**Math 55400** **Final Exam** **May 4, 2010** **W. Heinzer**

**Name:** ______________________________

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Instructions:

1. The point value of each exercise occurs to the left of the problem.
2. Write your answer in the box, if one is provided.
3. No books or notes or calculators are allowed.

| **Page** | **Points Possible** | **Points** |
|---|---|---|
| 2 | 16 | |
| 3 | 19 | |
| 4 | 21 | |
| 5 | 20 | |
| 6 | 16 | |
| 7 | 20 | |
| 8 | 18 | |
| 9 | 16 | |
| 10 | 18 | |
| 11 | 20 | |
| 12 | 16 | |
| Total | 200 | |

**1.** (16 pts) Let $p$ be a prime integer and let $G = \frac{\mathbb{Z}}{(p^2)} \oplus \frac{\mathbb{Z}}{(p)}$ be the direct sum of a cyclic group of order $p^2$ and a cyclic group of order $p$.

(a) How many subgroups of $G$ have order $p$?

(b) How many subgroups $H$ of order $p$ of $G$ are direct summands of $G$, i.e., are such that $G = H \oplus H'$ for some subgroup $H'$ of $G$? Justify your answer.

**2.** (6 pts ) State true or false and justify: If $G$ and $G'$ are abelian groups of order 42, then $G$ and $G'$ are isomorphic.

**3.** (6 pts ) If $t_0, t_1, \dots, t_n$ are $n+1$ distinct elements of a field $F$ and $c_0, c_1, \dots, c_n$ are elements of $F$, write down a polynomial $g(x) \in F[x]$ of degree $\le n$ such that $g(t_i) = c_i$ for each $i \in \{0, 1, \dots, n\}$.

**4.** ( 7 pts ) Let $\mathbb{Q}$ denote the field of rational numbers. Give an example of a linear operator $T : \mathbb{Q}^3 \to \mathbb{Q}^3$ having the property that the only $T$-invariant subspaces are the whole space and the zero subspace. Explain why your example has this property.

**5.** (21 pts ) Let $A \in \mathbb{C}^{5\times 5}$ be a diagonal matrix with exactly four distinct entries on its main diagonal.

(a) What is the dimension of the vector space over $\mathbb{C}$ of matrices that are polynomials in $A$?

(b) What is the dimension of the vector space over $\mathbb{C}$ of matrices $B \in \mathbb{C}^{5\times 5}$ such that $AB = BA$?

(c) If $B \in \mathbb{C}^{5\times 5}$ is a diagonal matrix with exactly four distinct entries on its main diagonal, is $B$ similar to a polynomial in $A$? Justify your answer.

**6.** (20 pts ) Let $A$ and $B$ in $\mathbb{Q}^{n\times n}$ be $n \times n$ matrices and let $I \in \mathbb{Q}^{n\times n}$ denote the identity matrix.

(a) State true or false and justify: If $A$ and $B$ are similar over an extension field $F$ of $\mathbb{Q}$, then $A$ and $B$ are similar over $\mathbb{Q}$.

(b) Let $M$ and $N$ be $n \times n$ matrices over the polynomial ring $\mathbb{Q}[x]$. Define "$M$ and $N$ are equivalent over $\mathbb{Q}[x]$".

(c) State true or false and justify: If $\det(xI - A) = \det(xI - B)$, then $xI - A$ and $xI - B$ are equivalent.

(d) State true or false and justify: If $xI - A$ and $xI - B$ are equivalent over $\mathbb{Q}[x]$, then $A$ and $B$ are similar over $\mathbb{Q}$.

**7.** (8 pts ) Let $V$ be a five-dimensional vector space over the field $F$ and let $T : V \to V$ be a linear operator such that $\operatorname{rank} T = 1$. List all polynomials $p(x) \in F[x]$ that are possibly the minimal polynomial of $T$. Explain.

**8.** (8 pts ) List up to isomorphism all abelian groups of order 16.

**9.** (20 pts ) Let $V$ be an abelian group generated by elements $a, b, c$. Assume that $2a = 6b, 2b = 6c, 2c = 6a$, and that these three relations generate all the relations on $a, b, c$.

(a) Write down a relation matrix for $V$.

(b) Find generators $x, y, z$ for $V$ such that $V = \langle x \rangle \oplus \langle y \rangle \oplus \langle z \rangle$ is the direct sum of cyclic subgroups generated by $x, y, z$.

(c) Express your generators $x, y, z$ in terms of $a, b, c$.

(d) What is the order of $V$?

**10.** (18 pts) Let $V$ be a finite-dimensional vector space over an infinite field $F$ and let $T : V \to V$ be a linear operator. Give to $V$ the structure of a module over the polynomial ring $F[x]$ by defining $x\alpha = T(\alpha)$ for each $\alpha \in V$.

(a) Outline a proof that $V$ is a direct sum of cyclic $F[x]$-modules.

(b) In terms of the expression for $V$ as a direct sum of cyclic $F[x]$-modules, what are necessary and sufficient conditions in order that $V$ have only finitely many $T$-invariant $F$-subspaces? Explain.

**11.** (16 pts) Let $V$ be a finite dimensional inner product space over $\mathbb{C}$ and let $T : V \to V$ be a linear operator.

(a) Define the adjoint $T^*$ of $T$.

(b) If $T = T^*$, prove that every characteristic value of $T$ is a real number.

(c) Assume that $T = T^*$ and that $c$ and $d$ are distinct characteristic values of $T$. If $\alpha$ and $\beta$ in $V$ are such that $T\alpha = c\alpha$ and $T\beta = d\beta$, prove that $\alpha$ and $\beta$ are orthogonal.

(d) If $W$ is a $T$-invariant subspace of $V$, prove or disprove that the orthogonal complement $W^\perp$ must be $T^*$-invariant.

**12.** (9 pts) Let $V$ be a vector space over an infinite field $F$. Prove that $V$ is not the union of finitely many proper subspaces.

**13.** (9 pts) Let $V$ be a finite-dimensional vector space over an infinite field $F$ and let $\alpha_1, \ldots, \alpha_m$ be finitely many nonzero vectors in $V$. Prove that there exists a linear functional $f$ on $V$ such that $f(\alpha_i) \neq 0$ for each $i$ with $1 \leq i \leq m$.

**14.** (5 pts) State true or false and justify: If $N_1$ and $N_2$ are $4 \times 4$ nilpotent matrices over the field $F$ and if $N_1$ and $N_2$ have the same minimal polynomial, then $N_1$ and $N_2$ are similar.

**15.** (5 pts) State true or false and justify: If $A$ and $B$ are $n \times n$ matrices over a field $F$, then $AB$ and $BA$ have the same characteristic polynomial.

**16.** (5 pts) State true or false and justify: If $A$ and $B$ are $n \times n$ matrices over a field $F$, then $AB$ and $BA$ have the same minimal polynomial.

**17.** (5 pts) State true or false and justify: If $V$ is a finite-dimensional vector space and $W_1$ and $W_2$ are subspaces of $V$ such that $V = W_1 \oplus W_2$, then for any subspace $W$ of $V$ we have $W = (W \cap W_1) \oplus (W \cap W_2)$.

**18.** (5 pts) State true or false and justify: For every symmetric matrix $A \in \mathbb{R}^{n\times n}$ there exists a matrix $B \in \mathbb{R}^{n\times n}$ such that $B^3 = A$.

**19.** (11 pts) Classify up to similarity all matrices $A \in \mathbb{C}^{3\times 3}$ such that $A^3 = I$, where $I$ is the identity matrix, i.e., write down all possibilities for the Jordan form of $A$.