PROJECTIVELY FULL IDEALS IN NOETHERIAN RINGS

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SETUP AND DEFINITIONS

Let I be a regular proper ideal of a Noetherian ring R.

DEFINITION. An ideal J of R is **projectively equivalent** to I if there exist positive integers m and n such that I^m and J^n have the same integral closure, i.e., $(I^m)_a = (J^n)_a$.

NOTATION. Let $\mathbf{P}(I)$ denote the set of integrally closed ideals projectively equivalent to I.

FACT. The set $\mathbf{P}(I)$ is linearly ordered and discrete with respect to inclusion.

MORE SETUP AND DEFINITIONS

Since $\mathbf{P}(I)$ is discrete and since $J \in \mathbf{P}(I)$ implies $(J^n)_a \in \mathbf{P}(I)$, there is naturally associated to I and $\mathbf{P}(I)$ a numerical semigroup, i.e., a subsemigroup $S(I)$ of the additive semigroup of nonnegative integers $\mathbb{N}_0$ that contains all sufficiently large integers.

DEFINITION. The set $\mathbf{P}(I)$ is said to be **projectively full** if $S(I) = \mathbb{N}_0$, or equivalently, if every element of $\mathbf{P}(I)$ is the integral closure of a power of the largest element K of $\mathbf{P}(I)$, i.e., every element of $\mathbf{P}(I)$ has the form $(K^n)_a$, for some positive integer n. If this holds, the ideal K is said to be **projectively full**.

MAIN GOALS

1. Describe ideals K that are projectively full.

2. Describe ideals I such that $\mathbf{P}(I)$ is projectively full.

3. Can things be improved by passing to an integral extension?

THEOREM. If R contains the field of rational numbers, then there exists a finite free integral extension ring A of R such that $\mathbf{P}(IA)$ is projectively full; and if R is an integral domain, then there also exists a finite integral extension domain B of R such that $\mathbf{P}(IB)$ is projectively full.

EXAMPLES

EXAMPLE 1. Let (R, M) be a Noetherian local ring having the property

$$a \in M^i \setminus M^{i+1} \text{ and } b \in M^j \setminus M^{j+1} \implies ab \notin M^{i+j+1},$$

then M is projectively full. Thus if the associated graded ring

$$\mathbf{G}(R, M) = R/M \oplus M/M^2 \oplus \cdots \oplus M^n/M^{n+1} \oplus \cdots$$

is an integral domain, then M is projectively full.

EXAMPLE 2. Let k be a field and let $R = k[[x^2, x^3]]$. Then the maximal ideal $M = (x^2, x^3)R$ is not projectively full. The numerical semigroup $S(M)$ is generated by 2 and 3.

DISCUSSION

With $R = k[[x^2, x^3]]$ as in Example 2, R is not integrally closed. Things improve by passing to the integral extension $R[x] = k[[x]]$. For each regular proper ideal I of R, $\mathbf{P}(IR[x])$ is projectively full.

Want to give an example of a normal local domain (R, M) such that M is not projectively full?

EXAMPLE 3. Let F be a field and let x, y, z, w be variables. Let

$$R_0 = F[x, y]_{(x,y)} \text{ and } R = \frac{R_0[z, w]}{(z^2 - x^3 - y^3,\ w^2 - x^3 + y^3)}.$$

If 2 and 3 are units of F, then R is a 2-dim normal local domain, and the maximal ideal $M = (x, y, z, w)R$ is not projectively full.

SOME HISTORY

The concept of projective equivalence of ideals and the study of ideals projectively equivalent to I was introduced by Samuel in

Some asymptotic properties of powers of ideals, Annals of Math 56 (1952), 11-21.

and further developed by Nagata in

Note on a paper of Samuel concerning asymptotic properties of ideals, Mem. Coll. Sci. Univ. Kyoto, Ser. A Math. 30 (1957), 165-175.

MORE HISTORY

Rather than 'projectively equivalent', Hartmut Göhner uses the term 'asymptotically equivalent' in

Semifactoriality and Muhly's condition (N) *in two-dimensional local rings*, J. Algebra 34 (1975), 403-429.

