PROJECTIVELY EQUIVALENT IDEALS AND REES VALUATIONS

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Abstract

Let R be a Noetherian ring. Two ideals I and J in R are projectively equivalent in case the integral closure of I^i is equal to the integral closure of J^j for some $i, j \in \mathbb{N}_+$. It is known that if I and J are projectively equivalent, then the set Rees I of Rees valuation rings of I is equal to the set Rees J of Rees valuation rings of J and the values of I and J with respect to these Rees valuation rings are proportional. We observe that the converse also holds. In particular, if the ideal I has only one Rees valuation ring V, then the ideals J projectively equivalent to I are precisely the ideals J such that Rees $J = \{V\}$. In certain cases such as: (i) $\dim R = 1$, or (ii) R is a two-dimensional regular local domain, we observe that if I has more than one Rees valuation ring, then there exist ideals J such that Rees $I =$ Rees J, but J is not projectively equivalent to I. If I and J are regular ideals of R, we prove that Rees $I \cup$ Rees $J \subseteq$ Rees IJ with equality holding if $\dim R \leq 2$, but not holding in general if $\dim R \geq 3$. We associate to I and to the set $\mathbf{P}(I)$ of integrally closed ideals projectively equivalent to I a numerical semigroup $S(I) \subseteq \mathbb{N}$ such that $S(I) = \mathbb{N}$ if and only if there exists $J \in \mathbf{P}(I)$ for which $\mathbf{P}(I) = \{(J^n)_a : n \in \mathbb{N}_+\}$.

1 INTRODUCTION.

All rings in this paper are commutative with a unit $1 \neq 0$. Let I be a regular ideal of the Noetherian ring R (that is, I contains a regular element of R). The concept of projective equivalence of ideals and the study of ideals projectively equivalent to I was introduced by Samuel in [24] and further developed by Nagata in [15]. Making use of interesting work of Rees in [23], McAdam, Ratliff and Sally in [14, Corollary 2.4] prove that the set of integrally closed ideals projectively equivalent to I is linearly ordered by inclusion and eventually periodic. They also prove [14, Proposition 2.10] that if an ideal J is projectively equivalent to I, then I and J have the same Rees valuations and the values of I and J with respect to these Rees valuations are proportional. Our goal in the present paper is to build on the work in [14] and further develop the relationship between projective equivalence of ideals and Rees valuations.

2 THE REES VALUATION RINGS OF AN IDEAL.

In this section we review a description of the Rees valuations (and their valuation rings) associated to an ideal I in a Noetherian ring R. For this, we need the following definitions. (Throughout, $\mathbb{N}$ denotes the set of nonnegative integers, and $\mathbb{N}_+$ (resp., $\mathbb{Q}_+$, $\mathbb{R}_+$) denotes the set of positive integers (resp., rational numbers, real numbers).)

Definition 2.1 Let I be an ideal in a Noetherian ring R.

(2.1.1) I_a denotes the **integral closure of** I **in** R, so $I_a = \{b \in R \mid b$ satisfies an equation of the form $b^n + i_1 b^{n-1} + \cdots + i_n = 0\}$, where $i_k \in I^k$ for $k = 1, \ldots, n$. The ideal I is said to be **integrally closed** in case $I = I_a$.

(2.1.2) R' denotes the **integral closure** of R in its total quotient ring.

(2.1.3) For each $x \in R$, let $v_I(x) = \max\{k \in \mathbb{N} \mid x \in I^k\}$ (as usual, $I^0 = R$). (Let $v_I(x) = \infty$ in case $x \in I^k$ for all $k \in \mathbb{N}$.)

(2.1.4) For each $x \in R$, let $\overline{v}_I(x) = \lim_{k \to \infty}(\frac{v_I(x^k)}{k})$ (see (2.1.3) and Remark 2.2).

Remark 2.2 Concerning (2.1.4), Rees shows in [23] that: (a) $\overline{v}_I(x)$ is well defined; (b) for each $k \in \mathbb{N}$ and $x \in R$, $\overline{v}_I(x) \geq k$ if and only if $x \in (I^k)_a$ (as usual, $(I^0)_a = R$); and, (c) there exist valuations $v_1, \ldots, v_g$ defined on R (with values in $\mathbb{N} \cup \{\infty\}$) and positive integers $e_1, \ldots, e_g$ such that, for each $x \in R$, $\overline{v}_I(x) = \min\{\frac{v_i(x)}{e_i} \mid i = 1, \ldots, g\}$. In the case where R is not an integral domain, we say that v is a **valuation on** R if $\{x \in R \mid v(x) = \infty\}$ is a prime ideal P of R, $v(x) = v(y)$ if $x + P = y + P$, and the induced function $\overline{v}$ on the integral domain R/P is a valuation.

To describe (2.2)(c) in more detail and to define the Rees valuation rings of I we need the following definition and notation.

Definition 2.3 Let I be an ideal in a Noetherian ring R, let t be an indeterminate, and let $u = 1/t$. Then the **Rees ring R of** R **with respect to** I is the graded subring $\mathbf{R} = R[u, tI]$ of $R[u, t]$. ($\mathbf{R} = R[u]$, if $I = (0)$.)

Notation 2.4 Let I be an ideal in a Noetherian ring R, let $z_1, \ldots, z_d$ be the minimal prime ideals z in R such that $z + I \neq R$, for $i = 1, \ldots, d$ let $R_i = R/z_i$, let F_i be the quotient field of R_i, let $\mathbf{R}_i = \mathbf{R}(R_i, (I + z_i)/z_i)$, let $p_{i,1}, \dots, p_{i,h_i}$ be the (height one) prime divisors of $u\mathbf{R}_i{}'$, let $w_{i,j}$ be the valuation of the discrete valuation ring $W_{i,j} = \mathbf{R}_i{}'_{p_{i,j}}$, let $e_{i,j} = w_{i,j}(u)$, let $V_{i,j} = W_{i,j} \cap F_i$, and define $v_{i,j}$ on R by $v_{i,j}(x) = w_{i,j}(x + z_i)$.

With this notation, Rees shows in [23] that $v_{i,j}$ is a **valuation on** R in the sense defined above and that $v_{i,j}(x) = \infty$ if and only if $x \in z_i$. Thus $v_1, \dots, v_g$ (see (2.2)) are the valuations $v_{1,1}, \dots, v_{d,h_d}$ resubscripted, and $e_1, \dots, e_g$ are the corresponding $e_{i,j}$ resubscripted.

Recall that if I is an ideal in R and v is a valuation on R, then $v(I) = \min\{v(b) \mid b \in I\}$, so $\overline{R}[\overline{I}/\overline{b}] \subseteq V$ if and only if $v(b) = v(I)$ (where V is the valuation ring of v and the overbar denotes residue class modulo the prime ideal $\{x \in R \mid v(x) = \infty\}$). We use this frequently in this paper.

Remark 2.5 With notation as in (2.4), we have:
(a) if $I \not\subseteq z_i$, then $e_{i,j} = w_{i,j}((I + z_i)/z_i)$ $(= v_{i,j}(I))$;
(b) if $I \subseteq z_i$, then $h_i = 1$, $e_{i,1} = 1$, and $w_{i,1}$ is a **trivial** valuation on R (defined by: $w_{i,1}(x) = 0$, if $x \notin z_i$; $w_{i,1}(x) = \infty$, if $x \in z_i$);
(c) if I is a regular ideal of R, then the Rees valuations of I are all nontrivial.

Proof. For (a), if $I \not\subseteq z_i$, then $t((I+z_i)/z_i) \subseteq \mathbf{R}_i{}' \setminus p_{i,j}$, by [18, (3.6)], so $t((I+z_i)/z_i)W_{i,j} \subseteq W_{i,j} \setminus p_{i,j}W_{i,j}$, so $w_{i,j}(t((I + z_i)/z_i)) = 0$. Therefore $w_{i,j}(u) = w_{i,j}((I + z_i)/z_i)$ (since $u = 1/t$), and $w_{i,j}(u) = e_{i,j}$ (by (2.4)), so $w_{i,j}((I + z_i)/z_i) = e_{i,j}$.

