UNIQUE IRREDUNDANT INTERSECTIONS OF COMPLETELY IRREDUCIBLE IDEALS

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Abstract. An ideal of a commutative ring is completely irreducible if it is not the intersection of any set of proper overideals. It is known that every ideal is an intersection of completely irreducible ideals. We characterize the rings for which every ideal can be represented uniquely as an irredundant intersection of completely irreducible ideals as precisely the rings in which every proper ideal is an irredundant intersection of powers of maximal ideals. We prove that every nonzero ideal of an integral domain R has a unique representation as an intersection of completely irreducible ideals if and only if R is an almost Dedekind domain with the property that for each proper ideal A the ring R/A has at least one finitely generated maximal ideal. We characterize the rings for which every proper ideal is an irredundant intersection of powers of prime ideals as precisely the rings R for which (i) R_M is a Noetherian valuation ring for each maximal ideal M, and (ii) every ideal of R is an irredundant intersection of irreducible ideals.

1. Introduction

Let R denote throughout a commutative ring with 1. An ideal of R is called *irreducible* if it is not the intersection of two proper overideals; it is called *completely irreducible* if it is not the intersection of any set of proper overideals. In this paper we characterize the rings for which every ideal can be represented uniquely as an irredundant intersection of completely irreducible ideals. We prove in Theorem 2.2 that such rings are necessarily arithmetical, and in Theorem 3.7 that the completely irreducible ideals of such a ring are precisely the powers of maximal ideals.

We recall that a ring is said to be *arithmetical* if its localization at each maximal ideal is a valuation ring, where by a *valuation ring* we mean a ring in which the ideals are linearly ordered with respect to inclusion, i.e., the ideals form a chain. An arithmetical integral domain is a *Prüfer domain*. An integral domain R is said to be *almost Dedekind* if R_M is a Noetherian valuation domain for each maximal ideal M. The completely irreducible ideals of an arithmetical ring are explicitly described in [4] (see Remark 3.1 below). Thus the ideals of a ring in which every ideal is uniquely

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represented as an irredundant intersection of completely irreducible ideals can be decomposed into "simpler" ideals belonging to a well-understood class.

Let $\mathcal{A}$ be the set of ideals of the ring R that are finite intersections of completely irreducible ideals. We prove in Theorem 2.2 that every ideal $A \in \mathcal{A}$ is a unique irredundant intersection of completely irreducible ideals if and only if R is arithmetical. Moreover, if R is arithmetical, then the components are unique in every irredundant intersection of irreducible ideals, even in the case of an infinite intersection.

In Corollary 2.9 we prove that in a zero-dimensional ring R every ideal is a unique irredundant intersection of irreducible ideals if and only if $\mathrm{Spec}(R)$ is a scattered topological space. Combining this with Theorem 2.8, we obtain that for a zero-dimensional ring R every ideal is a unique irredundant intersection of irreducible ideals if and only if R is an arithmetical ring such that for every radical ideal J of R, R/J has a principal maximal ideal. As we record in Question 2.12, the classification of rings of positive dimension for which every ideal is a unique irredundant intersection of irreducible (not necessarily *completely* irreducible) ideals remains an open problem.

In Theorem 3.5 we prove that every proper irreducible ideal of a ring R is a power of a maximal ideal if and only if R_M is a Noetherian valuation ring for every maximal ideal M of R. We then characterize in Theorem 3.7 the rings for which every ideal can be represented uniquely as an irredundant intersection of completely irreducible ideals. We show that this class of rings coincides with the class of rings for which every proper ideal is an irredundant intersection of powers of maximal ideals.

Theorem 3.7 motivates our consideration in Section 4 of the class of rings R in which every proper ideal is an irredundant intersection of powers of prime ideals. We observe that this class of rings properly includes the ZPI rings of classical interest, i.e., those rings for which every proper ideal is a product of prime ideals. We prove in Theorem 4.1 that the following two conditions are equivalent in a ring R: (i) every ideal of R can be represented as an irredundant intersection of powers of prime ideals, and (ii) R_M is a Noetherian valuation ring for each maximal ideal M of R and every ideal of R can be represented as an irredundant intersection of irreducible ideals.

Notation 1.1. For ideals I, J of the ring R, the residual $I : J$ is defined as usual by

$$I : J = \{x \in R \ : \ xJ \subseteq I\}.$$

For an ideal A and for a prime ideal P of R, we use the notation

$$A_{(P)} = \{x \in R \ : \ sx \in A \text{ for some } s \in R \setminus P\} = \bigcup_{s \in R \setminus P} A : s$$

to denote the *isolated P-component* (isoliertes Komponentenideal) of A in the sense of Krull [10, page 16]. Notice that $x \in A_{(P)}$ if and only if $A : x \not\subseteq P$. If R is a domain, then $A_{(P)} = AR_P \cap R$, where R_P denotes the localization of R at P.

Two different notions of associated primes of a proper ideal A of the ring R are useful for us. One of these was introduced by Krull [9, page 742], and following [7] we call a prime ideal P of R a *Krull associated prime* of A if for every $x \in P$, there exists $y \in R$ such that $x \in A : y \subseteq P$. The prime ideal P is said to be a *Zariski-Samuel* associated prime of A if there exists $x \in R$ such that $\sqrt{A : x} = P$. We denote by $\mathrm{Ass}(A)$ the set of Krull associated primes of A and by $\mathcal{Z}(A)$ the set of Zariski-Samuel associated primes of A. It is true in general that $\mathcal{Z}(A) \subseteq \mathrm{Ass}(A)$.

Remark 1.2. In [1] Fuchs defines a *primal ideal* of a ring R as an ideal A having the property that the zero divisors in R/A form an ideal. This ideal is necessarily prime and hence of the form P/A for some prime ideal P of R. The ideal P is called the *adjoint prime* of A. If A is a P-primal ideal (that is, A is a primal ideal with adjoint prime P) then $A = A_{(P)}$ [2, Theorem 3.4]. Moreover a prime ideal P of a ring R is a Krull associated prime of an ideal A if and only if $A_{(P)}$ is a P-primal ideal of R [2, Theorem 3.4]. Every irreducible ideal is primal [1]; hence every completely irreducible ideal is primal. In addition, the adjoint prime of a completely irreducible ideal is a maximal ideal [4, Proposition 1.2].

