COMMUTATIVE IDEAL THEORY WITHOUT FINITENESS CONDITIONS: COMPLETELY IRREDUCIBLE IDEALS

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Abstract. An ideal of a ring is completely irreducible if it is not the intersection of any set of proper overideals. We investigate the structure of completely irrreducible ideals in a commutative ring without finiteness conditions. It is known that every ideal of a ring is an intersection of completely irreducible ideals. We characterize in several ways those ideals that admit a representation as an irredundant intersection of completely irreducible ideals, and we study the question of uniqueness of such representations. We characterize those commutative rings in which every ideal is an irredundant intersection of completely irreducible ideals.

Introduction

Let R denote throughout a commutative ring with 1. An ideal of R is called *irreducible* if it is not the intersection of two proper overideals; it is called *completely irreducible* if it is not the intersection of any set of proper overideals. Our goal in this paper is to examine the structure of completely irreducible ideals of a commutative ring on which there are imposed no finiteness conditions. Other recent papers that address the structure and ideal theory of rings without finiteness conditions include [3], [4], [8], [10], [14], [15], [16], [19], [25], [26].

A proper ideal A of R is completely irreducible if and only if there is an element $x \in R$ such that A is maximal with respect to not containing x. Indeed, the condition is clearly sufficient for A to be completely irreducible since x is in the intersection of the proper overideals of A. On the other hand, if A is completely irreducible and x is an element that is not in A but is in the intersection of the proper overideals of A, then A is maximal with respect to not containing x. Evidently, maximal ideals are completely irreducible. If R is a domain (not a field), then the zero ideal of R is prime and irreducible, but it is not completely irreducible (it is the intersection of all nonzero ideals). More generally, a prime ideal is always

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irreducible and is completely irreducible if and only if it is a maximal ideal. In Theorem 1.3 we characterize completely irreducible ideals in various ways. We deduce in Corollary 1.5 that an irreducible ideal of a Noetherian ring is completely irreducible if and only if it is primary for a maximal ideal.

A great deal is known about the structure of the irreducible ideals of a Noetherian ring. Indeed, the first decomposition theorem established by Emmy Noether [27] states that each ideal of a Noetherian ring admits a representation as an irredundant intersection of finitely many irreducible ideals; moreover, the number of components appearing in such a representation is an invariant, and each such representation is *reduced* (i.e., no ideal in the representation can be replaced by a strictly larger ideal to obtain the same intersection). Another result due to Noether [27] is that a proper irreducible ideal of a Noetherian ring is a primary ideal. In a ring without finiteness assumptions there may exist proper irreducible ideals that are not primary. Fuchs in [12] introduced the concept of a primal ideal, where a proper ideal A of R is said to be *primal* if the zero-divisors in R/A form an ideal. The ideal of zero-divisors is then necessarily of the form P/A where P is a prime ideal of R called the *adjoint prime* of A. We also say that A is *P-primal.* In Fuchs [12] it is shown that proper irreducible ideals are primal. More can be said about completely irreducible ideals. We observe in Theorem 1.3 several equivalences to a proper ideal C of R being completely irreducible among which is that C is irreducible and R/C contains a simple submodule.

A ring in which every irreducible ideal is completely irreducible is zero-dimensional. Corollary 1.5 states that if $\dim R = 0$ and each primary ideal of R contains a power of its radical, then every irreducible ideal of R is completely irreducible. In Section 2 we address the question: Under what conditions is an ideal representable as an irredundant intersection of completely irreducible ideals? We consider in Section 3 the question of uniqueness of representation of the ideal A as an irredundant intersection of completely irreducible ideals. In Section 4 we characterize the rings in which every ideal is an irredundant intersection of completely irreducible ideals. We prove in Theorem 4.2 that every ideal of the ring R is an irredundant intersection of completely irreducible ideals exactly if the ring is semi-artinian, where a ring R is said to be *semi-artinian* if every nonzero R-module contains a simple R-module.

Proposition 1.4 and Corollary 1.5 characterize the completely irreducible ideals of a Noetherian ring. In Section 5 we give an explicit description of the completely irreducible ideals of an arithmetical ring, where the ring R is *arithmetical* if for every maximal ideal M the ideals of the localization R_M are linearly ordered with respect to inclusion. An arithmetical integral domain is a *Prüfer domain.* For a prime ideal P of a Prüfer domain, Fuchs and Mosteig prove in [17, Lemma 4.3] that the P-primal ideals form a semigroup under ideal multiplication. We generalize this result in Theorem 5.6. For a prime ideal P of an arithmetical ring, we show that the regular P-primal ideals form a semigroup under ideal multiplication, where an ideal is *regular* if it contains a nonzerodivisor. Theorem 5.8 states that if M is a maximal ideal of an arithmetical ring R, then the set $\mathcal{F}$ of completely irreducible regular ideals of R with adjoint maximal ideal M is closed under ideal-theoretic multiplication, and $\mathcal{F}$ with this multiplication is a totally ordered cancellative semigroup.

A good reference for our terminology and notation is [18]. For ideals I, J of the ring R, the *residual* $I : J$ is defined as usual by

$$I : J = \{x \in R \ : \ xJ \subseteq I\}.$$

For an ideal A and for a prime ideal P of R, we use the notation

$$A_{(P)} = \{x \in R \ : \ sx \in A \text{ for some } s \in R \setminus P\} = \bigcup_{s \in R \setminus P} A : s$$

to denote the *isolated P-component* (isoliertes Komponentenideal) of A in the sense of Krull [24, page 16]. Notice that $x \in A_{(P)}$ if and only if $A : x \not\subseteq P$. If R is a domain, then $A_{(P)} = AR_P \cap R$, where R_P denotes the localization of R at P.

Two different concepts of associated primes of a proper ideal A of the ring R are useful for us. One of these was introduced by Krull [23, page 742], and following [22] we call a prime ideal P of R a *Krull associated prime* of A if for every $x \in P$, there exists $y \in R$ such that $x \in A : y \subseteq P$. The prime ideal P is said to be a *strong Bourbaki associated prime* of A if $P = A : x$ for some $x \in R$.

1. Irreducible and Completely Irreducible Ideals

A ring is called *subdirectly irreducible* if in any of its representations as a subdirect product of rings, one of the projections to a component is an isomorphism. It is straightforward to see:

Lemma 1.1. *A proper ideal C of R is completely irreducible if and only if the factor ring R/C is subdirectly irreducible, or equivalently if and only if the R-module R/C has a simple essential socle.* □

Thus if C is completely irreducible, then R/C contains a minimal nonzero ideal C^*/C: the intersection of all nonzero ideals of R/C; this is then the essential socle of R/C. We shall call C^* the *cover of* C. Clearly, C^*/C is a simple R-module, so $C^*/C \cong R/M$ for a maximal ideal M of R. (If C is only assumed to be irreducible, then we can only claim that R/C is a *uniform* R-module, that is, the intersection of any two nonzero submodules of R/C is not zero.)

Proposition 1.2. *A completely irreducible proper ideal C of the ring R is a primal ideal whose adjoint prime is the maximal ideal M of R for which $C^*/C \cong R/M$. Furthermore, M is a strong Bourbaki associated prime of C.*

Proof. Let $x \in R$ be a representative of any coset of C generating $C^*/C \cong R/M$. Then $x \notin C$ and $xM \subseteq C$. Hence $M \subseteq C : x \subset R$ (proper containment); thus we necessarily have $M = C : x$, so M is a strong Bourbaki associated prime of C. Evidently, M has to be the adjoint prime of the primal ideal C. □

Completely irreducible ideals can be characterized in various ways as we demonstrate in Theorem 1.3.