For (b), if $I \subseteq z_i$, then $\mathbf{R}_i = (R/z_i)[u]$, so $p_{i,1} = u\mathbf{R}_i{}'$ is the only prime divisor of $u\mathbf{R}_i{}'$ and $W_{i,1} = \mathbf{R}_i{}'_{p_{i,1}}$, so $e_{i,1} = w_{i,1}(u) = 1$ and $w_{i,1}((I + z_i)/z_i) = w_{i,1}(0) = \infty$. It follows from this that $v_{i,1}$ is a trivial valuation on R.

Statement (c) follows from the fact that a regular ideal is not contained in any minimal prime ideal of the ring. ■

In the literature, the valuation rings $W_{i,j} = \mathbf{R}_i{}'_{p_{i,j}}$ of (2.4) are sometimes called the Rees valuation rings of I, and this causes no problems when only one ideal I is under consideration. However, when an ideal J that is projectively equivalent to I (see (3.1.1) below) is also considered, then this definition of the Rees valuation rings applied to J, in place of I, may yield different valuation rings from the Rees valuation rings $W_{i,j}$ of I (see

(3.3) below for a specific example). However, the rings $V_{1,1}, \ldots, V_{d,h_d}$ of (2.4) are the same for all ideals that are projectively equivalent to I (as is shown in (3.4) below), so we make the following definition.

Definition 2.6 The valuation rings $V_{1,1}, \ldots, V_{d,h_d}$ in (2.4) are the **Rees valuation rings** of I. The set of Rees valuation rings of I is denoted Rees I.

Remark 2.7 Concerning (2.6), notice that if $I \subseteq z_i$ for some $i = 1, \ldots, d$, then it follows from (2.5)(b) that $h_i = 1$, $V_{i,h_i} = V_{i,1} = F_i$, and $e_{i,1} = 1$.

Remark 2.8 The centers in R of the Rees valuation rings of I are the ideals ${\phi_i}^{-1}(p_{i,j} \cap (R/z_i))$, where ϕ_i is the natural homomorphism from R to R/z_i. Therefore these centers correspond to the prime divisors of $(u^n R[u, tI])_a \cap R$ for all large $n \in \mathbb{N}$, so they are the asymptotic prime divisors of I (see [10]). Therefore if these centers are the ideals $P_1, \ldots, P_f$, then $\mathrm{Ass}(R/(I^i)_a) \subseteq \{P_1, \ldots, P_f\}$ for all $i \in \mathbb{N}_+$ and equality holds for all large $i \in \mathbb{N}$.

In the next section we prove several results concerning the set of Rees valuations of ideals. Toward this end, the following alternate construction of the nontrivial Rees valuation rings of an ideal is helpful.

Construction 2.9 With the notation of (2.4), let z be a minimal prime ideal in R such that $I \not\subseteq z$ and $z + I \neq R$, let $b_1, \ldots, b_h$ be generators of I that are not in z, let an overbar denote residue class modulo z, and let F be the quotient field of $\overline{R}$. Let V be a discrete valuation ring such that $\overline{R} \subseteq V \subsetneq F$, and let N be the maximal ideal of V. Then V is a Rees valuation ring of I if and only if there exists $\overline{b} \in \{\overline{b_1}, \ldots, \overline{b_h}\}$ such that $N \cap A'$ is a height one prime ideal, where $A = \overline{R}[\overline{I}/\overline{b}]$. Moreover, if V is a Rees valuation ring of I and if $B = \overline{R}[\overline{I}/\overline{c}] \subseteq V$, then $V = {B'}_{N \cap B'}$.

Proof. By considering each of the rings R/z (with z a minimal prime ideal in R) separately it may be assumed to begin with that R is a Noetherian integral domain. Therefore [14, Proposition 3.1] applies to establish this equivalent way to define the set of Rees valuations of I.

For the final statement, it suffices to observe that if V is a valuation domain with maximal ideal N that has R as a subring and b and c are elements of I such that $IV = bV = cV$, then $R[I/b]_{N \cap R[I/b]} = R[I/c]_{N \cap R[I/c]}$; this equality is clear since c/b is a unit of $R[I/b]_{N \cap R[I/b]}$ and b/c is a unit of $R[I/c]_{N \cap R[I/c]}$. ■

Remark 2.10 (a) If $I = bR$ is a regular principal ideal in R, then it follows from (2.9) that Rees $I = \{R'_{p_1}/z_1, \ldots, R'_{p_g}/z_g\}$, where $p_1, \ldots, p_g$ are the prime divisors of bR' and z_i = rad (R'_{p_i}) for $i = 1, \ldots, g$ (possibly $z_i \cap R' = z_j \cap R'$ for some i, j).
(b) Every minimal prime divisor of an ideal I is the center of at least one Rees valuation of I. Therefore for ideals I and J of R, if Rees I = Rees J, then I and J have the same radical.
(c) For a fixed ideal I, let $\Gamma(I)$ denote the set of ideals J such that Rees I = Rees J. It would be interesting to have conditions on I or on the ring R in order that the set $\Gamma(I)$ have a unique maximal element with respect to inclusion. This is true for all ideals I of a one-dimensional Noetherian integral domain R, for in this case, by statement (1) of Example 3.5, rad (I) is the unique largest ideal of R having the same Rees valuations as I. It is also true for all ideals I of a two-dimensional regular local domain R, for in this case, as discussed in Example 3.8, every integrally closed ideal of R is uniquely a finite product of simple complete ideals, and the product of the simple complete factors of I_a with no repeated factors is the largest ideal having the same Rees valuations as I. It is also true for an ideal I of a general Noetherian domain R if I has only one Rees valuation ring V, for the integrally closed ideals J such that Rees $J = \{V\}$ are all contracted from V and thus are linearly ordered with respect to inclusion.

Remark 2.11 Let (R, M) be a Noetherian local domain and let $\widehat{R}$ denote the M-adic completion of R. It follows from [10, (3.19)] and (2.8) that the following are equivalent: (a) there exists a valuation domain V dominating R such that $V \in$ Rees I for every nonzero proper ideal I of R; (b) there exists a minimal prime ideal z of $\widehat{R}$ such that $\dim(\widehat{R}/z) = 1$; and, (c) there exists a height one maximal ideal in R'.

3 REES VALUATION RINGS AND PROJECTIVELY EQUIVALENT IDEALS.

It is shown in [14, Proposition 2.10] that if I is a regular ideal of a Noetherian ring, then every ideal J projectively equivalent to I satisfies Rees $I =$ Rees J and the values of I and J with respect to these Rees valuation rings are proportional. We prove in Theorem 3.4 that the converse also holds. In particular, if I has only one Rees valuation ring V, then the ideals J projectively equivalent to I are precisely the ideals J such that Rees $J = \{V\}$. In Example 3.5, we consider projective equivalence and Rees valuation rings of ideals of a one-dimensional Noetherian integral domain R. For an ideal I of R such that Rees I has cardinality greater than one, we prove there exist ideals J of R such that Rees $I =$ Rees J, but J is not projectively equivalent to I. In Proposition 3.6, we prove that if I and J are regular ideals of a Noetherian ring, then Rees $I \cup$ Rees $J \subseteq$ Rees IJ, with equality holding if $\dim R \leq 2$. We observe in Remark (3.7.3) that equality does not hold in general if $\dim R \geq 3$.

We recall the following definition.

Definition 3.1 Let I be an ideal in a Noetherian ring R. An ideal J in R is **projectively equivalent to** I in case $(J^j)_a = (I^i)_a$ (see (2.1.1)) for some $i, j \in \mathbb{N}_+$.

Samuel introduced projectively equivalent ideals in 1952 in [24]. A number of properties of projective equivalence can be found in [7], [8], [11], [12], [13], [14], [20], [21]. In this section we explore the relation between projectively equivalent ideals and Rees valuation rings.

Remark 3.2 Let R be a Noetherian ring. Then
(3.2.1) The relation "I is projectively equivalent to J" is an equivalence relation on $\mathbf{I} = \{I \mid I \text{ is an ideal of } R\}$.
(3.2.2) If I and J are ideals in R and if $i, j, k, l \in \mathbb{N}_+$ with $\frac{i}{j} = \frac{k}{l}$, then $(I^i)_a = (J^j)_a$ if and only if $(I^k)_a = (J^l)_a$.

