Math 554 Exercise 10 on page 213 Heinzer March 10, 2010

This is a nice exercise about projections. We are given that F is a field of characteristic zero, V is a finite-dimensional vector space over F, say $\dim V = n$, and $E_1, \dots, E_k$ are projections of V such that $E_1 + \cdots + E_k = I$, the identity map on V. We want to prove that $E_iE_j = 0$ for $i \neq j$.

It is interesting to observe that in the special case where $k = 2$, the exercise is easy and does not use the assumption that the field F has characteristic zero. In general if E is a projection operator, then $I - E$ is also a projection and is the unique projection E' such that $I = E + E'$. Note that if $I = E_1 + E_2$, with $E_1^2 = E_1$, then $E_2 = I - E_1$ has the property that $E_2^2 = E_2$ and $E_1 = E_1 I = E_1^2 + E_1E_2 = E_1 + E_1E_2$, so $E_1E_2 = 0$. Similarly, $E_2E_1 = 0$. Everything is easy if $k = 2$.

Let $W_i := E_iV$ denote the range of E_i, and let $r_i := \dim W_i$. Since E_i is a projection, E_i is the identity map when restricted to W_i. Using that F has characteristic zero, we see that the trace of E_i is r_i. Since the trace of a sum of linear operators is the sum of the traces, we have

$$n = \mathrm{tr}(I) = \mathrm{tr}(E_1 + \cdots + E_k) = \sum_{i=1}^{k} \mathrm{tr}(E_i) = r_1 + \cdots + r_k.$$

Since $I = E_1 + \cdots + E_k$, we have $V = W_1 + \cdots + W_k$. Since the dimension of the W_i add to n, the sum is direct, if $\mathbf{B}_i$ is an ordered basis for W_i, then $\mathbf{B} = (\mathbf{B}_1, \dots, \mathbf{B}_k)$ is an ordered basis for V..

To prove that $E_iE_j = 0$ for $i \neq j$, observe that for $\alpha \in V$, we have

$$E_i(\alpha) = IE_i(\alpha) = (E_1 + \cdots + E_k)E_i(\alpha) = \sum_{j=1}^{k} E_j(E_i(\alpha)).$$

Since $E_i^2(\alpha) = E_i(\alpha)$ and $E_j(E_i(\alpha) \in W_j$ for each j, and since $V = W_1 \oplus \cdots \oplus W_k$, we have $E_j(E_i(\alpha)) = 0$ for each $j \neq i$. Therefore $E_jE_i = 0$ for each $j \neq i$.

Even in the case where V is infinite dimensional, if $V = W_1 \oplus \cdots \oplus W_k$ and if E_i are projection operators such that $I = E_1 + \cdots + E_k$ and $E_iV = W_i$, then the above argument shows that $E_iE_j = 0$ for $i \neq j$.

To see that Exercise 10 is not true if the field F has characteristic $p > 0$, notice that $I = (p+1)I$, a sum of $p+1$ copies of I.

Another exercise on page 213 that deserves mention is Exercise 4. It is possible to have projections E_1 and E_2 onto independent subspaces such that $E_1 + E_2$ is not a projection. A simple way to illustrate this is by means of projection matrices in $\mathbb{R}^{2\times 2}$. Let $E_1 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $E_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Then E_1 is a projection of $\mathbb{R}^2$ onto its first coordinate and E_2 a projection of $\mathbb{R}^2$ onto its second coordinate, but $E_1 + E_2$ is not equal to its square and hence is not a projection. Notice also that

$$E_1E_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = E_2E_1.$$