# Coefficient Ideals in and Blowups of a Commutative Noetherian Domain

by

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**Abstract.** The Ratliff–Rush ideal associated to a nonzero ideal $I$ in a commutative Noetherian domain $R$ with unity is $\widetilde{I} = \bigcup_{n=1}^{\infty}(I^{n+1} :_R I^n) = \bigcap\{IS \cap R : S \in \mathcal{B}(I)\}$, where $\mathcal{B}(I) = \{R[I/a]_P : a \in I - 0, P \in \operatorname{Spec}(R[I/a])\}$ is the blowup of $I$. We observe that certain ideals are minimal or even unique in the class of ideals having the same associated Ratliff–Rush ideal. If $(R, M)$ is local, quasi-unmixed, and analytically unramified, and if $I$ is $M$-primary, then we show that the coefficient ideal $I_{\{k\}}$ of $I$, i.e., the largest ideal containing $I$ whose Hilbert polynomial agrees with that of $I$ in the highest $k$ terms, is also contracted from a blowup $\mathcal{B}(I)^{(k)}$, which is obtained from $\mathcal{B}(I)$ by a process similar to "$S_2$-ification". This allows us to generalize the notion of coefficient ideals. We investigate these ideals in the specific context of a two-dimensional regular local ring, observing the interaction of these notions with the Zariski theory of complete ideals.

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**1. Introduction: some results on Ratliff–Rush ideals.**

Let $R$ be a Noetherian ring and $I$ be a regular ideal in $R$. (By ring we mean a commutative ring with unity, and by regular ideal we mean one that contains a nonzerodivisor.) In [RR], Ratliff and Rush studied the ideal $\widetilde{I} = \bigcup_{n=1}^{\infty}(I^{n+1} :_R I^n)$ associated with $I$. They showed in particular that $\widetilde{I}$ is the largest ideal for which, for all sufficiently large positive integers $n$, $(\widetilde{I})^n = I^n$, and hence that $\widetilde{\widetilde{I}} = \widetilde{I}$. They noted that if $I$ is an invertible ideal, then $\widetilde{I^n} = I^n$ for each positive integer $n$; so that if $a$ is a nonzerodivisor in $R$, then $\widetilde{aR} = aR$. They also proved the interesting fact that, for any regular ideal $I$, there is a positive integer $m$ such that, for all $n \geq m$, $\widetilde{I^n} = I^n$.

In [HLS], a regular ideal for which $\widetilde{I} = I$ is called a *Ratliff–Rush ideal,* and the ideal $\widetilde{I}$ is called the *Ratliff–Rush ideal associated with* the regular ideal $I$. In the present paper, we pursue the study of these ideals. In this section, we recall and extend some of the results of [HLS], and we derive some general results used in the following sections. In Section 2, we consider the classes of regular ideals induced by the equivalence relation that they have the same associated Ratliff–Rush ideal. We find some conditions assuring that an ideal is minimal or unique in its Ratliff–Rush class.

Section 3 studies $M$-primary ideals in a quasi-unmixed (i.e., formally equidimensional, in the terminology of [Mt2, page 251]) local domain $(R, M)$. For such an ideal $I$, the "coefficient ideals" of $I$ are introduced in [Sh2] in relation to the coefficients of the Hilbert polynomial of $I$. We relate these coefficient ideals to properties of the blowup $\mathcal{B}(I)$ of $I$. It is shown, for example, in Corollary 3.12 that if $I$ is an $M$-primary ideal of a two-dimensional quasi-unmixed local domain $(R, M)$ with $R/M$ infinite, then the blowup $\mathcal{B}(I)$ of $I$ is Cohen–Macaulay iff all sufficiently high powers of $I$ are equal to their first coefficient ideals. We show in Example 3.22 that, for any integer $d \geq 2$, there exists an $M$-primary ideal $I$ in a $d$-dimensional regular local ring $(R, M)$ such that all the coefficient ideals $I_{\{k\}}, 0 \leq k \leq d$ are distinct.

Section 4 is devoted to results, shown to us by Craig Huneke, for computing the Hilbert polynomial and postulation number of an ideal primary for the maximal ideal in a two-dimensional Cohen–Macaulay local ring. These results are applied in Section 6 to investigate Ratliff–Rush ideals in a polynomial ring in two indeterminates over a field, its localization at the ideal generated by the indeterminates, and other two-dimensional regular local rings.

In the intervening Section 5, we attempt to relate Zariski's theory of complete ideals in a two-dimensional regular local ring $(R, M)$ to the other ideals associated to an $M$-primary ideal $I$, namely $\widetilde{I}$ and $I_{\{1\}}$. We find that if either of these latter ideals is actually equal to the integral closure of $I$ then the same is true for each transform $I^S$ of $I$ in any two-dimensional regular local ring $S$ birationally dominating $R$. We find that if $I$ is contracted in the sense of Zariski [ZS,

Appendix 5], then so are $\widetilde{I}$ and $I_{\{1\}}$. Finally, we give a characterization of the $M$-primary ideals of $R$ having the property that $\mathcal{B}(I)$ is nonsingular.

We now begin with some general remarks on Ratliff–Rush ideals. The use of the symbol $<$ between sets means proper inclusion.

(1.1) For nonzero ideals $J, I$ in a Noetherian domain, it can happen that $J < I$, but $\widetilde{J} \not\subseteq \widetilde{I}$. For example, if $R = k[[t^3, t^4]]$ as in [HLS, (1.11)], and $I = t^8 R$ and $J = (t^{11}, t^{12})R$, then $t^{13} \in \widetilde{J} - \widetilde{I}$. We can also find such examples as the above in a regular Noetherian domain. For example, if $k$ is a field, $R$ is the polynomial ring $k[x, y]$ and $I = (x^3, y^3)R$ and $J = (x^4, x^3y, xy^3, y^4)R$, then $J < I$, but $x^2y^2 \in \widetilde{J} - \widetilde{I}$. It is true, however, that the Ratliff–Rush property behaves well for powers of an ideal: If $n > m$, then $\widetilde{I^n} \subseteq \widetilde{I^m}$. It would be interesting to identify ideal pairs $J \subseteq I$ such that $\widetilde{J} \subseteq \widetilde{I}$. One situation where this is true is where $I$ is integral over $J$ (cf. (1.4) and Lemma 3.2).

Let $I$ be a proper regular ideal in a Noetherian ring $R$. We denote by $\mathrm{G}(I)$ the *associated graded ring* (or *form ring*) $R/I \oplus I/I^2 \oplus I^2/I^3 \oplus \cdots$ of $I$, and by $\mathrm{G}(I)^+$ the positively graded ideal $I/I^2 \oplus I^2/I^3 \oplus \cdots$ of $\mathrm{G}(I)$. By [HLS, (1.2)], $I$ and all its powers are Ratliff–Rush iff $\mathrm{G}(I)^+$ contains a nonzerodivisor. Hence, if $I$ is generated by a regular sequence, or more generally by a quasiregular sequence in the sense of Matsumura [Mt1, page 97], then since $\mathrm{G}(I)$ is isomorphic to a polynomial ring over $R/I$ with indeterminates the images of the generators, $I$ and all its powers are Ratliff–Rush ideals. However, an example of Huckaba and Marley [HM, Example 2.14; cf. Remark 2.6 below] is an almost complete intersection ideal, i.e., one generated by a regular sequence and one additional element, that is not Ratliff–Rush.

*Example 1.2.* (K. N. Raghavan) There exist parameter ideals that are not Ratliff–Rush. Let $R$ be the subring $k[x, y^2, y^7, x^2y^5, x^3y]$ of the polynomial ring $k[x, y]$. Then $I = (x, y^2)R$ is primary for a maximal ideal of $R$ of height two and $x^2y^5 \in (I^2 : I) - I$, so $I$ is not Ratliff–Rush. The regular local ring $k[x, y]_{(x,y)k[x,y]}$ is a finite integral extension of the localization of $R$ at the radical of $I$, so the completion of this localization is a domain; and the essential properties of the example just described continue to hold in this completion. Thus, even in a two-dimensional complete local domain, a parameter ideal need not be Ratliff–Rush.

(1.3) Let $I$ be a regular ideal in a Noetherian ring $R$. The fact that all powers of $I$ are Ratliff–Rush iff $\mathrm{G}(I)^+$ contains a nonzerodivisor can be refined as follows: The ideal $I$ is Ratliff–Rush iff there is no nonzero element of degree zero in $\mathrm{G}(I)$ that annihilates a power of $\mathrm{G}(I)^+$. More generally, in terms of the 0-th local cohomology of $\mathrm{G}(I) = G$ with respect to $G^+$, we have $(H^0_{G^+}(G))_n = (\widetilde{I^{n+1}} \cap I^n)/I^{n+1}$. Therefore, if $I^n$ is a Ratliff–Rush ideal, then, since $\widetilde{I^{n+1}} \subseteq \widetilde{I^n} = I^n$, we have $I^{n+1}$ is Ratliff–Rush iff $(H^0_{G^+}(G))_n = 0$. A good reference for the use of the local cohomology modules of $G$ in studying the reduction numbers (see below) of $I$ is [T].

(1.4) Let $I$ be a proper regular ideal in a local ring $(R, M)$. It is easy to see that an element $a$ of $(I^{n+1} :_R I^n)$ is integral over $I$, in the sense that there is an equation of the form $a^k + b_1 a^{k-1} + \cdots + b_k = 0$, where $b_j \in I^j$ for $j = 1, \ldots, k$. Thus, the ideal $\widetilde{I}$ is always between $I$ and the integral closure $I'$ of $I$; in particular, integrally closed ideals are Ratliff–Rush ideals. We use the following classical results of Northcott and Rees [NR] concerning reductions and integral dependence. If $\mathbf{q} \subseteq I$ are ideals in a Noetherian ring $R$, then $\mathbf{q}$ is said to be a *reduction* of $I$ iff $\mathbf{q}I^n = I^{n+1}$ for some positive integer $n$; or equivalently iff $I \subseteq \mathbf{q}'$. An ideal $\mathbf{q} \subseteq I$ in a local ring $(R, M)$ is a reduction of $I$ iff $\mathbf{q} + MI$ is a reduction of $I$. As in [Sh3], we denote by $\mathrm{F}(I)$ the *fiber ring* $\mathrm{G}(I)/M\,\mathrm{G}(I) = R/M \oplus I/IM \oplus I^2/I^2M \oplus \cdots$. With this notation, we see that $\mathbf{q} \subseteq I$ is a reduction of $I$ iff the image of $\mathbf{q}$ in the homogeneous degree-one piece $I/MI$ of $\mathrm{F}(I)$ generates an ideal that, for sufficiently large $n$, contains all of the degree-$n$ piece $I^n/MI^n$ of $\mathrm{F}(I)$. If $R/M$ is infinite, we can choose $\dim(\mathrm{F}(I))$ elements in $I$ that generate a *minimal reduction* $\mathbf{q}$ of $I$. The *analytic spread* of $I$ is $\dim(\mathrm{F}(I))$, i.e., the minimal number of generators of a minimal reduction of $I$ if $R/M$ is infinite. The *reduction number of $I$ with respect to* a reduction $\mathbf{q}$ is the smallest integer $n$ for which $\mathbf{q}I^n = I^{n+1}$. The smallest among the reduction numbers of $I$ with respect to all its minimal reductions is called the *reduction number of $I$*.

(1.5) Let $I$ be a regular ideal in a Noetherian ring $R$. For any reduction $\mathbf{q}$ of $I$ and positive integer $n$, the elements in $(I^n : \mathbf{q}) - I^{n-1}$ map to elements in $\mathrm{G}(I)$ that annihilate a power of $\mathrm{G}(I)^+$ and hence have bounded degree. Take a nonzerodivisor $x$ in $\mathbf{q}$. Then, using the Artin–Rees lemma, there exists a positive integer $k$ such that $x(I^n : x) = I^n \cap xR = I^{n-k}(I^k \cap xR) \subseteq xI^{n-k}$, so $(I^n : \mathbf{q}) \subseteq (I^n : x) \subseteq I^{n-k}$. Thus, for sufficiently large $n$, the images of the elements of $(I^n : \mathbf{q})$ do not lie in the low-degree pieces of $\mathrm{G}(I)$, so we have $(I^n : \mathbf{q}) = I^{n-1}$. See, e.g., [Mc, Lemma 8.1, page 61] for a related result.

It is shown in [HLS, Fact 2.1] that the Ratliff–Rush ideal $\widetilde{I}$ associated with the ideal $I$ in an integral domain $R$ is the contraction of $I$ from the model over $R$ obtained by blowing up $I$, i.e., the *blowup of $I$*: $\mathcal{B}(I) = \{R[I/a]_P : a \in I - 0 \text{ and } P \in \mathrm{Spec}(R[I/a])\}$; in symbols,

$$\widetilde{I} = \bigcap\{IS \cap R : S \in \mathcal{B}(I)\} = \bigcap\{IR[I/a] \cap R : a \in I - 0\} \quad .$$

We refer the reader to [ZS, Chapter VI, Section 17], [A2, Chapter 5] and [A3, Section 6] for the basic facts on models. In particular, $\mathcal{B}(I)$ is the set of all local rings $S$ between $R$ and its field of fractions minimal with respect to domination in which $IS$ is a principal ideal. In [HLS, (2.2)] it is shown that $\widetilde{I} = IR[I^n/a] \cap R$ for a single element $a$ of $I^n$; but it is not easy to identify the element $a$ in that result. There are small and more readily identifiable sets over which $a$ can vary that allow us to describe $\widetilde{I}$ in a similar way:

**Lemma 1.6.** *Let $I$ be a nonzero ideal in a Noetherian domain $R$, and let $a_1, \dots, a_n$ generate a reduction of $I$. Then $\mathcal{B}(I) = \{R[I/a_i]_P : i = 1, \dots, n$ and $P \in \operatorname{Spec}(R[I/a_i])\}$, and hence $\widetilde{I} = \bigcap_{i=1}^{n}(IR[I/a_i] \cap R)$.*

*Proof.* Let $S \in \mathcal{B}(I)$. Since $IS$ is principal and hence invertible and

$$((a_1, \dots, a_n)S)(IS)^k = (IS)^{k+1}$$

for some positive integer $k$, we conclude that $(a_1, \dots, a_n)S = IS$, and hence (since $S$ is local) that $a_iS = IS$ for some $i$ in $\{1, \dots, n\}$. Thus, $R[I/a_i] \subseteq S$. Denote by $Q$ the intersection of the maximal ideal of $S$ with $R[I/a_i]$, and suppose $S = R[I/b]_P$ where $b \in I - 0$ and $P \in \operatorname{Spec}(R[I/b])$. Then $a_iS = IS = bS$, so $a_i/b$ is a unit in $S$ and hence not in $Q$. Since $R[I/b] \subseteq R[I/a_i]_Q \subseteq S = R[I/b]_P$, $S$ is a localization of $R[I/a_i]_Q$ that dominates $R[I/a_i]_Q$, so they are equal. $\square$

**Proposition 1.7.** *In the Noetherian domain $R$, let $(a_1, \dots, a_n)R$ be a reduction of each of the ideals $I$ and $J$. Then $\widetilde{J} = \widetilde{I}$ iff, for each $i = 1, \dots, n$, $R[J/a_i] = R[I/a_i]$.*

*Proof.* ($\Leftarrow$): By Lemma 1.6, $\mathcal{B}(J) = \mathcal{B}(I)$, and each $S$ in this blowup has the form $R[I/a_i]_P = R[J/a_i]_P$. Thus, $IS = a_iS = JS$, and hence, by [HLS, Fact 2.1], $\widetilde{J} = \bigcap_{S \in \mathcal{B}(J)}(JS \cap R) = \bigcap_{S \in \mathcal{B}(I)}(IS \cap R) = \widetilde{I}$.

