GENERIC FIBER RINGS OF MIXED POLYNOMIAL/POWER SERIES RINGS

William Heinzer

Purdue University

joint with Christel Rotthaus and Sylvia Wiegand

MOTIVATION

A question raised by Mel Hochster:

Question. Can one describe or somehow classify the local maps $R \hookrightarrow S$ of complete local domains R and S such that

$$P \in \operatorname{Spec}(S), \quad P \neq (0) \implies P \cap R \neq (0)?$$

Hochster remarks that if, for example, R is equal characteristic zero, such extensions arise as

$$R = K[[x_1, \ldots, x_m]] \hookrightarrow T = L[[x_1, \ldots, x_m, y_1, \ldots, y_n]] \to T/P = S,$$

where K is a subfield of L and $P \in \operatorname{Spec}(T)$ is maximal with respect to $P \cap R = \{0\}$.

Partial Answer. In Hochster's set-up,

$$[L : K] < \infty \implies \dim S = 2 \text{ or } \dim S = m.$$

Main Results

Let K be a field, m and n positive integers, $X = \{x_1, \dots, x_m\}$, $Y = \{y_1, \dots, y_n\}$ sets of independent variables over K.

Let A be the localized polynomial ring $K[X]_{(X)}$.

Theorem 1. Every prime ideal P of $\widehat{A} = K[[X]]$ that is maximal with respect to $P \cap A = (0)$ has height $m - 1$.

Let $\quad B := K[[X]][Y]_{(X,Y)} \quad$ and $\quad C := K[Y]_{(Y)}[[X]]$.

Theorem 2. Every prime ideal P of $\widehat{B} = \widehat{C}$ that is maximal with respect to either $P \cap B = (0)$ or $P \cap C = (0)$, has height $m + n - 2$.

Theorem 3. Every prime ideal P of $K[[X, Y]]$ that is maximal with respect to $P \cap K[[X]] = (0)$ has height either n or $m + n - 2$.

Local Embeddings

There exist local embeddings:

$$A = K[X]_{(X)} \hookrightarrow \widehat{A} := K[[X]], \qquad \widehat{A} \hookrightarrow \widehat{B} = \widehat{C} = K[[X,Y]]$$

$$B = K[[X]]\,[Y]_{(X,Y)} \hookrightarrow C = K[Y]_{(Y)}[[X]] \hookrightarrow \widehat{B} = \widehat{C} = K[[X]]\,[[Y]].$$

Matsumura observes there exist $P \in \operatorname{Spec} \widehat{A}$ with $\operatorname{ht} P = m - 1$ and $P \cap A = (0)$, and also that there exist $P \in \operatorname{Spec} K[[X,Y]]$ with $\operatorname{ht} P = m + n - 2$ such that

$$P \cap C = (0) \quad \text{or} \quad P \cap B = (0).$$

However he does not address the question of whether all primes maximal with these properties have the same height.

Generic Fiber Rings

Let $R \hookrightarrow S$ be an injective homomorphism of commutative rings with R an integral domain. The **generic fiber ring** of the map $R \hookrightarrow S$ is the localization $(R \setminus (0))^{-1}S$ of S.

Let $(R, \mathbf{m})$ be a Noetherian local integral domain and let $\widehat{R}$ denote the $\mathbf{m}$-adic completion of R.

The **generic formal fiber ring** of R is the localization $(R \setminus (0))^{-1}\widehat{R}$ of $\widehat{R}$.

The **formal fibers** of R are the fibers of the map $\operatorname{Spec}\widehat{R} \to \operatorname{Spec} R$. For $P \in \operatorname{Spec} R$, the **formal fiber** over P is $\operatorname{Spec}(\,(R_P/PR_P) \otimes_R \widehat{R}\,)$.

Trivial Generic Fiber Extensions

Let R be a subring of an integral domain S.

Definition. $R \hookrightarrow S$ is a **trivial generic fiber** extension or a **TGF** extension if

$$(0) \neq P \in \operatorname{Spec} S \implies P \cap R \neq (0).$$

One obtains a TGF extension S of R by considering

$$R \hookrightarrow T \to T/P := S,$$

where T is an extension ring of R and $P \in \operatorname{Spec} T$ is maximal with respect to $P \cap R = (0)$.