Proof. (3.2.1) follows readily from basic properties of integral closures of ideals, and (3.2.2) is proved in [14, (2.1)(b)]. ■

The following example shows that projectively equivalent ideals may yield different valuation rings $W_{i,j}$ as in (2.4).

Example 3.3 Let $R = K[[X]]$, where K is a field and X is an indeterminate, let $I = XR$, and let $J = X^2R$, so I and J are projectively equivalent (since $(I^2)_a = I^2 = J = (J^1)_a$). In this case, $d = 1$, $z_1 = (0)$, $\mathbf{R}(R,I) = R[u,tX] = \mathbf{R}(R,I)'$, $p_{1,1} = uR[u,tX]$. Thus $W_{1,1} = R[u,tX]_{p_{1,1}}$ is the only valuation ring W of I as in (2.4), and $w_{1,1}(u) = 1$. On the other hand, $\mathbf{R}(R,J) = R[u,tX^2] = \mathbf{R}(R,J)'$, $p_{1,1} = (u,X)R[u,tX^2]$. Thus ${W_{1,1}}^* = R[u,tX^2]_{p_{1,1}}$ is the only valuation ring W^* of J as in (2.4), and ${w_{1,1}}^*(u) = 2$, since $tX^2 \in R[u,tX^2] \setminus p_{1,1}$. Therefore the valuation rings $W_{1,1}$ and ${W_{1,1}}^*$ differ, while $V_{1,1} = W_{1,1} \cap K((X)) = R = W^*_{1,1} \cap K((X)) = V^*_{1,1}$, so Rees I = Rees J.

With the definition of Rees valuation rings in (2.6), we have the following.

Theorem 3.4 *Let I and J be regular ideals of the Noetherian ring R. The following are equivalent:*

1. *I and J are projectively equivalent.*

2. Rees I = Rees J *and the values of I and J with respect to these Rees valuation rings are proportional.*

In particular, if the ideal I has only one Rees valuation ring V, then the ideals J projectively equivalent to I are precisely the ideals J such that Rees $J = \{V\}$.

Proof. It is shown in [14, Proposition 2.10] that (1) implies (2). To prove that (2) implies (1), notice first that by considering each of the rings R/z (with z a minimal prime ideal in R), it suffices to prove (2) implies (1) in the case where R is a Noetherian integral domain. Since $V_1, \dots, V_g$ are the Rees valuation rings of I it follows from (2.8) that, for all $i \in \mathbb{N}_+$, $(I^i)_a = \cap\{I^iV_h \cap R \mid h = 1, \dots, g\}$, and, similarly, for all $j \in \mathbb{N}_+$, $(J^j)_a = \cap\{J^jV_h \cap R \mid h = 1, \dots, g\}$. Therefore, if there exist $i, j \in \mathbb{N}_+$ such that $v_h(I) = (\frac{j}{i})v_h(J)$ for $h = 1, \dots, g$, then it follows that $v_h(I^i) = v_h(J^j)$ for $h = 1, \dots, g$, so $I^iV_h = J^jV_h$ for $h = 1, \dots, g$, hence $(I^i)_a = \cap\{I^iV_h \cap R \mid h = 1, \dots, g\} = \cap\{J^jV_h \cap R \mid h = 1, \dots, g\} = (J^j)_a$.

To prove the last statement, let v be the normalized valuation associated to the valuation ring V, let $j = v(I)$, and let $i = v(J)$. Then $v(I^i) = ij = v(J^j)$, so $I^iV = J^jV$, so $v(I) = (\frac{j}{i})v(J)$, so the conclusion follows from the equivalence of (1) and (2). ∎

Let I be a nonzero ideal in a Noetherian domain R. If $V_1, \ldots, V_g$ are the Rees valuation rings of I, then they may also be the Rees valuation rings of another ideal J of R such that J is not projectively equivalent to I. To illustrate the concepts of projective equivalence of ideals and Rees valuations rings, we consider in Example 3.5 the case where R is a Noetherian domain with $\dim R = 1$. In particular, Example 3.5 provides examples of ideals I and J such that Rees $I =$ Rees J, but I and J are not projectively equivalent.

Example 3.5 Let R be a Noetherian integral domain with $\dim R = 1$. It is well known that the integral closure R' of R is a Dedekind domain. If I is a nonzero proper ideal of R, then Rees $I = \{R'_P\}$, where P varies over the maximal ideals of R' such that $I \subseteq P$. Thus for ideals I and J of R, we have

1. Rees $I =$ Rees J if and only if rad $I =$ rad J.

2. Rees $IJ =$ Rees $I \cup$ Rees J.

3. For a nonzero ideal I of R, the set Rees I has cardinality greater than one if and only if I is contained in more than one maximal ideal of R'.

4. If Rees I has cardinality greater than one, then there exist ideals J of R such that Rees $I =$ Rees J, but J is not projectively equivalent to I.

To prove this last statement, notice that if $P, Q_1, \ldots, Q_s$ are distinct maximal ideals of R', then $PQ_1 \cdots Q_s \cap R$ is the set of elements of R having positive value in the normalized valuation v corresponding to R'_P and also positive value in the normalized valuation w_i corresponding to R'_{Q_i}, for $i = 1, \ldots, s$; while for each $n \in \mathbb{N}_+$, $P^nQ_1 \cdots Q_s \cap R$ is the set of elements of R having v-value at least n and positive value with respect to each w_i, $i = 1, \ldots, s$. Here we are using that the ideals $P^n, Q_1, \ldots, Q_s$ are pairwise comaximal in R'. Assume that Rees $I = \{R'_P, R'_{Q_1}, \ldots, R'_{Q_s}\}$, with $s \in \mathbb{N}_+$. To show there exists an ideal J of R with rad $I =$ rad J (so Rees $I =$ Rees J, by (3.5.1)) such that J is not projectively equivalent to I, it suffices to prove there exists $n \in \mathbb{N}_+$ such that $P^n Q_1 \cdots Q_s \cap R \subsetneq PQ_1 \cdots Q_s \cap R$. Since $PQ_1 \cdots Q_s \cap R \neq (0)$ and $\cap_{n=2}^{\infty}(P^n Q_1 \cdots Q_s \cap R) = (0)$, there must exist $n \in \mathbb{N}_+$ such that $P^n Q_1 \cdots Q_s \cap R \subsetneq PQ_1 \cdots Q_s \cap R$.

As suggested by the referee, Example (3.5.4) can be demonstrated concretely by taking R to be a one-dimensional semilocal normal Noetherian domain with distinct maximal ideals $\mathbf{m_1}$ and $\mathbf{m_2}$. Let $0 \neq a \in (\mathbf{m_1} \cap \mathbf{m_2})$ and $x \in \mathbf{m_1} \setminus \mathbf{m_2}$ and $y \in \mathbf{m_2} \setminus \mathbf{m_1}$. Then $I = axR$ and $J = ayR$ have the same Rees valuations, but are not projectively equivalent.

In Proposition 3.6, we relate the Rees valuation rings of the ideals I and J with those of their product IJ. In the case where the ambient ring is a two-dimensional pseudo-geometric normal Noetherian local domain, Proposition 3.6 is due to Göhner [2, Lemma 2.1]. The statement and proof of the first part of Proposition 3.6 are similar in spirit to a theorem of K.Whittington [10, Prop. 3.26]. See also the forthcoming book by Craig Huneke and Irena Swanson on integral closure of ideals for related information.

Proposition 3.6 *Let I and J be regular ideals in a Noetherian ring R. Then* Rees $I \cup$ Rees $J \subseteq$ Rees IJ*, with equality holding if* $\dim R \leq 2$.

Proof. To prove both statements it may be assumed (as in the proof of Theorem 3.4) that R is a Noetherian domain. And for the first statement it suffices (by symmetry) to show that if $V \in$ Rees I, then $V \in$ Rees IJ.