($\Rightarrow$): Since $I \subseteq \widetilde{J}$, for any $b$ in $I$, there is a positive integer $k$ for which $bJ^k \subseteq J^{k+1}$. In particular, for each $i = 1, \dots, n$, we have $ba_i^k \in J^{k+1}$, so $b/a_i \in (J/a_i)^{k+1} \subseteq R[J/a_i]$. The reverse inclusion is proved similarly. $\square$

(1.8) Suppose $I$ is a nonzero ideal in a Noetherian domain $R$. Certain properties of high powers of the ideal $I$ are naturally related to properties of the blowup $\mathcal{B}(I)$ of $I$. For example, suppose $R$ is normal; then all sufficiently high powers of $I$ are integrally closed ideals iff the local rings on $\mathcal{B}(I)$ are all normal. This follows from the following facts: (i) The (small) Rees ring $R[I^nt]$ of $I^n$ is a normal domain iff all powers of $I^n$ are integrally closed. (ii) $\mathcal{B}(I) = \mathcal{B}(I^n)$ is the set $\operatorname{Proj}(R[I^nt])$ of homogeneous localizations (i.e., the degree-zero pieces of rings of fractions with denominators only homogeneous elements) of $R[I^nt]$ at homogeneous primes that do not contain $I^nt$. (iii) Principal ideals in normal domains are integrally closed. (iv) All sufficiently high powers $I^m$ of $I$ are Ratliff–Rush, i.e., are the contractions of their extensions to $\mathcal{B}(I^m) = \mathcal{B}(I)$. We note the following consequence for Ratliff–Rush ideals:

**Proposition 1.9.** *Let $I$ be a Ratliff–Rush ideal in a normal Noetherian domain. If $I^n$ is integrally closed for all sufficiently large positive integers $n$, then $I$ is integrally closed.*

*Proof.* By [HLS, Fact 2.1], $I$ is the contraction of its extension to the local rings in $\mathcal{B}(I)$, which are normal because high powers of $I$ are integrally closed. Thus, $I$ is also integrally closed. $\square$

See Corollaries 3.10 and 3.12 below for related conclusions under different hypotheses.

*Definition 1.10.* (Cf. [ZS, page 115], [A2], [A3].) For nonzero ideals $I$ and $J$ in a Noetherian domain $R$, we say that the blowup $\mathcal{B}(I)$ of $I$ *dominates* the blowup $\mathcal{B}(J)$ of $J$, and write $\mathcal{B}(J) \preceq \mathcal{B}(I)$, iff each local ring in $\mathcal{B}(J)$ is dominated by a local ring in $\mathcal{B}(I)$ or, equivalently, iff $JS$ is principal for every local ring $S$ in $\mathcal{B}(I)$.

If $I, J$ are ideals in a Noetherian domain $R$, we may form first the complete model $\mathcal{B}(J)$ over $R$ and the sheaf of ideals, $I\mathcal{B}(J)$, obtained by extending $I$ to each local domain $S$ in $\mathcal{B}(J)$. Then for each such $S$, we may form the complete model over $S$, $\mathcal{B}(IS)$, the blowup over $S$ of the ideal $IS$. The union of these models is then again a complete model over $R$, namely the model obtained by blowing up $\mathcal{B}(J)$ at the sheaf of ideals $I\mathcal{B}(J)$. In fact, this model can be more concretely realized:

**Lemma 1.11.** *If $I$ and $J$ are ideals in a Noetherian domain $R$ and if $X$ is the complete model over $R$ obtained from $\mathcal{B}(J)$ by further blowing up the sheaf of ideals $I\mathcal{B}(J)$, then $X$ is both $\mathcal{B}(IJ)$ and the join of $\mathcal{B}(J)$ with $\mathcal{B}(I)$ (i.e., the smallest model dominating both $B(J)$ and $B(I)$; cf.* [ZS, page 121 and Lemma 6, page 120]*).*

*Proof.* The blowup of an ideal is characterized as the unique model minimal with respect to domination among the set of all models in which the extension of the given ideal is locally principal. From this it immediately follows that the join of $\mathcal{B}(I)$ with $\mathcal{B}(J)$ is a model which dominates $\mathcal{B}(IJ)$. From the other side, if in a local domain the product of two ideals is principal, then each of the ideals must themselves be principal. From this it follows that $I$ extended to any local ring in $\mathcal{B}(IJ)$ must be principal, and similarly with $J$. Therefore $\mathcal{B}(IJ)$ must dominate both $\mathcal{B}(I)$ and $\mathcal{B}(J)$ and hence their join. Thus we conclude that it is the join.

Concerning the blowup of the sheaf of ideals $I\mathcal{B}(J)$, we can again note that it is a model which dominates the model $\mathcal{B}(J)$ and which is minimal with respect to the property that $I$ extended to each of the local rings in this model becomes principal, then we can argue exactly as above to conclude that this model dominates and is dominated by $\mathcal{B}(IJ)$ and so must equal it. $\square$

Now let us recall some of the discussion before Proposition 1.9. If we start with a Noetherian domain $R$ and an ideal $I$ in it, and form the normalized blowup $\mathcal{B}(I)'$, i.e., the family of localizations at maximal ideals of the integral closures of the rings in $\mathcal{B}(I)$, we again obtain a complete model over $R$. Extending $I^n$ to any local ring $S$ on $\mathcal{B}(I)'$, we see that $I^nS$ is principal in a normal local domain and hence is integrally closed. But then $I^nS$ contracted to $R$ must also be integrally closed; hence the intersection of all of these ideals $\bigcap(I^nS \cap R)$, as $S$ varies over the local rings on $\mathcal{B}(I)'$, must be an integrally closed ideal of $R$ containing $I$. On the other hand, this ideal must itself be contained in the integral closure $(I^n)'$ of $I^n$, by the completeness of the model $\mathcal{B}(I)'$. We record this:

**Proposition 1.12.** *If $I$ is an ideal in a Noetherian domain $R$, then for all positive integers $n$, $(I^n)'$ is the intersection of the contraction to $R$ of the extension to $\mathcal{B}(I)'$ of $I^n$. On the other hand, if $R$ is a normal, local, analytically unramified domain, then for all large $n$, $\mathcal{B}(I)' = \mathcal{B}((I^n)')$.*

*Proof.* We have only to see the converse. Let $R[I/a]'$ be any affine piece of the normalized blowup. We claim that

$$R[I/a]' = \bigcup_{n \geq 1} ((I^n)'/a^n).$$

It is clear that $(I^n)'/a^n$ is contained in $R[I/a]'$ for all $n$. For the other inclusion, if $z$ is an element of $R[I/a]'$, then by writing out an integral equation for $z$, say of degree $m$, with coefficients in $I^n/a^n$ for some large $n$, and then multiplying through by $a^{nm}$, one concludes that $a^nz$ is integral over $I^n$. That $R$ is integrally closed then implies that $a^nz \in R$ and hence that $z \in (I^n)'/a^n$.

Since $R$ is analytically unramified, $R[I/a]'$ is a finitely generated $R[I/a]$-module [Re, Theorem 1.5, page 27], and if $m \leq n$, then $((I^m)'/a^m) \subseteq ((I^n)'/a^n)$, so for some $n$ we have $R[I/a]' = R[(I^n)'/a^n]$. As the model $\mathcal{B}(I)'$ is covered by a finite collection of such affine pieces, this completes the proof. $\square$

The following extension of [HLS, Fact 2.1] will prove useful later.

**Proposition 1.13.** *Let $R$ be a Noetherian domain and let $I$ be a nonzero proper ideal of $R$. Then:*

*(a) For any ideal $J$ containing and integral over $I$, we have $\bigcap\{IS \cap R : S \in \mathcal{B}(J)\} = \widetilde{J}$, i.e., $\widetilde{J}$ is the contraction to $R$ of the extension of $I$ to $\mathcal{B}(J)$.*

*(b) For an ideal $J$ as in (a), we have $\mathcal{B}(I) \preceq \mathcal{B}(J) \preceq \mathcal{B}(I)'$, and $\mathcal{B}(I) = \mathcal{B}(J)$ iff $\widetilde{I} = \widetilde{J}$.*

*(c) More generally, if $J_1, J_2$ are ideals containing and integral over the powers $I^{n(1)}, I^{n(2)}$ (respectively) of $I$, then the following conditions are equivalent:*

*(i) $\mathcal{B}(J_1) = \mathcal{B}(J_2)$.*

*(ii) $\widetilde{J_1^{n(2)}} = \widetilde{J_2^{n(1)}}$.*

*(iii) For some positive integers $m(1), m(2)$, we have $J_1^{m(1)} = J_2^{m(2)}$.*

*Proof.* (a): For every $S$ in $\mathcal{B}(J)$, $JS$ is principal, and $J$ is integral over $I$, so $JS$ is generated by an element of $I$. The result then follows from [HLS, Fact 2.1].

(b): Take any $T$ in $\mathcal{B}(I)$, say $T = R[I/a]_P$. Since $J$ is integral over $I$, the elements of $J/a$ are integral over $R[I/a]$, so there is a prime $Q$ in $R[J/a]$ lying over $P$. The element $R[J/a]_Q$ of $\mathcal{B}(J)$ dominates $T$. If $\widetilde{I} = \widetilde{J}$, then for some positive integer $k$, $I^k = J^k$, so $\mathcal{B}(I) = \mathcal{B}(I^k) = \mathcal{B}(J^k) = \mathcal{B}(J)$. Conversely if $\mathcal{B}(I) = \mathcal{B}(J)$, then $\widetilde{I} = \widetilde{J}$ by (a).

(c): (iii) $\Longrightarrow$ (i): Clear. (i) $\Longrightarrow$ (ii): Similar to the proof of (b), since $J_1^{n(2)}, J_2^{n(1)}$ are both integral over $I^{n(1)n(2)}$. (ii) $\Longrightarrow$ (iii): Take $n$ sufficiently large that $J_1^{n(2)n}, J_2^{n(1)n}$ are Ratliff–Rush. Since the Ratliff–Rush ideals associated to $J_1^{n(2)n}, J_2^{n(1)n}$ are equal, as above, the result follows by setting $m(1) = n(2)n$ and $m(2) = n(1)n$. $\square$

(1.14) Since the condition that the associated graded ring $\mathrm{G}(I)$ of a regular ideal $I$ is Cohen–Macaulay implies that $I$ and the powers of $I$ are Ratliff–Rush, we recall here some results in the literature related to this condition:

(a) [V, Lemma 1 and Theorem 1] Let $(R, M)$ be a Cohen–Macaulay local ring with $R/M$ infinite, and let $I$ be an $M$-primary ideal in $R$. If $I$ has reduction number at most one, then $\mathrm{G}(I)$ is Cohen–Macaulay.

(b) [Sh3, Theorem 4] [KV, Theorem 4.1] Let $(R, M)$ be a Cohen–Macaulay local ring with $R/M$ infinite, and $\mathbf{q}$ be a minimal reduction of an $M$-primary ideal $I$. Then the extended Rees ring $R[It, t^{-1}]$, or equivalently the graded ring $\mathrm{G}(I)$, is Cohen–Macaulay iff $\mathbf{q}I^n \cap I^{n+2} = \mathbf{q}I^{n+1}$ for all nonnegative integers $n$.

(c) [HM, Proposition 2.6] [JV, Theorem 4.1] [Sh3, Corollary 4(f)] Let $(R, M)$ be a two-dimensional regular local ring and $I$ be an $M$-primary ideal. Then the following conditions are equivalent: (i) The (small) Rees ring $R[It]$ is Cohen–Macaulay. (ii) $\mathrm{G}(I)$ is Cohen–Macaulay. (iii) $I$ has reduction number at most one.

(1.15) If $R$ is a one-dimensional semilocal domain and $I$ is an ideal of $R$ contained in the conductor of the integral closure $R'$ into $R$, then $I$ is *stable*, i.e., has a principal reduction and reduction number at most one — and hence is Ratliff–Rush (or the zero ideal), by [HLS, (1.1)]. For, $R'$ is a semilocal Dedekind domain and hence a principal ideal domain, so $I = IR' = aR'$ for some $a \in IR' = I$, and hence $I^2 = (aR')^2 = aI$. In general, in a Noetherian domain $R$, if $I$ is an ideal contained in the conductor of $R'$ into $R$ and having a principal reduction, then $I$ is stable. For, if $aR$ is a reduction of $I$, then since $aR'$ is integrally closed, $aR' = IR' = I$, so again $I^2 = aI$.

We end this section with some general questions.

*Questions 1.16.*

(Q1) Is the minimal number of generators $\mu(I)$ for a regular ideal $I$ always less than or equal to $\mu(\widetilde{I})$? By [HLS, (2.11)], this is true in a one-dimensional local domain.

(Q2) When is it the case that $I^n \cap \widetilde{I^{n+1}} = I^n \widetilde{I}$?

## 2. The equivalence classes of Ratliff–Rush ideals.

We define a binary relation on the set of regular ideals in a Noetherian ring by declaring that two such ideals are related iff they have the same associated Ratliff–Rush ideal. This is clearly an equivalence relation on this set of ideals. We are interested in the equivalence classes of Ratliff–Rush ideals in various Noetherian rings $R$. A regular ideal $I$ of $R$ is *minimal Ratliff–Rush* iff it is minimal in its Ratliff–Rush class, i.e., for any regular ideal $J < I$, $\widetilde{J} \neq \widetilde{I}$. The regular ideal $I$ is *uniquely Ratliff–Rush* iff $I$ is the only element in its Ratliff–Rush class, i.e., $I = \widetilde{I}$ and $I$ is minimal Ratliff–Rush. We are interested in particular in determining uniquely Ratliff–Rush and minimal Ratliff–Rush ideals.