Thus the generic fiber ring of $R \hookrightarrow T$ is relevant to constructing TGF extensions S of R.

Variations on a theme of Weierstrass

The Weierstrass Preparation Theorem is our main technical tool.

Let P be a prime ideal in the power series ring $\widehat{A} = K[[X]]$, where $X = \{x_1, \ldots, x_n\}$ is a set of n variables over the field K.

Here $A = K[X]_{(X)}$ is the localized polynomial ring.

Notation. By a **change of variables**, we mean a finite sequence of 'polynomial' change of variables over K of the form

$$x_1 \mapsto x_1 + x_n^{e_1}, \quad \ldots \quad x_{n-1} \mapsto x_{n-1} + x_n^{e_{n-1}}, \quad x_n \mapsto x_n,$$

perhaps followed by

$$z_1 \mapsto z_1, \quad z_2 \mapsto z_2 + z_1^{f_2}, \quad \ldots \quad z_n \mapsto z_n + z_1^{f_n},$$

where $e_i, f_i \in \mathbb{N}$ and z_i denotes the image of x_i under the first map.

The Weierstrass Preparation Theorem

Theorem (Weierstrass) Let $(R, \mathbf{m})$ be a complete local ring, let $f \in R[[x]]$ be a formal power series and let $\overline{f}$ denote the image of f in $(R/\mathbf{m})[[x]]$. Assume that $\overline{f} \neq 0$ and that $\operatorname{ord} \overline{f} = s > 0$. There exists a unique ordered pair (u, F) such that u is a unit in $R[[x]]$ and $F \in R[x]$ is a distinguished monic polynomial of degree s such that $f = uF$.

Here $F = x^s + a_{s-1}x^{s-1} + \cdots + a_0 \in R[x]$ is **distinguished** if $a_i \in \mathbf{m}$ for $0 \leq i \leq s-1$.

Corollary. The ideal $fR[[x]]$ is extended from $R[x]$ and $R[[x]]/(f)$ is a free R-module of rank s. Every $g \in R[[x]]$ is of the form $g = qf + r$, where $q \in R[[x]]$ and $r \in R[x]$ is a polynomial with $\deg r \leq s - 1$.

Embeddings of Power Series Rings

Theorem (Classical) There exists a K-algebra embedding of the formal power series ring $R := K[[x_1, \dots, x_n]]$ into the formal power series ring in two variables $K[[y, z]]$.

Proof. Let $f_1, \dots, f_n$ be power series in $K[[y]]$ that are algebraically independent over K. Define $\varphi : R \to K[[y, z]]$ by $\varphi(x_i) = f_i z$, for $1 \le i \le n$. Suppose $g \in R$ is such that $\varphi(g) = 0$. Write $g = \bigoplus_{k=0}^{\infty} g_k$, where g_k is a form in $K[x_1, \dots, x_n]$ of degree k. Then $\varphi(g_k) = z^k g_k(f_1, \dots, f_n)$ with $g_k(f_1, \dots, f_n) \in K[[y]]$. Hence

$$\varphi(g) = \bigoplus_{k=0}^{\infty} g_k(f_1, \dots, f_n) z^k = 0$$

implies $g_k(f_1, \dots, f_n) = 0$ for each $k \in \mathbb{N}$. Since $f_1, \dots, f_n$ are algebraically independent over K, each $g_k = 0$, so $g = 0$.

Cohen's Theorem 8

Theorem (Classical) Let I be an ideal of a ring R and let M be an R-module. Assume that R is complete in the I-adic topology and $\bigcap_{n=1}^{\infty} I^n M = (0)$. If M/I is generated over R/I by elements $\overline{w}_1, \dots, \overline{w}_s$ and w_i is a preimage in M of $\overline{w}_i$ for $1 \leq i \leq s$, then M is generated over R by $w_1, \dots, w_s$.

Corollary. Assume $\varphi : (R, \mathbf{m}) \to (S, \mathbf{n})$ is a local homomorphism. If R is $\mathbf{m}$-adically complete, $\mathbf{m}\, S$ is $\mathbf{n}$-primary and $S/\,\mathbf{n}$ is finite over $R/\,\mathbf{m}$, then S is a finitely generated R-module.