Theorem 1.3. *For a proper ideal C of R, the following conditions are equivalent.*

(i) *C is completely irreducible;*

(ii) *the factor module R/C is an essential extension of a simple module;*

(iii) *C is an irreducible ideal and R/C contains a simple R-submodule;*

(iv) *C is an irreducible ideal and C is properly contained in $C : M$ for some maximal ideal M of R;*

(v) *C is irreducible with adjoint prime a maximal ideal M of R, and $M = C : x$ for some $x \in R \setminus C$;*

(vi) *$C = C_{(M)}$ for some maximal ideal M of R and CR_M is a completely irreducible ideal of R_M.*

Proof. The equivalence of (i) and (ii) and of (iii) and (iv) is obvious, and so is the implication (i) $\Rightarrow$ (iv). To prove (iii) $\Rightarrow$ (i), observe that a simple submodule of a uniform module is an essential socle of the module.

By Proposition 1.2, (i) $\Rightarrow$ (v) is clear. On the other hand, Condition (v) implies that $x + C$ generates a simple R-submodule in R/C. By the uniformity of R/C this simple submodule is an essential socle of R/C, and (i) holds.

The equivalence of (v) and (vi) follows from the following two observations: (a) if M is a maximal ideal containing C, then $(C : x)R_M = CR_M :_{R_M} x$ for any

$x \in R$; and (b) $C_{(M)}$ is irreducible if and only if CR_M is an irreducible ideal of R_M (see Remark 1.6 of [14]). □

A ring is *Laskerian* if every ideal has a finite primary decomposition.

Proposition 1.4. *If every proper ideal of the ring R is an intersection of primary ideals (possibly an infinite intersection), then every completely irreducible proper ideal of R is primary for a maximal ideal. In particular, if R is a Laskerian ring (or a Noetherian ring), then every completely irreducible proper ideal of R is primary for a maximal ideal.*

Proof. Let C be a completely irreducible ideal of R with adjoint prime M and let $x \in R$ be a representative of any coset of C generating $C^*/C \cong R/M$. Then $x \notin C$ and $xM \subseteq C$. Since C is an intersection of primary ideals, there exists a primary ideal Q of R such that $C \subseteq Q$ and $x \notin Q$. Since C is completely irreducible, it follows that $C = Q$. Therefore C is a primary ideal that is M-primal, so C is M-primary. □

Corollary 1.5. *Let M be a maximal ideal of the ring R and let C be an irreducible M-primary ideal. If $M^n \subseteq C$ for some positive integer n, then C is completely irreducible. Thus if M is finitely generated, then every irreducible M-primary ideal is completely irreducible. Therefore an irreducible proper ideal of a Noetherian ring is completely irreducible if and only if it is primary for a maximal ideal.*

Proof. If $M^n \subseteq C$ for some positive integer n, then $C \subset (C : M)$. Hence by Theorem 1.3(iv), C is completely irreducible. The last statement now follows from Proposition 1.4. □

Remark 1.6. We are interested in describing the rings in which every irreducible ideal is completely irreducible. Since prime ideals are irreducible and a prime ideal is completely irreducible if and only if it is maximal, the condition that every irreducible ideal in R is completely irreducible implies that all prime ideals of R are maximal and $\dim R = 0$. If R is a reduced zero-dimensional ring, then R_P is a field for each $P \in \operatorname{Spec} R$ and every primal ideal of R is maximal. Therefore every irreducible ideal of a reduced zero-dimensional ring is completely irreducible.

Recall that a ring R is *semi-artinian* if every nonzero R-module contains a simple R-module. If R is semi-artinian, then all irreducible ideals of R are completely irreducible (see also Lemma 2.4 in Dilworth-Crawley [7]). The existence of a reduced zero-dimensional ring R having no principal maximal ideals shows that a ring in which all irreducible ideals are completely irreducible need not be semi-artinian. In Example 1.7 we give two specific ways to obtain an example of such a ring R.

Example 1.7. (1) Let L denote the algebraic closure of the field $\mathbb{Q}$ of rational numbers and let D be the integral closure of the ring $\mathbb{Z}$ of integers in the field L. Fix a prime integer p and let $R = D/\sqrt{pD}$. Then R is a reduced zero-dimensional ring having the property that no maximal ideal of R is finitely generated.

(2) Let $\mathbb{Q}^{\omega}$ denote the product of countably infinitely many copies of the field $\mathbb{Q}$ of rational numbers and let I denote the direct sum ideal of $\mathbb{Q}^{\omega}$. Then $R = \mathbb{Q}^{\omega}/I$ is a reduced zero-dimensional ring in which no maximal ideal is finitely generated.

Remark 1.8. Let M be a maximal ideal of the ring R. By Theorem 1.3, an irreducible M-primary ideal A is completely irreducible if and only if $A \subset (A : M)$. If (R, M) is a rank-one nondiscrete valuation domain and $A = xR$ is a principal M-primary ideal, then A is irreducible, but not completely irreducible. (Since the value group G of R is a dense subset of $\mathbb{R}$, the value of x is the limit of smaller elements of G.) On the other hand, if $\dim R = 0$ and if each primary ideal of R contains a power of its radical, then Corollary 1.5 implies that all irreducible ideals of R are completely irreducible.

Let $A \subseteq C$ be ideals of the ring R. If C is completely irreducible, we call C a *relevant completely irreducible divisor* of A if A has a decomposition as the intersection of completely irreducible ideals in which C is relevant (i.e. it cannot be omitted). If A admits an irredundant decomposition with completely irreducible ideals, then all the ideals in this decomposition are relevant. On the other hand, a prime ideal that is not a maximal ideal has no relevant completely irreducible divisors. In Proposition 1.9 we characterize the relevant completely irreducible divisors of an ideal.

Proposition 1.9. *A completely irreducible ideal C containing the ideal A is a relevant completely irreducible divisor of A if and only if the submodule C/A of R/A is not essential.*

Proof. First suppose C is a relevant completely irreducible divisor of A, and $A = C \cap \bigcap_{i \in I} C_i$ is a decomposition with completely irreducible ideals C_i, where C

cannot be omitted. Then there exist an element $x \in \bigcap_{i\in I} C_i \setminus C$ and an $r \in R$ such that $rx \in C^* \setminus C$. Let $u = rx$ and let $\overline{u}$ denote the image of u in R/A. Then $\overline{u} \notin C/A$, while if M denotes the maximal ideal of R such that $C^*/C \cong R/M$, then $Mu \subseteq C \cap \bigcap_{i\in I} C_i = A$ implies $R\overline{u}$ is a simple submodule of R/A. Hence the submodule C/A of R/A is not essential.

Conversely, assume C/A is not essential in R/A, i.e. there exists $x \in R \setminus A$ such that $(Rx + A) \cap C = A$. Write the ideal $(Rx + A)$ as an intersection of completely irreducible ideals C_i $(i \in I)$. Then $A = C \cap \bigcap_{i\in I} C_i$, where C is relevant. $\square$

Corollary 1.10. *For every completely irreducible ideal C containing the ideal A, the cover $C^\star$ satisfies: $C^\star/A$ is an essential submodule in R/A.*

Proof. If C is not a relevant completely irreducible divisor of A, then C/A is an essential submodule of R/A, so C^*/A is essential in this case. On the other hand, if C is relevant, then for any $x \in R \setminus C$, there exists $r \in R$ such that $rx \in C^* \setminus C$. $\square$

Corollary 1.11. *Let A be a proper ideal of the ring R. There exists a relevant completely irreducible divisor of A if and only if the socle* $\mathrm{Soc}(R/A) \neq 0$.

Proof. Assume that C is a relevant completely irreducible divisor of A. If $A = C$, then $\mathrm{Soc}(R/A) = \mathrm{Soc}(R/C) = C^*/C \neq 0$. If $A \subset C$, then $A = C \cap B$, where $A \subset B$. Let $b \in B \setminus A$. Then $b \notin C$, so there exists $r \in R$ such that $rb \in C^* \setminus C$. It follows that $Mrb \subseteq C \cap B = A$, where M is the adjoint prime of C. Therefore $(A + rbR)/A \subseteq \mathrm{Soc}(R/A)$ and $\mathrm{Soc}(R/A) \neq 0$.