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2. Unique irredundant intersections of irreducible ideals

In this section we consider irredundant intersections of irreducible ideals. This allows us to develop several technical characterizations needed in Section 3 and

Section 4. We give sufficient conditions for a ring to have the property that every ideal is a unique irredundant intersection of irreducible ideals, and we characterize this class of rings in the zero-dimensional case. We observe in Theorem 2.2 that the requirement of uniqueness places this problem in the setting of arithmetical rings, that is: *If every ideal of a ring R can be represented uniquely as an irredundant intersection of irreducible ideals, then R is arithmetical.*

Lemma 2.1. *Let (R, M) be a quasilocal ring. If R is not a valuation ring, then there exists an ideal of R that has two different representations as an irredundant intersection of two completely irreducible ideals. More precisely, there exist an ideal A of R and distinct completely irreducible ideals C_1, C_2, C_3 of R such that $A = C_1 \cap C_2 = C_1 \cap C_3$ are two different irredundant representations of A.*

Proof. Assume R is not a valuation ring. Since every ideal in R is an intersection of completely irreducible ideals, there exist incomparable completely irreducible ideals C_1 and C_2 of R. Let $A = C_1 \cap C_2$. Let C_1^* and C_2^* denote the unique minimal overideals to the completely irreducible ideals C_1 and C_2, respectively. There exist elements $x \in (C_1^* \cap C_2) \setminus A$ and $y \in (C_1 \cap C_2^*) \setminus A$. Since $x \in C_2$ and $y \in C_1$, we have $\operatorname{Soc} R/A = (A + (x, y)R)/A$ is a 2-dimensional vector space over R/M and $x + y \notin C_1 \cup C_2$. Let C_3 be an ideal containing $A + (x + y)R$ that is maximal with respect to $x \notin R$. Then C_3 is completely irreducible, distinct from C_1 and C_2, and $A = C_1 \cap C_3$. □

Theorem 2.2. *Let $\mathcal{A}$ be the set of ideals of the ring R that are finite intersections of completely irreducible ideals. Every ideal $A \in \mathcal{A}$ has a unique representation as an irredundant intersection of completely irreducible ideals if and only if R is arithmetical. Moreover, if R is arithmetical, then the components are unique in every irredundant intersection of irreducible ideals, even in the case of an infinite intersection.*

Proof. By Corollary 5.6 of [2] in an arithmetical ring the components in any irredundant intersection of irreducible ideals are unique. Assume, conversely, that for each $A \in \mathcal{A}$, the representation of A as an irredundant intersection of completely irreducible ideals is unique. This property then also holds in R_M for each maximal ideal M of R. For if M is a maximal ideal of R and C is a completely irreducible ideal of R_M, then the preimage B of C in R under the mapping $R \to R_M$ is completely irreducible and $C = BR_M$ (see [4, Theorem 1.3]). By Lemma 2.1, R_M is a valuation ring. Therefore R is arithmetical. $\square$

Lemma 2.3. *Let R be a ring in which every proper ideal has a Zariski-Samuel associated prime ideal. If A is a proper ideal of R, then $A = \bigcap_{P \in \mathcal{Z}(A)} A_{(P)}$.*

Proof. To show that $A = \bigcap_{P \in \mathcal{Z}(A)} A_{(P)}$, it suffices to verify the inclusion $\supseteq$, so suppose $x \in R \setminus A$. Then the proper ideal $A : x$ has a Zariski-Samuel associated prime ideal; that is, there exist a prime ideal P and $y \in R \setminus (A : x)$ such that $P = \sqrt{(A : x) : y}$. Since $(A : x) : y = A : xy$, we have $P \in \mathcal{Z}(A)$. Moreover, $A : xy \subseteq P$ implies that $xy \notin A_{(P)}$. Thus $x \notin A_{(P)}$. It follows that $\bigcap_{P \in \mathcal{Z}(A)} A_{(P)} \subseteq A$, and the proof is complete. $\square$

Lemma 2.4. *(i) Let R be a ring in which every radical ideal J has a minimal prime divisor P such that P/J is the radical of a finitely generated ideal of R/J. If A is a proper ideal of R, then $A = \bigcap_{P \in \mathcal{Z}(A)} A_{(P)}$.*

(ii) Assume the ring R satisfies the ascending chain condition on prime ideals. For a proper ideal A of R, let $\mathcal{Z}^*(A)$ denote the maximal elements of $\mathcal{Z}(A)$. If $A = \bigcap_{P \in \mathcal{Z}(A)} A_{(P)}$, then $A = \bigcap_{P \in \mathcal{Z}^*(A)} A_{(P)}$ and this second intersection is irredundant. If also R is arithmetical, then this second intersection is the unique representation of A as an irredundant intersection of irreducible ideals.

Proof. (i) To prove the first claim it suffices by Lemma 2.3 to show that every proper ideal B of R has a Zariski-Samuel associated prime ideal. Set $J = \sqrt{B}$. By assumption there exists a minimal prime divisor P of J such that $P = \sqrt{J + C}$ for some finitely generated ideal C of R. Since P is minimal over J, we have $JR_P = PR_P$. Thus $J_{(P)} = P$, so $C \subseteq J_{(P)}$ and since C is finitely generated, there exists $x \in R \setminus P$ such that $xC \subseteq J$. Hence $J + C \subseteq J : x$, and since $x \notin P$ we have $J : x \subseteq P$. Thus $P = \sqrt{J : x}$. It follows that $P = \bigcup_{n>0} \sqrt{B : x^n}$. For $x \notin P$ and $B \subseteq P$ implies $\sqrt{B : x^n} \subseteq P$ for all $n > 0$, and if $a \in P = \sqrt{J : x}$, then there exists $k > 0$ such that $a^k x \in J$, so $a^{kn} x^n \in B$ for some $n > 0$; hence $a \in \sqrt{B : x^n}$. Since C is finitely generated and contained in P, we have $C \subseteq \sqrt{B : x^n}$ for some $n > 0$. Also, $J = \sqrt{B} \subseteq \sqrt{B : x^n}$, so we have $P = \sqrt{C + J} \subseteq \sqrt{B : x^n} \subseteq P$, proving that P is a Zariski-Samuel associated prime of B.

(ii) Now assume that A is a proper ideal of a ring R that satisfies the ascending chain condition on prime ideals. Then for each $Q \in \mathcal{Z}(A)$ there exists $P \in \mathcal{Z}^*(A)$ such that $Q \subseteq P$. It follows that $A_{(P)} \subseteq A_{(Q)}$. Therefore $A = \bigcap_{P \in \mathcal{Z}(A)} A_{(P)}$ implies that $A = \bigcap_{P \in \mathcal{Z}^*(A)} A_{(P)}$. Moreover, this second intersection is irredundant. For if $P \in \mathcal{Z}^*(A)$ and $x \in R \setminus A$ is such that $P = \sqrt{A : x}$, then $x \in \bigcap_{Q \in \mathcal{Z}^*(A) \setminus \{P\}} A_{(Q)} \setminus A_{(P)}$. If we also assume that R is arithmetical, then each $A_{(P)}$ is irreducible (cf. Remark 1.6 of [2]), and as we have observed above in Theorem 2.2, in an arithmetical ring any representation of an ideal as an irredundant intersection of irreducible ideals is unique, so the proof of Lemma 2.4 is complete. □

Remark 2.5. The prime spectrum of a ring R is Noetherian as a topological space if and only if every radical ideal of R is the radical of a finitely generated ideal [12]. Thus if R is an arithmetical ring with Noetherian prime spectrum, then by Lemma 2.4 every ideal of R can be represented uniquely as an irredundant intersection of irreducible ideals. However, even for zero-dimensional arithmetical rings, there exist examples of rings without Noetherian prime spectrum for which every ideal is a unique irredundant intersection of irreducible ideals. We postpone a discussion of such examples until Remark 2.11.