Elements $a_1, \ldots, a_n$ of the maximal ideal $M$ of a local ring $R$ are said to be *analytically independent* iff any homogeneous polynomial $f(X_1, \ldots, X_n)$ in $R[X_1, \ldots, X_n]$ such that $f(a_1, \ldots, a_n) = 0$ has all of its coefficients in $M$. This condition is equivalent to the condition that the fiber ring $\mathrm{F}(I)$ of $I = (a_1, \ldots, a_n)R$ is isomorphic to a polynomial ring in $n$ indeterminates over the field $R/M$. Thus, in particular, a regular sequence of elements forms an analytically independent set.

Suppose now that $a_1, \ldots, a_n$ are analytically independent elements in $R$, $I = (a_1, \ldots, a_n)R$, and $J$ is an ideal of $R$ with $J < I$. Then the image of $J$ in the degree-one piece of $\mathrm{F}(I)$ is a proper $R/M$-subspace of $I/MI$, so the ideal it generates in this polynomial ring contains no power of the maximal graded ideal of $\mathrm{F}(I)$; so $J$ is not a reduction of $I$. Thus we have:

(2.1) A nonzero ideal $I$ in a local domain generated by analytically independent elements is not integral over any properly smaller ideal and so, in particular, is minimal Ratliff–Rush. If, in addition, $I$ is itself Ratliff–Rush, then $I$ is uniquely Ratliff–Rush. Thus if $I$ is generated by a regular sequence, then $I$ is uniquely Ratliff–Rush.

**Theorem 2.2.** *Let $(R, M)$ be a local domain with $R/M$ infinite, and let $I$ be a nonzero ideal of $R$ of height $n$. Assume that $I$ can be generated by $n+1$ elements and that any minimal reduction of $I$ generated by $n$ elements is generated by a regular sequence. Then $I$ is minimal Ratliff–Rush, i.e., for any ideal $J$ properly contained in $I$, $\widetilde{J} \neq \widetilde{I}$.*

*Proof.* The analytic spread of $I$ is bounded below by the height of $I$ and above by the minimal number of generators of $I$. If it is equal to the latter, then $I$ is a minimal reduction of itself, so it is generated by analytically independent elements, and then the result follows from (2.1). Thus we may assume that the analytic spread of $I$ is equal to the height of $I$ and that $I$ is minimally generated by $n+1$ elements. We only need to consider the case that $J$ is a reduction of $I$, so that $J$ contains a minimal reduction $(a_1, \ldots, a_n)R$ of $I$. Then there is an element $b$ of $I$ for which $I = (a_1, \ldots, a_n, b)R$. By applying Proposition 1.7, it suffices to see that $R[J/a_1] < R[I/a_1]$.

To show that $R[J/a_1] < R[I/a_1]$, it is enough to assume $J = (a_1, \ldots, a_n, Mb)R$. Since $a_1, \ldots, a_n$ is an $R$-sequence, the kernel of the mapping $R[X_2, \ldots, X_n] \to R[a_2/a_1, \ldots, a_n/a_1]$ determined by $X_i \mapsto a_i/a_1$ is generated by the polynomials $a_1X_i - a_i$ [D, Proposition 2, page 201] [Ra, Lemma 2.3, page 400], so it is contained in $MR[X_2, \ldots, X_n]$. Thus, the extension of $M$ to $R[a_2/a_1, \ldots, a_n/a_1]$ is a prime ideal; so we can localize this ring at the complement of this prime in $R[a_2/a_1, \ldots, a_n/a_1]$. Denote the resulting ring by $S$; it is local with maximal ideal $MS$. The mapping above extends to a local epimorphism $R(X_2, \ldots, X_n) \to S$. Setting $K = (a_1, \ldots, a_n)R$, we note that the kernel of this extension is contained in $KR(X_2, \ldots, X_n)$, so the fact that $R(X_2, \ldots, X_n)$ is faithfully flat over $R$ yields that $K = KS \cap R = a_1S \cap R$. Thus, $b \notin a_1S$, i.e., $b/a_1 \notin S$.

Assume by way of contradiction that $b/a_1 \in R[J/a_1]$; then *a fortiori* $b/a_1 \in S[Mb/a_1] = \bigcup_{k=1}^{\infty}(JS/a_1)^k$. An equation showing that $b/a_1 \in (JS/a_1)^k$ shows that $S[b/a_1] = S + MS[b/a_1]$. Since $a_1S = (a_1, a_2, \ldots, a_n)S$ is a reduction of $IS$, $b/a_1$ is integral over $S$, so $S[b/a_1]$ is a finitely generated $S$-module. Thus by Nakayama's lemma, $S = S[b/a_1]$, i.e., $b/a_1 \in S$, a contradiction. $\square$

In view of the good behavior of the Ratliff–Rush property with respect to faithfully flat ring extensions noted in [HLS, (1.7)], we have: The condition that an ideal be minimal Ratliff–Rush descends with respect to a faithfully flat extension of rings. Thus, for example, in showing that an ideal $I$ in a local domain $(R, M)$ is minimal Ratliff–Rush one may assume, by passing if necessary from $R$ to $R(X) = R[X]_{M[X]}$, that the field $R/M$ is infinite.

**Corollary 2.3.** *Let $I$ be a nonzero ideal of a Cohen–Macaulay local domain $(R, M)$. If $I$ is generated by $n+1$ elements and if $\mathrm{ht}(I) \geq n$, then $I$ is minimal Ratliff–Rush.*

*Proof.* By passing to $R(X)$ if necessary, we may assume $R/M$ is infinite. If $\mathrm{ht}(I) > n$, then $I$ is generated by a regular sequence, so it is uniquely Ratliff–Rush by (2.1). If $\mathrm{ht}(I) = n$, apply Theorem 2.2. $\square$

**Corollary 2.4.** *Let $I$ be a nonzero ideal in a local domain $(R, M)$. If $I$ can be generated by two elements, then $I$ is minimal Ratliff–Rush.*

*Proof.* Again we may assume that $R/M$ is infinite, and then the result is immediate from Theorem 2.2. $\square$

*Question 2.5.* For what Cohen-Macaulay local domains $(R, M)$ is it the case that the maximal ideal $M$ is uniquely Ratliff–Rush? By Theorem 2.2, if $M$ is generated by $\dim(R)+1$ elements, then $M$ is uniquely Ratliff–Rush. On the other hand, if $M$ is generated by $\dim(R)+2$ elements, then $M$ need not be uniquely Ratliff–Rush. For example, if $R = k[[t^3, t^4, t^5]]$ and $I = (t^3, t^4)R$, then $I < (t^3, t^4, t^5)R = M$, but $I^2 = M^2$, so $\widetilde{I} = M$.

(2.6) Indeed, even in a regular Noetherian domain, Theorem 2.2 cannot be generalized to the case where, in the notation of that theorem, $I$ is generated by $\mathrm{ht}(I)+2$ elements. For example, if $R$ is the polynomial ring $k[x, y]$, then Huckaba and Marley observe in [HM, Example 2.14] that if $J = (x^5, y^5, x^4y + x^2y^3)R$, then $x^2y^4 \in (J^2 : J) - J$. Using the computer algebra program MACAULAY written by David Bayer and Michael Stillman, we have checked that $I = J + x^2y^4R$ has the property that $\mathrm{G}(I)^+$ has positive depth, so $I$ and all its powers are Ratliff–Rush. Thus, $I = \widetilde{J}$.

*Question 2.7.* Suppose $I$ is an ideal primary for the maximal ideal in a local Noetherian domain. Is there a more general result on the length of the Ratliff–Rush class of $I$, in terms of the difference between the minimal number of generators of $I$ and the height of $I$? By the *length* of the class, we mean the length of the longest chain of ideals in the class.

## 3. Hilbert polynomials and coefficient ideals.

We denote the length of the $R$-module $A$ by $\lambda(A)$. It is well known that, if $I$ is an ideal in a Noetherian ring $R$ for which $R/I$ has (Krull) dimension zero, then the *Hilbert function* $H_I(n) = \lambda(R/I^n)$ of $I$ is, for all sufficiently large values of the positive integer $n$, a polynomial in $n$ of degree the dimension $d$ of the ring $R$, the *Hilbert polynomial* of $I$, which we denote by $P_I(n)$. We follow the convention of writing $P_I$ in terms of binomial coefficients:

$$(*) \qquad P_I(n) = e_0(I)\binom{n+d-1}{d} - e_1(I)\binom{n+d-2}{d-1} + \cdots + (-1)^d e_d(I) \quad .$$

Then the coefficients $e_i = e_i(I)$ are integers, the *Hilbert coefficients* of $I$. The leading coefficient $e_0$ is the *multiplicity* of $I$.

(3.1) Let $(R, M)$ be a Cohen–Macaulay local ring of dimension $d > 0$ and let $I$ be an $M$-primary ideal. Northcott observed [Nt, Theorem 3, page 214] that the Hilbert polynomial of $I$ has the form $e_0\binom{n+d-1}{d}$, if and only if $I$ is generated by a system of parameters.

In [Sh2], it is shown that if $(R, M)$ is a quasi-unmixed local ring of dimension $d > 0$, with $R/M$ infinite, and if $I$ is an $M$-primary ideal, then for each integer $k$ in $\{0, \ldots, d\}$ there exists a unique largest ideal $I_{\{k\}}$ containing $I$ such that $e_i(I_{\{k\}}) = e_i(I)$ for $i = 0, \ldots, k$. Then

$$I \subseteq I_{\{d\}} \subseteq \cdots \subseteq I_{\{1\}} \subseteq I_{\{0\}} \quad .$$

The ideal $I_{\{k\}}$ is called the *$k$-th coefficient ideal associated to $I$* or the *$e_k$-ideal associated to $I$*; and if $I_{\{k\}} = I$, then $I$ is called a *$k$-th coefficient ideal* or an *$e_k$-ideal.* The ideal $I_{\{0\}}$ is the integral closure $I'$ of $I$, and if $I$ is a regular ideal, the ideal $I_{\{d\}}$ is the Ratliff–Rush ideal $\widetilde{I}$ associated to $I$. Since an $e_k$-ideal is maximal among those sharing its Hilbert polynomial, any $e_k$-ideal is Ratliff–Rush.

**Proposition 3.2.** *With hypotheses as above, if $J$ is an ideal of $R$ with $I \subseteq J$, then:*

*(i) if $J \subseteq I_{\{k\}}$, then $I_{\{k\}} = J_{\{k\}}$;*

*(ii) if there exists one positive integer $m$ such that $J^m \subseteq (I^m)_{\{k\}}$, then for every positive integer $m$, $J^m \subseteq (I^m)_{\{k\}}$ (and of course conversely); and*

*(iii) for all positive integers $m$ and $n$, $(((I^m)_{\{k\}})^n)_{\{k\}} = (I^{mn})_{\{k\}}$.*

*Proof.* (i) holds because, for large $n$, $\lambda(R/I^n) - \lambda(R/J^n)$ is a polynomial in $n$ bounded below by the constant 0 and above by $\lambda(R/I^n) - \lambda(R/((I_{\{k\}})^n)$, a polynomial in $n$ of degree at most $k-1$. (ii) holds because if, in the usual expression (*) for Hilbert polynomials, $n$ is replaced with $mn$ and the result is rewritten in the form (*), the new coefficients $e_j(I^m)$, for $j = 0, \ldots, k$, are linear combinations of the original coefficients $e_j(I)$ for $j = 0, \ldots, k$ (independent of $e_j(I)$ for $j = k+1, \ldots, d$); and $e_0(I), \ldots, e_k(I)$ are linear combinations of $e_0(I^m), \ldots, e_k(I^m)$. (iii) follows from (ii) and (i). $\square$

**Proposition 3.3.** *Let $(R, M)$ be a two-dimensional Cohen–Macaulay local ring, and let $I$ be an $M$-primary ideal of $R$. Then $e_2(I_{\{1\}}) \leq e_2(I)$, and $e_2(I_{\{1\}}) = e_2(I)$ if and only if $\widetilde{I} = I_{\{1\}}$. In particular, if $e_2(I) = 0$, then $\widetilde{I} = I_{\{1\}}$.*

*Proof.* We have $P_I - P_{I_{\{1\}}} = e_2(I) - e_2(I_{\{1\}})$ and $I^n \subseteq (I_{\{1\}})^n$ for all positive integers $n$, so for large values of $n$,

$$e_2(I) - e_2(I_{\{1\}}) = \lambda(R/I^n) - \lambda(R/(I_{\{1\}})^n) \geq 0 \quad .$$

By [Nr], $e_2(I_{\{1\}}) \geq 0$, so if $e_2(I) = 0$, we have $e_2(I) = e_2(I_{\{1\}})$, and hence $P_I = P_{I_{\{1\}}}$, so that $\widetilde{I} = \widetilde{I_{\{1\}}} = I_{\{1\}}$. $\square$

(3.4) Let $I$ be an $M$-primary ideal in a two-dimensional Cohen–Macaulay local ring $(R, M)$ with $R/M$ infinite, and consider the following conditions on $I$:

(1) $I$ is Ratliff–Rush.

(1′) All powers of $I$ are Ratliff–Rush.

(2) $I$ is an $e_1$-ideal.

(2′) All powers of $I$ are $e_1$-ideals.

(3) $e_2(I) = 0$, or equivalently $e_2(I^m) = 0$ for all positive integers $m$.

(4) $H_I(n) = P_I(n)$ for all positive integers $n$.

(5) $I$ has reduction number at most one, or, equivalently by [H1, Theorem 2.1], $\lambda(R/I) = e_0(I) - e_1(I)$.

Of course, $(2') \implies (2) \implies (1)$ and $(2') \implies (1') \implies (1)$. Also by [H1, Theorem 2.1], $(5) \iff ((3)$ and $(4))$. Using the latter formulation, we see that if $I$ has reduction number at most one, then the same is true of every power of $I$. Using (1.14)(a), (5) implies that $G(I)$ is Cohen–Macaulay, so $(1')$ holds, and by Proposition 3.3 we see that $(2')$ holds. On the other hand, by [Sy2, (2.6)], (1) and (3) imply (5).