Conversely, assume that $\mathrm{Soc}(R/A) \neq 0$. Then there exists $x \in R \setminus A$ such that $xM \subseteq A$ for some maximal ideal M of R with $A \subseteq M$. Let $B = A + xR$. Then $B/A \cong R/M$. Let C be an ideal of R containing A and maximal with respect to $x \notin C$. Then C is completely irreducible and $A = C \cap B$. Therefore C is a relevant completely irreducible divisor of A. $\square$

Remark 1.12. The ring R of Example 1.7 is a zero-dimensional reduced ring for which $\mathrm{Soc}\, R = 0$. Thus in this ring, the ideal (0) has no relevant completely irreducible divisors.

In Proposition 1.13, we observe a connection between relevant completely irreducible divisors of an ideal and maximal ideals that are strong Bourbaki associated primes of the ideal.

Proposition 1.13. *Let A be a proper ideal of the ring R. A maximal ideal M of R is a strong Bourbaki associated prime of A if and only if there exists a relevant completely irreducible divisor of A that is M-primal.*

Proof. Suppose C is a relevant completely irreducible divisor of A that is M-primal. The proof of Proposition 1.9 establishes the existence of an element $u \in R \setminus A$ such that $Mu \subseteq A$. Hence $M = A : u$ is a strong Bourbaki associated prime of A. Conversely, if $M = A : x$, then $x \notin A$. Let C be an ideal of R that contains A and is maximal without x. Then $M = C : x$ and C is M-primal. Since every ideal is an intersection of completely irreducible ideals, there exist completely irreducible ideals C_i such that $A + Rx = \cap_{i \in I} C_i$. Then $(A + Rx)/A$ being simple implies that $A = C \cap (\cap_{i \in I} C_i)$, where C is clearly a relevant component. □

2. Irredundant Intersections

We consider under what conditions an ideal may be represented as an irredundant intersection of completely irreducible ideals, where, as usual, *irredundant* means that none of the components may be omitted without changing the intersection.

If $\{B_i\}_{i \in I}$ is a family of R-modules, then by an *interdirect product* of this family we mean an R-submodule of the direct product $\prod_{i \in I} B_i$ that contains the direct sum $\bigoplus_{i \in I} B_i$. Similarly, if $\{R_i\}_{i \in I}$ is a family of rings, by an *interdirect product* of this family of rings we mean a subring of the direct product $\prod_{i \in I} R_i$ that contains the direct sum $\bigoplus_{i \in I} R_i$ and the identity of the direct product.

In Lemma 2.1 and later in this section, we use the following notation. Let $\{C_i\}_{i \in I}$ be a family of completely irreducible ideals of the ring R and let $A = \bigcap_{i \in I} C_i$. For each $j \in I$ let $\overline{C}_j = \bigcap_{i \in I, i \neq j} C_i$. We frequently use the following basic fact.

Lemma 2.1. *Let $A = \bigcap_{i \in I} C_i$ be an irredundant representation of the proper ideal A with completely irreducible ideals C_i.*

(i) *There are elements $u_i \in R \setminus C_i$ $(i \in I)$ such that $C_i^* = C_i + Ru_i$ for $i \in I$ and $u_i \in C_j$ for all $j \neq i$.*

(ii) *For each $i \in I$, the R-module $\overline{C}_i/A$ has an essential socle generated by $u_i + A$.*

(iii) *In the representation $A = \bigcap_{i \in I} C_i$, no C_i can be replaced by a larger ideal and still have the intersection be equal to A.*

(iv) *With u_i as in (i), let $U_i = (Ru_i + A)/A$. Then U_i is a simple R-module and the socle of R/A is an interdirect product of the U_i.*

Proof. (i) Since the intersection of the C_i $(i \in I)$ is irredundant, for each $i \in I$ there exists an element $x_i \in \overline{C}_i$ such that $x_i \notin C_i$. By Lemma 1.1, R/C_i is subdirectly irreducible, so some multiple $rx_i \notin C_i$ has the property that its image in R/C_i is in Soc (R/C_i). If we choose $u_i = rx_i$ for each $i \in I$, then we obtain elements with the desired properties.

(ii) From (i) it is clear that $\langle u_i + A\rangle$ is in the socle of $\overline{C}_i/A$. By way of contradiction, suppose that there is a cyclic R-module $\langle v + A\rangle$ contained in $\overline{C}_i/A$ and independent of $\langle u_i + A\rangle$. As C_i is completely irreducible, there is an $r \in R$ such that $rv - u_i \in C_i$. Then $rv - u_i \in C_i \cap \overline{C}_i = A$, contradicting the independence of $\langle v + A\rangle$ and $\langle u_i + A\rangle$.

(iii) If we replace C_i by a larger ideal, then the intersection will contain u_i, so it will no longer represent A.

(iv) As $M_i u_i \subseteq A$ for a maximal ideal M_i, it is clear that the $U_i = \langle u_i + A\rangle$ are simple submodules of R/A. From (i) it follows that they are independent, so Soc (R/A) contains their direct sum $D = \oplus_{i\in I} U_i$. Suppose there is a $v \in R$ such that $v + A \in$ Soc (R/A), but not in D, say $Mv \subseteq A$ for a maximal ideal M of R. Thus $v \notin C_j$ for some $j \in I$. If $v \notin C_j + Ru_j$, then $Mv \subseteq C_j$ implies $C_j = (C_j + Ru_j) \cap (C_j + Rv)$, contradicting the irreducibility of C_j. Hence $v \in C_j + Ru_j$ for all j with $v \notin C_j$. This means that the canonical injection $R/A \to \prod_{j\in I} R/C_j$ maps v to an element of $\prod_{j\in I}(Ru_j + C_j)/C_j \cong \prod_{j\in I} U_j$. $\square$

We remark that part (iii) of Lemma 2.1 is also a consequence of the stronger statement: if $A = B \cap C$, where C is irreducible and relevant for the decomposition of A, then C cannot be replaced by any proper overideal (see Noether [27, Hilfssatz II]). In fact, if C is properly contained in C', then $A = B \cap C'$ would lead to $C = C + (B \cap C') = (C + B) \cap C'$, contradicting the irreducibility of C.

Remark 2.2. It will be useful to keep in mind that (iv) of Lemma 2.1 implies that

$$\bigoplus_{i\in I} \mathrm{Soc}(\overline{C}_i/A) \subseteq \mathrm{Soc}\ (R/A) \subseteq \prod_{i\in I} \mathrm{Soc}(\overline{C}_i/A). \tag{1}$$

Notice also that $\mathrm{Soc}(\overline{C}_i/A) \cong \mathrm{Soc}(C_i + \overline{C}_i)/C_i = \mathrm{Soc}\ (R/C_i) = C_i^*/C_i$.

Next we exhibit an example where $\mathrm{Soc}(R/A)$ is the direct product of the $\mathrm{Soc}(\overline{C}_i/A)$.

Example 2.3. For each nonnegative integer n, let $Z_n = \langle x_n\rangle$ be a cyclic group of order p, where p is a fixed prime integer. Consider the product $\prod_{n<\omega} Z_n$ with trivial multiplication and let R be the ring obtained by adjoining an identity to $\prod_{n<\omega} Z_n$

in such a way that as an additive group $R = \mathbb{Z} \oplus \prod_{n<\omega} Z_n$ and multiplication is defined as $(a, b) \times (c, d) = (ac, ad + cb)$, where $a, c \in \mathbb{Z}$ and $b, d \in \prod_{n<\omega} Z_n$.

For each nonnegative integer m, let $C_m \cong \prod_{n<\omega, n\neq m} Z_n$ denote the subgroup of $\prod_{n<\omega} Z_n$ of elements having zero coordinate in component Z_m. Then C_m is an ideal of R that is maximal with respect to not containing the element that is x_m in component Z_m and zero in all other components. Thus C_m is completely irreducible. We have $\bigcap_{n<\omega} C_n = 0$ is an irredundant intersection, and in this example, the first containment relation in (1) is strict, while the second becomes equality.

