Math 554 Commuting Matrices Heinzer March 10, 2010

Let $\mathcal{F} \subseteq \mathbb{C}^{3\times 3}$ be a subspace of $\mathbb{C}^{3\times 3}$ such that for each $A, B \in \mathcal{F}$ we have $AB = BA$. In Exercise 2 on page 208 we are asked about the maximum possible dimension of $\mathcal{F}$, and then also asked "What about the case where $\mathcal{F} \subseteq \mathbb{C}^{n\times n}$?"

Let me try to make some general remarks about this exercise. We may as well assume that $\mathcal{F}$ contains the scalar matrices cI since the scalar matrices commute with everything. This exercise comes in the section where we prove we can simultaneously upper triangulate commuting operators over an algebraically closed field, so we can assume if we want that all the matrices in $\mathcal{F}$ are upper triangular. To get further with the exercise, I think we want to make use of invariant subspaces.

Before going further into invariant subspaces related to Exercise 2, let me discuss their relevance to Exercise 3 on page 208 which says that if V is an n-dimensional vector space over a field F and $T : V \to V$ is a linear operator having n distinct characteristic values, then every operator that commutes with T is a polynomial in T. If $c_1, \ldots, c_n$ are the characteristic values of T and W_i is the nullspace of $T - c_i I$, then $V = W_1 \oplus \cdots \oplus W_n$ and each W_i is invariant under every operator S that commutes with T. Since $\dim W_i = 1$, if W_i is invariant for an operator S, then the nonzero vectors in W_i are characteristic vectors for S. Let $\mathbf{B} = \{\alpha_1, \ldots, \alpha_n\}$, where α_i is a nonzero vector in W_i. Then $\mathbf{B}$ is an ordered basis for V that consists of characteristic vectors for T and also for S, where S is any operator that commutes with T. Therefore each operator that commutes with T is represented by a diagonal matrix with respect to the basis $\mathbf{B}$. It follows that the family of operators that commute with T has dimension at most n. It also follows that if S and U are operators that commute with T, then S and U commute. Notice that for certain operators it happens that there exist operators S and U that commute with the given operator, but $SU \neq US$.

Another basic fact is that the dimension of the family of operators that are polynomials in T is equal to the degree of the minimal polynomial for T. Since the minimal polynomial for T has degree n, every operator that commutes with T is a polynomial in T.

Now let us go back to $\mathcal{F} \subseteq \mathbb{C}^{3\times 3}$ and consider the minimal polynomials of the matrices $A \in \mathcal{F}$. If there exists $A \in \mathcal{F}$ having 3 distinct characteristic values, then what we have said above implies the matrices in $\mathcal{F}$ can be simultaneously diagonalized and hence $\dim \mathcal{F} \leq 3$.

If each $A \in \mathcal{F}$ has only one distinct characteristic value, then when we simultaneously upper triangularize the elements of $\mathcal{F}$ we get a subspace of the 4-dimensional space $\mathcal{G} = \left\{ \begin{bmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{bmatrix} \mid a, b, c, d \in \mathbb{C} \right\}$ of upper triangular 3×3 matrices that have the same entry on the main diagonal. Since there exist two matrices in $\mathcal{G}$ that do not commute, e.g., $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $\mathcal{F}$ has to be a proper subspace of $\mathcal{G}$ and hence has dimension at most 3 in this case.

Suppose there exists $A \in \mathcal{F}$ that has 2 distinct characteristic values c_1 and c_2. We may assume the characteristic polynomial for A has the form $(x - c_1)(x - c_2)^2$. Let W_1 be the null space of $A - c_1 I$ and let W_2 be the null space of $(A - c_2 I)^2$. It is not hard to see directly that $\dim W_1 = 1$ and $\dim W_2 = 2$ and that $W_1 \cap W_2 = 0$. Hence $V = \mathbb{C}^3$ is the direct sum $V = W_1 \oplus W_2$. (We will see in the proof of the primary decomposition theorem a more general version of this result.) Since each $B \in \mathcal{F}$ commutes with A the subspaces W_1 and W_2 are invariant for B. The dimension of $\mathcal{F}$ is the sum of the dimension of the restrictions of the matrices acting via left multiplication on W_1 plus their similar restriction to W_2. The restriction to W_1 has dimension at most one and the restriction to W_2 gives a family of commuting linear operators on a 2-dimensional vector space over $\mathbb{C}$. The dimension of such a family is at most 2. Indeed, with the reductions we have made above, each of these operators on W_2 has a repeated characteristic value. In a suitable basis, the operators are represented by 2×2 upper triangular matrices of the form $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$.

Suppose now that $\mathcal{F} \subseteq \mathbb{C}^{4\times 4}$ is a vector space of commuting matrices. If there exists $A \in \mathcal{F}$ having at least two distinct characteristic values, I suggest you write out a proof that $\mathcal{F}$ has dimension at most 4. However, without this assumption, give an example of a space $\mathcal{F}$ of commuting 4×4 matrices that has dimension 5.