For this, let $I = (b_1, \ldots, b_h)R$ and by (2.9) let $b \in \{b_1, \ldots, b_h\}$ such that $V = A'_{N \cap A'}$, where $A = R[I/b]$ (so $v(I) = v(b)$) and N is the maximal ideal of V. Let $J = (c_1, \ldots, c_k)R$ and let $c \in \{c_1, \ldots, c_k\})$ such that $v(c) = v(J)$. Then $v(bc) = v(I) + v(J) = v(IJ)$, so $C = R[(IJ)/(bc)] \subseteq V$, hence $C'_{N \cap C'} \subseteq V$. However, $R[I/b] = A \subseteq C = R[(IJ)/(bc)]$ (since $IJ = (c_1 I, \ldots, c_k I)R$ and $c \in \{c_1, \ldots, c_k\}$ imply that $I/b \subseteq (IJ)/(bc)$), so it follows that $V = C'_{N \cap C'}$, so $V \in$ Rees IJ by (2.9).

In view of Example 3.5 (and the first paragraph of this proof), the second statement is clear if $\dim R = 1$, so we assume $\dim R = 2$. Let $V \in$ Rees IJ and let N denote the maximal ideal of V. Also, let $b_1, \ldots, b_h$ (resp., $c_1, \ldots, c_k$) be generators of I (resp., J), where the b_i and c_j may be assumed to be nonzero. Then $b_1 c_1, \ldots, b_h c_k$ generate IJ, so by

(2.9) there exists b_ic_j ($i \in \{1,\dots,h\}$ and $j \in \{1,\dots,k\}$) such that $V = C'_{p'}$, where $C = R[(IJ)/(b_ic_j)]$ and $p' = N \cap C'$. Let $b := b_i$ and $c := c_j$. Now $v(IJ) = v(bc)$ (since $(IJ)/(bc) \subseteq V) = v(b) + v(c) \geq v(I) + v(J) = v(IJ)$, so it follows that: $v(I) = v(b)$; $v(J) = v(c)$; $A = R[I/b] \subseteq C$; and, $B = R[J/c] \subseteq C$. Let $q = p' \cap A'$ and $w = p' \cap B'$, so $A'_q \subseteq C'_{p'} = V$ and $B'_w \subseteq C'_{p'} = V$, so: if ht $q = 1$, then $V = A'_q \in$ Rees I (by (2.9)); and, if ht $w = 1$, then $V = B'_w \in$ Rees J (by (2.9)). Therefore to show that Rees $IJ \subseteq$ Rees $I \cup$ Rees J it suffices to show that either: (a) ht $q = 1$; or, (b) ht $w = 1$. We now show that either (a) or (b) holds.

By the Mori-Nagata theorem [16, (33.12)], A', B' and C' are normal Noetherian domains of dimension at most two and hence are Cohen-Macaulay and therefore universally catenary. Let $D = A'[(IJ)/(bc)] = A'[J/c] = B'[I/b]$. Notice that $D' = C'$. Also D is universally catenary since it is a finitely generated integral domain over A'. Therefore A', B' and D satisfy the altitude formula, (or in other terminology the dimension formula [9, Theorem 15.6]). Let $p := p' \cap D$. Then ht $p =$ ht $p' = 1$. Assume that ht $q = 2$. Since $D = A'[J/c]$, the altitude formula implies for some $n \in \{1,\dots,k\}$ that the image of c_n/c in D/p is transcendental over A'/q. It follows that the image of c_n/c in $B/(p \cap B)$ is transcendental over $R/(p \cap R)$. This implies that ht $(p \cap B) <$ ht $(p \cap R)$. Therefore ht $(p \cap B) = 1$. Notice that $p \cap B = w \cap B$. Since ht $w \leq$ ht $(w \cap B)$, it follows that ht $w = 1$, so (b) of the preceding paragraph holds, hence $V \in$ Rees J. A similar argument shows that if ht $w = 2$, then (a) of the preceding paragraph holds, so $V \in$ Rees I. It follows that Rees $IJ \subseteq$ Rees $I \cup$ Rees J, so equality holds by the first statement. ■

Remark 3.7 Let I and J be regular ideals in a Noetherian ring R. Then:

(3.7.1) If $V \in$ Rees IJ, and if $J \not\subseteq N \cap R$, where N is the maximal ideal in V, then $V \in$ Rees I.

(3.7.2) If x is a regular element of R, then Rees $(xI) =$ Rees $(xR) \cup$ Rees I.

(3.7.3) Let (R, M) be a 3-dimensional regular local domain where $M = (x,y,z)R$ and let $I = (x,y)R$ and $J = (x,z)R$. Then the ord-valuation domain defined by the powers of M is a Rees valuation ring of $IJ = (x^2, xy, xz, yz)R$, but is not in Rees $I \cup$ Rees J. Therefore the equality statement of Proposition 3.6 fails if $\dim R = 3$.

Proof. For (3.7.1), let $V \in$ Rees IJ. Then it follows from (2.9) that there exist $b \in I$ and $c \in J$ such that $V = C'_{N \cap C'}$, where $C = R[(IJ)/(bc)]$ (and then $v(bc) = v(IJ) = v(I) + v(J) \leq v(b) + v(c) = v(bc)$, so $v(I) = v(b)$ and $v(J) = v(c)$, where v is the valuation of V). It follows that $A = R[I/b] \subseteq V$. Now, if $J \nsubseteq N \cap R$, then $v(J) = 0$ (so $v(c) = 0$), so $J/c \subseteq R_{N \cap R}$, so it follows that $C \subseteq A'_{N \cap A'}$. Since $V = C'_{N \cap C'}$, it follows that $V = A'_{N \cap A'}$, so $V \in$ Rees I by (2.9).

For (3.7.2), it suffices, by (3.6), to show that Rees $xI \subseteq$ Rees $xR \cup$ Rees I. For this, by (2.9) there exists $b \in I$ such that $V = C'_p$, where $C = R[(xI)/(xb)]$ and p is a height one prime divisor of xbC' (so $v(xb) = v(xI)$, so $v(b) = v(I)$). If $b \notin p$, then $v(I) = v(b) = 0$, so $I \nsubseteq p \cap R$, hence $V \in$ Rees xR by (3.7.1). On the other hand if $b \in p$, then since it is clear that $C = A$, where $A = R[I/b]$, it follows that p is a minimal prime divisor of $bA' = bC'$, hence $V \in$ Rees I by (2.9).

For (3.7.3), notice that the powers of I and the powers of J each define a valuation and these are the unique Rees valuations of I and J. Indeed, Rees $I = \{R[y/x]_{xR[y/x]}\}$ and Rees $J = \{R[z/x]_{xR[z/x]}\}$. On the other hand, the ord-valuation domain defined by the powers of M is $V = R[y/x, z/x]_{xR[y/x,z/x]}$. Since $R[xy/x^2, xz/x^2, yz/x^2] = R[y/x, z/x]$, we see that $V \in$ Rees IJ. ■

Example 3.8 (cf. [14, (3.6)].) Let R be a two-dimensional regular local domain. Zariski develops in [26, Appendix 5] the theory of complete (or integrally closed) ideals of R. He proves that in R a product of complete ideals is again complete, and establishes a unique factorization theorem: every complete ideal of R is uniquely expressible as a product of simple complete ideals [26, pages 385-386]. Here an ideal I is said to be simple if it is not the unit ideal and has no nontrivial factorizations. Since R is a unique factorization domain, every nonzero ideal I of R is of the form $I = xJ$, where either $J = R$ or J is primary for the maximal ideal M of R. Since principal ideals of R are complete, the theory reduces to a consideration of complete M-primary ideals. The simple complete M-primary ideals of R are in one-to-one correspondence with the DVRs that birationally dominate R and have the property that their residue field as an extension of R/M is not algebraic. If I is a simple complete ideal of R, then I has a unique Rees valuation domain V [5, Theorem 4.2].