Some of the other possible implications do not hold. For instance, in the polynomial ring $k[x, y]$ over the field $k$, the ideal $I = (x^4, x^3y, xy^3, y^4)k[x, y]$ satisfies (3) but not (1). An example satisfying (2) and even $(2')$ but not (3) is obtained as follows:

*Example 3.5.* Let $k$ be a field of characteristic different from 3, let $D$ denote the ring $k[x, y, z]$, subject to the single relation $z^3 = x^3 + y^3$, and let $R$ be the localization of $D$ at the maximal ideal generated by $x, y, z$. Since the associated graded ring of the maximal ideal $M$ of $R$ is isomorphic to $D$, a domain, it follows from [ZS, Theorem 1, page 249] that the powers of $M$ are all valuation ideals and hence $e_1$-ideals; but $e_2(M) = 1$. Note that the blowup $\mathcal{B}(M)$ of $M$ is nonsingular and so in particular satisfies condition $\mathrm{E}_1$ (Definition 3.8 below).

To show the existence of an $e_1$-ideal $I$ such that for all sufficiently large positive integers $n$, the Ratliff–Rush ideal $I^n$ is not an $e_1$-ideal, we use the following example of Sam Huckaba [Hc3, Example 1.6, page 183]. It yields a two-dimensional Cohen–Macaulay local domain for which the blowup $\mathcal{B}(M)$ of the maximal ideal $M$ is not Cohen–Macaulay. It follows from Corollary 3.18 below that all sufficiently high powers of $M$ fail to be $e_1$-ideals.

*Example 3.6.* Let $k$ be a field and let $R = k[x^3, x^5y^3, x^{15}y^2, y^7]$, where $x$ and $y$ are indeterminates. Then $R$ is a complete intersection,

$$R \cong k[u, v, w, z]/(w^7 - u^{35}z^2, v^3 - wz)$$

where $u, v, w, z$ are indeterminates. Therefore $R$ is Cohen–Macaulay. But the affine piece of the blowup of the maximal ideal $(x^3, x^5y^3, x^{15}y^2, y^7)R$ obtained by dividing by $x^3$ gives a ring isomorphic to $k[x^3, x^5y^3, x^{14}y^2, x^4y^7]$, which is not Cohen–Macaulay. Therefore the two-dimensional Cohen–Macaulay local domain obtained by localizing $R$ at the maximal ideal $(x^3, x^5y^3, x^{15}y^2, y^7)R$ gives the desired example.

(3.7) Several interesting questions concerning these implications remain open, particularly in the case of a two-dimensional regular local ring. For instance:

(a) In a two-dimensional regular local ring, does the conjunction of $(1')$ and (2) imply $(2')$? We have already seen in Example 3.6 that this need not be true for an arbitrary two-dimensional Cohen–Macaulay local domain.

(b) In a two-dimensional regular local ring, does (2) imply (3)? Or equivalently, for any $M$-primary ideal $I$ in such a ring, must we have $e_2(I) = \lambda((I_{\{1\}})^n/I^n)$ for all large $n$? A positive answer would imply, by [Sy2, (2.6)], that (2) even implies (5). Thus, it would also imply a positive answer to (Q3) in Questions 6.2 below: the ideal $J$ would be $I_{\{1\}}$.

(c) Inspired by (1.14)(c) above: For a Ratliff–Rush $M$-primary ideal $I$ of a two-dimensional regular local ring $(R, M)$, can it happen that $R[It]$ is not Cohen–Macaulay, but the blowup of $I$ is Cohen–Macaulay? (Cf. Questions 6.2 (Q5) below.) If we do not assume the ideal $I$ is Ratliff–Rush, then ideals such as $I = (x^4, x^3y, xy^3, y^4)R$ show this can happen.

*Definition 3.8.* Let $R$ be a Noetherian domain of dimension $d$, $I$ be a nonzero proper ideal of $R$, $X$ be a complete model over $R$ that dominates the blowup of $I$ (so that the extension of $I$ to every ring in $X$ is principal) and $k \in \{1, \dots, d-1\}$. We will say that $X$ satisfies *condition* $E_k$ *for* $I$ if, in every ring in $X$, all the associated primes of the extension of $I$ have height at most $k$. If $(R, M)$ is local and $I$ is $M$-primary, then we simply say $X$ satisfies condition $\mathrm{E}_k$. This is unambiguous because, if $X$ also dominates the blowup of the $M$-primary ideal $J$, then for any prime $P$ in any ring $S$ of $X$, if $P$ is an associated prime of the principal ideal $IS$, then $P$ contracts in $R$ to $M$, so $JS$ is also a principal ideal contained in $P$, so $P$ is also an associated prime of $JS$ [Ng, (12.6)].

**Theorem 3.9.** *Let $(R, M)$ be a quasi-unmixed local domain of dimension $d$, with $R/M$ infinite, let $I$ be an $M$-primary ideal in $R$, and let $k \in \{1, \dots, d-1\}$. Then the following conditions are equivalent:*

*(i) All powers of $I$ are Ratliff–Rush ideals, and $\mathcal{B}(I)$ satisfies condition $E_k$.*

*(ii) All powers of $I$ are $e_k$-ideals.*

*Proof.* (i) $\Rightarrow$ (ii): By [Sh1, Main Theorem 4], it suffices to show that the associated primes of the zero ideal in $\mathrm{G}(I)$ all have height less than $k$. Since $\mathrm{G}(I)^+$ contains a nonzerodivisor, the homogeneous maximal ideal of $\mathrm{G}(I)$ is not an embedded component of 0, so we must show that none of the other primes of height at least $k$ are associated to 0. Since $\mathrm{G}(I)$ is the factor ring $R[It]/IR[It]$ of the (small) Rees ring $R[It]$, we must show that none of the primes in $R[It]$ of height strictly greater than $k$ are associated primes of $IR[It]$, and as noted above, the maximal homogeneous ideal is not associated to $IR[It]$. So consider a prime $Q$ in $R[It]$ that contains $IR[It]$, has height greater than $k$, and is not the homogeneous maximal ideal. Then $Q$ cannot contain all of the degree-one summand $It$ of $R[It]$, so suppose $at \notin Q$ for some $a$ in $I - 0$. The ring of fractions of $R[It]$ with respect to the powers of the element $at$ is a Laurent polynomial ring $R[I/a][s, s^{-1}]$ in the indeterminate $s = at$ over $R[I/a]$; and the contraction to $R[I/a]$ of the extension of $Q$ to this Laurent polynomial ring is a prime $P$ containing the principal ideal $IR[I/a]$. By the hypothesis on the blowup, the associated primes of $IR[I/a]_P$ have height at most $k$, so the same is true of $IR[I/a]_P[s, s^{-1}]$. Since the extension of $Q$ to the ring of fractions $R[I/a]_P[s, s^{-1}]$ of $R[It]$ has height greater than $k$, it is not an associated prime of this principal ideal, so $Q$ is not an associated prime of $IR[It]$.

(ii) $\Rightarrow$ (i): Since $(I^n)_{\{k\}} = I^n$ for all $n$, all powers of $I$ are Ratliff–Rush. Again, by [Sh1, Main Theorem 4], no prime of $\mathrm{G}(I)$ of height at least $k$ is associated to 0; so $IR[It]$ has no associated primes of height greater than $k$. By reversing the argument in the first part of the proof, we see that for any nonzero element $a$ of $I$ and any prime $P$ in $R[I/a]$, the associated primes of $IR[I/a]_P$ all have height at most $k$. $\square$

Since all sufficiently high powers of an ideal are Ratliff–Rush, and since an ideal and its powers have the same blowup, it follows that:

**Corollary 3.10.** *Let $(R, M)$ be a quasi-unmixed local domain of dimension $d$ with $R/M$ infinite, let $I$ be an $M$-primary ideal in $R$, and let $k \in \{1, \ldots, d-1\}$. Then $\mathcal{B}(I)$ satisfies condition $E_k$ iff all sufficiently high powers of $I$ are $e_k$-ideals.* $\square$

Suppose $(R, M)$ is a local domain. Condition $\mathrm{E}_1$ (for an $M$-primary ideal) on a model over $R$ is equivalent to the condition that all the local rings in the model that dominate $R$ satisfy Serre's condition $\mathrm{S}_2$; and if $R$ is of dimension two, these conditions are equivalent to the condition that the rings in the model (which have dimension at most two) are Cohen–Macaulay, i.e., that the model is Cohen–Macaulay. Thus, special cases of Theorem 3.9 and Corollary 3.10 respectively are:

**Corollary 3.11.** *Let $(R, M)$ be a two-dimensional quasi-unmixed local domain with $R/M$ infinite, and let $I$ be an $M$-primary ideal. Then the following conditions are equivalent:*

*(i) All powers of $I$ are Ratliff–Rush, and $\mathcal{B}(I)$ is Cohen–Macaulay (i.e., all the rings in $\mathcal{B}(I)$ are Cohen–Macaulay).*

*(ii) All powers of $I$ are $e_1$-ideals.* $\square$

**Corollary 3.12.** *If $I$ is an $M$-primary ideal in a two-dimensional quasi-unmixed local domain $(R, M)$ with $R/M$ infinite, then the blowup $\mathcal{B}(I)$ is Cohen–Macaulay if and only if all sufficiently high powers of $I$ are $e_1$-ideals.* $\square$

(3.13) The hypothesis in Corollary 3.12 that all sufficiently high powers of $I$ are $e_1$-ideals cannot be replaced by the assumption that $I$ is an $e_1$-ideal. For example, the subring $k[x^3y, x^2y, xy^2, y]$

of the polynomial ring $k[x, y]$, when localized at the maximal ideal generated by $\{x^3y, x^2y, xy^2, y\}$, gives a two-dimensional local domain $(R, M)$ for which the blowup $\mathcal{B}(M)$ is not Cohen–Macaulay, but the maximal ideal $M$ is integrally closed and hence an $e_1$-ideal.

For a Noetherian domain $R$, we write, as usual,

$$R^{(1)} = \bigcap \{R_P : P \in \operatorname{Spec}(R) \text{ and } \operatorname{ht}(P) = 1\};$$

and if $X$ is a model over $R$, then we denote by $X^{(1)}$ the set of localizations at maximal ideals of the rings $S^{(1)}$ as $S$ varies over $X$. If we assume that $R$ has the property that every height-one prime $P'$ of the integral closure $R'$ of $R$ is such that $P' \cap R$ is a height-one prime of $R$, then $R^{(1)} \subseteq R'$. Thus, for example, if $R$ is universally catenary, or, equivalently, if the dimension formula holds for $R$, then $R^{(1)} \subseteq R'$. Since $R^{(1)}$ is an intersection of its localizations at height-one primes, the principal ideals in $R^{(1)}$ do not have embedded primes, i.e., $R^{(1)}$ satisfies Serre's condition $\mathrm{S}_2$. In particular, if $R$ has dimension two, then $R^{(1)}$ is Cohen–Macaulay, and if $R^{(1)} \subseteq R'$, then it is the smallest subring of $R'$ containing $R$ that is so.

As we saw in Theorem 3.9 and Corollary 3.10, it will be convenient to have an extension of the familiar concept $R^{(1)}$ to primes of greater height:

*Notation 3.14.* Let $R$ be a Noetherian domain, $I$ be a nonzero ideal in $R$, and $k$ be a positive integer. If $I$ is principal, we denote by $R^{(k,I)}$ the intersection of the localizations $R_P$ of $R$ at primes $P$ for which either $I \not\subseteq P$ or $\operatorname{ht}(P) \leq k$. For any $I$ and a model $X$ over $R$ that dominates $\mathcal{B}(I)$, we denote by $X^{(k,I)}$ the collection of localizations of the rings $S^{(k,IS)}$ at their maximal ideals, as $S$ varies over $X$.

Suppose that $I$ is principal. The family of localizations in the description of $R^{(k,I)}$ above need not be locally finite, i.e., a nonzero element of $R$ may be a nonunit in infinitely many of the $R_P$. But $R^{(k,I)}$ can also be represented as an irredundant locally finite intersection of $R_P$'s, as follows. Recall [K, Theorem 53] that $R$ can be represented as the intersection of its localizations $R_P$, where $P$ varies over all primes maximal with respect to the property of being associated to a principal ideal. Thus, in the definition of $R^{(k,I)}$, it is harmless to discard those localizations $R_P$ in which $P$ is not maximal among the primes that are associated to principal ideals. If, however, $P$ is an associated prime of any principal ideal and $I \subseteq P$, then $P$ is also an associated prime of $I$. Hence $R^{(k,I)}$ is exactly the intersection of the localizations $R_P$ as $P$ varies over the associated primes of principal ideals of $R$ *omitting the associated primes of $I$ having height greater than $k$.*

Similarly $R^{(k,I)}$ has a representation as a finite intersection of rings: If $a$ is a generator of $I$, then $R^{(k,I)} = R[1/a] \cap \bigcap_P R_P$, where the intersection is taken as $P$ varies over the maximal associated primes of $I$ of height at most $k$. Since multiplication by $a$ distributes over this intersection, we can see that the associated primes of $IR^{(k,I)}$ are precisely the primes $R^{(k,I)} \cap PR_P$ as $P$ varies over the associated primes of $I$ of height at most $k$. Applying this to the case of a general $I$ and a model $X$ dominating $\mathcal{B}(I)$, we see that the following conditions on a local ring $(S, N)$ are equivalent: (1) $S \in X^{(k,I)}$ and $N$ is an associated prime of $IS$. (2) $S \in X$ and $N$ is an associated prime of $IS$ of height at most $k$. In particular, the contraction $\bigcap\{IS \cap R : S \in \mathcal{B}(I)^{(k,I)}\}$ to $R$ of the extension of $I$ to $\mathcal{B}(I)^{(k,I)}$ is $\bigcap\{IS \cap R : (S,N) \in \mathcal{B}(I), N \in \operatorname{Ass}(S/IS), \operatorname{ht}(N) \leq k\}$. Thus, if all of the local rings $(S, N)$ in $\mathcal{B}(I)$ for which $N$ is an associated prime of $IS$ are localizations of the same "affine piece" $R[I/a]$ of $\mathcal{B}(I)$, then $\widetilde{I} = \bigcap\{IS \cap R : S \in \mathcal{B}(I)\} = IR[I/a] \cap R$ and $\bigcap\{IS \cap R : S \in \mathcal{B}(I)^{(k,I)}\} = IR[I/a]^{(k,I)} \cap R$.

**Lemma 3.15.** *If a Noetherian domain $R$ has the property that, for each prime ideal $P'$ in the integral closure $R'$ of $R$, $\operatorname{ht}(P' \cap R) = \operatorname{ht}(P')$, then for any principal ideal $I$ of $R$, $R^{(k,I)}$ is the smallest $R$-subalgebra $S$ of $R'$ such that all of the associated primes of $IS$ have height at most $k$.*

*Remark.* The hypothesis that contraction preserves heights is satisfied, for example, if $R$ is universally catenary; in particular, if $R$ is local and quasi-unmixed.