For an irredundant intersection $A = \bigcap_{i\in I} C_i$ with completely irreducible ideals C_i, we demonstrate in Example 2.4 that Soc (R/A) may fail to be an essential submodule of R/A.

Example 2.4. There is a ring R in which the zero ideal is an irredundant intersection of completely irreducible ideals and $\operatorname{Soc} R$ is not an essential submodule of R. Let $\{p_n \mid n < \omega\}$ be the set of prime numbers, and let Z_n denote the ring $\mathbb{Z}/p_n^2\mathbb{Z}$ $(n < \omega)$. In the direct product $P = \prod_{n<\omega} Z_n$ consider the subring R generated by $1 = (1, 1, \ldots, 1, \ldots), e = (p_0, p_1, \ldots, p_n, \ldots)$, and $r_n = (0, \ldots, 0, p_n, 0, \ldots)$ for each $n < \omega$. Then $e^2 = 0, er_n = 0$ and $r_n r_m = 0$ for all $n, m < \omega$. Consequently, a typical element of R has the form $a = m\cdot 1 + se + n_0 r_0 + \cdots + n_k r_k$ for some $k \geq 0$, where $m, s, n_i \in \mathbb{Z}$. The ideal Re is just the infinite cyclic subgroup generated by e. Define C_n as an ideal containing all r_i $(i < \omega, i \neq n)$, $r_n - p_0^n p_1^n \cdots p_n^n e$, and being maximal with respect to $r_n \notin C_n$. Clearly, the intersection $\bigcap_{n<\omega} C_n$ contains no nonzero element of the socle $\operatorname{Soc} R = \oplus_{n<\omega} Rr_n$, and is evidently irredundant. To see that it cannot contain any element outside the socle either, assume by way of contradiction that a as above is contained in every C_n. Then either a multiple of se or $ae = me$ is contained in every C_n, which is impossible, since $p_0^n p_1^n \cdots p_n^n e \notin C_n$. However, the socle of R is not essential in R, as the ideal Re intersects $\operatorname{Soc} R$ trivially.

Let A be a proper ideal of the ring R. In Theorem 2.5 we relate representations of A as an intersection of bigger ideals with representations of R/A as a subdirect product.

Theorem 2.5. *For a proper ideal A of the ring R the following are equivalent:*

(i) *A is the irredundant intersection of completely irreducible ideals;*

(ii) *R/A is an irredundant subdirect product of subdirectly irreducible R-algebras R_i.*

Proof. (i) $\Rightarrow$ (ii) Let $\{C_i\}_{i\in I}$ be a family of completely irreducible ideals of R such that $A = \bigcap_{i\in I} C_i$ is an irredundant intersection. Then R/A can be identified via the canonical map $\phi : R \to \prod_{i\in I} R/C_i$ with the subring $\phi(R)$ of the direct product $\prod_{i\in I} R/C_i$, where each component R/C_i contains a simple R-module C_i^*/C_i as an essential socle, so each component is subdirectly irreducible. If we drop a component R/C_j, then ϕ will no longer induce an embedding of R/A in the product $\prod_{i\in I, i\neq j} R/C_i$, so the subdirect representation is irredundant.

(ii) $\Rightarrow$ (i) Assume that R/A is an irredundant subdirect product of the subdirectly irreducible R-algebras R_i $(i \in I)$. Let $T = \prod_{i\in I} R_i$, and let $\pi_i : R/A \to R_i$ be the ith canonical projection. The kernel of π_i is C_i/A, where C_i is a completely irreducible ideal of R properly containing A, and R_i is isomorphic to R/C_i. Clearly, $A = \bigcap_{i\in I} C_i$ and this representation is irredundant. □

Example 2.6. Let F_p denote the prime field of characteristic p, for a prime p. Define R as the direct product $\prod_p F_p$ with p running over an infinite set of primes. Identify the product C_q of the F_p with $p \neq q$ (q a fixed prime) as a subset of R consisting of all tuples having a zero in position q. Under this identification, C_q is a completely irreducible ideal of R, and $0 = \bigcap_q C_q$ is an irredundant intersection. (In this example, $\operatorname{Soc} R$ is an essential submodule of R.)

In the preceding example, we could have used equally well several subrings (containing $\oplus_p F_p$) of R, e.g. the von Neumann regular ring over which all von Neumann regular rings are unital algebras; see Fuchs-Halperin [13].

Another criterion for the existence of an irredundant intersection representation of an ideal A with completely irreducible ideals can be given in terms of the injective hull of R/A.

Theorem 2.7. *The proper ideal A of the ring R admits an irredundant representation as an intersection of completely irreducible ideals C_i $(i \in I)$ if and only if the injective hull $E(R/A)$ of R/A is an interdirect product of injective hulls of simple R-modules.*

Proof. The 'only if' part of the theorem is an immediate consequence of Theorem 2.5, since the components R/C_i in the subdirect product are R-modules each with a simple essential submodule.

Conversely, assume $E(R/A)$ is an interdirect product of injective hulls $E(S_i)$ of simple R-modules S_i. Define $R_i = \pi_i(R/A)$, where π_i denotes the ith coordinate projection of the product of the $E(S_i)$. Then R/A is evidently a subdirect product of the R_i, and the R_i are subdirectly irreducible. For each i, the submodule $E(S_i)$ of $E(R/A)$ has nonzero intersection with R/A, and this intersection is contained in the kernel of π_j for each $j \neq i$. This shows that the representation is irredundant. By Theorem 2.5, A admits an irredundant representation as an intersection of completely irreducible ideals. $\square$

For a proper ideal A of R, let $S(A)$ denote the set of elements $r \in R$ that are not prime to A, i.e. that satisfy $A \subset A : r$.

Proposition 2.8. *Let A be a proper ideal of a ring R. If A is an irredundant intersection of completely irreducible ideals C_i, then $S(A)$ is the union of maximal ideals of R, each of which is a strong Bourbaki associated prime of A.*

Proof. Given $r \in R$, we have $r \in S(A) \iff A \subset A : r$. Since $A : r = \bigcap_{i \in I}(C_i : r)$, we have, by Lemma 2.1(iii), $r \in S(A) \iff C_i \subset C_i : r$ for some i. Letting $M_i = \{t \in R \,|\, C_i \subset C_i : t\}$, the adjoint prime of C_i, we see that $S(A) = \bigcup_{i \in I} M_i$.

The elements $u_i \in R$ introduced in Lemma 2.1 satisfy $M_i = C_i : u_i$ and $u_i \in C_j$ for $j \neq i$. Thus $A : u_i = M_i$, showing that M_i is a strong Bourbaki associated prime of the ideal A. $\square$

We can now prove:

Corollary 2.9. *Let $A = \bigcap_{i \in I} C_i$ be an irredundant intersection, where the C_i are completely irreducible ideals. Then A is an M-primal ideal if and only if all the C_i are M-primal.*

Proof. If all the completely irreducible ideals C_i are M-primal, then by Proposition 2.8 $S(A) = M$, so A is M-primal. On the other hand, if A is M-primal, i.e. if $S(A) = M$, then again by Proposition 2.8, all the adjoint primes of the C_i must be equal to M. $\square$

This leads to a coarser intersection decomposition of ideals admitting irredundant intersections with completely irreducible ideals.

Corollary 2.10. *Assume that the proper ideal A of R has an irredundant representation as an intersection of completely irreducible ideals. Then A is an irredundant* *intersection of primal ideals with distinct adjoint maximal ideals such that each primal component is an irredundant intersection of completely irreducible ideals.*

Proof. Let $A = \bigcap_{i \in I} C_i$ be an irredundant intersection, where the C_i are completely irreducible ideals. For each maximal ideal M that is the adjoint prime of some C_i, , let $A^{[M]}$ denote the intersection of the C_i that are M-primal. By Corollary 2.9, $A^{[M]}$ is M-primal. Thus $A = \bigcap_{M \in \mathrm{Max}\, R} A^{[M]}$ is a decomposition of A into an irredundant intersection of primal ideals with distinct adjoint primes M, and each $A^{[M]}$ is an irredundant intersection of completely irreducible ideals. □

Since the isolated M-component $A_{(M)}$ of an ideal A is the intersection of all M-primal overideals of A, it is clear that $A_{(M)} \subseteq A^{[M]}$ for each M.