It follows that the integrally closed ideals that are projectively equivalent to I are precisely the ideals I^n for $n \in \mathbb{N}_+$. In particular, every simple complete ideal of R is projectively full in the sense of Definition 4.9. For an arbitrary nonzero proper ideal I of R, the Rees valuation rings of I are in one-to-one correspondence with the distinct simple complete ideals that are factors of the integral closure of I. We have $I_a = J_1^{e_1} \cdots J_n^{e_n}$, where the J_i are simple complete ideals and the e_i are positive integers. If J_i is a height-one prime of R, the Rees valuation ring associated to J_i is $V_i = R_{J_i}$, while if J_i is M-primary, then V_i is the DVR that birationally dominates R described above. Let K be a nonzero proper ideal of R and let $K_a = L_1^{f_1} \cdots L_m^{f_m}$, where the L_i are distinct simple complete ideals and the f_i are positive integers. Then Rees $I =$ Rees K if and only if $n = m$ and the set of simple complete ideals $\{J_i\}_{i=1}^n$ is the same as the set $\{L_i\}_{i=1}^m$. Necessary and sufficient conditions for I and K to be projectively equivalent are that Rees $I =$ Rees K, so $I_a = J_1^{e_1} \cdots J_n^{e_n}$ and $K_a = J_1^{f_1} \cdots J_n^{f_n}$, and, in addition, the n-tuples $e_1, \dots, e_n$ and $f_1, \dots, f_n$ are proportional, i.e., there exist positive integers a and b such that $ae_i = bf_i$ for $i = 1, \dots, n$. In particular, if I has more than one Rees valuation ring and if J_1 and J_2 are distinct simple complete ideals of R that are factors of I_a, then $J_1 I_a$ and $J_2 I_a$ are complete ideals that are not projectively equivalent, but have the same Rees valuation rings.

Remark 3.9 Since an ideal and its powers have the same blowup, if I and J are normal projectively equivalent ideals of a Noetherian domain R, then the blowups of I and J are equal. In the case where I and J are normal ideals of a two-dimensional regular local domain, then I and J have the same blowup if Rees $I =$ Rees J. However, Cutkosky shows in [1, Example 2] the existence of an infinite set of normal ideals in a three-dimensional regular local domain that have the same Rees valuations but have pairwise distinct blowups.

4 NUMERICAL SEMIGROUPS AND PROJECTIVELY FULL IDEALS.

Let I be a regular ideal of the Noetherian ring R. In [14], McAdam, Ratliff and Sally prove that the set $\mathbf{P}(I)$ of integrally closed ideals projectively equivalent to I is linearly ordered by inclusion and eventually periodic. ($\mathbf{P}(I)$ is **eventually periodic** means there exist $I_1, \ldots, I_h \in \mathbf{P}(I)$ such that $\mathbf{P}(I) = \{({I_1}^{k_1} \cdots {I_h}^{k_h})_a \mid$; each k_i is a nonnegative integer$\}$. (It can be shown that this definition of "eventually periodic" is equivalent to the definition, given in [14], that the set $\mathbf{U}$ in (4.1.3) is eventually periodic; see (4.2)(d).)) They also prove the existence of a fixed $d \in \mathbb{N}_+$ such that for every ideal J projectively equivalent to I there exists $n \in \mathbb{N}_+$ such that $(I^n)_a = (J^d)_a$. As we note in Remark 4.3, using results proved in [14], there is naturally associated to I or to the projective equivalence class of I a unique numerical semigroup. Here we are using the term numerical semigroup in the sense of Herzog-Kunz [4] and Watanabe [25].

We recall the following definitions from [14].

Definition 4.1 *Let I be a regular ideal in a Noetherian ring R.*

(4.1.1) For $\alpha \in \mathbb{R}_+$ let $I_\alpha = \{x \in R \mid \overline{v}_I(x) \geq \alpha\}$.

(4.1.2) $\mathbf{W} = \{\alpha \in \mathbb{R}_+ \mid \overline{v}_I(x) = \alpha$ for some $x \in R\}$ (see (2.1.4)).

(4.1.3) $\mathbf{U} = \{\alpha \in \mathbf{W} \mid I_\alpha$ is projectively equivalent to $I\}$ (see (4.1.1) and (3.1)), and $\mathbf{P} = \mathbf{P}(I) = \{I_\alpha \mid \alpha \in \mathbf{U}\}$.

Remark 4.2 Let R be a Noetherian ring and let I be a regular ideal in R. Then:

(a) for each $\alpha \in \mathbb{R}_+$, the ideal I_α of (4.1.1) is an integrally closed ideal ($= (I_\alpha)_a$) in R, and for all $k \in \mathbb{N}_+$ and for all $I_\alpha \in \mathbf{P}(I)$ it holds that $({I_\alpha}^k)_a = I_{k\alpha}$, by [14, (2.1)(g) and (2.6)].

(b) for the set $\mathbf{P}$ of (4.1.3), $\mathbf{P} = \{J \mid J$ is an integrally closed ideal in R that is projectively equivalent to $I\}$, and $\mathbf{P}$ is linearly ordered by inclusion, by [14, (2.4)];

(c) $\mathbf{W}$ and $\mathbf{U}$ are discrete subsets of $\mathbb{Q}_+$, by [14, (1.1) and (2.8)];

(d) there exist $n^* \in \mathbb{N}_+$ and a *unique* $d \in \mathbb{N}_+$ such that: $\{\alpha \in \mathbf{U} \mid \alpha \geq n^*\} = \{n^* + \frac{k}{d} \mid k \in \mathbb{N}\}$; d is a common divisor (but not necessarily the greatest common divisor) of the integers $e_1, \ldots, e_g$ of (2.2)(c); $d\alpha \in \mathbb{N}_+$ for all $\alpha \in \mathbf{U}$; and, for each $J \in \mathbf{P}$, there exists $n \in \mathbb{N}_+$ such that $(J^d)_a = (I^n)_a$, by [14, (2.8) and (2.9)].

Remark 4.3 With d as in (4.2)(d), the set of integers $d\mathbf{U} \cup \{0\}$ is a numerical semigroup in the sense of Herzog-Kunz [4] and Watanabe [25] that is naturally associated to I. We denote this semigroup by $S(I)$. It is an invariant of the projective equivalence class of I

in the sense that if J is projectively equivalent to I, then $S(J) = S(I)$. Thus $S(I)$ is an invariant of $\mathbf{P}(I)$. We are interested in considering properties of this semigroup.

In [6, Section 2], Itoh gave the following construction, that will be used below to gain some information concerning $\mathbf{W}$ and $\mathbf{U}$.

Proposition 4.4 *Let I be a regular ideal in a Noetherian ring R, let $n \in \mathbb{N}_+$, let $\mathbf{R} = R[u, tI]$, let $\mathbf{S} = \mathbf{R}[u^{1/n}]$, let $\mathbf{T} = \mathbf{S}' \cap R[u^{1/n}, t^{1/n}]$, and let $I_{[k/n]} = u^{k/n}\mathbf{T} \cap R$. Then $I_{[k/n]} = I_{k/n}$ for all $k \in \mathbb{N}_+$.*

Proof. $\mathbf{T}$ is a graded subring of $R[u^{1/n}, t^{1/n}]$. Also, if $x \in R$ and $k \in \mathbb{N}_+$, then $x \in I_{[k/n]} = u^{k/n}\mathbf{T} \cap R$ if and only if $xt^{k/n} \in \mathbf{T}$ if and only if $x^n t^k \in \mathbf{R}' \cap R[u, t]$ (for $\mathbf{T}$ is an integral extension of $\mathbf{R}$, so $x^n t^k$ is integral over $\mathbf{R}$ and is in $R[u, t]$). And $x^n t^k \in \mathbf{R}' \cap R[u, t]$ if and only if $x^n \in u^k(\mathbf{R}' \cap R[u, t]) \cap R = (I^k)_a$ if and only if $x \in I_{k/n}$ (since $x^n \in (I^k)_a = I_k$ (by (4.2)(a)) if and only if $\overline{v}_I(x^n) \geq k$ (by (4.1.1)) if and only if $n\overline{v}_I(x) \geq k$ if and only if $\overline{v}_I(x) \geq k/n$ if and only if $x \in I_{k/n}$, by (4.1.1))). ■

Remark 4.5 Let I be a regular ideal in a Noetherian ring R. Then:
(4.5.1) In (4.4) let $n = d$ (with d as in (4.2)(d)) and let $\mathbf{V}_1 = \{k/d \mid k \in \mathbb{N}_+\}$. Then $\mathbf{U} \subseteq \mathbf{V}_1$ and $\mathbf{V}_1 \setminus \mathbf{U}$ is a finite set.
(4.5.2) In (4.4) let $n = e_1 \cdots e_g$ (with $e_1, \ldots, e_g$ as in (2.2)(c)) and let $\mathbf{V}_2 = \{k/(e_1 \cdots e_g) \mid k \in \mathbb{N}_+\}$. Then $\mathbf{W} \subseteq \mathbf{V}_2$.
(4.5.3) $\{I_{[k/n]}\}_{k \geq 1}$ is a filtration of integrally closed ideals on R. Therefore if R is an analytically unramified semi-local ring, then for all large $k \in \mathbb{N}_+$ it holds that $I_{mk/n} = {I_{k/n}}^m$ for all $m \in \mathbb{N}_+$, so $I_{k/n}$ is a normal ideal (that is, all powers of $I_{k/n}$ are integrally closed).

