Math 554 Assignment 11 To hand in Friday April 2, 2010 Heinzer

These are exercises from Chapter 3 of Jacobson. By a basis or base for a module is meant a linearly independent subset which spans the module, cf. page 164 of Hoffman and Kunze. From Section 3.6 of Jacobson.

1. Find a base for the submodule of $\mathbb{Z}^{(3)}$ generated by
$f_1 = (1, 0, -1), \quad f_2 = (2, -3, 1), \quad f_3 = (0, 3, 1), \quad f_4 = (3, 1, 5)$.

2. Find a base for the submodule of $\mathbb{Q}[\lambda]^{(3)}$ generated by
$f_1 = (2\lambda - 1, \lambda, \lambda^2 + 3), \quad f_2 = (\lambda, \lambda, \lambda^2), \quad f_3 = (\lambda + 1, 2\lambda, 2\lambda^2 - 3)$.

3. Find a base for the $\mathbb{Z}$-submodule of $\mathbb{Z}^{(3)}$ consisting of all (x_1, x_2, x_3) satisfying the conditions $x_1 + 2x_2 + 3x_3 = 0, \quad x_1 + 4x_2 + 9x_3 = 0$.

From Section 3.7, 1. Obtain a normal form over $\mathbb{Z}$ for the integral matrix

$$B = \begin{bmatrix} 6 & 2 & 3 & 0 \\ 2 & 3 & -4 & 1 \\ -3 & 3 & 1 & 2 \\ -1 & 2 & -3 & 5 \end{bmatrix}$$

From Section 3.8, 1. Determine the structure of $\mathbb{Z}^{(3)}/K$ where K is generated by $f_1 = (2, 1, -3)$, and $f_2 = (1, -1, 2)$.

From Section 3.9, 1. Let $D = \mathbb{R}[\lambda]$ and suppose M is a direct sum of cyclic D-modules whose order ideals are the ideals generated by the polynomials $(\lambda-1)^3, \quad (\lambda^2+1)^2, \quad (\lambda-1)(\lambda^2+1)^4, \quad (\lambda+2)(\lambda^2+1)^2$. Determine the elementary divisors and invariant factors of M.

Let D be a principal ideal domain (PID) and let M be a D-module. A submodule N of M is said to be *pure in* M if for any $y \in N$ and $a \in D$, if there exists $x \in M$ with $ax = y$, then there exists $x' \in N$ with $ax' = y$. The module M is said to be a *torsion* module if for each $m \in M$ there exists a nonzero $d \in D$ such that $dm = 0$.

7. Show that if N is a direct summand of M, then N is pure in M. Show that if N is a pure submodule of M and $\operatorname{ann}(x + N) = (d)$ then x can be chosen in its coset $x + N$ so that $\operatorname{ann} x = (d)$.

8. Show that if N is a pure submodule of a finitely generated torsion module M over a PID, then N is a direct summand of M.

Some remarks about T-annihilators.

In connection with Exercise 4 on page 225 of Hoffman and Kunze, I suggest you go back and review the remark on page 202 concerning a vector $\alpha \in V$ and W a T-invariant subspace of V. The *T-conductor of α into W* is by definition

$$S_T(\alpha; W) = \{\, g(x) \in F[x] \,:\, g(T)(\alpha) \in W \,\}.$$

$S_T(\alpha; W)$ is an ideal of the polynomial ring $F[x]$. One also calls the unique monic generator of $S_T(\alpha; W)$ the T-conductor of α into W. A useful fact is that for every $\alpha \in V$ and T-invariant subspace W of V, the T-conductor of α into W divides the minimal polynomial of T.

In Exercise 4 on page 225, we are given that $p = (x - c_1)^{r_1} \cdots (x - c_k)^{r_k}$ is the minimal polynomial of T. If $\alpha \in V$ is such that $(T - c_i I)^m(\alpha) = 0$ for some positive integer m, then the T-annihilator of α is a divisor of $(x - c_i)^m$ and of p and therefore has the form $(x - c_i)^s$, where $s \le r_i$ and $s \le m$. Thus $(T - c_i I)^{r_i}(\alpha) = 0$.

Given a vector space V over a field F and $T : V \to V$ a linear operator, we give to V the structure of a module over the polynomial ring $F[x]$ by defining $g(x)(\alpha) = g(T)(\alpha)$ for each $g(x) \in F[x]$ and $\alpha \in V$. The submodules of V are precisely the T-invariant subspaces of V. Suppose V is finite-dimensional and $p = p_1^{r_1} \cdots p_k^{r_k}$ is the minimal polynomial for T where the p_i are distinct monic irreducible polynomials in $F[x]$. Let W_i be the null space of $p_i(T)^{r_i}$. The primary decomposition theorem tells us that $V = W_1 \oplus \cdots \oplus W_k$. Moreover, as is asserted in Exercise 10 on page 226 of Hoffman and Kunze, if W is a T invariant subspace of V, then

$$W = (W \cap W_1) \oplus \cdots \oplus (W \cap W_k).$$

Note that this tells us a great deal about the submodules of V. It says, for example, that if $\dim(V) = n$ and if T has n distinct characteristic values, then V has precisely 2^n submodules. Thus an easy way to prove Exercise 7 (b) on page 231 of Hoffman and Kunze (which asks to show that if $\{\alpha_1, \ldots, \alpha_n\}$ is a basis for V of characteristic vectors having distinct characteristic values, then $\alpha = \alpha_1 + \cdots + \alpha_n$ is a cyclic vector for V) is to observe that α is in no proper submodule of V.