Göhner mentions that the expression 'projective asymptotic equivalence' is used by David Rees in

Valuations associated with ideals (II), J. London Math. Soc. 36 (1956), 221-228.

and by H. T. Muhly in

On the existence of asymptotically irreducible ideals, J. London Math. Soc. 40 (1965), 99-107.

RESULTS OF REES

Let I be a regular proper ideal of a Noetherian ring R.

For each $x \in R$, let $v_I(x) = \max\{k \in \mathbb{N} \mid x \in I^k\}$, and define

$$\overline{v}_I(x) = \lim_{k\to\infty}(\frac{v_I(x^k)}{k}).$$

Rees established that:

(a) $\overline{v}_I(x)$ is well defined;

(b) for each $k \in \mathbb{N}$ and $x \in R$, $\overline{v}_I(x) \geq k \iff x \in (I^k)_a$

(c) there exist discrete valuations $v_1, \ldots, v_g$ defined on R, and positive integers $e_1, \ldots, e_g$ such that, for each $x \in R$,

$\overline{v}_I(x) = \min\{\frac{v_i(x)}{e_i} \mid i = 1, \ldots, g\}$.

Describing the Rees Valuation Rings

For simplicity, assume R is a Noetherian integral domain with field of fractions F. Let t be an indeterminate, The **Rees ring** of R with respect to I is the graded subring

$$\mathbf{R} = R[t^{-1}, It] = \bigoplus_{n \in \mathbb{Z}} I^n t^n$$

of the Laurent polynomial ring $R[t^{-1}, t]$. The integral closure B of $R[t^{-1}, It]$ is a Krull domain, and B_P is a DVR for each minimal prime P of $t^{-1}B$. The set $\text{Rees}\, I$ of **Rees valuation rings** of I is precisely the set of rings $V = B_P \cap F$, where P is a minimal prime of $t^{-1}B$.

THE REES INTEGERS OF I

Let $(V_1, N_1), \dots, (V_g, N_g)$ be the Rees valuation rings of I. The integers $(e_1, \dots, e_g)$, where $IV_i = N_i^{e_i}$, are the **Rees integers** of I.

PROPOSITION. A sufficient condition for I to be projectively full is that $\gcd(e_1, \dots, e_g) = 1$. This is not a necessary condition.

EXAMPLE 4. Let (R, M) be a 2-dimensional regular local ring with $M = (x, y)R$. The ideal $I = (x, y^2)R$ is integrally closed with unique Rees valuation ring $V = R[x/y^2]_{MR[x/y^2]}$. The integrally closed ideals projectively equivalent to I are precisely the powers I^n of I. Thus I is projectively full. The maximal ideal of V is $N = yV$ and $IV = N^2$, so the gcd is two not one.

Projective Equivalence and Rees Valuation Rings

Recall that ideals I and J are projectively equivalent if $(I^m)_a = (J^n)_a$ for some $m, n \in \mathbb{N}$. If I and J are projectively equivalent, then $\operatorname{Rees} I = \operatorname{Rees} J$. The converse holds if I or J has only one Rees valuation ring, but fails in general. Steve McAdam, Jack Ratliff and Judy Sally show in

Integrally closed projectively equivalent ideals, in Commutative Algebra, MSRI Pub. 15, 1988, 391-405

that if I and J are projectively equivalent, then the Rees integers of I and J are proportional. The converse also holds: if $\operatorname{Rees} I = \operatorname{Rees} J$ and the Rees integers of I and J are proportional, then I and J are projectively equivalent.

Projectively full ideals of a 2-dim RLR

EXAMPLE 4. (continued) (R, M) is a 2-dim regular local ring with $M = (x, y)R$. Zariski's theory of unique factorization of complete (= integrally closed) ideals of R as finite products of simple complete ideals implies $\mathbf{P}(I)$ is projectively full for each nonzero proper ideal I of R. The ideal I has a unique Rees valuation ring if and only if I is a power of a simple complete ideal. If I factors as $I = I_1^{f_1} \cdots I_g^{f_g}$,

where $I_1, \ldots, I_g$ are distinct simple complete ideals, then I is projectively full if and only if $\gcd(f_1, \ldots, f_g) = 1$.