Proof. (4.5.1) is clear by (4.2)(d) together with (4.4), and (4.5.2) is clear by the definitions of $e_1, \ldots, e_g$ and $\mathbf{W}$ together with (4.4).

For (4.5.3), $u^{k/n}\mathbf{T}$ is integrally closed, so $I_{[k/n]}$ is integrally closed, so the conclusion follows from (4.2)(a), (4.5.1), [17, (4.4.3)], and [19, Theorem (5.2) and (4.5)]. ■

Remark 4.6 It is shown in [22] that the set $\mathbf{P}(I)$ (together with R) forms a subfiltration f^* of the filtration $e = \{I_{i/d}\}_{i \geq 0}$ of Proposition 4.4, as does $f = \{(I^i)_a\}_{i \geq 0}$, and the graded subring of $\mathbf{R} = R[u, te]$ ($= R[u, tI_{1/d}, t^2 I_{2/d}, \dots]$) generated by either of the filtrations f^* and g has homogeneous prime spectrum isomorphic to the homogeneous prime spectrum of $\mathbf{R}$; however, if I is not projectively full (see Definition 4.9), then the homogeneous prime spectra of the Rees rings of f^* and f are not isomorphic.

We next note some things concerning $\mathbf{U}$ (see (4.1.3)) and n^* and d (with n^* and d as in (4.2)(d)) (recall that $\mathbf{U}$, n^*, and d depend on I). For this, let $\mathbf{U} = \{\alpha_1, \alpha_2, \dots\}$ (with $\alpha_1 < \alpha_2 < \cdots$), let $\mathbf{P} = \{I_{\alpha_1}, I_{\alpha_2}, \dots\}$ (so $I_{\alpha_1} \supsetneqq I_{\alpha_2} \supsetneqq \cdots$), and assume that $\alpha_i = 1$ (that is, assume that I_{α_i} is the ideal in $\mathbf{P}$ that is the integral closure of I). Then in what follows $\mathbf{U}$, n^*, d, and $\mathbf{P}$ will be denoted $\mathbf{U}(I)$, $n^*(I)$, $d(I)$, and $\mathbf{P}(I)$ (so $\{\alpha \in \mathbf{U}(I) \mid \alpha \geq n^*(I)\}$ $= \{n^*(I) + \frac{k}{d(I)} \mid k \in \mathbb{N}\}$ and for each $\alpha \in \mathbf{U}(I)$ it holds that $({I_\alpha}^{d(I)})_a = (I^n)_a$ for some $n \in \mathbb{N}_+$ (by (4.2)(d)). For $J \in \mathbf{P}(I)$ let $\mathbf{U}(J) = \{\beta_1, \beta_2, \dots\}$ (with $\beta_1 < \beta_2 < \cdots$), $n^*(J)$, $d(J)$, and $\mathbf{P}(J) = \{J_{\beta_1}, J_{\beta_2}, \dots\}$ (so $J_{\beta_1} \supsetneqq J_{\beta_2} \supsetneqq \cdots$) be defined analogously (so $\{\beta \in \mathbf{U}(J) \mid \beta \geq n^*(J)\} = \{n^*(J) + \frac{k}{d(J)} \mid k \in \mathbb{N}\}$ and for each $\beta \in \mathbf{U}(J)$ it holds that $({J_\beta}^{d(J)})_a = (J^n)_a$ for some $n \in \mathbb{N}_+$).

Remark 4.7 **(4.7.1)** [14, (2.11)] If $\delta \in \mathbf{U}(I)$ and $J = I_\delta \in \mathbf{P}(I)$, then $\mathbf{U}(J) = \{\frac{\alpha}{\delta} \mid \alpha \in \mathbf{U}(I)\}$.

(4.7.2) [14, (2.3)] If $(I^i)_a = (J^j)_a$, then $\frac{i}{j} \in \mathbf{U}(I)$ and $J = I_{\frac{i}{j}}$. Also, if $m, n \in \mathbb{N}_+$ and if $\frac{m}{n} \in \mathbf{U}(I)$, then $({I_{\frac{n}{m}}}^m)_a = (I^n)_a$.

Proposition 4.8 *Let I be a regular ideal in a Noetherian ring R and let $J \in \mathbf{P}(I)$. Then:*

(4.8.1) $\mathbf{P}(I) = \mathbf{P}(J)$.

(4.8.2) $(I^{d(J)})_a = (J^{d(I)})_a$.

(4.8.3) *If $H, J \in \mathbf{P}(I)$ and if $(J^j)_a = (H^h)_a$, then $jd(J) = hd(H)$ and $(J^{d(H)})_a = (H^{d(J)})_a$.*

(4.8.4) *If $\delta \in \mathbf{U}(I)$, then $\delta = \frac{d(I_\delta)}{d(I)}$.*

(4.8.5) *If $H, J \in \mathbf{P}(I)$ and if $J \subsetneqq H$, then $d(J) > d(H)$.*

(4.8.6) *If $H, J \in \mathbf{P}(I)$ and if $J \subsetneqq H$, then $n^*(H) \geq n^*(J)$. Also, $n^*(I_{\alpha_j}) = 1$ for all $\alpha_j \geq n^*(I)$.*

Proof. For (4.8.1), by definition $\mathbf{P}(I)$ (resp., $\mathbf{P}(J)$) is the set of integrally closed ideals in R that are projectively equivalent to I (resp., J). Since projective equivalence is an equivalence relation, and since I and J are projectively equivalent, it follows that $\mathbf{P}(I) = \mathbf{P}(J)$.

For (4.8.2), by (4.2)(d) it follows that $(J^{d(I)})_a = (I^n)_a$ for some $n \in \mathbb{N}_+$. By raising both sides of this equality to a large power (say the k-th power) it may be assumed that $kd(I) \geq n^*(J)$ and $kn \geq n^*(I)$. Therefore if $m \in \mathbb{N}_+$ is such that $m \geq min\{kd(I), kn\}$, then there are exactly $d(J) - 1$ ideals in $\mathbf{P}(J)$ strictly between $(J^m)_a$ and $(J^{m+1})_a$ (by (4.2)(d)) and there are exactly $d(I) - 1$ ideals in $\mathbf{P}(I)$ strictly between $(I^m)_a$ and $(I^{m+1})_a$ (by (4.2)(d)). Therefore, it follows that in the chain $(J^{kd(I)})_a \supsetneqq (J^{kd(I)+1})_a \supsetneqq \cdots \supsetneqq (J^{(k+1)d(I)})_a$ there are exactly $d(I)d(J) - 1$ ideals in $\mathbf{P}(J)$ strictly between $(J^{kd(I)})_a$ and $(J^{(k+1)d(I)})_a$, and in the chain $(I^{kn})_a \supsetneqq (I^{kn+1})_a \supsetneqq \cdots \supsetneqq (I^{(k+1)n})_a$ there are exactly $nd(I) - 1$ ideals in $\mathbf{P}(I)$ strictly between $(I^{kn})_a$ and $(I^{(k+1)n})_a$. Since the first and last ideals in these two chains are the same (namely, $(I^{kn})_a = (J^{kd(I)})_a$ and $(I^{(k+1)n})_a = (J^{(k+1)d(I)})_a$), and since $\mathbf{P}(J) = \mathbf{P}(I)$ (by (4.8.1)) (and since $\mathbf{P}(J)$ and $\mathbf{P}(I)$ are linearly ordered by inclusion), it follows that $nd(I) - 1 = d(J)d(I) - 1$, hence $n = d(J)$.