If $\varphi : R \to K[[y, z]] = S$ is an embedding of the power series ring in $n > 2$ variables, then $\mathbf{m}\, S$ is not $\mathbf{n}$-primary.

The Generic Formal Fiber Rings of B and C

Since $B := K[[X]][Y]_{(X,Y)} \hookrightarrow K[Y]_{(Y)}[[X]] := C$,

if $P \in \operatorname{Spec} K[[X, Y]]$ and $P \cap C = (0)$, then $P \cap B = (0)$.

We know $P \in \operatorname{Spec} K[[X, Y]]$ maximal with respect to $P \cap B = (0)$

or maximal with respect to $P \cap C = (0) \implies \operatorname{ht} P = n + m - 2$.

Hence if $P \in \operatorname{Spec} K[[X, Y]]$ is maximal with $P \cap C = (0)$,

then P is also maximal with respect to $P \cap B = (0)$.

However, if $n \geq 2$, the generic fiber of $B \hookrightarrow C$ is nonzero.

This means $\exists \quad P \in \operatorname{Spec} K[[X, Y]]$ maximal in the generic formal

fiber of B but P not in the generic formal fiber of C.

The Generic Fiber of $B \hookrightarrow C$

Lemma. $B = K[[x]][y_1, y_2] \hookrightarrow K[y_1, y_2][[x]] = C$ is not TGF.

Proof. There exists $\sigma \in K[y_1][[x]]$ that is transcendental over $K[[x]][y_1]$. Let $\mathbf{q} = (y_2 - \sigma x)K[y_1, y_2][[x]]$ and define $\pi : K[y_1, y_2][[x]] \to K[y_1, y_2][[x]]/\mathbf{q} \cong K[y_1][[x]]$. Thus $\pi(y_2) = \sigma x$. If $h \in \mathbf{q} \cap (K[[x]][y_1, y_2])$, then $\exists\ s, t \in \mathbb{N}$ so that

$h = \sum_{i=0}^{s} \sum_{j=0}^{t} (\sum_{k \in \mathbb{N}} a_{ijk} x^k) y_1^i y_2^j, \quad$ where $a_{ijk} \in K$.

Hence $0 = \pi(h) = \sum_{i=0}^{s} \sum_{j=0}^{t} (\sum_{k \in \mathbb{N}} a_{ijk} x^k) y_1^i (\sigma x)^j$.

Since σ is transcendental over $K[[x]][y_1]$, each $a_{ijk} = 0$.

Therefore $\mathbf{q} \cap (K[[x]][y_1, y_2]) = (0)$, and $B \hookrightarrow C$ is not TGF.

Lemma. Let $R := K[[X]]$ and let $P \in \operatorname{Spec} R$ with $x_1 \notin P$ and $\operatorname{ht} P = r$, where $1 \le r \le n-1$. There exists a change of variables $x_1 \mapsto z_1 := x_1$ (x_1 is fixed), $x_2 \mapsto z_2, \ldots, x_n \mapsto z_n$ and a regular sequence $f_1, \ldots, f_r \in P$ so that, upon setting $Z_1 = \{z_1, \ldots, z_{n-r}\}$, $Z_2 = \{z_{n-r+1}, \ldots, z_n\}$ and $Z = Z_1 \cup Z_2$,

1. $(f_1, \ldots, f_r) \subset K[[Z_1]][Z_2]$ and P is a minimal prime of $(f_1, \ldots, f_r)R$.
2. The (Z_2)-adic completion of $K[[Z_1]]\,[Z_2]_{(Z)}$ is identical to the $(f_1, \ldots, f_r)$-adic completion and is $R = K[[X]] = K[[Z]]$.
3. If $P_1 := P \cap K[[Z_1]]\,[Z_2]_{(Z)}$, then $P_1 R = P$, that is, P is extended from $K[[Z_1]]\,[Z_2]_{(Z)}$.
4. The extension: $K[[Z_1]] \hookrightarrow K[[Z_1]]\,[Z_2]_{(Z)}/P_1 \cong K[[Z]]/P$ is finite (and integral).