Projective fullness and Rees integers

EXAMPLE 4. (continued) (R, M) is a 2-dim regular local ring with $M = (x, y)R$, and $I = I_1^{f_1} \cdots I_g^{f_g}$ is the factorization of the M-primary ideal I as a product of distinct simple complete ideals.

How do the integers $f_1, \ldots, f_g$ relate to the Rees integers of I?

The simple complete M-primary ideals of R are in one-to-one correspondence with the prime divisors birationally dominating R.

Thus the Rees valuation rings of I are $(V_1, N_1), \ldots, (V_g, N_g)$, where (V_j, N_j) corresponds to I_j. If $I_j V_j = N_j^{c_j}$, then the Rees integers of I are $\quad e_1 = c_1 f_1, \ \ldots, \ e_g = c_g f_g.$

EXAMPLE 4. (continued) (R, M) is a 2-dim regular local ring with $M = (x, y)R$. Let $I = (x^2, xy^2, y^3)R$. Notice that $J = (x^2, y^3)R$ is a reduction of I and $JI = I^2$, so the reduction number $r_J(I) = 1$. Let

$$V = R[\frac{xy^2}{x^2}, \frac{y^3}{x^2}]_{MR[\frac{xy^2}{x^2}, \frac{y^3}{x^2}]} = R[\frac{y^2}{x}, \frac{y^3}{x^2}]_{MR[\frac{y^2}{x}, \frac{y^3}{x^2}]}.$$

One sees that V is a valuation ring with maximal ideal $N = (y^2/x)V$, and I is a simple complete ideal. The ideals of R that are contracted from V descend as follows: $M = N^2 \cap R \supsetneq (x, y^2)R = N^3 \cap R \supsetneq M^2 = N^4 \cap R \supsetneq (x, y^2)M$ $= N^5 \cap R \supsetneq I = N^6 \cap R$. The ideals in $\mathbf{P}(I)$ are precisely the ideals $I^m = N^{6m} \cap R$, for $m \in \mathbb{N}$.

THE ASSOCIATED GRADED RING $\mathbf{G}(R, I)$

PROPOSITION. If $\mathbf{G}(R, I) = R/I \oplus I/I^2 \oplus \cdots \oplus I^n/I^{n+1} \oplus \cdots$ has a minimal prime p such that p is its own p-primary component of (0), then I has a Rees integer equal to one. Therefore I is projectively full.

More can be said using the Rees ring $\mathbf{R} = R[t^{-1}, It]$, and the identification $\mathbf{G}(R, I) = \mathbf{R}/t^{-1}\mathbf{R}$. Let $\mathbf{R}'$ denote the integral closure of $\mathbf{R}$.

PROPOSITION. The ideal I has a Rees integer equal to one if and only if $t^{-1}\mathbf{R}'$ has a minimal prime p such that $t^{-1}\mathbf{R}'_p = p\mathbf{R}'_p$.

EXAMPLE 5. An example of a 2-dim normal local domain (R, M) such that M is projectively full and the associated graded ring $\mathbf{G}(R, M)$ is not reduced. Let F be an algebraically closed field with $\operatorname{char} F = 0$, and let R_0 be a 2-dim regular local domain with maximal ideal $M_0 = (x, y)R_0$ and coefficient field F, e.g., $R_0 = F[x, y]_{(x,y)}$, or $R_0 = F[[x, y]]$. Then $V_0 = R_0[y/x]_{xR_0[y/x]}$ is the unique Rees valuation ring of M_0. Let

$$R = R_0[z], \quad \text{where} \quad z^2 = x^3 + y^j \quad \text{with} \quad j \geq 3.$$

It is readily checked that R is 2-dim normal local with maximal ideal $M = (x, y, z)R$. Notice that $I = (x, y)R$ is a reduction of M since z is integral over I.