Lemma 2.6. *Let A be a proper ideal of the ring R and assume that $J = \sqrt{A}$ is an irredundant intersection $J = M \cap B$, where M is a maximal ideal of R and B is a proper ideal not contained in M. Then M is a minimal prime divisor of A and M/A is the radical of a principal ideal of R/A. Moreover M/J is a principal maximal ideal of R/J.*

Proof. Since $J = M \cap B$ is irredundant and M is maximal we have $R = M + B$, so $J = MB$. Thus there exist $x \in M \setminus B$ and $y \in B \setminus M$ such that $1 = x + y$. It follows that $M = xR + J$, for if $w \in M$, then $w = wx + wy \in xR + MB = xR + J$. Hence M/J is a principal maximal ideal of R/J. To show that M is minimal over A is equivalent to showing that M is minimal over J, and for this it is enough to show that R_M/JR_M is a field. Since the maximal ideal of R_M/JR_M is generated by $x/1 + JR_M$, it suffices to show $x/1$ is in the ideal JR_M. This is indeed the case since $yx \in J$ and $y \notin M$.

Since the prime ideals containing A are precisely those that contain J, and M is the only prime ideal that contains both J and x, we have $M = \sqrt{xR + A}$. It follows that M/A is the radical of a principal ideal of R/A. $\square$

Lemma 2.7. *Let A be an ideal of the ring R. If M and N are distinct maximal ideals of R with $A \subseteq M \cap N$ and if $A_{(M)} \subseteq N$, then there is a prime ideal P of R such that $A_{(M)} \subseteq P \subsetneq M \cap N$.*

Proof. Define S to be the multiplicatively closed set $\{xy : x \in R \setminus M, y \in R \setminus N\}$. We observe that $A_{(M)} \subseteq N$ implies $A_{(M)} \cap S = \emptyset$. For suppose there exists an element $r \in A_{(M)} \cap S$. Then $r = xy$, with $x \in R \setminus M$ and $y \in R \setminus N$. Also $r \in A_{(M)}$ implies there exists $x' \in R \setminus M$ such that $x'r = a \in A$, and $a = x'xy$ implies $y \in A_{(M)}$. But $y \notin N$. Thus $A_{(M)} \subseteq N$ implies $A_{(M)} \cap S = \emptyset$. Hence there is a prime ideal P of R containing $A_{(M)}$ such that $P \cap S = \emptyset$. It follows that $A_{(M)} \subseteq P \subsetneq M \cap N$. $\square$

A topological space X is *scattered* if every nonempty subset of X contains a point that is isolated in the relative topology. If R is a zero-dimensional ring, then $\mathrm{Spec}(R)$ is scattered if and only if for each nonempty family $\{M_i : i \in I\}$ of maximal ideals of R, there exists $i \in I$ such that $\bigcap_{j \neq i} M_j \not\subseteq M_i$.

Theorem 2.8. *Let R be a zero-dimensional ring. The following statements are equivalent.*

(i) $\mathrm{Spec}(R)$ *is a scattered space.*

(ii) *For every radical ideal J of R, there is a maximal ideal M containing J such that M/J is a principal ideal of R/J.*

(iii) *For every radical ideal J of R, there is a maximal ideal M containing J such that M/J is the radical of a finitely generated ideal of R/J.*

(iv) *For every proper ideal A of R, the set $\mathcal{Z}(A)$ of Zariski-Samuel associated primes of A is nonempty.*

(v) *For every proper ideal A of R, $A = \bigcap_{P \in \mathcal{Z}(A)} A_{(P)}$.*

(vi) *Every radical ideal of R is an irredundant intersection of maximal ideals.*

Moreover if R satisfies (i)-(vi) and A is a proper ideal of R, then $A = \bigcap_{P \in \mathcal{Z}(A)} A_{(P)}$ is an irredundant intersection.

Proof. First observe that if A is a proper ideal of R, then each member of $\mathcal{Z}(A)$ is a maximal ideal of R since R is zero-dimensional. Therefore by Lemma 2.4, if $A = \bigcap_{P \in \mathcal{Z}(A)} A_{(P)}$, then this intersection is irredundant.

(i) ⇒ (ii) If $J = \bigcap_{i \in I} M_i$ is an intersection of maximal ideals M_i of R, then by (i) there exists $i \in I$ such that $\bigcap_{j \neq i} M_j \not\subseteq M_i$. By Lemma 2.6, M_i/J is a principal ideal of R/J.

(ii) ⇒ (iii) This is clear.

(iii) ⇒ (iv) Apply Lemma 2.4.

(iv) ⇒ (v) Apply Lemma 2.3.

(v) ⇒ (vi) Let J be a radical ideal of R. By (v) $J = \bigcap_{P \in \mathcal{Z}(J)} J_{(P)}$, and by the remark at the beginning of the proof this intersection is irredundant, so to complete the proof it suffices to observe that each $J_{(P)}$, $P \in \mathcal{Z}(J)$, is a maximal ideal of R. Indeed, since J is a radical ideal, so is $J : x$ for every $x \in R$. It follows that $J_{(P)}$, as a set union of radical ideals, is a radical ideal. Since R is zero-dimensional, $J_{(P)}$ is contained only in P (Lemma 2.7). Thus $J_{(P)} = P$, and it follows that J is an irredundant intersection of maximal ideals of R.