*Proof.* The hypothesis that contraction preserves heights assures that $R^{(k,I)} \subseteq R^{(1)} \subseteq R'$. The principal ideal $IR^{(k,I)}$ is the intersection of its extensions to the local rings in the locally finite intersection described in the paragraph following (3.14), so each of its primary components must survive in at least one of these local rings. If the primary component $\mathbf{q}$ of $IR^{(k,I)}$ does not extend to the unit ideal in one of these local rings $R_P$, then $I$ is contained in $P$, so $P$ is an associated prime of $I$, so $P$ has height at most $k$, so the same is true of the radical of $\mathbf{q}$.Thus, none of the associated primes of $IR^{(k,I)}$ can have height greater than $k$. On the other hand, if $S$ is an $R$-subalgebra of $R'$ in which all of the associated primes of $IS$ have height at most $k$, then $S^{(k,IS)}$ is the intersection of all the localizations of $S$ at associated primes of principal ideals, so it is just $S$, so $R^{(k,I)} \subseteq S$. $\square$

**Corollary 3.16.** *Let $I$ be a nonzero proper ideal in a universally catenary Noetherian domain. Then $\mathcal{B}(I)^{(k,I)}$ is the unique model dominating $\mathcal{B}(I)$ and dominated by the normalization $\mathcal{B}(I)'$ that is minimal (with respect to domination) among the models that satisfy condition $\mathrm{E}_k$ for $I$.* $\square$

As in the terminology "condition $\mathrm{E}_k$", if we restrict our attention to a local domain and ideals primary for the maximal ideal, then we may safely omit the mention of the ideal $I$ in the superscript of $\mathcal{B}(I)^{(k,I)}$. In the sequel, we will simply write $\mathcal{B}(I)^{(k)}$ in this context.

**Theorem 3.17.** *Let $(R, M)$ be a $d$-dimensional, quasi-unmixed, analytically unramified local domain with $R/M$ infinite, let $I$ be an $M$-primary ideal, and let $k \in \{1, \ldots, d\}$. Then for each positive* *integer $n$, $(I^n)_{\{k\}}$ is the contraction of $I^n$ from $\mathcal{B}(I)^{(k)}$, i.e., $(I^n)_{\{k\}} = \bigcap\{I^nS \cap R : S \in \mathcal{B}(I)^{(k)}\}$. Moreover, for all sufficiently large integers $n$, the blowup of $(I^n)_{\{k\}}$ is $\mathcal{B}(I)^{(k)}$, and $(I^n)_{\{k\}}$ has the property that all its powers are $e_k$-ideals.*

*Proof.* The fiber ring $R[It]/MR[It] = \bigoplus_{n=0}^{\infty} I^n/MI^n$ of $I$ is generated as an algebra over its degree-0 piece, the field $R/M$, by its degree-1 piece $I/MI$; and the images in $R[It]/MR[It]$ of the degree-1 pieces of the associated primes in $R[It]$ of $IR[It]$, not including the homogeneous maximal ideal, are proper subspaces of the degree-1 piece of the fiber ring. Since $R/M$ is infinite, by arguing as in the proof of Noether normalization [ZS, Theorem 25, page 200], we can find elements $a_1, \dots, a_d$ of $I$ such that their images in $I/MI$ are not in any of these proper subspaces and generate an $R/M$-algebra over which the fiber ring is integral. This means that (1) the ideal $(a_1, \dots, a_d)R$ is a reduction of $I$, so that $\mathcal{B}(I)$ is the union of the spectra of the rings $R[I/a_i]$, and (2) for each $i = 1, \dots, d$, all the local rings $(S, N)$ in $\mathcal{B}(I)$ such that $N$ is an associated prime of $IS$ are in the affine piece $\mathrm{Spec}(R[I/a_i])$ of this covering of $\mathcal{B}(I)$. (Throughout the rest of the proof, we have suppressed the mention of the extension of $I$ in the superscript of $R[I/a_i]^{(k)}$, etc.) Note that, for each $i = 1, \dots, d$,

$$R[I/a_i]^{(k)} \subseteq \bigcap\{R[I/a]_P : P \in \mathrm{Spec}(R[I/a_i]) \text{ and } a_i \notin P\} = R[1/a_i] \quad .$$

For each positive integer $n$, let $J_n = \bigcap\{I^nS \cap R : S \in \mathcal{B}(I)^{(k)}\}$. Since all the powers of the extension of $I$ to one of the elements $S$ of $\mathcal{B}(I)^{(k)}$ are principal, the maximal ideal of $S$ is associated to one of the powers of $I$ iff it is associated to all; so in view of the discussion following (3.14), our choice of $a_1, \dots, a_d$ guarantees that, for each $i = 1, \dots, d$, $J_n = I^nR[I/a_i]^{(k)} \cap R$.

Indeed, for each $i = 1, \dots, d$, $R[I/a_i]^{(k)} = \bigcup_{n \geq 1}(J_n/a_i^n)$: It is clear that each $J_n/a_i^n$ is contained in $R[I/a_i]^{(k)}$; and for an element $z$ of $R[I/a_i]^{(k)}$, since $R[I/a_i]^{(k)} \subseteq R[1/a_i]$, for some $n$ we have $a_i^n z \in R \cap I^nR[I/a_i]^{(k)} = J_n$. Since $R$ is analytically unramified, $R[I/a_i]'$ is a finitely generated $R[I/a_i]$-module [Re, Theorem 1.5, page 27]. Since $R$ is quasi-unmixed, $R[I/a_i]^{(k)} \subseteq R[I/a_i]'$ by Lemma 3.15, so $R[I/a_i]^{(k)}$ is a finitely generated $R[I/a_i]$-module. If $m \leq n$, then $(J_m/a_i^m) \subseteq (J_n/a_i^n)$; so for some $m_i$ we have $R[I/a_i]^{(k)} = R[J_n/a_i^n]$ for all $n \geq m_i$. Thus, if $m = \max\{m_1, \dots, m_d\}$, then $R[I/a_i]^{(k)} = R[J_n/a_i^n]$ for all $n \geq m$. Note that we also have $I^n \subseteq J_n \subseteq (I^n)'$ for each positive integer $n$.

For all $n$ and for each $S$ in $\mathcal{B}(I)^{(k)}$, $J_nS = I^nS$ is principal, so $\mathcal{B}(J_n) \preceq \mathcal{B}(I)^{(k)}$. Now suppose $n \geq m$; then for all $i = 1, \dots, d$, we have $R[I/a_i]^{(k)} = R[J_n/a_i^n]$, so $\mathcal{B}(I)^{(k)} = \bigcup_{i=1}^{d} \mathrm{Spec}(R[I/a_i]^{(k)}) = \bigcup_{i=1}^{d} \mathrm{Spec}(R[J_n/a_i^n]) = \mathcal{B}(J_n)$, the last equality because $(a_1^n, \dots, a_d^n)R$ is also a reduction of $J_n$. Since $\mathcal{B}(J_m)$ satisfies condition $\mathrm{E}_k$, all sufficiently high powers of $J_m$ are $e_k$-ideals by Corollary 3.10.

We next observe that there exist arbitrarily large positive integers $n$ such that $\mathcal{B}(I)^{(k)} \preceq \mathcal{B}((I^n)_{\{k\}})$: Let $J$ be an ideal containing and integral over a power $I^e$ of $I$. Then $J$ is integral over $(a_1^e, \dots, a_d^e)R$, and since $R[I^e/a_i^e] = R[I/a_i]$, we have $R[I/a_i] \subseteq R[J/a_i^e] \subseteq R[I/a_i]'$. Again by [Re, Theorem 1.5, page 27], $R[I/a_i]'$ is a finitely generated $R[I/a_i]$-module, so the family of rings between $R[I/a_i]$ and $R[I/a_i]'$ satisfy the ascending chain condition (with respect to inclusion). Thus the blowups of ideals integral over powers of $I$ satisfy the ascending chain condition with respect to domination. Now if $\mathcal{B}(I_{\{k\}})$ does not dominate $\mathcal{B}(I)^{(k)}$, then by Corollary 3.16 $\mathcal{B}(I_{\{k\}})$ does not satisfy condition $\mathrm{E}_k$, so by Corollary 3.10 some power $(I_{\{k\}})^{n_1}$ (in fact, an arbitrarily high power) of $I_{\{k\}}$ is not an $e_k$-ideal. By Proposition 3.2(iii), $((I_{\{k\}})^{n_1})_{\{k\}} = (I^{n_1})_{\{k\}}$. If $\mathcal{B}((I^{n_1})_{\{k\}})$ does not satisfy condition $\mathrm{E}_k$, then some power $((I^{n_1})_{\{k\}})^{n_2}$ of $(I^{n_1})_{\{k\}}$ is not an $e_k$-ideal, and we continue by considering the blowup of $(((I^{n_1})_{\{k\}})^{n_2})_{\{k\}} = (I^{n_1 n_2})_{\{k\}}$. By the ascending chain condition, for some finite $n = n_1 n_2 \cdots n_t$, $\mathcal{B}((I^n)_{\{k\}})$ satisfies condition $\mathrm{E}_k$. By Corollary 3.16, $\mathcal{B}(I)^{(k)} \preceq \mathcal{B}((I^n)_{\{k\}})$. By increasing, if necessary, the $m$ chosen earlier in the proof, we may assume that $((I^n)_{\{k\}})^m$ is an $e_k$-ideal, so that $\mathcal{B}((I^{mn})_{\{k\}}) = \mathcal{B}((((I^n)_{\{k\}})^m)_{\{k\}}) = \mathcal{B}(((I^n)_{\{k\}})^m) = \mathcal{B}((I^n)_{\{k\}})$.

Now we show that $(J_m)^n \subseteq (I^{mn})_{\{k\}}$ for the $m$ and $n$ chosen earlier: Since both $(J_m)^n$ and $(I^{mn})_{\{k\}}$ contain and are integral over $I^{mn}$, their associated Ratliff–Rush ideals are the contractions to $R$ of the extensions of $I^{mn}$ to their respective blowups, and $(I^{mn})_{\{k\}}$, as an $e_k$-ideal, is itself Ratliff–Rush. Since $\mathcal{B}((J_m)^n) = \mathcal{B}(J_m) = \mathcal{B}(I)^{(k)} \preceq \mathcal{B}((I^n)_{\{k\}}) = \mathcal{B}((I^{mn})_{\{k\}})$, it follows that $(J_m)^n \subseteq (I^{mn})_{\{k\}}$. Therefore, by Proposition 3.2(ii), $(J_m)^i \subseteq (I^{mi})_{\{k\}}$ for every positive integer $i$. With $h$ chosen so that $(J_m)^i$ is an $e_k$-ideal for all $i \geq h$ (using Corollary 3.10), it follows from Proposition 3.2(i) that $(J_m)^i = (I^{mi})_{\{k\}}$.

We now show that $(I^n)_{\{k\}} = \bigcap\{IS \cap R : S \in \mathcal{B}(I)^{(k)}\} = J_n$ for any positive integer $n$. Take the positive integer $h$ sufficiently large that $(J_m)^i = (I^{mi})_{\{k\}}$ for all $i \geq h$. Then since $J_n = \bigcap\{I^n S \cap R : S \in \mathcal{B}(J_m)\}$,

$$
\begin{aligned}
(J_n)^{mh} &\subseteq J_{nmh} \\
&= \bigcap\{I^{nmh} S \cap R : S \in \mathcal{B}(J_m)\} \\
&= \bigcap\{(J_m)^{nh} S \cap R : S \in \mathcal{B}(J_m)\} \\
&= \bigcap\{(J_m)^{nh} S \cap R : S \in \mathcal{B}((J_m)^{nh})\} \\
&= \widetilde{(J_m)^{nh}} = ((I^n)^{mh})_{\{k\}} \quad ,
\end{aligned}
$$

so $J_n \subseteq (I^n)_{\{k\}}$ by Proposition 3.2(ii). Conversely, since $I^{nmh} \subseteq ((I^n)_{\{k\}})^{mh} \subseteq (I^{nmh})_{\{k\}} = (J_m)^{nh}$ by Proposition 3.2, $(I^n)_{\{k\}} S$ is principal generated by an element of $I^n$ for each $S$ in

$\mathcal{B}(J_m) = \mathcal{B}(I)^{(k)}$, so $(I^n)_{\{k\}} \subseteq J_n$ also.

It remains to show that, for sufficiently large $n$, $((I^n)_{\{k\}})^i = (J_n)^i$ is an $e_k$-ideal for all $i$. Since $(((I^n)_{\{k\}})^i)_{\{k\}} = (I^{ni})_{\{k\}} = J_{ni}$, this is equivalent to showing that $(J_n)^i = J_{ni}$. We consider the Rees-like ring $S = R \oplus J_1 t \oplus J_2 t^2 \oplus \dots$, which contains $R[It]$ and is contained in the integral closure $R[It]' = R' \oplus I' t \oplus (I^2)' t^2 \oplus \dots$, where $(I^j)'$ means the integral closure of $I^j$ in $R'$. We want to show first that $R[It]'$ and hence $S$ are finitely generated $R[It]$-modules, and since $R'$ is finitely generated over $R$, it is safe to assume that $R' = R$. Then by [Re, proof of Theorem 1.4, page 28] there is a positive integer $s$ for which $(I^{s+j})' \subseteq I^j$ for all $j$, so $I^s R[It]$ is a nonzero conductor ideal of $R[It]'$ into $R[It]$; so $R[It]'$ and $S$ are indeed finitely generated over $R[It]$. Now take a finite set of homogeneous elements of $S$ that generate $S$ over $R[It]$ and let $n$ be any integer greater than all the degrees of these generators. Then for all positive integers $i$, $J_{ni} = J_n I^{n(i-1)} \subseteq J_n (J_n)^{i-1} = (J_n)^i \subseteq J_{ni}$; so the proof is complete. $\square$

It follows that, in the analytically unramified case, we can strengthen Corollaries 3.10 and 3.12:

**Corollary 3.18.** *Let $R, M, I, k$ be as in Theorem 3.17. If the set of positive integers $n$ for which $I^n$ is an $e_k$-ideal is infinite, then $\mathcal{B}(I)$ satisfies condition $E_k$ and all sufficiently high powers of $I$ are $e_k$-ideals (and of course conversely).*

*Proof.* Let $m$ be a positive integer for which, for all $n \geq m$, the blowup of $(I^n)_{\{k\}}$ is $\mathcal{B}(I)^{(k)}$, and choose an $n \geq m$ for which $I^n$ is an $e_k$-ideal. Then $\mathcal{B}(I) = \mathcal{B}(I^n) = \mathcal{B}((I^n)_{\{k\}}) = \mathcal{B}(I)^{(k)}$, so the result follows from Corollary 3.16. $\square$

**Corollary 3.19.** *Let $I$ be a Ratliff–Rush ideal in a $d$-dimensional, quasi-unmixed, analytically unramified, local domain with infinite residue field, and let $k \in \{1, \dots, d\}$. If $I^n$ is an $e_k$-ideal for sufficiently large positive integers $n$, then $I_{\{k\}} = I$.*

*Proof.* By Corollary 3.10, $\mathcal{B}(I)$ satisfies condition $\mathrm{E}_k$, so $\mathcal{B}(I) = \mathcal{B}(I)^{(k)}$, so by Theorem 3.17, $I_{\{k\}} = \widetilde{I} = I$, using [HLS, Fact 2.1]. $\square$

**Corollary 3.20.** *With notation as in Theorem 3.17, if $J$ is an ideal containing and integral over $I$, then $\mathcal{B}(I)^{(k)} \preceq \mathcal{B}(J)^{(k)}$, and hence $I_{\{k\}} \subseteq J_{\{k\}}$.*

*Proof.* Apply Proposition 1.13, Corollary 3.16, and Theorem 3.17. $\square$

In view of Theorem 3.17, it seems natural to extend the definition of the coefficient ideals to an ideal $I$ that is not necessarily $M$-primary in a local ring $(R, M)$, as follows:

*Definition 3.21.* Let $R$ be a Noetherian domain of dimension $d$, $I$ be a nonzero proper ideal in $R$, and $k \in \{1, \dots, d\}$. Then $I_{\{k\}}$ is the contraction to $R$ of the extension of $I$ to $\mathcal{B}(I)^{(k,I)}$, and if

$I_{\{k\}} = I$, then $I$ is an $e_k$-ideal or a $k$-th coefficient ideal.