EXAMPLE 5 (continued). Since $I = (x, y)R$ is a reduction of M, every Rees valuation ring of M is an extension of V_0. Let V be a Rees valuation ring of M and let v denote the normalized valuation with value group $\mathbb{Z}$ corresponding to V. Then $v(x) = v(y)$ and the image of y/x in the residue field of V is transcendental over F. Since $z^2 = x^3 + y^j$ and $j \geq 3$, we have

$$2v(z) = v(z^2) = v(x^3 + y^j) = 3v(x).$$

We conclude that $v(x) = 2$ and $v(z) = 3$. Therefore V is ramified over V_0. This implies that V is the unique extension of V_0 and thus the unique Rees valuation ring of M.

EXAMPLE 5 (continued). For each positive integer n, let $I_n = \{r \in R \,|\, v(r) \geq n\}$. Thus $I_2 = M$. Since V is the unique Rees valuation ring of M, we have $I_{2n} = (M^n)_a$ for each $n \in \mathbb{N}$. To show M is projectively full, we prove that V is not the unique Rees valuation ring of I_{2n+1} for each $n \in \mathbb{N}$.

Consider the inclusions

$$M^2 \subseteq I_4 \subset (z, x^2, xy, y^2)R := J \subseteq I_3 \subset M.$$

Since $\lambda(M/M^2) = 3$ and since the images of x and y in M/M^2 are F-linearly independent, $J = I_3$ and $M^2 = I_4 = (M^2)_a$.

EXAMPLE 5 (continued). Since $x^3 = z^2 - y^j$ and $j \geq 3$,

$L = (z, y^2)R$ is a reduction of $I_3 = (z, x^2, xy, y^2)R$. Indeed,

$(x^2)^3 \in L^3$ and $(xy)^3 \in L^3$ implies x^2 and xy are integral over L.

It follows that V is not a Rees valuation of I_3, for $zV \neq y^2V$.

Consider $M^3 \subset I_3M \subseteq I_5 \subset I_4 = M^2$. Since the images of

x^2, xy, y^2, xz, yz in M^2/M^3 are an F-basis, it follows that

$I_3M = I_5$ and $M^3 = (M^3)_a = I_6$. Proceeding by induction,

we assume $M^{n+1} = (M^{n+1})_a = I_{2n+2}$, and consider

$$M^{n+2} \subset I_3M^n \subseteq I_{2n+3} \subset M^{n+1} = I_{2n+2}.$$

EXAMPLE 5 (continued). Since the images in M^{n+1}/M^{n+2} of $\{x^a y^b \mid a+b=n+1\} \cup \{zx^a y^b \mid a+b=n\}$ is an F-basis,

$$\lambda(M^{n+1}/M^{n+2}) = 2n+3, \quad \text{and the inequalities}$$

$$\lambda(M^{n+1}/I_{2n+3}) \geq n+2 \quad \text{and} \quad \lambda(I_3 M^n/M^{n+2}) \geq n+1$$

imply $I_3 M^n = I_{2n+3}$ and $M^{2n+2} = (M^{2n+2})_a$.

Therefore the ideal I_{2n+3} has a Rees valuation ring different from V, and thus is not projectively equivalent to M. We conclude that M is projectively full. We have also shown that M is a normal ideal.

Questions

Let (R, M) be a complete normal Noetherian local domain of altitude two.

1. What are necessary and sufficient conditions in order that M is projectively full?

2. What are necessary and sufficient conditions in order that $\mathbf{P}(I)$ is projectively full for each nonzero proper ideal I of R?

EXAMPLE 6. An example of a (complete) 2-dim normal local domain (R, M) such that M is not projectively full. Let F be an algebraically closed field with $\operatorname{char} F = 0$, and let R_0 be a 2-dim regular local domain with maximal ideal $M_0 = (x, y)R_0$ and coefficient field F, e.g., $R_0 = F[x, y]_{(x,y)}$, or $R_0 = F[[x, y]]$.