(vi) ⇒ (i) Let $\{M_i : i \in I_1\}$ be a collection of maximal ideals of R, and let $J = \bigcap_{i \in I_1} M_i$. By (vi) there is a collection $\{N_i : i \in I_2\}$ of maximal ideals such that $J = \bigcap_{i \in I_2} N_i$ is an irredundant intersection. Fix $i_2 \in I_2$, and let $J_2 = \bigcap_{i \neq i_2} N_i$. The intersection $J = N_{i_2} \cap J_2$ is irredundant, and $J \subseteq M_i$ implies either $N_{i_2} = M_i$ or $J_2 \subseteq M_i$. If $N_{i_2} \neq M_i$ for all $i \in I_1$, then $J_2 \subseteq \bigcap_{i \in I_1} M_i = J$, a contradiction. Hence $N_{i_2} = M_{i_1}$ for some $i_1 \in I_1$. Set $J_1 = \bigcap_{i \neq i_1} M_i$. Since $N_{i_2} \cap J_2 = J \subseteq J_1$ and $N_{i_2} \not\subseteq M_i$ for any $i \in I_1$ with $i \neq i_1$, it follows that $J_2 \subseteq J_1$. Thus $J = M_{i_1} \cap J_1$ and $J_1 \not\subseteq M_{i_1}$ since otherwise $J_2 \subseteq J_1 \subseteq M_{i_1} = N_{i_2}$, a contradiction. This proves Spec(R) is scattered. □

Corollary 2.9. *Let R be a zero-dimensional ring. The following statements are equivalent.*

(i) *Every ideal of R can be represented uniquely as an irredundant intersection of irreducible ideals.*

(ii) *R is an arithmetical ring with scattered prime spectrum.*

Proof. (i) ⇒ (ii) By Theorem 2.2, R is arithmetical. If an ideal A of an arithmetical ring R is an irredundant intersection $A = \bigcap_i A_i$ of irreducible ideals A_i, then each A_i is an isolated component of A, that is, $A_i = A_{(P_i)}$ for some prime ideal P_i of R (cf. Lemma 5.5 of [2]). If A is a radical ideal, then $A_{(P_i)}$ is a prime ideal by

Lemma 2.7, so it follows that A is an irredundant intersection of prime ideals. Since R is zero-dimensional, R satisfies the equivalent conditions of Theorem 2.8.

(ii) $\Rightarrow$ (i) Apply Lemma 2.4 and Theorem 2.8. $\square$

In Corollary 2.9 we have characterized the zero-dimensional rings for which every ideal can be represented uniquely as an irredundant intersection of irreducible ideals. We record the following characterization for one-dimensional integral domains.

Corollary 2.10. *Let R be a one-dimensional integral domain. The following statements are equivalent.*

(i) *Every ideal of R can be represented uniquely as an irredundant intersection of irreducible ideals.*

(ii) *R is a Prüfer domain and for each nonzero proper ideal A, the ring R/A has a scattered spectrum.*

(iii) *R is a Prüfer domain and for each proper ideal A, the ring R/A contains at least one maximal ideal that is the radical of a finitely generated ideal.*

Remark 2.11. (i) An almost Dedekind domain R with at most finitely many maximal ideals that are not finitely generated satisfies (i)-(iii) of Corollary 2.10. Thus if A is any proper nonzero ideal of R, then R/A is satisfies the equivalent conditions of Corollary 2.9. One reference for such an example that is non-Noetherian is [6, Example 2.2].

(ii) In [13], the example of (i) is generalized to show that if X is a compact totally disconnected topological space, then there is an almost Dedekind domain R such that $\mathrm{Max}(R)$ is homeomorphic to X. Since a compact scattered space is totally disconnected, such a space can be realized as the maximal spectrum of an almost Dedekind domain. Several examples are discussed in [13]; we mention one here. Let $(X, \leq)$ be a well-ordered set. Then X is a compact scattered space with respect to the order topology on X, and the isolated points of X are precisely the smallest element of X and the immediate successors of elements in X [8, Example 17.3, p. 272]. Hence by suitable choices of the space X, one obtains examples of scattered spaces having infinitely many non-isolated points. If R is an almost Dedekind domain with $\mathrm{Max}(R)$ homeomorphic to X, then R has nonzero Jacobson radical and the isolated points of X correspond to finitely generated maximal ideals of R [13]. The ring R satisfies the equivalent conditions of Corollary 2.10 and every proper homomorphic image of R satisfies the conditions of Corollary 2.9.

It would be interesting to discover how to extend the characterizations of this section to resolve the following:

Question 2.12. *What rings R have the property that every ideal of R can be represented uniquely as an irredundant intersection of irreducible ideals?*

Necessarily such rings are arithmetical. Moreover this question is equivalent to Question 5.17(1) of [2], which asks for a classification of the arithmetical rings R having the property that every proper ideal A can be written as an irredundant intersection $A = \bigcap_{P \in C} A_{(P)}$ for some set C of Krull associated primes of A. (Apply Lemma 5.5 and Corollary 5.6 of [2].)

3. IRREDUNDANT INTERSECTIONS OF COMPLETELY IRREDUCIBLE IDEALS

In this section we characterize the rings in which every ideal is a unique irredundant intersection of completely irreducible ideals.

Remark 3.1. In [4, Theorem 4.3] the completely irreducible ideals of an arithmetical ring are explicitly described: *A ring R is arithmetical if and only if the proper completely irreducible ideals of R are precisely the ideals of the form $MB_{(M)}$, where M is a maximal ideal and B is a principal ideal having the property that $BR_M \neq 0$.*

Lemma 3.2. *If R is a valuation ring having a proper nonzero principal ideal rR that is completely irreducible, then the maximal ideal M of R is principal and every M-primary ideal of R is a power of M.*

Proof. By Remark 3.1, $rR = sM$ for some $s \in R$. If $M = M^2$, then $rM = sM^2 = sM = rR$, but this implies $r \in rM$ which contradicts the fact that $r \neq 0$. Hence $M \neq M^2$, and since M^2 is irreducible, it follows that the R/M-vector space M/M^2 has dimension one. Consequently, M is a principal ideal of R. It follows that every M-primary ideal of R is a power of M. □

We recall that a principal ideal ring R is called a *special* PIR if R is a local ring with maximal ideal M and every ideal of R is a power of M [14, page 245].

Lemma 3.3. *Every ideal of a ring R is completely irreducible if and only if R is a special PIR.*

Proof. Suppose that every ideal of R is completely irreducible. Then any two ideals of R are comparable, so R is a valuation ring and hence is quasilocal. In Remark 1.6 of [4] it is noted that if every irreducible ideal of a ring is completely irreducible, then the ring is zero-dimensional. Thus R is zero-dimensional, and the claim follows from Lemma 3.2.

Conversely, suppose that R is local with maximal ideal M and every ideal of R is a power of M. Then R is a valuation ring and either R is a field or $M \neq M^2$, so as in the proof of the preceding lemma, M is a principal ideal of R. The claim now follows from Remark 3.1. □

Proposition 3.4. *Every primal ideal of a ring R is completely irreducible if and only if R_M is a special PIR for each maximal ideal M of R.*

Proof. Assume every primal ideal of R is completely irreducible. By Theorem 1.8 of [2], R is arithmetical. Let M be a maximal ideal of R. If B is an ideal of R_M and A is the preimage of B in R under the canonical mapping $R \to R_M$, then $AR_M = B$ and $A = A_{(M)}$ is irreducible and hence primal and therefore completely irreducible. (The irreduciblity of $A = A_{(M)}$ follows because R is arithmetical, so AR_M is irreducible; see Remark 1.6 of [2].) It is shown in Theorem 1.3 of [4] that an ideal C of (any ring) R is completely irreducible if and only if for some maximal ideal N of R, $C = C_{(N)}$ and CR_N is a completely irreducible ideal of R_N. Thus we conclude that B is a completely irreducible ideal of R_M. Consequently, every ideal of R_M is completely irreducible. By Lemma 3.3, R_M is a special PIR.

