomorphic.

Questions 7.2. What integral domains R admit a completely Q-irreducible ideal? a nonzero Q-irreducible ideal?

The Noetherian and Prüfer cases of Question 7.2 are settled in Proposition 3.5 and Theorem 6.3, respectively.

Questions 7.3. If R admits a nonzero Q-irreducible ideal, does R also admit a completely Q-irreducible ideal?

The answer to Question 7.3 is yes if R is Prüfer or Noetherian.

Questions 7.4. If A is a (completely) irreducible submodule of the quotient field of a quasilocal domain R, what can be said about $\mathrm{End}(A)$? For a completely Q-irreducible ideal A of a quasilocal domain R does it follow that $\mathrm{End}(A)$ is integral over R?

Theorem 6.3 along with the fact that if A is completely irreducible, then $\mathrm{End}(A)$ is quasilocal, shows that if R is not quasilocal, then $\mathrm{End}(A)$ need not be integral over R even if R is a Prüfer domain.

Theorem 2.11, Example 3.10 and Example 3.13 are relevant to Question 7.4.

Questions 7.5. If R is a (Noetherian) domain, what are the completely irreducible submodules of Q?

Theorem 6.2 answers Question 7.5 in the case where R is Prüfer.

Questions 7.6. If A is a completely Q-irreducible R-submodule of Q, when is A a fractional ideal of R? of $\mathrm{End}(A)$?

If R is a valuation domain, then every proper submodule of Q is a fractional ideal of R. The case where R is a one-dimensional Noetherian domain is deeper, but has been resolved independently by Bazzoni and Goeters. A consequence of Theorem 3.4 of [3] is that if A is a completely Q-irreducible submodule of Q such that $\mathrm{End}(A)$ is Noetherian and has Krull dimension 1, then (by Theorem 2.11) $\mathrm{End}(A)$ is local and (by the cited result of Bazzoni) A is a fractional ideal of $\mathrm{End}(A)$. Indeed, a more general result due to H. P. Goeters is true: If A is a submodule of the quotient field of a local Noetherian domain of Krull dimension 1, then A is a fractional ideal of $\mathrm{End}(A)$ [22, Lemma 1]. Recently, Goeters has extended this to all quasilocal Matlis domains [23].

8. Appendix: Corrections to [17]

In this appendix we correct several mistakes from our earlier paper [17]. We include also a stronger version of Lemma 3.2 of this paper. The main corrections concern Lemmas 2.1(iv) and 3.2 of [17]. The notation and terminology of this appendix is that of [17].

The proof of statement (iv) of Lemma 2.1 of [17] is incorrect. Statement (iv) should be modified in the following way:

(iv) *For each nonzero nonmaximal prime ideal P of R, if $\{M_i\}$ is the collection of maximal ideals of R not containing P, then $R_P \subseteq (\bigcap_i R_{M_i})R_M$ for each maximal ideal M of R containing P.*

Having changed statement (iv), we modify now the original proofs of (iii) $\Rightarrow$ (iv) and (iv) $\Rightarrow$ (v) in the following way. For (iii) $\Rightarrow$ (iv) we note that by Theorem 3.2.6 of [11] $\mathrm{End}(P) = R_P \cap (\bigcap_i R_{M_i})$ and $\mathrm{End}(P_M) = R_P$. Thus by (iii) $R_P = \mathrm{End}(P_M) = \mathrm{End}(P)_M = R_P \cap (\bigcap_i R_{M_i})R_M$, and (iv) follows.

For the proof of (iv) $\Rightarrow$ (v), we have as in the original proof that

$$R_P = \mathrm{End}(A)_M = (\bigcap_{Q\in\mathcal{X}_A} R_Q)R_M \cap (\bigcap_N R_N)R_M.$$

We claim that $\bigcap_{Q\in\mathcal{X}_A} R_Q \subseteq R_P$. If this is not the case then since R_M is a valuation domain it must be that $R_P \subset (\bigcap_{Q\in\mathcal{X}_A} R_Q)R_M$ (proper containment). Hence from the above representation of $\mathrm{End}(A)_M$ we deduce that since R_P is a valuation domain, $R_P = (\bigcap_N R_N)R_M$. Thus $(\bigcap_N R_N)R_M \subset (\bigcap_{Q\in\mathcal{X}_A} R_Q)R_M$. By (iv),

$R_{Q'} \subseteq (\bigcap_N R_N)R_M$ since no N contains Q'. However $Q' \in \mathcal{X}_A$, so this implies $R_{Q'} \subset R_{Q'}R_M$, but since M contains Q', $R_{Q'}R_M = R_{Q'}$. This contradiction implies that $\bigcap_{Q \in \mathcal{X}_A} R_Q \subseteq R_P$, so every element $r \in P$ is contained in some $Q \in \mathcal{X}_A$. Consequently, no element of P is prime to A.