Another consequence of Theorem 3.17 is as follows: Suppose we can find a Noetherian domain $S$ of dimension $d$ with an element $a$ for which $S^{(d,aS)} < S^{(d-1,aS)} < \cdots < S^{(1,aS)} < S'$, and realize $S$ as an affine piece of the blowup of an ideal $I$ in a quasi-unmixed, analytically irreducible local domain $R$ with infinite residue field. Then because all of the models $\mathcal{B}(I)^{(k,I)}$ are distinct, it will follow that for each $I^n$ for sufficiently large $n$, all of the coefficient ideals $(I^n)_{\{k\}}$ are distinct. For $d = 1$, essentially all that is required is a one-dimensional Noetherian domain that is not normal. We now construct such a domain $S$ of dimension $d > 1$:

*Example 3.22.* Let $F$ be an infinite field and let $x, y_2, \ldots, y_d$ be indeterminates over $F$. Consider the affine domain $S = F[x, \{xy_i, y_i^4, y_i^6\}_{i=2}^d]$. Then

$$S^{(1,xS)} = S^{(1)} = F[x, \{xy_i, y_i^2\}_{i=2}^d] < F[x, \{y_i\}_{i=2}^d] = S'.$$

To see that, for $k$ in $\{1, \ldots, d-1\}$, we have $S^{(k+1,xS)} < S^{(k,xS)}$, we note that the product of $x^{2k-2}$ and $y_i^2$ for $k$ distinct values of $i$ is an element of $S^{(k,xS)}$, since any prime $P$ in $S$ of height at most $k$ and containing $x$ does not contain $y_i^4, y_i^6$ for at least one of the $y_i$'s that appear in the product, so one of the factors $y_i^2$ is a unit in $S_P$, and the product of the remaining factors is an element of $S$. But this element is not in $S^{(k+1,xS)}$ because it is not in the localization of $S$ at the prime ideal of height $k+1$ generated by $x$ and $xy_i, y_i^4, y_i^6$ for the $y_i$'s that appear in the product.

The domain $S$ is an affine piece of the blowup of the ideal $I$ generated by $x^6, x^4(xy_i)^2, (xy_i)^6$, $i = 2, \ldots, 6$ in the polynomial ring $R = F[x, \{xy_i\}_{i=2}^d]$. Forming the rings of fractions of $R, S$ with respect to the complement in $R$ of the maximal ideal $M = (x, \{xy_i\}_{i=2}^d)R$ yields a regular local ring $R_M$, and $(R - M)^{-1}S$ is an affine piece of the blowup of $IR_M$ which retains the properties verified in the last paragraph.

**Proposition 3.23.** *Let $I$ be an $M$-primary ideal in a $d$-dimensional, quasi-unmixed, analytically unramified local domain $(R, M)$ with $R/M$ infinite and $k \in \{1, \ldots, d-1\}$. If $\mathcal{B}(I)^{(k)} = \mathcal{B}(I)'$, then for each positive integer $n$, $(I^n)_{\{k\}} = (I^n)'$. Conversely, if $R$ is normal and the set of positive integers $n$ for which $(I^n)_{\{k\}} = (I^n)'$ is infinite, then $\mathcal{B}(I)^{(k)} = \mathcal{B}(I)'$.*

*Proof.* By Proposition 1.12, $(I^n)'$ is the intersection of the contractions to $R$ of each of the extensions of $I^n$ to each of the local rings on the model $\mathcal{B}(I)'$. By Theorem 3.17, $(I^n)_{\{k\}}$ is the contraction of $I^n$ from $\mathcal{B}(I)^{(k)}$. Thus it is clear that the equality of these two models implies that of the corresponding ideals.

For the converse, we simply take a sufficiently high power of $I$ such that $\mathcal{B}(I)' = \mathcal{B}((I^n)')$, using Proposition 1.12, and such that $\mathcal{B}(I)^{(k)} = \mathcal{B}((I^n)_{\{k\}})$, using Theorem 3.17. $\square$

**4. Interlude: Computing the Hilbert polynomial in Cohen–Macaulay local rings of low dimension.**

It would be desirable to be able to compute the Hilbert polynomial of an $M$-primary ideal $I$ in a local ring $(R, M)$. For this, it would be sufficient to find an upper bound on its *postulation number*, i.e., the smallest integer $k$ such that $H_I(n) = P_I(n)$ for all $n > k$. For, if the ring is of dimension $d$, then any $d+1$ consecutive values of the Hilbert function $H_I$ determines a polynomial in $n$ of degree $d$; the $d+1$ values of $H_I$ just above our bound give the correct polynomial, i.e., the Hilbert polynomial $P_I$. The following results, shown to us by Craig Huneke, are what we need in dimensions one and two, since their hypotheses can be verified at one or a few values. We are grateful for his permission to include these results here.

**Proposition 4.1.** *(Huneke) Suppose $I$ is an $M$-primary ideal of a one-dimensional Cohen–Macaulay local ring $(R, M)$ and that $bR$ is a principal reduction of $I$. Then $e_0 = \lambda(R/bR)$ is the multiplicity of $I$. If $m$ is a positive integer such that $H_I(n) = e_0 n - g_1$ for $n = m+1, m+2$, then $e_0 n - g_1$ is the Hilbert polynomial of $I$, and $H_I(n) = e_0 n - g_1$ for all $n > m$, i.e., the postulation number of $I$ is at most $m$.*

*Proof.* Consider the map $I^n \to I^{n+1}$ given by multiplication by $b$. Since $b$ is a regular element, this induces an injection $I^n/b^nR \to I^{n+1}/b^{n+1}R$. Note that $H_I(n) = \lambda(R/I^n) = \lambda(R/b^nR) - \lambda(I^n/b^nR) = e_0 n - \lambda(I^n/b^nR)$, so the hypothesis implies that $\lambda(I^{m+1}/b^{m+1}R) = g_1 = \lambda(I^{m+2}/b^{m+2}R)$; hence, for $n = m+1$ this injection is an isomorphism. It follows that $I^{m+2} = bI^{m+1}$, and hence the map is an isomorphism, i.e., $H_I(n) = e_0 n - g_1$, for each positive integer $n > m$. $\square$

The proof of the next result is very close to that of [H1, Fundamental Lemma 2.4]. The difference is that that lemma uses a polynomial that is already known to be the Hilbert polynomial.

**Proposition 4.2.** *(Huneke) Suppose $I$ is an $M$-primary ideal of a two-dimensional Cohen–Macaulay local ring $(R, M)$ and that $(a, b)R$ is a minimal reduction of $I$. Let $m$ be a positive integer for which:*

*(a) $I^{m+2} = (a, b)I^{m+1}$ (i.e., $m+1$ is at least as large as the reduction number of $I$ with respect to $(a, b)R$), and*

*(b) $(I^{n+1} : (a, b)R) = I^n$ for all $n \geq m$.*

*Let $e_0 = \lambda(R/(a, b)R)$ (the multiplicity of $I$), and let $Q(n) = e_0\binom{n+1}{2} - f_1\binom{n}{1} + f_2$ be a polynomial for which $Q(n) = \lambda(R/I^n)$ for $n = m, m+1$. Then $Q$ is the Hilbert polynomial of $I$, and $Q(n) = \lambda(R/I^n)$ for all $n \geq m$, i.e., $m$ is greater than the postulation number of $I$.*

*Proof.* It is enough to show that $Q(m+2) = \lambda(R/I^{m+2})$. The sequence

$$0 \to R/I^m \xrightarrow{\psi} R/I^{m+1} \oplus R/I^{m+1} \xrightarrow{\varphi} (a, b)R/I^{m+2} \to 0 \quad ,$$

where $\psi(t+I^m)=(bt+I^{m+1},-at+I^{m+1})$ and $\varphi(r+I^{m+1},s+I^{m+1})=ar+bs+I^{m+2}$, is exact; and the second difference $Q(m+2)-2Q(m+1)+Q(m)$ of the polynomial $Q$ is its leading coefficient (when written in terms of binomial coefficients) $e_0$; so

$$\begin{aligned}\lambda(R/I^{m+2}) &= 2\lambda(R/I^{m+1})-\lambda(R/I^m)+\lambda(R/(a,b)R)\\ &= 2Q(m+1)-Q(m)+e_0\\ &= Q(m+2)\quad ,\end{aligned}$$

as required. □

It remains to find a computable way to be sure that $m$ satisfies condition (b) of Proposition 4.2. It is possible that, under the hypotheses of that result,

$$(I^n:(a,b)R)=I^{n-1}\qquad\text{but}\qquad(I^{n+1}:(a,b)R)\neq I^n\quad;$$

see, for example, Example 6.1 (E3) below. To find a computable $m$ for which condition (b) of that result holds, we can use the following result, since $(I^{n+1}:(a,b)R)\subseteq(I^{n+1}:a)$. Note that the hypothesis that $I^{m+2}=(a,b)I^{m+1}$ is just condition (a) of Proposition 4.2.

**Proposition 4.3.** *(Huneke) Let $(R,M)$ be a two-dimensional Cohen–Macaulay local ring, $I$ be an $M$-primary ideal, and $(a,b)R$ be a reduction of $I$. If $I^{m+2}=(a,b)I^{m+1}$ and $(I^{m+1}:a)=I^m$, then $(I^{n+1}:a)=I^n$ for all $n\geq m$.*

*Proof.* It suffices to show this for $n=m+1$, and clearly $I^{m+1}\subseteq(I^{m+2}:a)$. Suppose $au\in I^{m+2}$; then $a(u-r)\in bI^{m+1}$ for some $r$ in $I^{m+1}$, and since $a,b$ is a regular sequence, we get $u-r=bv$ for some $v$ in $R$. Since $abv=a(u-r)\in bI^{m+1}$ and $b$ is a nonzerodivisor, $v\in(I^{m+1}:a)=I^m$, and hence $u=bv+r\in I^{m+1}$. □

We conclude the section with a result providing a connection to the concept of Ratliff–Rush ideal:

**Proposition 4.4.** *Let $(R,M)$ be a two-dimensional Cohen-Macaulay local ring, let $I$ be an $M$-primary ideal, and let $P_I$ and $H_I$ denote the Hilbert polynomial and Hilbert function of $I$.*

*(i) If $P_I(1)=H_I(1)$ and $P_I(2)=H_I(2)$, then for any minimal reduction $\mathbf{q}$ of $I$, we have $e_2(I)=\lambda(I^2/\mathbf{q}I)$. In particular, if we also have $e_2(I)=0$, then $I$ is an $e_1$-ideal.*

*(ii) If for some positive integer $m$, we have that $I^n=\widetilde{I^n}$ for all $n\geq m$, and if $P_I(n)=H_I(n)$ for $n=m,m+1,m+2$, then the reduction number of $I$ is bounded by $m+1$.*

*Proof.* (i) Apply [H1, Fundamental Lemma 2.4] with $n=1$.

(ii) The hypothesis on $I$ implies that $(I^{m+1} : \mathbf{q}) = I^m$. Apply [H1, Fundamental Lemma 2.4] with $n = m + 1$. $\square$

(4.5) With the notation of Proposition 4.4, it can happen that $P_I(n) = H_I(n)$, for $n = 2, 3$ and yet the reduction number of $I$ is greater than 2, and $e_2(I) = 3$ while $\lambda(I^2/\mathbf{q}I) = 1 = \lambda(I^3/\mathbf{q}I^2)$. This is illustrated by the example $R = k[x, y]$ and $I = (x^6, x^4y, x^3y^4, xy^5, y^6)R$. In this case, $I$ and all its powers are Ratliff–Rush. See Examples 6.1, (E2).

**5. Coefficient ideals in a two-dimensional regular local ring.**

In the context of a two-dimensional regular local ring, there exists an extensive theory, due mainly to Zariski [ZS, Appendix 5] [H2], of complete (=integrally closed) ideals and the closely associated notion of contracted ideals. In this section we seek to investigate to what extent it may be possible to extend some of this theory to Ratliff–Rush ideals and coefficient ideals.

(5.1) Unlike the case of two complete ideals in a two-dimensional regular local ring, the product of two Ratliff–Rush ideals need not be Ratliff–Rush. In fact, neither the product of two parameter ideals, nor the product of a parameter ideal and a complete ideal, nor the intersection of parameter ideals, need be Ratliff–Rush. For example, if $x, y$ is a regular system of parameters in $R$, the ideal $I = (x^4, x^3y, xy^3, y^4)R$ is not Ratliff–Rush, but it can be expressed as $I = (x^3, y^3)R(x, y)R = (x^3, y^3)M = (x^3, y^3)R \cap (x, y^4)R \cap (x^4, y)R$.

Since $I = (x^3, y^3)M$, the product of an $e_1$-ideal and $M$, is not Ratliff–Rush, the standard tool in the Zariski theory of taking the transform with respect to the blowup of the maximal ideal cannot be immediately applied to Ratliff–Rush ideals or coefficient ideals.