$$
\begin{tikzcd}
 & R = K[[X]] = K[[Z_1, Z_2]] \arrow[dl, dash] \arrow[ddr, dash] & \\
(X)R \arrow[dd, dash] & & \\
 & & D = K[[Z_1]]\,[Z_2]_{(Z)} \arrow[ddl, dash] \\
P = P_1 R \arrow[dr, dash] & & \\
 & P_1 = P \cap D &
\end{tikzcd}
$$

$(f_1, \ldots, f_r) \subset P_1$ and $K[[Z_1]] \hookrightarrow D/P_1 \cong K[[Z]]/P$ is finite.

Theorem. Let y and $X = \{x_1, \ldots, x_m\}$ be variables over the field K. Assume V is a DVR with $K[y] \subseteq V \subseteq K[[y]]$ such that V has completion $K[[y]]$. Also assume the field $K((y)) = K[[y]]\,[1/y]$ has **uncountable** transcendence degree over the fraction field $\mathcal{Q}(V)$ of V. Set $R_0 := V[[X]]$ and $R = \widehat{R_0} = K[[y, X]]$. Let P be a prime ideal of R such that:

$$P \subseteq (X)R \text{ (so } y \notin P) \quad \text{and} \quad \dim(R/P) > 2.$$

Then there exists a prime ideal Q of R such that

1. $P \subset Q \subset XR$,
2. $\dim(R/Q) = 2$, and
3. $P \cap R_0 = Q \cap R_0$.

In particular, $P \cap K[[X]] = Q \cap K[[X]]$.

$$
\begin{tikzcd}[column sep=small]
R = K[[y,Z]] \arrow[dd, dash] & & \\
& & \mathcal{Q}(K[[y]]) = K[[y]]\,[1/y] = K((y)) \arrow[dd, dash] \\
D := K[[y,Z_1]]\,[Z_2]_{(Z)} \arrow[d, dash] & & \\
K[[y]] \arrow[uurr, dash] \arrow[dr, dash] & & E := \mathcal{Q}(T) = L(\gamma_2, \ldots, \gamma_{n-r}) \arrow[dl, dash] \arrow[dd, dash] \\
& T := L(\gamma_2, \ldots, \gamma_{n-r}) \cap K[[y]] \arrow[dd, dash] & \\
& & L := \mathcal{Q}(S) = \mathcal{Q}(V(\Delta)) \arrow[dl, dash] \arrow[dd, dash] \\
& S := \mathcal{Q}(V(\Delta)) \cap K[[y]] \arrow[dd, dash] & \\
& & \mathcal{Q}(V) \arrow[dl, dash] \\
& K[y] \subseteq V &
\end{tikzcd}
$$

Prime Ideals With The Same Contraction

Corollary. Let K be a field, y and $X = \{x_1, \dots, x_m\}$ variables over K and let $R = K[[y, X]]$. Assume $P \in \operatorname{Spec} R$ is such that:

(i) $P \subseteq (x_1, \dots, x_m)R$ and

(ii) $\dim(R/P) > 2$.

Then there exists a prime ideal Q of R such that

1. $P \subset Q \subset XR$,
2. $\dim(R/Q) = 2$, and
3. $P \cap K[y]_{(y)}[[X]] = Q \cap K[y]_{(y)}[[X]]$.

In particular, $P \cap K[[x_1, \dots, x_m]] = Q \cap K[[x_1, \dots, x_m]]$.

$$
\begin{tikzcd}
 & R = K[[y, X]] \arrow[dl, dash] \arrow[ddr, dash] & \\
(X)R \arrow[dd, dash] & & \\
 & & R_0 = K[y]_{(y)}\,[[X]] \arrow[ddl, dash] \\
P \subset Q \arrow[dr, dash] & & \\
 & P \cap R_0 = Q \cap R_0 &
\end{tikzcd}
$$

$$\dim R/Q = 2 \quad \text{and} \quad P \cap K[[X]] = Q \cap K[[X]]$$

Generic Fibers of Power Series Extensions

Theorem. Let y and $X = \{x_1, \ldots, x_m\}$ be variables over the field K. Consider the extension $\widehat{A} = K[[X]] \hookrightarrow K[[X]][[y]] = \widehat{B}$. Let P be a prime ideal of $\widehat{B}$ such that $P \cap \widehat{A} = (0)$. Then