For (4.8.3), it follows from (4.8.2) that $(J^{d(I)})_a = (I^{d(J)})_a$ and $(H^{d(I)})_a = (I^{d(H)})_a$. Also, $(J^j)_a = (H^h)_a$, by hypothesis, so it follows that $(I^{jd(J)})_a = (J^{jd(I)})_a = (H^{hd(I)})_a = (I^{hd(H)})_a$. Therefore $jd(J) = hd(H)$, and $(J^j)_a = (H^h)_a$ (by hypothesis), so (3.2.2) shows that $(J^{d(H)})_a = (H^{d(J)})_a$.

For (4.8.4), let $\delta \in \mathbf{U}(I)$, so $\delta = \frac{m}{d(I)}$ for some $m \in \mathbb{N}_+$, by (4.2)(d). Then $I_\delta = I_{\frac{m}{d(I)}} \in \mathbf{P}(I)$ and $({I_\delta}^{d(I)})_a = (I^m)_a$ (by (4.7.2)) and $({I_\delta}^{d(I)})_a = (I^{d(I_\delta)})_a$ (by (4.8.3)), so $m = d(I_\delta)$, hence $\delta = \frac{d(I_\delta)}{d(I)}$.

For (4.8.5), since $J \subsetneqq H$, it follows from (4.8.3) that $(H^{d(J)})_a = (J^{d(H)})_a \subsetneqq (H^{d(H)})_a$, so $d(J) > d(H)$.

For (4.8.6), let $\delta < \gamma$ in $\mathbf{U}(I)$ and let $H = I_\delta$ and $J = I_\gamma$ in $\mathbf{P}(I)$. Then $H \supsetneqq J$, $\delta = \frac{d(H)}{d(I)}$ and $\gamma = \frac{d(J)}{d(I)}$ (by (4.8.4)), and $d(H) < d(J)$. Let $\alpha \in \mathbf{U}(I)$, so $\alpha = \frac{d(I_\alpha)}{d(I)}$ (by (4.8.4)), and (4.7.1) shows that $\frac{\alpha}{\delta} = \frac{(d(I_\alpha))/(d(I))}{(d(H))/(d(I))} = \frac{d(I_\alpha)}{d(H)} \in \mathbf{U}(H)$ and $\frac{\alpha}{\gamma} = \frac{(d(I_\alpha))/(d(I))}{(d(J))/(d(I))} = \frac{d(I_\alpha)}{d(J)} \in \mathbf{U}(J)$.

Now for $\beta \geq n^*(H)$ in $\mathbf{U}(H)$ there exists $m \in \mathbb{N}$ such that $\beta = n^*(H) + \frac{m}{d(H)}$. Therefore for $m \in \mathbb{N}$ define β_m by $\beta_m = n^*(H)+\frac{m}{d(H)} = \frac{n^*(H)d(H)+m}{d(H)}$, so $\beta_m \in \mathbf{U}(H)$ (by the definition of $n^*(H)$ and $d(H)$), $\beta_m\frac{\delta}{\gamma} \in \mathbf{U}(J)$ (by (4.7.1)), and $\beta_m\frac{\delta}{\gamma} = \beta_m\frac{d(H)}{d(J)} = \frac{n^*(H)d(H)+m}{d(J)}$. Since $d(H) < d(J)$, let $z \in \mathbb{N}_+$ such that $d(H) = d(J) - z$, so $\frac{n^*(H)d(H)+m}{d(J)} = \frac{n^*(H)(d(J)-z)+m}{d(J)} = (n^*(H)+\frac{-zd(H)}{d(J)})+\frac{m}{d(J)}$, and this holds for all $m \in \mathbb{N}$. Therefore define n to be $n^*(H)-w$, where w is defined by $wd(J) = zn^*(H) - r$ with $r \in \mathbb{N}$ such that $0 \le r < d(J)$ (note that $n = n^*(H) - w \le n^*(H)$). Then it follows (from the preceding computation) that if $m \ge r$ (say $m = r + k$ with $k \in \mathbb{N}$), then

$$(*) \quad \textit{for all } k \in \mathbb{N},\ \beta_{r+k} = \frac{n^*(H)d(H) + (r+k)}{d(H)} \in \mathbf{U}(H) \textit{ and } \beta_{r+k}\frac{\delta}{\gamma} = n + \frac{k}{d(J)} \in \mathbf{U}(J).$$

Therefore, to show that n is the desired $n^*(J)$, it remains to show that: (i) for each $k \in \mathbb{N}$, $n + \frac{k}{d(J)} \in \mathbf{U}(J)$; and, (ii) if $\sigma \in \mathbf{U}(J)$ is such that $\sigma \ge n$, then $\sigma = n + \frac{k}{d(J)}$ for some $k \in \mathbb{N}$. However, (i) follows immediately from (*). And for (ii), σ can be written in the form $\frac{g}{d(J)}$ for some $g \in \mathbb{N}_+$, so since $\sigma \ge n$ it follows that $g = d(J)n + k$ for some $k \in \mathbb{N}$, hence $\sigma = \beta_{r+k}\frac{\delta}{\gamma}$ (by (*)). Therefore $n^*(J)$ may be taken to be n, and then $n^*(J) \le n^*(H)$.

For the final statement assume that $\delta \ge n^*(I) \in \mathbf{U}(I)$ and let $\beta \ge 1$ in $\mathbf{U}(I_\delta)$. Then $\delta = \frac{d(I_\delta)}{d(I)}$, by (4.8.4), and $\beta = \frac{\alpha}{\delta}$ for some $\alpha \ge \delta$ (with $\alpha \in \mathbf{U}(I)$). Therefore $\alpha = \delta + \frac{m}{d(I)}$ for some $m \in \mathbb{N}$ (since $\delta \ge n^*(I)$), so $\beta = \frac{\alpha}{\delta} = 1 + \frac{m}{\delta d(I)} = 1 + \frac{m}{d(I_\delta)} \in \mathbf{U}(I_\delta)$. It follows that $n^*(\delta)$ may be chosen to be 1. ■

Definition 4.9 A regular ideal I in a Noetherian ring R is said to be **projectively full** in case the only integrally closed ideals that are projectively equivalent to I are the ideals $(I^k)_a$ with $k \in \mathbb{N}_+$. If there exists $J \in \mathbf{P}(I)$ such that J is projectively full, then we say that $\mathbf{P}(I)$ is **projectively full**. Such an ideal J, if it exists, must be the largest element of $\mathbf{P}(I)$.

Remark 4.10 Concerning (4.9), note that it follows from (4.2)(d) that if the greatest common divisor of the integers $e_1, \dots, e_g$ of (2.2) is 1, then I is projectively full. In particular, if P is a prime ideal in R such that R_P is a regular local ring, then P is projectively full (since the integer e of (2.2) is 1 for the order valuation of R_P).

Proposition 4.11 *The following are equivalent for a regular ideal I in a Noetherian ring R:*

(4.11.1) *There exists $K \in \mathbf{P}(I)$ that is projectively full.*

(4.11.2) *There exists $K \in \mathbf{P}(I)$ such that $\mathbf{U}(K) = \mathbb{N}_+$.*

(4.11.3) *There exists $K \in \mathbf{P}(I)$ such that $d(K) = 1$.*

(4.11.4) $I_{1/d(I)} \in \mathbf{P}(I)$.

Proof. Assume that (4.11.1) holds, let $\beta \in \mathbf{U}(K)$, and let $J = K_\beta$. Then J is an integrally closed ideal that is projectively equivalent to K, so there exists $k \in \mathbb{N}_+$ such that $J = (K^k)_a$ (by hypothesis) and $(K^k)_a = K_k$ (by (4.7.2)), hence $\beta = k$ (since $J = K_\beta$). Therefore $\mathbf{U}(K) \subseteq \mathbb{N}_+$, and the opposite inclusion is clear (since $K_k = (K^k)_a \in \mathbf{U}(K)$ for all $k \in \mathbb{N}_+$), hence (4.11.1) $\Rightarrow$ (4.11.2).