Assume, conversely, that R_M is a special PIR for every maximal ideal M of R. Then R is arithmetical and zero-dimensional. If A is a primal ideal of R, then A is M-primal for some maximal ideal M. By Lemma 3.3, every ideal of R_M is completely irreducible. In particular, AR_M is completely irreducible, and it follows (as above) from Theorem 1.3 of [4] that $A = A_{(M)}$ is completely irreducible. □

Theorem 3.5. *The following statements are equivalent for a ring R.*

(i) *Every proper irreducible ideal of R is a power of a prime ideal.*

(ii) *For every maximal ideal M of R, R_M is a Noetherian valuation ring.*

Proof. (i) ⇒ (ii) We first establish the following three claims.

Claim (1) *If R is a quasilocal ring with maximal ideal $M \neq M^2$ and every proper irreducible ideal of R is a power of a prime ideal, then R is a Noetherian valuation ring.* We show first that M is a principal ideal. Let $m \in M \setminus M^2$. Since every ideal is an intersection of completely irreducible ideals, it follows from our assumption that $mR = \bigcap_{i\in I} P_i^{e_i}$ for some prime ideals P_i of R such that for each i, $P_i^{e_i}$ is a completely irreducible ideal of R. Since the only completely irreducible prime ideal of R is M, it follows that for each i with $P_i \neq M$, it must be that $e_i > 1$. In fact, if $e_i > 1$, then $m \in P_i^{e_i} \subseteq P_i^2 \subseteq M^2$, contrary to assumption. This forces $\{P_i\}_{i\in I} = \{M\}$ and $mR = M$, proving that M is a principal ideal of R.

Under the assumptions of claim (1), we show that R is a zero-dimensional ring or a one-dimensional domain. If P is a prime ideal of R properly contained in $M = mR$, then $P \subsetneq mR$, so that $P = mA$ for some proper ideal A of R. Since P is a prime ideal of R and $m \notin P$, it follows that $A = P$. Thus for all prime ideals P properly contained in M, $P = mP$ and $P \subseteq \bigcap_{k=1}^{\infty} M^k$. Now suppose that there exists a nonmaximal prime ideal P of R. We claim that $P = (0)$. Let $y \in P$. Since every ideal of R is an intersection of irreducible ideals, $yR = \bigcap_{i\in I} P_i^{e_i}$ for some prime ideals P_i of R. If $\{P_i\}_{i\in I} = \{M\}$, then $\bigcap_{k=1}^{\infty} M^k \subseteq yR \subseteq P$. As already noted, $P \subseteq \bigcap_{k=1}^{\infty} M^k$, so we have in this case that $yR = P$. Thus $P = mP$ implies $y = ymr$ for some $r \in R$ and $y(1 - mr) = 0$. Since R is quasilocal, $1 - mr$ is a unit and $y = 0$. Assume $\{P_i\}_{i\in I} \neq \{M\}$. Since $P_i \neq M$ implies $P_i \subseteq M^k$ for all $k > 0$, we may assume that $P_i \neq M$ for each i. In particular for each i, $P_i = mP_i$ and we have $yR = \bigcap_{i\in I} P_i^{e_i} = m(\bigcap_{i\in I} P_i^{e_i}) = ymR$. From the fact that R is quasilocal, we conclude $y = 0$. This shows that if there exists a nonmaximal prime ideal P of R, then $P = 0$. We conclude that R is either a zero-dimensional ring or a one-dimensional domain. In the case that R is zero-dimensional, by assumption every irreducible ideal of R is a power of M; in the case that R is one-dimensional, every nonzero irreducible ideal of R is a power of M. Since every ideal of a ring is an intersection of irreducible ideals, we conclude that R is a Noetherian valuation ring.

Claim (2) *If R is a ring such that every proper irreducible ideal is a power of a prime ideal, then for each prime ideal Q of R, every proper irreducible ideal of R_Q is a power of a prime ideal.* Indeed, if A is a proper irreducible ideal of R_Q, let B denote the preimage of A under the mapping $R \to R_Q$. Then B is an irreducible ideal of R, and by assumption $B = P^e$ for some prime ideal P of R and $e > 0$. Thus $A = BR_Q = P^e R_Q$, so that every proper irreducible ideal of R_Q is a power of a prime ideal of R_Q.

Claim (3) *If R is a quasilocal domain in which every proper irreducible ideal is a power of a prime ideal, then R is either a field or a discrete rank-one valuation domain (DVR).* Suppose that R is not a field and let M be the maximal ideal of R. Suppose that there exists a prime ideal P of R properly contained in M. We claim that $P = (0)$. Let $y \in M \setminus P$, and let Q be a prime ideal of R minimal over $P + Ry$. Since QR_Q is the radical of $PR_Q + yR_Q$, it is also the radical of $A := PR_Q + y^2R_Q$. Furthermore $A \neq QR_Q$, so by claim (2) $A = Q^eR_Q$ for some $e > 1$ since A is an intersection of irreducible ideals and the only prime ideal containing A is QR_Q. In particular $QR_Q \neq Q^2R_Q$. Thus R_Q is a quasilocal ring such that every proper irreducible ideal of R_Q is a power of a prime ideal and $QR_Q \neq Q^2R_Q$. This places us in the setting of claim (1), so we conclude that R_Q is a DVR. In particular, $PR_Q = 0$. Thus (since R is a domain) $P = 0$ and R is a one-dimensional domain. Now if B is a nonzero irreducible ideal of R properly contained in M, then $B = M^k$ for some $k > 1$. Hence $M \neq M^2$ and by claim (1), R is a DVR.

We now prove that (i) $\Rightarrow$ (ii) in full generality. Let R be a ring satisfying (i) and let M be a maximal ideal of R. By claim (2), every proper irreducible ideal of R_M is a power of a prime ideal. Thus to prove that R_M is a Noetherian valuation ring, it suffices by claim (1) to show that R_M is a field or $MR_M \neq M^2R_M$. Suppose there exists a prime ideal P of R properly contained in M. Then every proper irreducible ideal of R_M/P_M is a power of a prime ideal, so by claim (3) R_M/P_M is a DVR. In particular, $MR_M \neq M^2R_M$, as claimed. Otherwise, if R_M is zero-dimensional and if R_M is not a field, then there is an irreducible ideal B of R_M properly contained in MR_M. By assumption, $B = M^kR_M$ for some $k > 1$. Hence $MR_M \neq M^2R_M$ in this case also. Thus by claim (1) R_M is a Noetherian valuation ring.