Let $(R, M)$ be a two-dimensional regular local ring, and let $I$ be an $M$-primary ideal. Assume that $R/M$ is infinite. Recall that $I$ is said to be *contracted* (in the sense of [ZS, page 368]) iff it is the contraction of its extension to an affine piece of the blowup of $M$, i.e., $I = IR[M/x] \cap R$ for some $x$ in $M - M^2$, or equivalently [H2, Proposition 2.1, page 326] iff $(I : M) = (I : x)$. Such an ideal $I$ is contracted iff its minimal number of generators is one more than its order [H2, Proposition 2.3].

**Proposition 5.2.** *If $I$ is a contracted ideal in a two-dimensional regular local ring $(R, M)$, say from $S = R[M/x]$, then $\widetilde{I}$ and $I_{\{1\}}$ are also contracted from $S$.*

*Proof.* The powers of $I$ are also contracted from $S$ [H2, Proposition 2.6, page 328], so for sufficiently large $n$ we have

$$\widetilde{I} = (I^{n+1} :_R I^n) = (((IS)^{n+1} \cap R) :_R I^n) = (((IS)^{n+1} :_S (IS)^n) \cap R \quad ,$$

so $\widetilde{I}$ is contracted from $S$ also. Moreover, by [Sh2, Theorem 3], there is a positive integer $n$ and an element $a$ of $I^n$ for which $I_{\{1\}} = (I^n : a)$; since $I^n$ is contracted from $S$, so is $I_{\{1\}}$. $\square$

We now turn to consideration of some of the effect on coefficient ideals of the process of taking the transform of an $M$-primary ideal with respect to the blowup of the maximal ideal. Recall that if $I$ is an $M$-primary ideal in a two-dimensional regular local ring $(R, M)$, and if $(S, N)$ is any two-dimensional regular local ring on the blowup $\mathcal{B}(M)$ (in which case we say that $S$ is a *quadratic transform of* $R$), then the transform of $I$ in $S$ is obtained as follows: The extension $IS$ of $I$ to $S$ can be factored as $IS = x^r I^S$, where $x$ is a regular parameter in $R$, $r = \mathrm{ord}_R(I)$, and $I^S$, the *transform of* $I$ *in* $S$, is either $N$-primary or equal to $S$. More generally, any two-dimensional regular local ring $(S, N)$ having the same field of fractions as $R$ which contains $R$ must, by the Zariski–Abhyankar Factorization Theorem [A1,Theorem 3, p. 343], be obtainable from $R$ by a unique finite sequence of quadratic transforms, in which case we say that $S$ is an *iterated quadratic transform of* $R$. We can then iterate the process above to obtain a well-defined transform, $I^S$, of the ideal $I$ in any such $S$.

We now proceed to show that if $I$ is an $M$-primary ideal in a two-dimensional regular local ring $(R, M)$ such that either $I_{\{1\}} = I'$ or $\widetilde{I} = I'$, then for any two-dimensional regular local ring $(S, N)$ birationally dominating $R$, $I^S$ possesses the same property. It is interesting that such ideals are completely determined by the corresponding properties of their blowups.

**Theorem 5.3.** *If $I$ is an $M$-primary ideal in a two-dimensional regular local ring $(R, M)$, the following statements are equivalent:*

*(i)* $I' = I_{\{1\}}$*;*

*(ii)* $(I^m)' = (I^m)_{\{1\}}$ *for all positive integers $m$;*

*(iii)* $\mathcal{B}(I)' = \mathcal{B}(I)^{(1)}$.

*Proof.* For any $M$-primary ideal $I$ in any $d$-dimensional quasi-unmixed local ring $(R, M)$, an ideal $J$ containing $I$ is contained in $I_{\{d-1\}}$ if and only if $\lambda(J^n/I^n)$ is constant for all $n$ sufficiently large. In particular, $I' = I_{\{d-1\}}$ if and only if $\lambda((I')^n/I^n)$ is constant for all $n$ sufficiently large.

Assuming now that we are back in the two-dimensional regular local ring case, let $I^m$ be any fixed power of the $M$-primary ideal $I$. Suppose that (i) holds, i.e., that $\lambda((I')^n/I^n)$ is constant for all sufficiently large $n$, from which it follows that $\lambda((I')^{mn}/I^{mn})$ is constant for all large $n$. However, in this regular case, $I'$ is a normal ideal (i.e., all powers of $I'$ are integrally closed), so that $(I')^m = (I^m)'$ for all $m$. We conclude that $(I^m)' = (I^m)_{\{1\}}$. This shows (i)$\Rightarrow$(ii).

(ii)$\Rightarrow$(iii) and (iii)$\Rightarrow$(i) are immediate consequences of Proposition 3.23. $\square$

**Proposition 5.4.** *If an $M$-primary ideal $I$ in a two-dimensional regular local ring $(R, M)$ enjoys the properties of Theorem 5.3, then so does the ideal $MI$.*

*Proof.* It suffices to show that $(MI)' = (MI)_{\{1\}}$. We utilize the characterization given in the first paragraph of the proof of Theorem 5.3. Thus, we first fix $n$ sufficiently large so that $\lambda((I')^n/I^n)$ is a constant.

We claim that $\lambda(M(I')^n/MI^n) \leq \lambda((I')^n/I^n)$. To see this, it suffices to show that $\lambda(I^n/MI^n) \leq \lambda((I')^n/M(I')^n)$. However, in these latter two terms we are just considering the minimal numbers of generators $\mu(I^n), \mu((I')^n)$ of the corresponding ideals. From the general theory of complete ideals, we know that $\mu(I^n) \leq \operatorname{ord}(I^n) + 1$ and that $\mu((I')^n) = \operatorname{ord}((I')^n) + 1$, because $(I')^n$ is still integrally closed [H2]. But $\operatorname{ord}((I')^n) = \operatorname{ord}(I^n)$, which proves the claim.

Now we may repeat the argument of the above claim $n$ times, as multiplication of $M^j(I')^n$ by $M$ leaves the ideal complete. Putting all of the resulting inequalities together, we conclude that $\lambda((MI')^n)/(MI)^n) \leq \lambda((I')^n/I^n)$, which is constant. As the left side of this latter inequality is known to be polynomial in $n$ for large $n$, we conclude that it must actually be constant. □

**Corollary 5.5.** *If $I$ is an $M$-primary ideal in a two-dimensional regular local ring $(R, M)$ such that $I_{\{1\}} = I'$, and if $(S, N)$ is any two-dimensional regular local ring birationally dominating $R$, then $(I^S)_{\{1\}} = (I^S)'$.*

*Proof.* By the Zariski-Abhyankar Factorization Theorem, $S$ is an iterated quadratic transform of $R$, so it suffices to assume that in fact $S$ is a first quadratic transform of $R$, i.e., $S = T_Q$, where $T = R[M/x]$ and $Q$ is a height-two maximal ideal of $T$. By Theorem 5.3 we need to show that $\mathcal{B}(I^S)^{(1)} = \mathcal{B}(I^S)'$. To see this, it suffices to show the more general statement that, if $X$ is the complete model over $R$ obtained by blowing up the model $\mathcal{B}(M)$ at the sheaf of ideals $I\mathcal{B}(M)$, then $X^{(1)} = X'$. By Lemma 1.11, this model $X$ is $\mathcal{B}(MI)$. That this model has the requisite property is then the statement of Proposition 5.4. □

**Theorem 5.6.** *If $(R, M)$ is a two-dimensional regular local ring and $I$ is an $M$-primary ideal in $R$, the following are equivalent:*

*(i)* $\widetilde{I} = I'$;

*(ii)* $\widetilde{I^m} = (I^m)'$ *for all positive $m$;*

*(iii)* $\mathcal{B}(I) = \mathcal{B}(I)'$.

*Proof.* Clearly (ii) $\Longrightarrow$ (i), and if (i) holds, then for all large $n$ we have $I^n = (I')^n$, so for each positive $m$ we have $I^{mn} = (I')^{mn}$ for all large $n$, and (ii) holds. And (ii) $\Longleftrightarrow$ (iii) follows immediately from Proposition 1.12 and [HLS, Fact 2.1]. □

**Proposition 5.7.** *If an $M$-primary ideal $I$ in a two-dimensional regular local ring enjoys the properties stated in Theorem 5.6, then so does the ideal $MI$.*

*Proof.* We are given that $I^n = (I')^n$ for all large $n$, i.e., that $I^n$ is integrally closed for large $n$. We must then see that $(MI)^n = ((MI)')^n$ for all large $n$. But $((MI)')^n = M^n(I')^n = (MI)^n$ for large $n$, using our hypothesis, the fact that $M$ is integrally closed, and the fact that in this ring, the product of integrally closed ideals is integrally closed. □

With these results in hand, there is now no trouble repeating the argument of Corollary 5.5 in order to obtain the analogous statement for the Ratliff-Rush ideal associated to $I$.

**Corollary 5.8.** *If $I$ is an $M$-primary ideal in a two-dimensional regular local ring $(R, M)$ such that $\widetilde{I} = I'$ and if $(S, N)$ is any two-dimensional regular local ring birationally dominating $R$, then $\widetilde{(I^S)} = (I^S)'$.* □

It remains unclear whether the corresponding relation holds between $\widetilde{I}$ and $I_{\{1\}}$, so we state this as a formal question.

*Questions 5.9.*

(Q1) If an $M$-primary ideal $I$ in a two-dimensional regular local ring $(R, M)$ has the property that $\widetilde{I} = I_{\{1\}}$, does the transform, $I^S$, of $I$ in a quadratic transform, $S$, of $R$ again have this property?

(Q2) Does $\widetilde{I} = I_{\{1\}}$ imply the same property for $MI$?

By using the Hoskin-Deligne Formula for the length of $R/I$, where $I$ is complete, we are able to produce an explicit formula for $e_1(I)$, in the special case that $I' = I_{\{1\}}$, in terms of infinitely near points (=two-dimensional regular local rings birationally dominating $R$).

**Proposition 5.10.** *If $I$ is an $M$-primary ideal in a two-dimensional regular local ring $(R, M)$ having infinite residue field such that $I_{\{1\}} = I'$, then*

$$e_1(I) = \sum [S/N : R/M] \binom{r_S}{2} .$$

*Here, $(S, N)$ varies over all regular local rings which birationally dominate $R$, $r_S$ is the order of the transform $I^S$ in the $N$-adic valuation, and $[S/N : R/M]$ is the algebraic residue field extension degree.*

*Proof.* Every integrally closed $M$-primary ideal in $R$ has reduction number equal to one [H2, Theorem 5.1], and so $e_1(I) = e_1(I') = e_0(I') - \lambda(R/I')$ [H1, Theorem 2.1]. However,

$$e_0(I') = \sum [S/N : R/M] (r_S)^2$$

and
$$\lambda(R/I') = \sum [S/N : R/M] \binom{r_S + 1}{2} ,$$
where $S$ varies over all two-dimensional regular local rings birationally dominating $R$. These are the "Multiplicity Formula" [JV, Theorem 3.7] and the "Hoskin–Deligne Formula" [JV, Theorem 3.11], [L, Theorem (3.1)] respectively. (We are using here the fact that $(I^S)' = (I')^S$, so that $r_S(I) = r_S(I')$, for all $S$.) Subtracting now yields the statement. $\square$

Suppose $I$ is an $M$-primary ideal of a two-dimensional regular local ring $(R, M)$. Since the powers of a complete ideal are again complete, the blowup $\mathcal{B}(I)$ of $I$ is normal iff the contraction $\widetilde{I}$ of $I$ extended to $\mathcal{B}(I)$ is a complete ideal. We have also seen, in Corollary 3.12, that $\mathcal{B}(I)$ is Cohen-Macaulay if and only if all sufficiently high powers of $I$ are $e_1$-ideals. When is $\mathcal{B}(I)$ nonsingular? That $\mathcal{B}(I)$ is nonsingular certainly implies that $\widetilde{I}$ is complete, and necessary and sufficient conditions in order that $\mathcal{B}(I)$ be nonsingular can then be read off from the factorization of $\widetilde{I}$ as a product of simple complete ideals [ZS, Appendix 5]. For ease of expression, we introduce some new terminology.

*Definition 5.11.* The complete $M$-primary ideal $I$ in the two-dimensional regular local ring $(R, M)$ has a *saturated factorization* iff in the unique expression for $I$ as a product of simple complete ideals the following condition holds: Whenever $J$ is a simple complete ideal which is a factor of $I$ having $V$ as its corresponding prime divisor (of the second kind), and whenever $K$ is any simple complete $V$-ideal containing $J$, then $K$ must also be a simple complete factor of $I$.

**Proposition 5.12.** *Let $I$ be an $M$-primary ideal in a two-dimensional regular local ring $(R, M)$. Then the blowup of $I$, $\mathcal{B}(I)$, is nonsingular if and only if $\widetilde{I} = I'$ has a saturated factorization.*

*Proof.* There are essentially two points to consider here. The first is that the set of *Rees valuations* of $I$, i.e., the set of prime divisors of $R$ obtained by localizing $\mathcal{B}(I)$ at the finite set of height one primes minimal over $M\mathcal{B}(I)$, is exactly the set of prime divisors assigned by the Zariski theory to each of the simple complete ideals in the factorization of $I'$. The other is that if the factorization of $I'$ into simple complete ideals is given by $I' = I_1^{e_1} \cdots I_m^{e_m}$, then $\mathcal{B}(I) = \mathcal{B}(I') = \mathcal{B}(I_1 \cdots I_m)$. This latter fact follows from the characterization of the blowup as the minimal complete model in which the given ideal is principal, and from the characterization of the join of two blowups as the blowup of the product, as in Lemma 1.11.

Now, if $\mathcal{B}(I)$ is nonsingular, it is a consequence of the Zariski–Abhyankar factorization theorem [A1, Theorem 3, page 343] that the domination map from $\mathcal{B}(I)$ to $\operatorname{Spec} R$ factors through a finite sequence of blowups of maximal ideals of simple points. Thus the set of Rees valuations must be precisely the set of prime divisors of the second kind on $R$ which "come out", i.e., which correspond to the order valuation given by the powers of one of these maximal ideals. But it is known in the general theory that if one makes a succession of quadratic transforms of a two-dimensional regular local ring along a prime divisor $V$ associated to the simple complete ideal $J$, then the prime divisors which come out are precisely those associated to the simple $V$-ideals containing $J$ [ZS, (F) on page 392]. Therefore the factorization of $I'$ must be saturated.