1. If $P \not\subseteq X\widehat{B}$, then $\dim \widehat{B}/P = m$ and P is maximal with respect to $P \cap \widehat{A} = (0)$.
2. If $P \subseteq X\widehat{B}$, then there exists Q with $P \subseteq Q$, $\dim \widehat{B}/Q = 2$, and Q is maximal with respect to $Q \cap \widehat{A} = (0)$.

If $m > 2$, then for each prime Q maximal with respect to $Q \cap \widehat{A} = (0)$, we have either

(i) $\dim \widehat{B}/Q = m$ and $\widehat{A} \hookrightarrow \widehat{B}/Q$ is finite, or

(ii) $\dim \widehat{B}/Q = 2$ and $Q \subset X\widehat{B}$.

Generic Fibers of Power Series Extensions 2

Theorem. Let K be a field, m and n positive integers, $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_n\}$ sets of independent variables over K. Consider the extension

$$\widehat{A} = K[[X]] \hookrightarrow K[[X]][[Y]] = \widehat{B}.$$

Let Q be a prime ideal of $\widehat{B}$ maximal with respect to $Q \cap \widehat{A} = (0)$.

1. If $m = 1$, then $\dim \widehat{B}/Q = 1$ and $\widehat{A} \hookrightarrow \widehat{B}/Q$ is finite.

2. If $m \geq 2$, there are two possibilities:

(i) $\widehat{A} \hookrightarrow \widehat{B}/Q$ is finite, in which case $\dim \widehat{B}/Q = \dim \widehat{A} = m$, or

(ii) $\dim \widehat{B}/Q = 2$.

Mixed Polynomial/Power Series Rings

Consider

$$A := K[x, y] \hookrightarrow B := K[[x]][y] \hookrightarrow C := K[y][[x]]$$

$$\hookrightarrow D := K[y][[x/y]] \hookrightarrow E := K[y, 1/y][[x]]$$

where K is a field and x and y are indeterminates over K.

We are interested in the prime spectra of these rings and the maps on the spectra determined by the inclusion maps on the rings. For example, do there exist nonzero primes of one of the larger rings that intersect a smaller ring in zero (i.e. non-TGF extensions).

Motivation 2

From the introduction to a paper by Alonzo-Tarrio, Jeremias-Lopez and Lipman:

If a map between noetherian formal schemes can be factored as a closed immersion followed by an open one, can it also be factored as an open immersion followed by a closed one?

Brian Conrad observed that a counterexample can be constructed for every triple (R, x, p), where

1. R is an adic domain, that is, R is a Noetherian domain that is separated and complete with respect to the powers of a proper ideal I.
2. $x \in R$ is nonzero, the completion of $R[1/x]$ with respect to the powers of $IR[1/x]$, denoted $S := R_{\{x\}}$, is an integral domain.
3. p is a nonzero prime ideal of S that intersects R in (0).

Motivation 2 continued

If (R, x, p) is such a triple with $S = R_{\{x\}}$, then the composition $R \to S \to S/p$ determines a map on formal spectra $\mathrm{Spf}(S/p) \to \mathrm{Spf}(S) \to \mathrm{Spf}(R)$ that is a closed immersion followed by an open one.

For a surjection such as $S \to S/p$ of adic rings gives rise to a closed immersion $\mathrm{Spf}(S/p) \to \mathrm{Spf}(S)$, while a localization like that of R with respect to x followed by the completion of $R[1/x]$ with respect to the powers of $IR[1/x]$ to obtain S gives rise to an open immersion $\mathrm{Spf}(S) \to \mathrm{Spf}(R)$.

However, the map $\mathrm{Spf}(S/p) \to \mathrm{Spf}(R)$ cannot be factored as an open immersion followed by a closed one. For a closed immersion into $\mathrm{Spf}(R)$ corresponds to a surjective map of adic rings $R \to R/J$, where J is an ideal of R.