It would be interesting to know if the converse of Corollary 2.10 is also true.

Example 2.11. It is possible that an ideal A is an irredundant intersection of completely irreducible ideals $\{C_i\}_{i \in I}$ and also the intersection of relevant completely irreducible divisors $\{C'_i\}_{i \in I'}$, where no subset of $\{C'_i\}_{i \in I'}$ gives an irredundant representation of A. Let R be the ring defined in Example 2.4. It has been proved there that its zero ideal admits a representation as an irredundant intersection of completely irreducible ideals. Define A_n $(n > 0)$ to be an ideal containing all r_i $(i < \omega, i \neq n)$, containing e, and maximal with respect to not containing r_n. Furthermore, for each $n < \omega$ let B_n $(n < \omega)$ be an ideal that contains all r_i with $i > 0$, contains the element $r_0 - p_0^n p_1^n \cdots p_n^n e$, and is maximal with respect to not containing r_0. In view of Proposition 1.9, all of A_n, B_n are relevant completely irreducible divisors of A. Evidently, $A = \bigcap_{n>0} A_n \cap \bigcap_{n<\omega} B_n$. Here each of the A_n, but none of B_n, is relevant for the intersection. Indeed, infinitely many B_n can be deleted from this intersection as long as infinitely many B_n remain. In view of this, this intersection cannot be made irredundant by canceling components.

3. The Question of Uniqueness

In her seminal paper [27] E. Noether proved that in a Noetherian ring every proper ideal A is the finite irredundant intersection of irreducible ideals, and this intersection has the following uniqueness properties: (1) the number of irreducible components is unique, (2) the components satisfy the Replacement Property as defined in (3.2), and (3) the components are primary ideals and their set of prime radicals (along with their multiplicities) is uniquely determined by A.

Our next goal is to prove, if possible, analogous results for ideals A that admit a representation as an irredundant intersection of completely irreducible ideals. Of course, the role of primary ideals will be taken over by primal ideals, and the set of prime radicals will be replaced by the set of adjoint maximal ideals.

In our attempt to generalize the Noetherian situation the first problem we face is that the cardinality of the set of completely irreducible components in an irredundant intersection need not be unique. This is illustrated by the following example.

Example 3.1. Let κ be an infinite cardinal and λ any cardinal such that $\kappa < \lambda \leq 2^{\kappa}$. There exists a ring R that contains an ideal which is the irredundant intersection of κ completely irreducible ideals and also the irredundant intersection of λ completely irreducible ideals. Fix a prime integer p and let $P = \prod_{\alpha<\kappa} Z_\alpha$ be the direct product of of κ cyclic groups Z_α of order p. Give P the trivial multiplication and make $R = \mathbb{Z} \oplus P$ into a ring as in Example 2.3. Evidently $P = \operatorname{Soc} R$ and $|P| = 2^{\kappa}$. For $\beta < \kappa$, the product $P_\beta = \prod_{\alpha<\kappa, \alpha\neq\beta} Z_\alpha$ may be viewed as an ideal in R; it is completely irreducible, since it is maximal with respect to intersecting Z_β in zero. Clearly, $0 = \bigcap_{\beta<\kappa} P_\beta$ is an irredundant intersection with κ components. Pick a subgroup P' of P that contains the direct sum $\oplus_{\alpha<\kappa} Z_\alpha$ and has index λ in P. Now $P = P' \oplus \bigoplus_{\gamma<\lambda} W_\gamma$ for cyclic groups W_γ of order p. Consider completely irreducible ideals C_α that contain $\bigoplus_{\gamma<\lambda} W_\gamma$ and are maximal with respect to intersecting Z_α in zero, and consider as well completely irreducible ideals C'_γ that contain P' and are maximal with respect to intersecting W_γ in zero. Then the intersection $0 = \bigcap_{\alpha<\kappa} C_\alpha \cap \bigcap_{\gamma<\lambda} C'_\gamma$ is irredundant with λ components.

As far as uniqueness of irredundant decompositions is concerned, we have the following:

Theorem 3.2. *Let*

$$A = \bigcap_{i\in I} C_i = \bigcap_{j\in J} B_j$$

be two irredundant intersection representations of the proper ideal A of the ring R with completely irreducible ideals C_i, B_j. Then:

(i) *Each $M \in \operatorname{Max} R$ that occurs as the adjoint prime of some primal ideal C_i also occurs as the adjoint prime of some B_j. If it occurs a finite number of times in one intersection, then it occurs the same number of times in the other intersection.*

(ii) *The intersections have the Replacement Property: for each C_k there is a B_j such that replacing C_k by B_j in the first intersection, we obtain an irredundant representation of A.*

Proof. We start the proof with (ii). (Note that in the first paragraph of its proof irredundancy is irrelevant.)

(ii) Write $\overline{C}_k = \bigcap_{i \neq k} C_i$ and $A_{kj} = \overline{C}_k \cap B_j$. The R-module $(C_k + \overline{C}_k)/C_k \cong \overline{C}_k/(C_k \cap \overline{C}_k) \cong \overline{C}_k/A$ has an essential simple socle. Therefore, $A = \bigcap_{j \in J} A_{kj}$ with $A_{kj} \subseteq \overline{C}_k$ implies that one of A_{kj} $(j \in J)$ must be equal to A, say, $\overline{C}_k \cap B_j = A$. Thus C_k can be replaced by B_j.

To show that if $A = \bigcap_{i \in I} C_i$ is irredundant, then so is the new intersection, assume by way of contradiction that C_ℓ $(\ell \neq k)$ can be omitted from the new intersection, i.e. $A = \overline{C}_k \cap \overline{C}_\ell \cap B_j$ (where C_k and C_ℓ are missing). What has been proved is applied to this decomposition and the original one to conclude that here B_j can be replaced by one of the C_i. After we do this, at least one of C_k and C_ℓ would be missing from the intersection, contradicting the irredundancy of the decomposition $A = \bigcap_{i \in I} C_i$.

(i) Both C_k and its replacement B_j satisfy $\overline{C}_k \cap C_k = A = \overline{C}_k \cap B_j$. Hence the covers satisfy $C_k^*/C_k \cong \mathrm{Soc}(\overline{C}_k/A) \cong B_j^*/B_j$. It follows that both C_k and B_j are M-primal for the maximal ideal M that satisfies $\mathrm{Soc}(\overline{C}_k/A) \cong R/M$. Thus an M-primal completely irreducible ideal is always replaced by another M-primal ideal in an irredundant intersection.

Now if the first intersection contains n completely irreducible ideals that are M-primal, then successively replacing them by (necessarily distinct) completely irreducible ideals from the second intersection, it follows that the second intersection contains at least n M-primal ideals. Hence the claim is evident. □

Dilworth-Crawley [7, Theorem 4.2] prove an analogue of (ii) for compactly generated modular lattices under the hypothesis that the lattice is atomic (in our setting, this corresponds to the case where Soc (R/A) is essential in R/A). They also characterize atomic lattices where the components in irredundant intersections are unique. Since we do not have atomicity in general (see Example 2.4), in our search for uniqueness an independent approach is needed.

Remark 3.3. In [21] the rings are characterized for which every ideal can be represented uniquely as an irredundant intersection of completely irreducible ideals.

These rings are necessarily arithmetical. Indeed, it is shown in [21] that if $\mathcal{A}$ is the set of ideals of a ring R that are finite intersections of completely irreducible ideals, then R is arithmetical if and only if for each $A \in \mathcal{A}$ the representation of A as an irredundant intersection of completely irreducible ideals is unique.