(ii) $\Rightarrow$ (i) Let A be a proper irreducible ideal of R with adjoint prime ideal P, and let M be a maximal ideal of R containing P. Then $A = A_{(M)}$ since A is P-primal. Now AR_M is the zero ideal of R_M or a power of MR_M. If $AR_M = M^kR_M$ for some k, then the preimages $A_{(M)}$ and $(M^k)_{(M)}$ of AR_M and M^kR_M, respectively, under the mapping $R \to R_M$ are equal; that is, $A = A_{(M)} = (M^k)_{(M)}$. Since M is a maximal ideal of R, then $(M^k)_{(M)} = M^k$. For it is enough to verify this equality locally and if N is a maximal ideal of R, then $(M^k)_{(M)}R_N = M^kR_N$. Thus $A = M^k$ in case AR_M is a power of MR_M. Otherwise, AR_M is not a power of MR_M, so by (ii) $AR_M = (0)R_M$, where $(0)R_M$ is a prime ideal of R_M; in particular, $AR_M = QR_M$ for some prime ideal Q of R. Now $A = A_{(M)} = Q_{(M)} = Q$, so that

A is a prime ideal of R. This proves that every irreducible ideal of R is a power of a prime ideal of R. □

Corollary 3.6. *If A is a proper ideal of an arithmetical ring R such that every primal ideal of R/A is completely irreducible, then A is an intersection $A = \bigcap_i M_i^{e_i}$ of powers of maximal ideals M_i of R.*

Proof. By Proposition 3.4 R_M/A_M is a special PIR for all maximal ideals M of R containing A. Since every ideal of a ring is an intersection of completely irreducible ideals, we may write $A = \bigcap_i A_i$, where each A_i is completely irreducible. Since A_i is completely irreducible, A_i is primal; hence $A_i = (A_i)_{(M_i)}$ for some maximal ideal M_i of R. Since R/A_i is a zero-dimensional ring, M_i is by Lemma 2.7 the unique maximal ideal of R containing A_i. Moreover by Theorem 3.5 for each i, $A_i = M_i^{e_i} + A$ for some $e_i > 0$. Hence $A_i = (M_i^{e_i})_{(M_i)} + A_{(M_i)}$. Since R_{M_i} is a valuation ring, the ideals $A_{(M_i)}$ and $(M_i^{e_i})_{(M_i)}$ are comparable, and since $A_{(M_i)} \subseteq A_i$, we conclude $A_i = (M_i^{e_i})_{(M_i)}$. As noted in the proof of (ii) ⇒ (i) of Theorem 3.5, $M_i^{e_i} = (M_i^{e_i})_{(M_i)}$. Thus $A_i = M_i^{e_i}$, proving that A is an intersection of powers of maximal ideals of R. □

Theorem 3.7. *The following statements are equivalent for a ring R.*

(i) *Every ideal of R can be represented uniquely as an irredundant intersection of completely irreducible ideals.*

(ii) *R has a scattered prime spectrum and R_M is a special PIR for each maximal ideal M of R.*

(iii) *For every proper ideal A of R, R/A has a finitely generated maximal ideal, and for every maximal ideal M of R, R_M is a special PIR.*

(iv) *Every proper ideal A of R is an irredundant intersection $A = \bigcap_i M_i^{e_i}$ of powers of maximal ideals M_i.*

Proof. (i) ⇒ (ii) Since an irreducible ideal cannot be expressed as an irredundant intersection of two distinct proper overideals, (i) implies that every irreducible ideal of R is completely irreducible, so by Proposition 3.4 R_M is a special PIR for every maximal ideal M of R. Thus R is zero-dimensional and by Corollary 2.9 Spec(R) is a scattered space.

(ii) ⇒ (iii) Let A be a proper ideal of R. By Theorem 2.8 there exists $y \in R$ and a maximal ideal M of R such that $M = yR + \sqrt{A}$. Since R_M is a special PIR, there exists $z \in M$ such that $MR_M = zR_M$. Define $B = (y, z)R + A$. We claim

$M = B$. Clearly $MR_M = BR_M$ and by construction B has radical M. Therefore $MR_N = BR_N$ for each maximal ideal N of R, so $B = M$. Hence M/A is a finitely generated ideal of R/A.

(iii) $\Rightarrow$ (iv) By (iii) R is an arithmetical ring since each localization at a maximal ideal is a valuation ring. Let A be a proper ideal of R. By Theorem 2.8 and Corollary 2.9 A is an irredundant intersection of irreducible ideals, and by Proposition 3.4 every primal (hence irreducible) ideal is completely irreducible. Also, it follows from Corollary 3.6 that every completely irreducible proper ideal of R is a power of a maximal ideal of R. Hence A is an irredundant intersection of powers of maximal ideals of R.

(iv) $\Rightarrow$ (i) From (iv) it follows that every proper irreducible ideal of R is a power of a maximal ideal of R, so R is zero-dimensional. Thus, by Theorem 3.5, R_M is a special PIR for each maximal ideal M of R. Now to show that (i) holds, it suffices by (iv) and Theorem 2.2 to show that M^e is completely irreducible for each maximal ideal M of R and integer $e > 0$. As noted in the proof of Theorem 3.5 (ii) $\Rightarrow$ (i), $M^e = (M^e)_{(M)}$. Since R_M is a valuation ring, the ideal M^eR_M of R_M is irreducible, so M^e, as the preimage of M^eR_M under the mapping $R \to R_M$, is an irreducible ideal of R. Thus every power of a maximal ideal of R is irreducible. Since for each maximal ideal M of R the ring R_M is as special PIR, we have by Proposition 3.4 that every irreducible ideal, hence every power of a maximal ideal, is completely irreducible. □

Remark 3.8. (i) A Noetherian ring satisfying the equivalent conditions of Theorem 3.7 is necessarily semilocal since it is zero-dimensional. Conversely, any semilocal ring satisfying the equivalent conditions of Theorem 3.7 is Noetherian. It follows that such a ring is a finite product of special PIRs (see for example Corollary 4.5).

(ii) Rings that are proper homomorphic images of the almost Dedekind domains described in Remark 2.11 (ii) satisfy the equivalent conditions of Theorem 3.7. In particular, if X is a compact scattered space having both isolated and non-isolated points, then there is an almost Dedekind domain R such that $\text{Max}(R)$ is homeomorphic to X. Since $\text{Max}(R)$ has an isolated point, the Jacobson radical J of R is nonzero. The ring R/J has a scattered prime spectrum and R/J satisfies the equivalent conditions of Theorem 3.7. Since $\text{Spec}(R/J)$ has a non-isolated point, R/J is not semilocal; hence by (i) R/J is not Noetherian.

A ring satisfying the equivalent conditions of Theorem 3.7 is locally Noetherian at each maximal ideal, but, as noted in Remark 3.8, the ring itself need not be Noetherian. However from Theorem 3.7 (iii) it follows that every non-unit in such a ring is contained in a finitely generated maximal ideal.