Conversely, if we assume that the factorization of $\widetilde{I} = I'$ into a product of simple complete ideals is saturated, then in particular $M$ must occur in the factorization. Then by using Lemma 1.11 we may conclude that $\mathcal{B}(I)$ may be obtained by first blowing up $M$ and then further blowing up the ideal-sheaf $I\mathcal{B}(M)$. However, if $I' = M^e J$ is the factorization of $I'$, where $J$ is again complete and $M$ does not divide $J$, then the blowup of $I\mathcal{B}(M)$ is the same thing as the blowup of $J\mathcal{B}(M)$. From the Zariski theory, we know that the transform process preserves products and preserves completion. We also know that the proper transform of $J$ to $\mathcal{B}(M)$ has finite support (cf. [L]), i.e., is not the whole ring $S$ for only finitely many two-dimensional regular local rings $S$ on $\mathcal{B}(M)$. Thus we may repeat the above argument on $JS$ for each of the finitely many $S$ in which the transform of $J$ is not all of $S$. By induction on the number of simple complete factors, we conclude that $\mathcal{B}(I)$ is obtainable via a finite process of blowings up of nonsingular points, and so must be nonsingular. $\square$

## 6. Examples and questions.

In this section, unless otherwise stated, $R$ will denote either the polynomial ring $k[x, y]$ over a field $k$ or a two-dimensional regular local ring with regular system of parameters $x, y$; $M$ will denote $(x, y)R$, and $I$ will be an $M$-primary ideal.

*Examples 6.1.*

(E1) It can happen that $P_I(n) = H_I(n)$ for all $n \geq 1$ and yet $I < \widetilde{I}$. An example given by Sally in [Sy2, Section 5] illustrates this: Let $R = k[x, y]$ and $I = (x^6, x^4y, xy^5, y^6)R$. Then $x^3y^4 \in (I^2 : I) - I$, so $I < \widetilde{I}$; but Tom Marley has pointed out to us using MACAULAY that $P_I(n) = H_I(n)$ for $1 \leq n \leq 8$. We have checked that $\mathbf{q} = (x^4y, x^6 + y^6)R$ is a minimal reduction of $I$ with reduction number 3, i.e., $\mathbf{q}I^3 = I^4$ while $\mathbf{q}I^2 < I^3$. Since $(I^3 : x^4y) = I^2$, Proposition 4.3 shows that $(I^{n+1} : \mathbf{q}) = I^n$, for all $n \geq 2$. Using Proposition 4.1 in the ring $R/x^4yR$, we found that the Hilbert polynomial of the image of $I$ is $30n - 10$ with postulation number 2. Therefore, we see that the postulation number of $I$ is at most two. Since $P_I(n) = H_I(n)$ for $n = 1, 2$, we see that the postulation number of $I$ is in fact less than or equal to 0. Since the Hilbert polynomial for $I$ is

$$P_I(n) = 30\binom{n+1}{2} - 10\binom{n}{1} + 3 \ ,$$

we see that the postulation number $n(I)$ of $I$ is equal to 0. Therefore the reduction number $r(I)$ is independent of the minimal reduction chosen for $I$ [W, Theorem 3.3]. Note that $r(I) = n(I) + 3$ in this example.

(E2) Since a complete ideal of $R$ has reduction number at most one, it is natural to ask about the reduction number of Ratliff–Rush and coefficient ideals. There is no connection between an ideal having reduction number at most two and it being Ratliff–Rush. An example of a Ratliff–Rush ideal of reduction number 3 (with a minimal reduction generated by $x^6 + y^6, x^4y$) is the Ratliff–Rush ideal associated to the ideal $I$ in (E1): Let $J = (x^6, x^4y, x^3y^4, xy^5, y^6)R$ in $R = k[x, y]$. Here in fact $J$ and all the powers of $J$ are Ratliff–Rush, since we have computed via MACAULAY that $\mathrm{G}(J)^+$ has a homogeneous nonzerodivisor of positive degree. Therefore by [Hc3, Theorem 2.1] the reduction number is independent of the minimal reduction chosen. Moreover, the postulation number for $J$ is 1 [My2, Theorem 2, page 5]. (We have confirmed this with MACAULAY and Proposition 4.3, computing $(J^3 : x^4y) = J^2$.)

On the other hand, an example of an ideal of reduction number 2 that is not Ratliff–Rush is $I = (x^4, x^3y, xy^3, y^4)R$; a minimal reduction $\mathbf{q}$ of $I$ for which $I^3 = \mathbf{q}I^2$ is $(x^4, y^4)R$. (Also, see (E7) below.)

(E3) If all the powers of $I$ are Ratliff–Rush, then $P_I(n) \geq H_I(n)$ for all $n \geq 0$ [My2, Theorem 1]. However, it can happen that $I$ is Ratliff–Rush and yet $P_I(1) - H_I(1) = 3$, $P_I(2) = H_I(2)$, $P_I(3) - H_I(3) = -1$, and $P_I(n) = H_I(n)$, for $n \geq 4$: Let $I = (x^3, y^3)(x^5, y^5)R = (x^8, x^5y^3, x^3y^5, y^8)R$, then the Hilbert polynomial of $I$ is

$$P_I(n) = 64\binom{n+1}{2} - 28\binom{n}{1} + 10 \quad ,$$

while the Hilbert function begins 43, 146, 311, 538, 830, 1186, 1606.

For this ideal $I$ we have confirmed with MACAULAY that $I$ is a Ratliff–Rush ideal, but $I^3$ is not Ratliff–Rush. Thus, even in the polynomial ring $R = k[x, y]$ a power of a Ratliff–Rush ideal need not be Ratliff–Rush.

The parameter ideal $\mathbf{q} = (x^8, y^8)R$ is a minimal reduction of $I$ for which the reduction number is 4, i.e., $\mathbf{q}I^4 = I^5$. We also have $(I^2 : \mathbf{q}) = I$ and $(I^3 : \mathbf{q}) = I^2$, while $(I^4 : \mathbf{q}) > I^3$. We then have $(I^n : \mathbf{q}) = I^{n-1}$ for all $n > 4$. We would like to know whether the reduction number is independent for this ideal $I$.

(E4) There are easy examples of non-integrally closed ideals of which all powers are $e_1$-ideals; e.g., a parameter ideal such as $(x^2, y^2)R$. A slightly more complicated example is $I = (x^2, xy^4, y^5)R$. Then $I$ has reduction number one with $(x^2, y^5)R$ as a minimal reduction, so all powers of $I$ are

$e_1$-ideals. But the integral closure $I'$ of $I$ is $(x^2, xy^3, y^5)R$. We compute the Hilbert polynomials for $I$ and $I'$: These ideals have multiplicity 10, and since they have reduction number one, $e_2(I) = e_2(I') = 0$. By (3.4)(5), $e_1(I) = e_0(I) - \lambda(R/I) = 10 - 9 = 1$ and $e_1(I') = e_0(I') - \lambda(R/I') = 2$.

(E5) Let $R = k[x, y]$. What are the possible Hilbert polynomials for ideal containing $(x^3, y^3)R$ and integral over this ideal? What can we get for the linear coefficient and the constant term of such ideals $I$? These polynomials all look like

$$9\binom{n+1}{2} - e_1\binom{n}{1} + e_2 \quad .$$

With $I = (x^3, y^3)R$ we have $e_1 = e_2 = 0$. With $I = (x^3, x^2y^2, y^3)R$ we have $e_1 = 1$ and $e_2 = 0$; this follows from $\lambda(R/I) = e_0 - e_1$ since this ideal has reduction number one. With $I = (x^3, x^2y, y^3)$ we have $e_1 = 3$ and $e_2 = 1$; we get this by comparison with the Hilbert polynomial for the integral closure $I'$ of $I$ which has $e_1 = 3$ and $e_2 = 0$. Finally with $I = (x^3, x^2y + xy^2, y^3)R$, we believe that we again get $e_1 = 3$ and $e_2 = 1$. Perhaps these are the only possible Hilbert polynomials of ideals between $I$ and $I'$.

(E6) We have seen in Proposition 5.2 that if $I$ is contracted, then $\widetilde{I}$ and $I_{\{1\}}$ are also contracted. But a contracted ideal need not be Ratliff–Rush. For instance, $I = (x^8, x^6y, x^5y^2, x^2y^3, y^4)R$ is contracted from $R[M/x]$, since $(I : M) = (I : x)$; but since $I < ((x^2, y)R)^4$ and $I^2 = ((x^2, y)R)^8$, $I$ is not a Ratliff–Rush ideal.

(E7) A Ratliff–Rush contracted ideal need not be an $e_1$-ideal. Let $I = (x^3, x^2y^4, xy^5, y^7)R$. Then $I$ is a contracted ideal with $\mathbf{q} = (x^3, y^7)R$ as a minimal reduction. We see that $(xy^5)^2 \notin \mathbf{q}I$. Therefore the reduction number of $I$ is greater than one. We have checked that the reduction number is two. Also we have checked that the Hilbert polynomial of the integral closure $I'$ of $I$ is

$$21\binom{n+1}{2} - 6\binom{n}{1} \quad ,$$

while the Hilbert polynomial of $I$ is

$$21\binom{n+1}{2} - 6\binom{n}{1} + 1 \quad .$$

By using MACAULAY, we have checked that $I$ and all the powers of $I$ are Ratliff–Rush.

We have also computed that the proper transform of $I$ is $J = (x_1^3, x_1y^3, y^4)S$, where $x_1 = x/y$ and $S = R[x_1]_{(x_1,y)R[x_1]}$. We get the Hilbert polynomial of $J$ to be

$$12\binom{n+1}{2} - 3\binom{n}{1} + 1 \; .$$

(E8) There can be ideals between a contracted ideal $I$ and its integral closure that are not contracted. For example, if $I = (x^3, x^2y^5, xy^6, y^7)R$, and $J = (x^3, xy^5, y^7)R$, then $I < J < I'$, and $J$ is not contracted because the minimal number of generators of a contracted ideal is one more than its order [H2, Proposition 2.3, page 327].

(E9) There can be dramatically different behavior of the Ratliff–Rush property in different characteristics. Sam Huckaba has pointed out to us that for $k$ a field of suitably large characteristic or of characteristic zero, and $R = k[x,y]$, then the ideal $I = (x^6, y^6, (x-y)^6)R$, has the property that $I = (I^2 : I)$ while $I < (I^3 : I^2)$. On the other hand, Mark Johnson has noted that if $k$ has characteristic 2, 3, or 7, then $I$ and all its powers are Ratliff–Rush. In the case where $k$ has characteristic 3, the ideal $I$ has reduction number one so that $\mathrm{G}(I)$ is Cohen–Macaulay, while if $k$ has characteristic 2 or 7, then $\mathrm{G}(I)$ is no longer Cohen–Macaulay, but $\mathrm{G}(I)^+$ contains a nonzerodivisor.

*Questions 6.2.*

(Q1) If $I$ is a simple complete ideal of $R$, is $I$ uniquely Ratliff-Rush? More generally, which complete ideals of $R$ are uniquely Ratliff–Rush?

(Q2) Example 6.1 (E3) shows that the product of the two parameter ideals $(x^3, y^3)R$ and $(x^5, y^5)R$ is a Ratliff–Rush ideal $I$ with the property that $I^3$ is not Ratliff–Rush. Thus the powers of a Ratliff–Rush ideal need not be Ratliff–Rush, even under the present hypotheses on $I$. It would be interesting to know more about the family of Ratliff–Rush ideals that are stable under powers. For example, is it true that if $I$ is an $e_1$-ideal, then $I^n$ is an $e_1$-ideal (and hence a Ratliff–Rush ideal) for all $n$?

(Q3) Is there a unique smallest overideal $J$ of $I$ that is integral over $I$ and has reduction number at most one? (If we do not require that $J$ be integral over $I$, then the statement fails: For example, if $R = k[x,y]$ and $I = (x^4, x^3y, xy^3, y^4)R$, then $I = (x,y)^4R \cap (x^3, y^3)R$, an intersection of two ideals having reduction number at most one, and $\lambda((x,y)^4R/I) = 1$.)

(Q4) If $(R, M)$ is a two-dimensional regular local ring and $I$ is an $M$-primary ideal of $R$, then it is known that $I$ has reduction number at most one iff the associated graded ring $\mathrm{G}(I)$ is Cohen–Macaulay iff the Rees ring $R[It]$ is Cohen–Macaulay [HM, Proposition 2.6] [JV, Theorem 4.1][Sh3, Corollary 4(f)]. From [Sh1, Main Theorem 4], all powers of $I$ are $e_1$-ideals iff $\mathrm{G}(I)$ is unmixed. So: Can it happen that $\mathrm{G}(I)$ is unmixed but not Cohen–Macaulay? [Note: Since this paper was submitted, we have discovered that the answer to this question is positive. This result will appear in a later paper.]

(Q5) When do two Ratliff–Rush ideals in $R$ give the same blowup? For example, if $I = (a,b)R$ and $J = (c,d)R$ are $M$-primary and give the same blowup, does it follow that $I = J$? These hypotheses guarantee that the integral closures of $I, J$ have the same simple complete ideals as factors. Moreover, the multiplicities of $I, J$ are equal to the multiplicities of the one-dimensional local domains $R[a/b]_{MR[a/b]}, R[c/d]_{MR[c/d]}$ respectively; assuming $\mathcal{B}(I) = \mathcal{B}(J)$, these rings are equal, so $I, J$ have the same multiplicity. (Note: If $C, D$ are distinct simple complete $M$-primary ideals, then $C^2D, CD^2$ are Ratliff–Rush — indeed, integrally closed — ideals with the same blowup, but they have no common power.)

(Q6) If $R$ is a two-dimensional RLR and $I$ is a simple complete ideal in $R$, is $I$ uniquely determined by its high powers? I.e., if $J$ is an ideal such that $\widetilde{J} = \widetilde{I}$, does it follow that $J = I$?

(6.3) If $R$ is the polynomial ring $k[x, y]$, then we believe that for each positive integer $r$, the ideal $I_r = (x^r, x^{r-1}y, y^r)R$ is such that $I_r$ and all powers of $I_r$ are Ratliff–Rush. (The ideal $I_3 = (x^3, x^2y, y^3)R$ is mentioned in example (E5) above. That all the powers of $I_3$ are Ratliff–Rush follows from Sally [Sy1, Theorem 1.4].) In general, we believe that the image of $y^r$ in $\mathrm{G}(I_r)$ is a regular element. It is shown in [Hc1, Example 5.9] that the reduction number of $I_r$ with respect to the minimal reduction $(x^r, y^r)R$ is $r-1$, and it follows from [HM, Corollary 3.3] that the reduction number of $I_r$ is independent of the minimal reduction chosen of $I$. Therefore there is no bound on the reduction number for Ratliff–Rush ideals in a two-dimensional regular local ring.

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