4. When Every Ideal is an Irredundant Intersection

In this section we characterize the rings in which every ideal is an irredundant intersection of completely irreducible ideals. We start with the following observation.

Lemma 4.1. *Consider the following statements about a proper ideal A of the ring R.*

(i) *A is an irredundant intersection of completely irreducible ideals.*

(ii) *Soc (R/A) is an essential submodule of R/A.*

Then (ii) implies (i). On the other hand, if (i) holds for every proper ideal of R, then also (ii) holds for every proper ideal of R.

Proof. Assume that (ii) holds and let $E(R/A)$ denote the injective hull of R/A. Express Soc (R/A) as the direct sum $\oplus_{i \in I} S_i$ of simple R-modules S_i, and let $E(S_i)$ be a maximal essential extension of S_i in $E(R/A)$. Then $E(S_i)$ is an injective hull of S_i and $\oplus_{i \in I} E(S_i)$ is an essential submodule of $E(R/A)$. The canonical map of the direct sum $\oplus_{i \in I} E(S_i)$ into the direct product $\prod_{i \in I} E(S_i)$ extends to an injection of $E(R/A)$ into $\prod_{i \in I} E(S_i)$. Thus $E(R/A)$ is an interdirect sum of the $E(S_i)$. By Theorem 2.7, A is an irredundant intersection of completely irreducible ideals.

Assume that (i) holds for every proper ideal of R and let A be a proper ideal of R. By Lemma 2.1, Soc (R/A) cannot be trivial. Let B/A be a maximal essential extension of Soc (R/A) in R/A. Then R/B has trivial socle, so B cannot admit an irredundant representation as intersection of completely irreducible ideals. It follows that $B = R$. This means that Soc (R/A) is an essential submodule in R/A. □

Recall that a ring R is semi-artinian if every nonzero R-module contains a simple R-module. We will call a domain *almost semi-artinian* if every proper factor ring is semi-artinian.

Theorem 4.2. *Every ideal of a ring is an irredundant intersection of completely irreducible ideals exactly if the ring is semi-artinian. Every nonzero ideal of an* *integral domain is an irredundant intersection of completely irreducible ideals if and only if the integral domain is almost semi-artinian.*

Proof. Lemma 4.1 implies that all the ideals of a ring R have the stated property if and only if Soc (R/A) is essential in R/A for all proper ideals A of R. This is the case exactly if R is semi-artinian.

The claim concerning domains is an immediate consequence of the first part. $\square$

Example 4.3. A ring that is perfect in the sense of Bass [2] is semi-artinian. Perfect rings can be defined in several equivalent ways; for example, the ring R is perfect is equivalent to each of the following statements: (i) every flat R-module is projective, (ii) R satisfies the minimum condition for principal ideals (see e.g. Anderson-Fuller [1, p. 315]).

Example 4.4. Consider the Bézout domain R of dimension 1 constructed by Heinzer-Ohm [20, Example 2.2]. For this ring R, $\operatorname{Spec} R$ is not Noetherian, but all of its localizations at maximal ideals are rank-one discrete valuation domains. Moreover, all but one of its maximal ideals are principal. It can be seen that this example is almost semi-artinian. Thus an almost semi-artinian domain need not have Noetherian spectrum.

More interesting examples are those almost semi-artinian domains that are also h-local, where an integral domain is *h-local* if each nonzero element is contained in only finitely many maximal ideals and each nonzero prime ideal is contained in a unique maximal ideal.

Example 4.5. The *almost perfect domains* introduced by Bazzoni-Salce [4] are defined as domains such that every proper factor ring is a perfect ring. They prove that an almost perfect domain can be characterized as an h-local domain R such that for every nonzero proper ideal A, the factor ring R/A contains a simple R-module. We are interested in obtaining more information about the ideal structure of almost perfect domains. An almost perfect domain is a Matlis domain of dimension ≤ 1 with Noetherian prime spectra, where an integral domain R is a *Matlis domain* if its field of fractions has projective dimension at most 1 as an R-module. The inclusion here is strict since a rank-one nondiscrete valuation domain is a Matlis domain of dimension 1 that fails to be almost perfect.

In an almost perfect domain, irreducible ideals are primary. Indeed, by [11] a necessary and sufficient condition for an irreducible ideal A of a ring R to be primary is that any strictly ascending chain of the form

$$A \subset A : b_1 \subset A : b_1 b_2 \subset \cdots \subset A : b_1 b_2 \cdots b_n \subset \ldots$$

for any sequence $b_1, b_2, \ldots, b_n, \ldots$ in R terminates. This is evidently true if the descending chain condition holds for principal ideals mod A.

Since an almost perfect domain R is h-local, every nonzero ideal A of R is contained in at most a finite number of maximal ideals, $M_1, \ldots, M_n$. Consequently, we can write

$$A = A_{(M_1)} \cap \cdots \cap A_{(M_n)}.$$

The localization at a maximal ideal of a domain of dimension one is again of dimension one, and if R is a one-dimensional quasilocal domain with maximal ideal M, then every nonzero proper ideal of R is M-primary. Consequently, the isolated M_i-component $A_{(M_i)}$ must be M_i-primary for each $i = 1, \ldots, n$.

We record as Theorem 4.6 an immediate consequence of [5, Theorem 8].

Theorem 4.6. *In an almost perfect domain R, every nonzero ideal A is the product of pairwise comaximal primary ideals:*

$$A = B_1 B_2 \cdots B_n,$$

where the B_i are primary ideals with distinct maximal ideals M_i as radicals. This product representation is unique up to the order of the factors. Since an almost perfect domain is almost semi-artinian, the B_i are irredundant intersections of completely irreducible M_i-primary ideals. $\square$

In particular, it follows that an almost perfect domain is Laskerian. We recall that a ring is said to be *strongly Laskerian* if it is Laskerian and every primary ideal contains a power of its radical. If A is a nonzero ideal of a one-dimensional strongly Laskerian domain R, then R/A satisfies the minimum condition for principal ideals, for $R/A \cong \prod_{i=1}^{n} R_i$, where each R_i is a zero-dimensional ring with nilpotent maximal ideal M_i. If r_i is a positive integer such that $M_i^{r_i} = 0$, then a strictly descending chain of principal ideals in R_i has length at most r_i. It follows that a strictly descending chain of principal ideals in R/A has length at most $r_1 + \cdots + r_n$. Thus R is almost perfect.

Remark 4.7. The representations of the ideals B_i of Theorem 4.6 as irredundant intersections of completely irreducible M_i-primary ideals may be infinite intersections. For example, if R with maximal ideal M is a one-dimensional local strongly Laskerian domain that is not Noetherian, then M/M^2 is infinite dimensional as a vector space over R/M. Since R is almost semi-artinian, every nonzero ideal of R is an irredundant intersection of completely irreducible ideals. However, every representation of M^2 as an intersection of completely irreducible ideals is an infinite intersection (see for example part (iv) of Lemma 2.1).

5. Arithmetical Rings

Proposition 1.4 and Corollary 1.5 characterize the completely irreducible ideals of a Noetherian ring. In this section we give an explicit description of the completely irreducible ideals in a much different setting, that of an arithmetical ring. These rings arise naturally in the consideration of irreducible ideals. In Theorem 1.8 and Remark 1.6 of [14] it is shown that the following are equivalent: (i) the ring R is arithmetical, (ii) every primal ideal of R is irreducible, (iii) the ideal $A_{(P)}$ is irreducible for every ideal A and prime ideal P of R with $A \subseteq P$.

We first consider the special case of a valuation ring, where by a *valuation ring* we mean a ring in which the ideals are linearly ordered with respect to inclusion.