Let $k < i$ be relatively prime positive integers ≥ 2, and let

$$R = R_0[z, w], \quad \text{where} \quad z^k = x^i + y^i \;\text{ and }\; w^k = x^i - y^i.$$

It is readily checked that R is 2-dim normal local with maximal ideal $M = (x, y, z, w)R$. Also R is a free R_0-module of rank k^2.

EXAMPLE 6 (continued). With $R = R_0[z, w]$ as above, we want to show that $M = (x, y, z, w)R$ has a unique Rees valuation ring and that M is not projectively full. $L = (x, y)R$ is a reduction of M, for $z^k \in (x^i, y^i)R \subseteq L^k$ and $w^k \in (x^i, y^i)R \subseteq L^k$ imply z and w are integral over L. Thus each Rees valuation ring V of M is an extension of the order valuation ring $V_0 = R_0[y/x]_{xR_0[y/x]}$ of R_0. To show there exists a unique Rees valuation ring of M, we observe that V as an extension of V_0 ramifies of degree k and undergoes a residue field extension of degree $\geq k$.

EXAMPLE 6 (continued). To show V ramifies of degree k over V_0, observe that $kv(z) = v(z^k) = v(x^i + y^i) = iv(x) = iv(y)$ implies $v(z) = i$ and $v(x) = v(y) = k$. Similarly, $v(w) = i$. Let N denote the maximal ideal of V. We have $(z, w)V = N^i$ and $MV = (x, y)V = N^k$. The residue field of V_0 is $F(\overline{\tau})$, where $\overline{\tau}$ is the image of $\tau = y/x$ and is transcendental over F. Now w/z is a unit of V and $(\frac{w}{z})^k = \frac{x^i - y^i}{x^i + y^i} = \frac{1-\tau^i}{1+\tau^i}$. It follows that the residue class of w/z in V/N is algebraic of degree k over $F(\overline{\tau})$. This proves that V is the unique Rees valuation ring of M.

EXAMPLE 6 (continued). To show M is not projectively full, notice that $(z^k, w^k)R = (x^i + y^i, x^i - y^i)R$, and since $\operatorname{char} F \neq 2$, $(x^i + y^i,\ x^i - y^i)R = (x^i, y^i)R$. Since $(x^i, y^i)R$ is a reduction of M^i, we have $((z^k, w^k)R)_a = (M^i)_a$. Also $((z^k, w^k)R)_a = ((z, w)^k R)_a$. Therefore $(z, w)R$ and M are projectively equivalent, so V is the unique Rees valuation ring of $(z, w)R$, and $((z, w)R)_a = N^i \cap R$. We have $M = N^k \cap R$, $M^n V = N^{nk}$ and $(M^n)_a = N^{nk} \cap R$, for each positive integer n. Since i is not a multiple of k, M is not projectively full.

RATIONAL SINGULARITIES

Joe Lipman in his paper

Rational singularities, with applications to algebraic surfaces and unique factorization, Publ. Math. Inst. Hautes Études Sci. N^o 36 (1969), 195-279.

extended Zariski's theory of complete ideals in 2-dim regular local rings to 2-dim normal local rings R having a rational singularity. Lipman proved that R has unique factorization of complete ideals if and only if the completion of R is a UFD. For R having this property, it follows that $\mathbf{P}(I)$ is projectively full for each nonzero proper ideal I, e.g., $R = F[[x, y, z]]$, where $z^2 + y^3 + x^5 = 0$.

MORE ON RATIONAL SINGULARITIES

Let (R, M) be a normal local domain of altitude two. Göhner proves that if R has a rational singularity, then the set of complete asymptotically irreducible ideals associated to a prime R-divisor v consists of the powers of an ideal A_v which is uniquely determined by v. In our terminology, this says that if I is a nonzero proper ideal of R having only one Rees valuation ring, then $\mathbf{P}(I)$ is projectively full. Göhner's proof involves choosing a desingularization $f : X \to \operatorname{Spec} R$ such that v is centered on a component E_1 of the closed fiber on X.