Corollary 3.9. *The following statements are equivalent for a domain R.*

(i) *Every nonzero proper ideal of R has a unique representation as an irredundant intersection of powers of maximal ideals.*

(ii) *R is almost Dedekind and for every proper ideal A, the ring R/A has at least one finitely generated maximal ideal.*

(iii) *R is an almost Dedekind domain and for each proper nonzero ideal A of R,* $\mathrm{Spec}(R/A)$ *is a scattered space.*

(iv) *Every nonzero proper ideal of R can be represented uniquely as an irredundant intersection of completely irreducible ideals.*

Proof. (i) $\Rightarrow$ (ii) Statement (i) implies that R/A is zero-dimensional for every nonzero proper ideal A of R. Therefore $\dim R \leq 1$. If A and B are nonzero ideals contained in the maximal ideal M, then by Theorem 3.7, $R_M/(A \cap B)R_M$ is a special PIR. Thus AR_M and BR_M are comparable ideals of R_M. It follows that R_M is a valuation domain. Indeed, since R_M/AR_M is a special PIR for any nonzero ideal $A \subseteq M$, it follows that R_M is a Noetherian valuation domain. This proves that R is almost Dedekind. Now let A be a proper ideal of R. If A is nonzero, then by (i) every proper ideal of R/A can be represented as an irredundant intersection of powers of maximal ideals of R/A. Hence by Theorem 3.7 R/A has a finitely generated maximal ideal. On the other hand, if $\dim R = 1$ and $A = 0$, let a be a nonzero nonunit of R. Then R/aR has a finitely generated maximal ideal, say M/aR, where M is a maximal ideal of R. Thus M is a finitely generated ideal of R.

(ii) $\Rightarrow$ (iii) Apply Theorem 2.8.

(iii) $\Rightarrow$ (iv) If A is a nonzero proper ideal of R, then by (iii), R/A satisfies the equivalent conditions of Theorem 3.7. Hence A is an irredundant intersection of completely irreducible ideals.

(iv) $\Rightarrow$ (i) Suppose R is not a field. For any nonzero ideal B of R, R/B is by Theorem 3.7 a zero-dimensional ring. Hence R is a one-dimensional domain. Let A be a proper nonzero ideal of R. Then A is an irredundant intersection $A = \bigcap_{i \in I} A_i$ of completely irreducible ideals A_i of R. Let $i \in I$. For each maximal ideal M of

R containing A_i, $R_M/(A_i)_M$ is a special PIR, so by Proposition 3.4 every primal ideal of R/A_i is completely irreducible. Since A_i is completely irreducible, we have by Corollary 3.6 that A_i is a power of a maximal ideal of R. Thus (i) follows. □

Remark 3.10. As mentioned in Remark 2.11, it is shown in [13] that every compact scattered space can be realized as $\mathrm{Max}(R)$ for an almost Dedekind domain R. If R is such a domain, then for every nonzero ideal A, $\mathrm{Spec}(R/A)$ is a scattered space and R satisfies the equivalent conditions of Corollary 3.9.

Corollary 3.11. *The following statements are equivalent for a ring R.*

(i) *Every proper ideal of R can be represented uniquely as a finite irredundant intersection of completely irreducible ideals.*

(ii) *Every proper ideal of R is a finite product of maximal ideals.*

(iii) *R is isomorphic to a finite direct product of special PIRs.*

Proof. (i) ⇒ (ii) By Theorem 3.7 R is a zero-dimensional ring. Now $(0) = A_1 \cap \cdots \cap A_n$ for some completely irreducible ideals A_i of R. For each i, A_i is primal, so $A_i = (A_i)_{(M_i)}$ for some maximal ideal M_i of R. By Lemma 2.7 M_i is the unique maximal ideal of R containing A_i. It follows that the only maximal ideals of R are $M_1, M_2, \ldots, M_n$. Now apply Theorem 3.7 to obtain (ii).

(ii) ⇒ (iii) By (ii) the zero ideal (0) can be represented uniquely as a finite intersection of powers $M_i^{e_i}$, $i = 1, \ldots, n$, of distinct maximal ideals M_i of R. Thus $R \cong \prod_{i=1}^n R/M_i^{e_i}$. By Theorem 3.7, each $R/M_i^{e_i}$ is a special PIR.

(iii) ⇒ (i) Since R is Noetherian, every ideal of R is an intersection of finitely many irreducible ideals. Also, by Proposition 3.4 every irreducible ideal of R is completely irreducible, so (i) follows from Theorem 2.2. □

In view of the fact that every ideal of a ring is an intersection of irreducible ideals, if a ring has the property that every irreducible ideal is a power of a prime ideal, then every proper ideal is an intersection of powers of prime ideals. This motivates us to ask:

Question 3.12. *If every proper ideal A of a ring R is an intersection $A = \bigcap_i P_i^{e_i}$ of powers of prime ideals, does R satisfy the equivalent conditions of Theorem 3.5?*

Since every ideal of a ring is an intersection of completely irreducible ideals, this question is equivalent to: *If every completely irreducible proper ideal of R is a power of a prime ideal, is every irreducible proper ideal of R a power of a prime ideal?*

4. A generalization of ZPI rings

A ring is a ZPI (Zerlegung in Primideale) ring if every proper ideal in the ring is a product of prime ideals. It is well known that the ZPI domains are precisely the Dedekind domains. The general case is also well-understood: a ring is a ZPI ring if and only if it is isomorphic to a finite product of Dedekind domains and special PIRs (see Chapter IX, Section 2, of [11]). As an application of our previous results, we consider in this section a related class of rings, those for which every proper ideal is an irredundant intersection of powers of prime ideals. We observe in Remark 4.4 and Corollary 4.5 that this class of rings properly contains the class of ZPI rings.

Theorem 4.1. *The following statements are equivalent for a ring R.*

(i) *Every proper ideal A of R can be represented uniquely as an irredundant intersection $A = \bigcap_i P_i^{e_i}$ of powers of prime ideals P_i.*

(ii) *Every proper ideal A of R is an irredundant intersection $A = \bigcap_i P_i^{e_i}$ of powers of prime ideals P_i.*

(iii) *Every ideal of R can be represented as an irredundant intersection of irreducible ideals and R_M is a Noetherian valuation ring for each maximal ideal M of R.*

Proof. Before proving Theorem 4.1, we establish ($\star$): *If R is a ring in which every proper irreducible ideal is a power of a prime ideal, then every power of a prime ideal of R is an irreducible ideal.* By Theorem 3.5, R_M is a Noetherian valuation ring for each maximal ideal M of R. Let P be a prime ideal of R and let $k > 0$. Let M be a maximal ideal of R containing P. It suffices to prove that $(P^k)_{(M)} = P^k$, since this implies that P^k, as a preimage of an irreducible ideal under the mapping $R \to R_M$, is itself irreducible. If P is a maximal ideal of R (i.e. $P = M$), then as in the proof of (ii) $\Rightarrow$ (i) of Theorem 3.5, $(P^k)_{(M)} = P^k$. Otherwise if P is not a maximal ideal of R, then P is a minimal prime ideal and $PR_M = (0)R_M$. Hence $P^k R_M = PR_M$, so that $(P^k)_{(M)} = P_{(M)} = P$. Thus for any maximal ideal M containing P, $(P^k)_{(M)} = P$. Now $P^k = \bigcap_{N \supseteq P}(P^k)_{(N)}$, where N ranges over the set of maximal ideals of R containing P. Therefore we conclude $P^k = P$, so that P^k is clearly an irreducible ideal of R. This shows that every power of a prime ideal is an irreducible ideal.

