Math 554 Practice Problems for test April 14 Heinzer March 31, 2010

1. Let V be an abelian group and assume that $(v_1, \dots, v_n)$ are generators of V. Describe a process for obtaining for some positive integer m an $m \times n$ matrix $A \in \mathbb{Z}^{m\times n}$ such that if $\phi : \mathbb{Z}^m \to \mathbb{Z}^n$ is the $\mathbb{Z}$-module homomorphism defined by right multiplication by A (i.e., if $\vec{x} = (x_1, \dots, x_m) \in \mathbb{Z}^m$, then $\phi(\vec{x}) = \vec{x}A$), then $V \cong \mathbb{Z}^n/\phi(\mathbb{Z}^m)$. Recall that such a matrix A is said to be a relation matrix of V.

2. Consider the abelian group $V = \mathbb{Z}/(5^4) \oplus \mathbb{Z}/(5^3) \oplus \mathbb{Z}$.

(i) Write down a relation matrix for V as a $\mathbb{Z}$-module.

(ii) Let W be the cyclic subgroup of V generated by the image of the element $(5^2, 5, 5)$ in $\mathbb{Z}/(5^4) \oplus \mathbb{Z}/(5^3) \oplus \mathbb{Z} = V$. Write down a relation matrix for W.

(iii) Write down a relation matrix for the quotient module V/W.

3. Let A be an 5×4 matrix with integer coefficients and let $\phi : \mathbb{Z}^5 \to \mathbb{Z}^4$ be defined by right multipliation by A.

(i) Prove or disprove: if ϕ is surjective, then the determinants of the 4×4 minors of A generate the unit ideal of $\mathbb{Z}$.

(ii) Prove or disprove: if ϕ is surjective, then there exists a matrix $B \in \mathbb{Z}^{4\times 5}$ such that BA is the 4×4 identity matrix.

4. Let $V = \mathbb{Z}^2$ and let L be the submodule of V spanned by the rows of $A = \begin{bmatrix} 6 & 4 \\ 8 & 12 \end{bmatrix}$. Find a basis $(\vec{\alpha}_1, \vec{\alpha}_2)$ of V and integers c_1, c_2 so that $c_1\vec{\alpha}_1$, $c_2\vec{\alpha}_2$ is a basis for L.

With regard to this last exercise, we have seen in general that if D is a principal ideal domain (PID), n is a positive integer and $D^{(n)}$is a free D-module of rank n, then every submodule of $D^{(n)}$ is free of rank $\leq n$. Let $\{e_1, \dots, e_n\}$ be an ordered basis for $D^{(n)}$. Let L be a submodule of $D^{(n)}$ and let $\{f_1, \dots, f_m\}$, $m \leq n$ be an ordered basis for L. With respect to these bases, the inclusion map $L \hookrightarrow D^{(n)}$ is defined by right multiplication with respect to the matrix $A = (a_{ij})$, where $f_i = \sum_{j=1}^{n} a_{ij}e_j$, for $1 \leq i \leq m$. There exist invertible matrices $P \in D^{m\times m}$ and $Q \in D^{n\times n}$ such that $PAQ^{-1} = B \in D^{m\times n}$, where B is a 'diagonal' matrix in the sense that the only nonzero elements of B are on the main diagonal. Moreover, P

and Q can be chosen so that the diagonal entries of B are $d_1, d_2, \ldots, d_m$, where $d_1|d_2, d_2|d_3, \ldots, d_{m-1}|d_m$. Consider the equality $PA = BQ$. Since Q is invertible over D, the row vectors $\vec{q}_1, \ldots \vec{q}_n$ of Q are an ordered basis for $D^{(n)}$. The row vectors of BQ are $d_1\vec{q}_1, \ldots, d_m\vec{q}_m$, and the row vectors $f_1, \ldots, f_m$ of A are an ordered basis for L. Since $P \in D^{m\times m}$ is invertible over D, the row vectors of PA are an ordered basis for L. In view of the equality $PA = BQ$, the row vectors of PA are $d_1\vec{q}_1, \ldots, d_m\vec{q}_m$. It follows that $d_m \neq 0$. This procedure of finding an ordered basis $\{\vec{q}_1, \ldots, \vec{q}_n\}$ for $D^{(n)}$ such that $\{d_1\vec{q}_1, \ldots, d_m\vec{q}_m\}$ is an ordered basis for a given submodule L is sometimes called the 'stacked basis' property.

5. Let F be a field, let x be an indeterminate over F, and consider the polynomial ring $F[x]$. Let r and s and $a_1 \geq a_2 \geq \cdots \geq a_r$ and $b_1 \geq b_2 \geq \cdots \geq b_s$ be positive integers. Suppose

$$V = F[x]/(x^{a_1}) \oplus F[x]/(x^{a_2}) \oplus \cdots \oplus F[x]/(x^{a_r})$$

and

$$W = F[x]/(x^{b_1}) \oplus F[x]/(x^{b_2}) \oplus \cdots \oplus F[x]/(x^{b_s}).$$

If the $F[x]$-modules V and W are isomorphic, prove the structure theorem that asserts that $r = s$, and that $a_i = b_i$ for $i = 1, \ldots, r$.

6. Determine, up to isomorphism, the number of abelian groups of order 16.
7. Determine, up to isomorphism, the number of abelian groups of order 360.
8. For $D = \mathbb{R}[x]$, express the D-module $M = D^{(3)}/ < f_1, f_2, f_3 >$ as a direct sum of cyclic D-modules, where $f_1 = (2x-1, x, x^2+3)$, $f_2 = (x, x, x^2)$, and $f_3 = (x+1, 2x, 2x^2-3)$. Also determine the invariant factor ideals of the module M.
9. For $D = \mathbb{R}[x]$, express the D-module $V = D^{(3)}/ < g_1, g_2, g_3 >$ as a direct sum of cyclic D-modules, where $g_1 = (x+1, 2, -6)$, $g_2 = (1, x, -3)$, and $g_3 = (1, 1, x-4)$. Also determine the invariant factor ideals of the module V.
10. Express the abelian group $\mathbb{Z}^{(3)}/ < f_1, f_2, f_3 >$ as a direct sum of cyclic groups, where $f_1 = (4, 2, -6)$, $f_2 = (6, -6, 12)$, and $f_3 = (10, -4, 6)$. Also determine the invariant factor ideals of this group.