Motivation 2 continued

If the immersion $\mathrm{Spf}(S/p) \to \mathrm{Spf}(R)$ factored as an open immersion followed by a closed one, we would have R-algebra homomorphisms from $R \to R/J \to S/p$, where $\mathrm{Spf}(S/p) \to \mathrm{Spf}(R/J)$ is an open immersion. Since $p \cap R = (0)$, we must have $J = (0)$. This implies $\mathrm{Spf}(S/p) \to \mathrm{Spf}(R)$ is an open immersion, that is, the composite map $\mathrm{Spf}(S/p) \to \mathrm{Spf}(S) \to \mathrm{Spf}(R)$, is an open immersion. But also $\mathrm{Spf}(S) \to \mathrm{Spf}(R)$ is an open immersion. It follows that $\mathrm{Spf}(S/p) \to \mathrm{Spf}(S)$ is both open and closed. Since S is an integral domain this implies $\mathrm{Spf}(S/p) \cong \mathrm{Spf}(S)$. This is a contradiction since p is nonzero.

One of our motivations is to describe such triples (R, x, p).

An example of (R, x, p).

An example described in [AJL]: Let w, x, y, z be indeterminates over a field K and consider

$$R := K[w, x, z][[y]] \hookrightarrow S := R_{\{x\}} = K[w, x, 1/x, z][[y]].$$

Let $P = (w, z)R$ and $T := R_P \subset S_{PS} := G$.

Then $T \subset G$ are 2-dim regular local domains, and the residue field of G (i.e. the fraction field of $K[x, 1/x][[y]]$) is transcendental over that of T (i.e. the fraction field of $K[x][[y]]$).

It follows by [HR, p. 364, Theorem 1.12] that there exist infinitely many height-one prime ideals of G in the generic fiber over T. Any one of these prime ideals of G contracts in S to a prime p as above.

Is There A Simpler Example?

In some correspondence to Lipman, Conrad asked:

Question. Is there a nonzero prime ideal of $E := K[x, 1/x][[y]]$ that intersects $C = K[x][[y]]$ in zero?

If there were such a prime ideal, then

$$C := K[x][[y]] \hookrightarrow E := K[x, 1/x][[y]]$$

would be a simpler counterexample to the assertion that a closed immersion followed by an open one also has a factorization as an open immersion followed by a closed one.

We show there is no such prime ideal.

A two-dimensional TGF extension.

Let K be a field and let x and y be indeterminates over K.

Remark. $C := K[x][[y]] \hookrightarrow E := K[x, 1/x][[y]]$ is a TGF extension.

It is true that $\mathcal{Q}(E)$ has infinite transcendence degree over $\mathcal{Q}(C)$. Notice that $\dim C = \dim E = 2$ and y is in every maximal ideal of C or E. Let P be a nonzero prime of E. If $y \in P$, then $P \cap C \neq (0)$. Assume $y \notin P$ and $P \neq (0)$. Then $K[[y]] \hookrightarrow C/(P \cap C) \hookrightarrow E/P$ and E/P is finite over $K[[y]]$ and hence also over $C/(P \cap C)$, so $P \cap C \neq (0)$.

Question. Is $K[x, z][[y]] \hookrightarrow K[x, 1/x, z][[y]]$ a TGF extension?

An Example of a TGF extension.

Example. The extension $R := K[[x,y]] \hookrightarrow S := K[[x]][[\frac{y}{x}]]$ is a TGF extension of the type described in Hochster's question.

To see this extension is TGF, it suffices to show $P \cap R \neq (0)$ for each $P \in \operatorname{Spec} S$ with $\operatorname{ht} P = 1$.

This is clear if $x \in P$, while if $x \notin P$, then $K[[x]] \cap P = (0)$, so $K[[x]] \hookrightarrow R/(P \cap R) \hookrightarrow S/P$ and S/P is finite over $K[[x]]$. Therefore $\dim R/(P \cap R) = 1$, so $P \cap R \neq (0)$.

Remark. The extension $K[[x,y]] \hookrightarrow K[[x, \frac{y}{x}]]$ is, up to isomorphism, the same as the extension $K[[x,xy]] \hookrightarrow K[[x,y]]$.