We now verify the theorem.

(i) ⇒ (ii) This is clear.

(ii) ⇒ (iii) Since an irreducible ideal cannot be expressed as an irredundant intersection of distinct overideals, (ii) implies that every proper irreducible ideal is a power of a prime ideal. By Theorem 3.5, for each maximal ideal M, R_M is a Noetherian valuation ring. By our above observation ($\star$), every power of a prime ideal is irreducible so the assertion that every proper ideal of R can be represented as an irredundant intersection of irreducible ideals is a consequence of (ii).

(iii) ⇒ (i) By Theorem 3.5, every proper irreducible ideal of R is a power of a prime ideal of R. If A is a proper ideal of R, then by (iii) there is an irredundant intersection $A = \bigcap_i P_i^{e_i}$ of powers of prime ideals P_i of R. By ($\star$), the set of proper irreducible ideals of R is precisely the set of powers of prime ideals of R. Thus from Theorem 2.2 it follows that the representation $A = \bigcap_i P_i^{e_i}$ is unique among irredundant intersections of powers of prime ideals. □

Corollary 4.2. *A domain R satisfies the equivalent conditions of Theorem 4.1 if and only if R is an almost Dedekind domain such that for each proper ideal A of R, the ring R/A has at least one finitely generated maximal ideal.*

Proof. Apply Theorem 4.1 and Corollary 3.9. □

Remark 4.3. If P is a nonmaximal prime ideal of a ring R satisfying (i) - (iii) of Theorem 4.1, then as noted in the proof of ($\star$) of this theorem, $P^k = P$ for all $k > 0$. Thus by (i) every proper ideal A of R is an irredundant intersection: $A = (\bigcap_{i \in I} M_i^{e_i}) \cap (\bigcap_{j \in J} P_j)$, where each M_i is a maximal ideal of R and each P_j is a one-dimensional prime ideal of R. (We admit here the possibility that I or J is empty.) Moreover, since R is arithmetical, the set $\{M_i\} \cup \{P_j\}$ consists of pairwise comaximal prime ideals.

Remark 4.4. (i) By Theorem 3.7, if R is an almost Dedekind domain with nonzero Jacobson radical and R has a scattered prime spectrum and at least one non-isolated point, then R satisfies the conditions of Corollary 4.2 but is not a ZPI ring. See Remark 2.11 for such an example.

(ii) If $R = R_1 \times R_2 \times \cdots \times R_n$ is a finite product of rings satisfying the equivalent conditions of Theorem 4.1, then R also satisfies the conditions of the theorem. To prove this we may assume $n = 2$. Let e_1 and e_2 be idempotent elements of R such that $Re_i = R_i$, $i = 1, 2$. A maximal ideal M of R contains either e_1 or e_2, but not both. If $e_1 \in M$, then $M = R_1 \times M_2$, where M_2 is a maximal ideal of

R_2 and $R_M \cong (R_2)_{M_2}$. Similarly, if $e_2 \in M$, then $M = M_1 \times R_2$, where M_1 is a maximal ideal of R_1 and $R_M \cong (R_1)_{M_1}$. Thus each localization of R at a maximal ideal is a Noetherian valuation ring. If $B = B_1 \times B_2$ is an arbitrary ideal of R, we obtain an irredundant representation of B as an intersection of irreducible ideals as follows: there exist irreducible ideals A_{1i} of R_1 that intersect irredundantly in B_1 and irreducible ideals A_{2i} of R_2 that intersect irredundantly in B_2. Each of the ideals $A_{1i} \times R_2$ and $R_1 \times A_{2i}$ is irreducible in R and $B = B_1 \times B_2$ is represented irredundantly as the intersection of this collection of ideals. Therefore $R = R_1 \times R_2$ also satisfies the conditions of Theorem 4.1.

(iii) If R is a finite product of domains satisfying the conditions of Corollary 4.2 and rings satisfying the conditions of Theorem 3.7, then by (ii) R is a one-dimensional ring with zero-divisors having the property that every proper ideal is an irredundant intersection of powers of prime ideals.

Corollary 4.5. *The following statements are equivalent for a ring R.*

(i) *R is a ZPI ring.*

(ii) *There is an isomorphism of rings, $R \cong R_1 \times R_2 \times \cdots \times R_n$, where each R_i is a Dedekind domain or a special PIR.*

(iii) *Every proper ideal A of R is a finite intersection $A = \bigcap_{i=1}^{n} P_i^{e_i}$ of powers of prime ideals P_i of R.*

(iv) *Every proper ideal A of R is a finite irredundant intersection $A = \bigcap_{i=1}^{n} P_i^{e_i}$ of powers of prime ideals P_i of R, and this representation of A is unique among irredundant representations of A as an intersection of powers of prime ideals.*

Proof. (i) $\Leftrightarrow$ (ii) This equivalence can be found in Theorem IX.9.10 of [11].

(ii) $\Rightarrow$ (iii) Write $R = Re_1 \oplus \cdots \oplus Re_n$, where the e_i are orthogonal idempotents and for each i, $R_i \cong Re_i$. For each i define $B_i = \sum_{j \neq i} Re_j$. If A is an ideal of R, then $A = \bigcap_{i=1}^{n}(A + B_i)$, and since each R_i is a Dedekind domain or a special PIR, each ideal $A + B_i$ is a finite intersection of powers of prime ideals. Hence (iii) follows.

(iii) $\Rightarrow$ (iv) By (iii) R satisfies the equivalent conditions of Theorem 4.1, so that by property ($\star$) in the proof of this theorem, every power of a prime ideal of R is irreducible. In an arithmetical ring, an irredundant intersection of irreducible ideals of R is unique (Theorem 2.2). Thus (iv) follows.

(iv) $\Rightarrow$ (i) Let A be a proper ideal of R. By Theorem 4.1 and Remark 4.3, A is an irredundant intersection of comaximal powers of prime ideals. By the uniqueness assertion in (iv) this intersection must be finite, so A is a product of these same powers of prime ideals. □

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