Assume that (4.11.2) holds. Now for all $k \in \mathbb{N}_+$ it holds that $n^*(K) + \frac{k}{d(K)} \in \mathbf{U}(K) = \mathbb{N}_+$. Therefore $d(K) = 1$, so(4.11.2) $\Rightarrow$ (4.11.3).

Assume that (4.11.3) holds. Then $(I^{d(K)})_a = (K^{d(I)})_a$, by (4.8.2), so $I_{1/d(I)} = I_{d(K)/d(I)}$ (by hypothesis) $= K$, by (4.2)(a), and $K \in \mathbf{P}(I)$, by hypothesis. Therefore $I_{1/d(I)} \in \mathbf{P}(I)$, so (4.11.3) $\Rightarrow$ (4.11.4).

Finally, assume that (4.11.4) holds and let $J \in \mathbf{P}(I)$. Then $(J^{d(I)})_a = (I^{d(J)})_a$, by (4.8.2), and $(I^{d(J)})_a = I_{d(J)}$, by (4.2.2)(a), so $J = I_{d(J)/d(I)}$ (by (4.7.2)) $= ({I_{1/d(I)}}^{d(J)})_a$ (by (4.2.2)(a)). Therefore, since $I_{1/d(I)} \in \mathbf{P}(I)$, it follows from (4.9) that $\mathbf{P}(I)$ is projectively full, hence (4.11.4) $\Rightarrow$ (4.11.1). ■

Remark 4.12 Let I be a regular ideal of a Noetherian ring R. Then it follow from (4.2.2)(d) that:

(a) I is projectively full if and only if $d(I) = 1$.

(b) I is projectively full if and only if there exists a large $k \in \mathbb{N}_+$ such that $I_k \supsetneq I_{k+1}$ are consecutive in $\mathbf{P}(I)$ if and only if this holds for all $k \in \mathbb{N}_+$.

Remark 4.13 Let I be a nonzero proper ideal of a two-dimensional regular local domain R and let $I_a = J_1^{e_1} \cdots J_n^{e_n}$ be the factorization of I_a as a product of distinct simple complete ideals. Let d be the greatest common divisor of $e_1, \dots, e_n$ and let $f_i = e_i/d$ for $i = 1, \dots, n$.

Then $K = J_1^{f_1} \cdots J_n^{f_n}$ is projectively full and $K^d = I_a$, so $\mathbf{P}(I) = \mathbf{P}(K)$. Therefore $\mathbf{P}(I)$ is projectively full for every nonzero proper ideal of a two-dimensional regular local domain.

In general, if R is a two-dimensional normal local domain with maximal ideal M and I is an M-primary ideal, the set $\mathbf{P}(I)$ need not contain a projectively full ideal as we illustrate in Example 4.14. In this example, the ideal I has only one Rees valuation and the numerical semigroup $S(I)$ associated to I is $(2,3)\mathbb{N}$.

Example 4.14 Let k be an algebraically closed field of characteristic zero and let $R = k[[x,y,z]]$, where $z^2 = x^3 + y^7$. It is readily seen that R is a two-dimensional normal local domain. Consider the ideal $I = (x, y^2)R$. It is shown in [3, Example 16, page 300] that I has a unique Rees valuation ring V. Therefore $xV = y^2V$, $I_a = xV \cap R$, and the image of x/y^2 in the residue field of V is transcendental over k. The equality $(z/y^3)^2 = (x/y^2)^3 + y$ implies that z/y^3 is integral over $R[x/y^2]$. It also implies that $zV = y^3V$ and that the image of z/y^3 in the residue field of V is transcendental over k. Let $J = y^3V \cap R$. Then $(z, y^3, xy, x^2)R \subseteq J$. To show I and J are projectively equivalent it suffices to show that I^3 and J^2 have the same integral closure. Since $I = (x, y^2)R$ and V is the unique Rees valuation of I, the integral closure of I^3 is $I^3V \cap R$ and is the integral closure of $(x^3, y^6)R$. We have $J^2V = I^3V$. Therefore $(J^2)_a \subseteq (I^3)_a$. To show the reverse inclusion, it suffices to observe that x^3 and y^6 are in J^2. Since $y^3 \in J$, it is clear that $y^6 \in J^2$. Also we have $x^3 = z^2 - y^7$ and z^2 and y^7 are in J^2, so $x^3 \in J^2$. Therefore I and J are projectively equivalent. Notice that $I_a = (y^2, x, z)R \subsetneq M$ and there are no ideals properly between I_a and M. To complete the proof that $\mathbf{P}(I)$ is not projectively full, it suffices to observe that V is not a Rees valuation of M. Since $(x, y)R$ is a reduction of M, the Rees valuation rings of M are all extensions of the order valuation defined by the powers of the maximal ideal of the two-dimensional regular local subdomain $k[[x,y]]$ of R. In particular, if W is a Rees valuation ring of M, then $xW = yW$. Since $xV = y^2V$, V is not a Rees valuation ring of M. Let v denote the normalized valuation with value group $\mathbb{Z}$ associated to the valuation domain V. We have $v(y) = 1$, $v(x) = 2$, $v(z) = 3$, $v(I) = 2 = d$, $v(J) = 3$, and $\mathbf{P}(I) = \{I, J, (I^2)_a, (IJ)_a, (I^3)_a = (J^2)_a, (I^2J)_a, \dots\}$.

An interesting question we have not been successful in answering is whether for a regular ideal I in a Noetherian ring R there always exists a finite integral extension ring T of R such that $\mathbf{P}(IT)$ contains a projectively full ideal. If R is a one-dimensional Noetherian domain, then the integral closure R' of R is a Dedekind domain and it is easily seen that $\mathbf{P}(IR')$ contains a projectively full ideal.

Remark 4.15 Let I be a nonzero proper ideal of a Noetherian integrally closed domain R. With d as in (4.2.2)(d), it is natural to ask if there exist $x \in R$ such that $\overline{v}_I(x) = 1/d$. To illustrate that this is not true in general, let s, t be algebraically independent elements over the field k and let $R = k[s, t]$. Consider the ideal $I = (s^2, t^3)R$. We observe that there exists a unique $V \in \text{Rees } I$. Indeed, for $V \in \text{Rees } I$ we have $s^2V = t^3V$ and the image of s^2/t^3 in the residue field of V is transcendental over k. Therefore $z = s/t$ is in the maximal ideal of V. We have $s = tz$ and $R[z] = k[t, z]$. Moreover, $I_a = (s^2, st^2, t^3)R$ and $V \in \text{Rees } I$ is centered on a height-one prime ideal of $R[s^2/t^3, st^2/t^3 = s/t]$ that lies over the maximal ideal $(s, t)R$ of R. Since $s^2/t^3 = z^2t^2/t^3 = z^2/t$, we see that V is a localization of $k[t, z][z^2/t]$ at a height-one prime ideal that contains $M = (t, z)k[t, z]$. Since $P = Mk[t, z][z^2/t]$ is a height-one prime ideal, we see that V is the localization of $k[t, z][z^2/t]$ at P. Let v denote the normalized valuation associated to V. Then $v(t) = 2$, $v(s) = 3$, and $v(I) = 6$. The integer d of (4.2.2)(d) is 6, while for $x \in R$ the smallest possible positive value of $\overline{v}_I(x)$ is 1/3. Indeed, $\overline{v}_I(t) = 1/3$ and $\overline{v}_I(s) = 1/2$, and therefore $d = 6$.

Remark 4.16 Let H, I, K and J be ideals of a Noetherian domain R. In analogy to a result that holds for reductions of ideals, it is natural to ask whether H is projectively equivalent to I and K is projectively equivalent to J implies that $H + K$ is projectively equivalent to $I + J$. To illustrate that this is not true in general, let s, t be algebraically independent elements over the field k and let $R = k[s, t]$. Let $H = (s, t^2)R$ and $I = (s^2, t^4)R$. Also let $K = J = (s^2, t)R$. Then H and I are projectively equivalent as are also K and J, but $H + K = (s, t)R$ is not projectively equivalent to $I + J = (s^2, t)R$.

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