Lemma 5.1. *Let M be a maximal ideal of the ring R. The following statements are equivalent.*

(i) *R is a valuation ring with maximal ideal M.*
(ii) *The set of proper completely irreducible ideals of R is precisely the set of ideals of the form rM, where r is a nonzero element of R.*
(iii) *For every nonzero $r \in R$, the ideal rM is irreducible.*

Proof. (i) $\Rightarrow$ (ii) Let C be a completely irreducible ideal of R. Since C^*/C is a simple R-module and R is a valuation ring, C^* must be a principal ideal of R and $C = MC^*$. On the other hand, every ideal of the form rM, $r \in R$ and r nonzero, is completely irreducible by Theorem 1.3(v) and the fact that every ideal of a valuation ring is irreducible.

(ii) $\Rightarrow$ (iii) This is clear.

(iii) $\Rightarrow$ (i) We first observe that (iii) implies R is quasilocal. For if r is not in M but is in some other maximal ideal of R, then $rM = M \cap rR$ is an irredundant intersection, contradicting the fact that rM is irreducible.

Let x and y be nonzero elements of R. We show that the ideals xR and yR are comparable. Notice that $M = xM : x$ and $M = yM : y$. The ideals xM and yM are irreducible, so by Theorem 1.3(v) xM and yM are completely irreducible. Thus it follows that xR and yR are the respective covers (unique minimal overideals) of xM and yM. Let $I = (x, y)R$ and let A be an ideal of R with $MI = Mx+My \subset A$. Since xR is the cover of xM and yR is the cover of yM, we must have $x \in A$ and $y \in A$. Also $MI \subset I$. Therefore I is the unique minimal overideal of MI, and MI is completely irreducible with cover I. In particular, MI is irreducible, so I/MI has dimension one as an R/M-vector space. Thus $I = xR + MI$ or $I = yR + MI$. Since R is quasilocal, it follows that $I = xR$ or $I = yR$, proving that xR and yR are comparable. $\square$

Lemma 5.2. *Let A be an ideal of a ring R. If M is a maximal ideal of R, then $MA_{(M)} = (MA)_{(M)}$.*

Proof. Note first that $MA_{(M)} \subseteq (MA)_{(M)}$. For if $m \in M$ and $x \in A_{(M)}$, then there exists $y \in R \backslash M$ such that $xy \in A$, so that $mxy \in MA$ and $mx \in (MA)_{(M)}$. To complete the proof we verify locally that $(MA)_{(M)} = MA_{(M)}$. Clearly $(MA)_{(M)}R_M = MAR_M = MA_{(M)}R_M$, so suppose N is a maximal ideal of R distinct from M. Then $MA_{(M)} \subseteq (MA)_{(M)}$ implies that $A_{(M)}R_N = MA_{(M)}R_N \subseteq (MA)_{(M)}R_N \subseteq A_{(M)}R_N$; hence $MA_{(M)}R_N = (MA)_{(M)}R_N$. $\square$

Theorem 5.3. *A ring R is arithmetical if and only if the completely irreducible proper ideals of R are precisely the ideals of the form $MB_{(M)}$, where M is a maximal ideal and B is a principal ideal having the property that $BR_M \neq 0$.*

Proof. Suppose that R is an arithmetical ring. Let C be a completely irreducible proper ideal of R and let M be the adjoint (maximal) ideal of C. By Theorem 1.3, CR_M is completely irreducible, and by Lemma 5.1, $CR_M = (r/1)MR_M$, where $r \in R$ is such that its image $r/1$ in R_M is nonzero. Define $B = rR$. Since for every ideal A of R, the ideal $A_{(M)}$ is the preimage of AR_M under the canonical mapping $R \to R_M$, and since $C = C_{(M)}$, we have $C = (MB)_{(M)}$. Therefore by Lemma 5.2 $C = MB_{(M)}$, and every completely irreducible ideal of R has the stated form. By Lemmas 5.1 and 5.2 and Theorem 1.3 (vi), every ideal of the stated form is completely irreducible.

Conversely, assume that the completely irreducible proper ideals of R are precisely the ideals of the form $MB_{(M)}$, where M is a maximal ideal and B is a principal ideal of R having the property that $BR_M \neq 0$. It suffices to prove that R_M is a valuation ring for every maximal ideal M of R. By Lemma 5.1 it suffices to show that $(r/1)MR_M$ is irreducible for all elements $r \in R$ such that $r/1 \in R_M$ is nonzero. By assumption, $M(rR)_{(M)}$ is an irreducible ideal of R, and this implies that $(r/1)MR_M$ is an irreducible ideal of R_M (Remark 1.6 of [14]). □

An ideal A of a domain R is said to be *archimedean* if its only endomorphisms are multiplications by elements of R, i.e. if its endomorphism ring, $\mathrm{End}(A)$, is equal to R [18, page 71]. We observe in Remark 5.4 that completely irreducible ideals of a Prüfer domain need not be archimedean.

Remark 5.4. Let $R = \mathbb{Z} + x\mathbb{Q}[[x]]$ be the ring of power series in x with rational coefficients whose constant term is an integer. It is known that R is a Prüfer domain, $\dim R = 2$ and $P = x\mathbb{Q}[[x]]$ is the unique nonmaximal nonzero prime ideal of R. Moreover P is contained in each maximal ideal of R, and for $p \in \mathbb{Z}$ a prime integer, pR is a maximal ideal of R. Let V denote the valuation overring $R_{2R} = \mathbb{Z}_{2\mathbb{Z}} + P$, and consider the principal ideal $A = xV$ of V. Since $x \in P$, A is also an ideal of R and is completely irreducible as an ideal of R with adjoint maximal ideal $2R$ and cover $(x/2)V$. Observe that $\mathrm{End}(A)$ is the proper valuation overring V of R. Therefore A is not archimedean as an ideal of R. Since A is completely irreducible, it is not possible to represent A as an intersection of archimedean ideals of R.

For a prime ideal P of a Prüfer domain, Fuchs and Mosteig prove in [17, Lemma 4.3] that the P-primal ideals form a semigroup under ideal multiplication. We generalize this result in Theorem 5.6. In the proof of Theorem 5.6, we use the following lemma.

Lemma 5.5. *Let M be a maximal ideal of an arithmetical ring R. If A is a finitely generated ideal of R contained in M such that $AR_M \neq 0$, then $A_{(M)}$ is an M-primal ideal of R.*

Proof. Observe that $A_{(M)}$ is an M-primal ideal of R if and only if M is the unique maximal Krull associated prime of $A_{(M)}$. If P is a Krull associated prime of $A_{(M)}$, then necessarily $P \subseteq M$, so to show that $A_{(M)}$ is an M-primal ideal it suffices to show that M is a Krull associated prime of $A_{(M)}$. Now by Lemma 2.4 of [14], M is a Krull associated prime of $A_{(M)}$ if and only if MR_M is a Krull associated prime of AR_M. Also, a consequence of Lemma 1.5 in [15] is that every nonzero principal ideal of a valuation ring V has adjoint prime the maximal ideal of V. Thus AR_M is MR_M-primal, and it follows that $A_{(M)}$ is an M-primal ideal of R. □

Theorem 5.6. *Let P be a prime ideal of an arithmetical ring R. If A and B are P-primal ideals such that for all maximal ideals M of R, ABR_M is a nonzero ideal of R_M, then AB is a P-primal ideal of R. In particular the set of regular P-primal ideals of R is a semigroup under ideal multiplication.*

Proof. We first make two observations:

(a) *If C is an ideal of R, then P is a Krull associated prime of C if and only if PR_M is a Krull associated prime of CR_M for some (equivalently, all) maximal ideal(s) M containing P.* This is proved in Lemma 2.4 of [14].

(b) *If C is a primal ideal of R, then P is the adjoint prime of C if and only if for each maximal ideal M of R containing P, CR_M is a PR_M-primal ideal of R_M.* This is an application of (a) and the fact that an ideal C is P-primal if and only if P is the unique maximal member of the set of Krull associated primes of A.