Reference is made in the first paragraph of the proof of Lemma 3.3 of [16] to the original version of statement (iv). In particular it is claimed that since R has the separation property, P_iS is a maximal ideal of S. This can be justified now using the following more general fact, which does not appear explicitly in [17]:

Lemma 8.1. *A Prüfer domain R has the separation property if and only if for each collection $\{P_i : i \in I\}$ of incomparable prime ideals, the ideals P_i extend to maximal ideals of $S := \bigcap_{i \in I} R_{P_i}$.*

Proof. If R has the separation property, then for each $j \in I$, $\mathrm{End}(P_j) = R_{P_j} \cap (\bigcap_N R_N)$ by Theorem 3.2.6 of [11], where N ranges over the maximal ideals of R that do not contain P_j. Thus $\mathrm{End}(P_j) \subseteq S$ since the P_i's are comaximal. By Lemma 2.1(ii) of [17] P_j is a maximal ideal of $\mathrm{End}(P_j)$, and since R is a Prüfer domain, either P_j extends to a maximal ideal SP_j of S or $SP_j = S$. The latter case is impossible since $S \subseteq R_{P_j}$. Thus SP_j is a maximal ideal of S. The converse follows from Theorem 3.2.6 of [15] and Lemma 2.1(ii) of [17]. □

A second reference to the original version of Lemma 2.1(iv) is made in the first paragraph of the proof of (i) ⇒ (ii) of Theorem 3.7. In this paragraph it is claimed that since $\mathrm{End}(A)_M = R_P$, the elements of P are not prime to A. Since (by Theorem 2.3 of [17]) R has the separation property, this claim is immediate from Lemma 2.1(v) and the original argument that appealed to Lemma 2.1(iv) is unnecessary.

The argument in the third paragraph of the proof of Lemma 3.2 of [17] is incorrect, but rather than patch this argument we give below a stronger version of this lemma. It requires a slight strengthening of Lemma 3.1 of [17].

Lemma 8.2. *(cf. Lemma 3.1 of* [17]*) Let A be an ideal of a Prüfer domain R. Suppose Q is a prime ideal of R that contains A, and P is a prime ideal such that $\mathrm{End}(A)_Q = R_P$. If $P \in \mathrm{Ass}(A)$, then $\mathrm{End}(A)_Q = \mathrm{End}(A_Q)$.*

Proof. Since $P \in \mathrm{Ass}(A)$, $A_{(P)}$ is a primal ideal with adjoint prime P, and it follows that A_P is a P_P-primal ideal. By [17, Lemma 1.4], $\mathrm{End}(A_P) = R_P$. Thus $\mathrm{End}(A_P) = \mathrm{End}(A)_Q$, so $A\,\mathrm{End}(A_P) = A\,\mathrm{End}(A)_Q$ implies $A_P = A_Q$. Consequently, $\mathrm{End}(A_Q) = \mathrm{End}(A_P) = R_P = \mathrm{End}(A)_Q$. □

Lemma 8.3. *(cf. Lemma 3.2 of [17]) Let R be a Prüfer domain with field of fractions F, let X be an R-submodule of F, and let M be a maximal ideal of R. Then $\mathrm{End}(X)_M = R_P$ for some $P \in \operatorname{Spec} R$ with $P \subseteq M$. Assume that P is the union of prime ideals P_i, where each P_i is the radical of a finitely generated ideal. Then $\mathrm{End}(X)_Q = \mathrm{End}(X_Q)$ for all prime ideals Q such that $P \subseteq Q \subseteq M$.*

Proof. Since $R_M \subseteq \mathrm{End}(X)_M$ and R_M is a valuation domain, $\mathrm{End}(X)_M = R_P$ for some prime ideal $P \subseteq M$. If $\mathrm{End}(X)_M = F$, then clearly $\mathrm{End}(X)_M = \mathrm{End}(X_M)$, so we assume $\mathrm{End}(X)_M \neq F$ and thus $P \neq (0)$. Let Q be a prime ideal of R such that $P \subseteq Q \subseteq M$. Since $\mathrm{End}(X)_M = R_P$, we have $\mathrm{End}(X)_Q = R_P$. Now $R_P = \mathrm{End}(X)_Q \subseteq \mathrm{End}(X_Q) \subseteq \mathrm{End}(X_P)$, so to prove Lemma 8.3, it suffices to show that $\mathrm{End}(X_P) \subseteq R_P$.

Let $S = \mathrm{End}(X)$. Now $PS \subseteq PR_P$, so $PS \neq S$. Since S is an overring of the Prüfer domain R, S is a flat extension of R, so PS is a prime ideal of S and $S_{PS} = R_P$. Also, PS is the union of the prime ideals P_iS, and each P_iS is the radical of a finitely generated ideal of S.