THE CLOSED FIBER 1

Let $E_2, \dots, E_n$ be the other components of the closed fiber on X.

Let E_X denote the group of divisors having the form $\sum_{i=1}^n n_i E_i$, with $n_i \in \mathbb{Z}$, and consider

$$E_X^+ = \{D \in E_X \mid D \neq 0 \text{ and } (D \cdot E_i) \leq 0 \text{ for all } 1 \leq i \leq n\}, \quad \text{and}$$

$$E_X^\# = \{D \in E_X \mid D \neq 0 \text{ and } O(-D) \text{ is gen. by its sections over } X\}.$$

Lipman shows $E_X^\# \subseteq E_X^+$ and equality holds if R has a rational singularity. Also, if $D = \sum_i n_i E_i \in E_X^+$, then negative-definiteness of the intersection matrix $(E_i \cdot E_j)$ implies $n_i \geq 0$ for all i.

THE CLOSED FIBER 2

For if $D \in E_X^+$ and $D = A - B$, where A and B are effective, then

$(A - B \cdot B) \leq 0$ and $(A \cdot B) \geq 0$ imply $(B \cdot B) \geq 0, \quad$ so $B = 0$.

Let $v = v_1, v_2, \ldots, v_n$ denote the discrete valuations corresponding to $E_1, \ldots, E_n$. Associated with $D = \sum_i n_i E_i \in E_X^\#$ one defines the complete M-primary ideal $I_D = \{r \in R \mid v_i(r) \geq n_i \text{ for } 1 \leq i \leq n\}$.

This sets up a one-to-one correspondence between elements of $E_X^\#$ and complete M-primary ideals that generate invertible O_X-ideals.

THE CLOSED FIBER 3

Lipman suggested to me the following proof that $\mathbf{P}(I)$ is proj. full for each complete M-primary ideal I if R has a rational singularity. Fix a desingularization $f : X \to \operatorname{Spec} R$ such that I generates an invertible O_X-ideal and let $D = \sum_i n_i E_i \in E_X^{\#}$ be the divisor associated to I. Let $g = \gcd\{n_i\}$. Since $E_X^{+} = E_X^{\#}$, $(1/g)D \in E_X^{\#}$. The ideals $J \in \mathbf{P}(I)$ correspond to divisors in $E_X^{\#}$ that are integral multiples of $(1/g)D$. Thus if K is the complete M-primary ideal associated to $(1/g)D$, then each $J \in \mathbf{P}(I)$ is the integral closure of a power of K, so $\mathbf{P}(I)$ is projectively full.

INTEGRAL EXTENSIONS

QUESTION. Let I be a nonzero ideal of a Noetherian domain R. Does there always exist a finite integral extension domain B of R such that $\mathbf{P}(IB)$ is projectively full?

Let $I = (b_1, \ldots , b_g)R$ be a nonzero regular ideal of the Noetherian ring R, let $R_g = R[X_1, \ldots , X_g]$ and $K = (X_1^c - b_1, \ldots , X_g^c - b_g)R_g$, where c is a positive integer. Then $A = R_g/K$ is a finite free integral extension of rank c^g of R. Let $x_i = X_i \mod K$. Then $J = (x_1, \ldots , x_g)A$ is such that $(IA)_a = (J^c)_a$, so IA and J are projectively equivalent.

MAIN THEOREM

THEOREM. Let $I = (b_1, \dots, b_g)R$ be as above and let (V, N) be a Rees valuation ring of I. Assume that $b_iV = IV = N^c$ for each i with $1 \le i \le g$, and that c is a unit of R. Then the finite free integral extension $A = R_g/K$ is such that $J = (x_1, \dots, x_g)A$ is projectively full. Hence $\mathbf{P}(IA)$ is projectively full. If R is an integral domain and p is a minimal prime of A, then $B = A/p$ is an integral extension domain of R such that $\mathbf{P}(IB)$ is projectively full.