We now prove the theorem. Suppose first that R is a valuation ring. Then AB is irreducible, hence primal, with adjoint prime, say Q. We claim that $Q \subseteq P$. If this is not the case then $P \subset Q$, and since AB is Q-primal, there exists $q \in Q \setminus P$ such that $qx \in AB$ for some $x \in Q \setminus AB$. Since B is P-primal, $B : q = B$. Also, $B \subseteq qR$, so $B = qC$ for some ideal C. Necessarily $B \subseteq C \subseteq B : q = B$, so this forces $B = C$. Thus $qB = B$ and $qAB = AB$. Since R is a valuation ring and $x \notin AB$, we have $AB \subseteq xR$. Thus $xqR \subseteq AB = qAB \subseteq xqR$, and we conclude that $xqR = AB$. It follows that $xq^2R = qAB = AB = xqR$, so that $xq = rxq^2$ for some $r \in R$. But this implies that $(1 - rq)xq = 0$, and since $1 - rq$ is a unit in R, $xq = 0$, contradicting the assumption that $xqR = AB \neq 0$. Thus $Q \subseteq P$.

We show that in fact $Q = P$. If this is not the case, then $Q \subset P$ and there exists $p \in P \setminus Q$. Since AB is Q-primal, $AB : p^2 = AB$. Since A and B are P-primal, there exist $x \in P \setminus A$ and $y \in P \setminus B$ such that $xp \in A$ and $yp \in B$. Hence $xy \in AB : p^2 = AB$. Since R is a valuation ring, $A \subset xR$ and $B \subset yR$. We conclude that $xyR \subseteq AB \subseteq xyR$. Thus by Lemma 5.5 and the fact that $AB \neq 0$, we have that AB is an M-primal ideal of R. This forces $Q = P = M$, a contradiction to the assumption that $Q \subset P$. Hence $P = Q$ and AB is P-primal.

We consider now the general case in which R is not necessarily a valuation ring. As noted above in (b), to prove that AB is P-primal, it suffices to show that P is the unique maximal member of the set of Krull associated primes of AB. By

(b) AR_M and BR_M are PR_M-primal ideals of R, and since R_M is a valuation ring we have by the case considered above that ABR_M is a PR_M-primal ideal. By (a) P is a Krull associated prime of AB. To complete the proof it remains to show that P is the unique maximal member of the set of Krull associated primes of AB. Suppose that Q is a Krull associated prime of AB, and let M be a maximal ideal of R containing Q. By (a) QR_M is a Krull associated prime of ABR_M. Since ABR_M is PR_M-primal, $QR_M \subseteq PR_M$, so that $Q \subseteq P$. This completes the proof that every Krull associated prime of AB is contained in P. □

Remark 5.7. Simple examples show that even in a valuation ring R with maximal ideal M it is possible to have a proper ideal A that is completely irreducible (and therefore M-primal) such that A^2 is not M-primal. Let V be a valuation domain whose value group is $\Gamma = \mathbb{Z} \oplus \mathbb{Z}$ with lexicographic ordering. Then the maximal ideal of V is principal generated by an element x where the value $v(x) = (0,1)$ is the smallest positive element in Γ. Let $y \in V$ have value $(1,0)$, and let Q be the prime ideal of V of height one. Thus Q is generated by elements z having value $v(z) = (1,b)$, where $b \in \mathbb{Z}$ may be negative. Notice that $Q^2 \subset yV \subset Q$ and that Q^2 is Q-primary. Define $R = V/Q^2$. Then R is a valuation ring with maximal ideal $M = xR$. The ideal $A = yR$ is completely irreducible, so M-primal. However, $A^2 = (0)$ is not M-primal. For a specific realization of this example, let x, y be indeterminates over a field k, let $W = k(x)[y]_{(y)}$, let $Q = yW$, and set $V = k[x]_{(x)} + Q$. Then V is a rank-two valuation domain with principal maximal ideal xV and height-one prime Q. The valuation ring $R = V/Q^2$ has maximal ideal $M = xR$. Since Q^2 is Q-primary, the ideal (0) of R is not M-primal. The principal ideal $A = yR$ is a completely irreducible ideal and therefore is M-primal. However, $A^2 = (0)$ is not M-primal.

If we consider only those completely irreducible regular ideals whose adjoint prime is a fixed maximal ideal M, then we can verify:

Theorem 5.8. *Let R be an arithmetical ring and let M be a maximal ideal of R. The set $\mathcal{F}$ of completely irreducible regular ideals of R with adjoint maximal ideal M is closed under ideal-theoretic multiplication, and $\mathcal{F}$ with this multiplication is a totally ordered cancellative semigroup.*

Proof. We show first that: $(\star)$ *If I and J are principal ideals of R and $I_{(M)}$ and $J_{(M)}$ are regular ideals of R, then $(IJ)_{(M)} = I_{(M)}J_{(M)}$.* Suppose that I or J, say

I, is not contained in M. Then $IJR_M = JR_M$, so that $(IJ)_{(M)} = J_{(M)}$, and since $I \not\subseteq M$, $I_{(M)} = R$. Thus if I or J is not contained in M, then $(\star)$ holds. Otherwise, both I and J are contained in M, and by Lemma 5.5 $I_{(M)}$ and $J_{(M)}$ are M-primal ideals of R. As a product of regular M-primal ideals, $I_{(M)}J_{(M)}$ is M-primal by Theorem 5.6. Thus $(I_{(M)}J_{(M)})_{(M)} = I_{(M)}J_{(M)}$, so since $IJ \subseteq I_{(M)}J_{(M)}$ we have that $(IJ)_{(M)} \subseteq I_{(M)}J_{(M)}$, and it remains to verify the reverse inclusion. Suppose that $x \in I : b$ and $y \in J : c$ for some $b, c \in R \setminus M$. Then $xy \in IJ : bc \subseteq (IJ)_{(M)}$ since $bc \notin M$. It follows then that $I_{(M)}J_{(M)} \subseteq (IJ)_{(M)}$, and this proves $(\star)$.

Now let A and B be in $\mathcal{F}$. Then AB is an M-primal ideal by Theorem 5.6. By Theorem 5.3, $A = MJ_{(M)}$ and $B = MK_{(M)}$ for some principal ideals J and K of R. Since A and B are regular ideals, so are $J_{(M)}$ and $K_{(M)}$. Now if $M \neq M^2$, then since by Theorem 5.6, M^2 is primal, hence irreducible, it follows that M/M^2 is a one-dimensional R/M-vector space. Hence if $M \neq M^2$, $M = I_{(M)}$ for some principal ideal $I \subseteq M$. In this case we have by $(\star)$ (and the fact that M is regular) that $AB = I_{(M)}J_{(M)}K_{(M)}M = (IJK)_{(M)}M$, so $AB \in \mathcal{F}$ by Theorem 5.3. Otherwise, if $M = M^2$, then again by $(\star)$ $AB = (IJ)_{(M)}M$ and by Theorem 5.3, $AB \in \mathcal{F}$.

Now assume $AC = BC$, where A, B, C are in $\mathcal{F}$. Using Theorem 5.3, write $A = (MJ)_{(M)}$, $B = (MK)_{(M)}$ and $C = (ML)_{(M)}$ for principal ideals J, K, and L of R. Then $ACR_M = BCR_M$ implies that $M^2JLR_M = M^2KLR_M$. Moreover LR_M is a regular principal ideal of R_M since C is regular and $C \subseteq L_{(M)}$ implies $CR_M \subseteq LR_M$. Thus LR_M is a cancellation ideal of R_M and $JM^2R_M = KM^2R_M$. If $MR_M = M^2R_M$, then clearly $AR_M = CR_M$. Otherwise, if $MR_M \neq M^2R_M$, then the irreduciblity of M^2R_M implies (as above) that MR_M is principal and regular, in which case also $AR_M = CR_M$. Therefore $A = A_{(M)} = C_{(M)} = C$.

Since R is an arithmetical ring, the ideals of R_M are linearly ordered with respect to inclusion and thus so are their preimages under the canonical map $R \to R_M$. Therefore for ideals A and B of R, the ideals $A_{(M)}$ and $B_{(M)}$ of R are comparable. It follows that the ideals in $\mathcal{F}$ are totally ordered under inclusion. □

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