Let L be a prime ideal of S such that $L \subseteq PS$ and such that $L = \sqrt{I}$, where I is a finitely generated ideal of S. We prove there exists a nonzero $q \in F$ such that qX_L is an ideal of S_L that is primary for L_L. The invertible ideal I^2 of S is an intersection of principal fractional ideals of S. Since $\mathrm{End}(X) = S$, each principal fractional ideal of S is an intersection of S-submodules of F of the form qX, $q \in F$. Since $I^2 \subseteq L$, I^2 is an intersection of ideals of S of the form $L \cap qX$, where $q \in F$. Since $I^2 \subset I \subseteq L$ (where $\subset$ denotes proper containment), there exists $q \in F$ such that $I^2 \subseteq L \cap qX \subset L$. Hence there exists a maximal ideal N of S with $L \subseteq N$ such that $I_N^2 \subseteq L_N \cap qX_N \subset L_N$. Since S_N is a valuation domain, the S_N-modules qX_N and L_N are comparable and $I_N^2 \subseteq L_N \cap qX_N \subset L_N$ implies $I_N^2 \subseteq qX_N \subset L_N$. Now $\sqrt{I^2} = \sqrt{I} = L$ and $I^2 \subseteq N$ implies $L \subseteq N$. Thus $I_L^2 \subseteq qX_L \subseteq L_L$, and we conclude that $\sqrt{qX_L} = L_L$.

We observe next that $X_P \neq F$. Since $P \neq 0$, there exists i such that $P_i \neq 0$ and $L := P_iS \subseteq PS$, where $L = \sqrt{I}$ for some finitely generated ideal I of S. As we have established in the paragraph above, there exists a nonzero $q \in F$ such that qX_L is an ideal of S_L. Thus $qX_P \subseteq qX_L \subseteq S_L$, so it is not possible that $X_P = F$.

Fix some member L of the chain $\{P_iS\}$. Since $X_P \neq F$, $L \subseteq PS$ and R_P is a valuation domain, there exists a nonzero element s of S such that $sX \subseteq L_L$. Since $\mathrm{End}(X_P) = \mathrm{End}(sX_P)$ and we wish to show that $\mathrm{End}(X_P) \subseteq R_P$ we may assume without loss of generality that $s = 1$; that is, we assume for the rest of the proof that $X \subseteq L_L$. Define $A = X \cap S$. Then A is an ideal of S. Moreover A is contained in L since $A_L \subseteq X_L \subseteq L_L$.

With the aim of applying Lemma 8.2, we show that $PS \in \mathrm{Ass}(A)$. For each i define $L_i = P_i S$. It suffices to show each L_i with $L \subseteq L_i \subseteq PS$ is in $\mathrm{Ass}(A)$, since this implies that $PS = \bigcup_{L_i \supseteq L} L_i$ is a union of members of $\mathrm{Ass}(A)$. Let i be such that $L \subseteq L_i$. Since L_i is the radical of a finitely generated ideal of S, there exists (as we have established above) a nonzero $q \in F$ such that qX_{L_i} is an ideal of S_{L_i} that is primary for $(L_i)_{L_i}$. Now $A_{L_i} = X_{L_i} \cap S_{L_i}$. Since S_{L_i} is a valuation domain, $A_{L_i} = X_{L_i}$ or $S_{L_i} \subseteq X_{L_i}$. By assumption, $X \subseteq L_L$. Since $L \subseteq L_i$, it follows that $X_{L_i} \subseteq L_L$, so it is impossible that $S_{L_i} \subseteq X_{L_i}$. Thus $A_{L_i} = X_{L_i}$. Consequently, $qX_{L_i} = qA_{L_i}$ and qA_{L_i} is an ideal of S_{L_i} that is primary for $(L_i)_{L_i}$. Since S_{L_i} is a valuation domain, it follows that $qA_{L_i} = A_{L_i} : s$ for some $s \in S$. Thus $(L_i)_{L_i} \in \mathrm{Ass}(A_{L_i})$, so $L_i \in \mathrm{Ass}(A)$. This proves $PS \in \mathrm{Ass}(A)$.

Since $A = X \cap S$ is an ideal of S, $S \subseteq \mathrm{End}(A)$. For each maximal ideal N of S, either $A_N = X_N$ or $A_N = S_N$ It follows that $\mathrm{End}(A) \subseteq \mathrm{End}(X) = S$, so $\mathrm{End}(A) = S$. Thus $\mathrm{End}(A)_P = S_P = R_P$, and by Lemma 8.2, $\mathrm{End}(A_P) = R_P$. (We have used here that $S_{SP} = R_P$.) Now $A_P = X_P \cap S_P = X_P \cap R_P$. Since R_P is a valuation domain, $A_P = X_P$ or $R_P \subseteq X_P$. The latter case is impossible since $X_P \subseteq X_L \subseteq L_L$. Thus $A_P = X_P$. We conclude that $\mathrm{End}(X_P) = \mathrm{End}(A_P) = R_P$. □

Finally we make two corrections to the proof of Lemma 3.3. The third paragraph should read: *Define $A = JR_Q \cap R$. Then $AS = JR_Q \cap S$ is QS-primary. In particular, QS is the unique minimal prime of AS and $A \not\subseteq P_i S \cap R = P_i$ for each $i \geq 1$.*

Also, in the fifth paragraph an exponent is incorrect: x_i needs to be chosen in $A_i \setminus (P_1 \cup \cdots \cup P_i \cup A^{i+1})$. Then in the eighth paragraph, we have $x_{i+1} S_N \subset x_i S_N$ since $x_i \in A^i \setminus A^{i+1}$ and $A^{i+1} S_N \cap R = A^{i+1} R_Q \cap R = A^{i+1}$.

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