Math 558 Exercises

(1) Let R be a commutative ring and let P be a prime ideal of the power series ring $R[[x]]$. Let $P(0)$ denote the ideal in R of constant terms of elements of P.

(i) If $x \notin P$ and $P(0)$ is generated by n elements of R, prove that P is generated by n elements of $R[[x]]$.

(ii) If $x \in P$ and $P(0)$ is generated by n elements of R, prove that P is generated by $n+1$ elements of $R[[x]]$.

(iii) If R is a PID, prove that every prime ideal of $R[[x]]$ of height one is principal.

(2) Let R be a DVR with maximal ideal yR and let $S = R[[x]]$ be the formal power series ring over R in the variable x. Let $f \in S$. Recall that f is a unit in S if and only if the constant term of f is a unit in R.

(a) If g is a factor of f and S/fS is a finite R-module, then S/gS is a finite R-module.

(b) If n is a positive integer and $f := x^n + y$, then S/fS is a DVR that is a finite R-module if and only if $R = \widehat{R}$, i.e., R is complete.

(c) If f is irreducible and $fS \neq xS$, then S/fS is a finite R-module implies that R is complete.

(d) If R is complete, then S/fS is a finite R-module for each nonzero f in S such that $fS \cap R = (0)$.

Suggestion: For item (c) use that if R is not complete, then by Nakayama's lemma, the completion of R is not a finite R-module.

(3) Let R be an integral domain and let $f \in R[[x]]$ be a nonzero nonunit of the formal power series ring $R[[x]]$. Prove that the principal ideal $fR[[x]]$ is closed in the (x)-adic topology, that is, $fR[[x]] = \bigcap_{m \geq 0}(f, x^m)R[[x]]$.

Suggestion: Reduce to the case where $c = f(0)$ is nonzero. Then f is a unit in the formal power series ring $R[\frac{1}{c}][[x]]$. If $g \in \bigcap_{m \geq 0}(f, x^m)R[[x]]$, then $g = fh$ for some $h \in R[\frac{1}{c}][[x]]$, say $h = \sum_{n \geq 0} h_n x^n$, with $h_n \in R[\frac{1}{c}]$. Let $m \geq 1$. As $g \in (f, x^m)R[[x]]$, $g = fq + x^m r$, for some $q, r \in R[[x]]$. Thus $g = fh = fq + x^m r$, hence $f(h - q) = x^m r$. As $f(0) \neq 0$, $h - q = x^m s$, for some $s \in R[\frac{1}{c}[[x]]$. Hence $h_0, h_1, \ldots, h_{m-1} \in R$.

(4) Let R be a commutative ring and let $f = \sum_{n \geq 0} f_n x^n \in R[[x]]$ be a power series having the property that its leading form f_r is a regular element of R, that is, $\operatorname{ord} f = r$, so $f_0 = f_1 = \cdots = f_{r-1} = 0$, and f_r is a regular element of R. As in the previous exercise, prove that the principal ideal $fR[[x]]$ is closed in the (x)-adic topology.

(5) Let a be a nonunit in an integral domain R and let $b \in \bigcap_{n=1}^{\infty} a^n R$. Prove that $b \in (a-x)R[[x]] \cap R$.

(6) Let $\phi : (R, \mathbf{m}) \hookrightarrow (S, \mathbf{n})$ be an injective local map of the Noetherian local ring $(R, \mathbf{m})$ into the Noetherian local ring $(S, \mathbf{n})$. Let $\widehat{R} = \varprojlim_n R/\mathbf{m}^n$ denote the $\mathbf{m}$-adic completion of R and let $\widehat{S} = \varprojlim_n S/\mathbf{n}^n$ denote the $\mathbf{n}$-adic completion of S.

(i) Prove that there exists a map $\widehat{\phi} : \widehat{R} \to \widehat{S}$ that extends the map $\phi : R \hookrightarrow S$.

(ii) Prove that $\widehat{\phi}$ is injective if and only if for each positive integer n there exists a positive integer s_n such that $\mathbf{n}^{s_n} \cap R \subseteq \mathbf{m}^n$.

(iii) Prove that $\widehat{\phi}$ is injective if and only if for each positive integer n the ideal $\mathbf{m}^n$ is closed in the topology on R defined by the ideals $\{\mathbf{n}^n \cap R\}_{n \in \mathbb{N}}$, i.e., the topology on R that defines R as a subspace of S.

(7) Let $(R, \mathbf{m})$ and $(S, \mathbf{n})$ be Noetherian local rings such that S dominates R and the $\mathbf{m}$-adic completion $\widehat{R}$ of R dominates S.

(i) Prove that R is a subspace of S.

(ii) Prove that $\widehat{R}$ is an algebraic retract of $\widehat{S}$, i.e., $\widehat{R} \hookrightarrow \widehat{S}$ and there exists a surjective map $\pi : \widehat{S} \to \widehat{R}$ such that π restricts to the identity map on the subring $\widehat{R}$ of $\widehat{S}$.

(8) Let k be a field and let R be the localized polynomial ring $k[x]_{xk[x]}$, and thus $\widehat{R} = k[[x]]$. Let $n \geq 2$ be a positive integer. If the field k has characteristic $p > 0$, assume that n is not a multiple of p.

(i) Prove that there exists $y \in k[[x]]$ such that $y^n = 1 + x$.

(ii) For y as in (i), let $S := R[yx] \hookrightarrow k[[x]]$. Prove that S is a local ring integral over R with maximal ideal $(x, yx)S$. By the previous exercise, $\widehat{R} = k[[x]]$ is an algebraic retract of $\widehat{S}$.

(iii) Prove that the integral closure $\overline{S}$ of S is not local. Indeed, if the field k contains a primitive n-th root of unity, then $\overline{S}$ has n distinct maximal ideals. Deduce that $\widehat{R} \neq \widehat{S}$, so $\widehat{R}$ is a nontrivial algebraic retract of $\widehat{S}$.

(9) Let $I = (a_1, \ldots, a_n)A$ be an ideal in a Noetherian ring A and let $\widehat{A}$ denote the I-adic completion of A. Prove that up to isomorphism

$$\widehat{A} \;=\; \frac{A[[x_1, \ldots, x_n]]}{(x_1 - a_1, \ldots, x_n - a_n)A[[x_1, \ldots, x_n]]},$$

where $A[[x_1, \ldots, x_n]]$ is the formal power series ring in the variables $x_1, \ldots, x_n$ over A.