NOTES ON BLOWUPS

I would like to expand on these notes from last summer as class notes for what we cover this fall.

1. May 19, 2009

1.1. **Motivation.**

Proposition 1.1. *Let $(R, \mathbf{m})$ be a two-dimensional regular local ring with $R/\mathbf{m} = k = \overline{k}$ algebraically closed. Let I be a complete $\mathbf{m}$-primary ideal in R and $\lambda(\mathbf{m}/I) = 1$. Then I is projectively full.*

Definition 1.2.

(1) Let I and J be any ideals in a commutative ring R. Then I and J are *projectively equivalent* if there exist positive integers m and n such that $\overline{I^m} = \overline{J^n}$.

(2) A regular ideal I in a Noetherian ring R is *projectively full* if the only integrally closed ideals which are projectively equivalent to I are of the form $\overline{I^n}$ for a positive integer n.

Let $(R, \mathbf{m})$ be a two-dimensional Muhly local domain, i.e. a two-dimensional integrally closed Noetherian local domain with $k = \overline{k}$ and $\mathrm{gr}_{\mathbf{m}}(R)$ an integrally closed domain.

Remark 1.3. If $(R, \mathbf{m})$ is a two-dimensional Muhly local domain, then the blowup $\mathrm{Bl}_{\mathbf{m}}(R)$ at $\mathbf{m}$ is a desingularization of R. We want to understand the blowup.

1.2. **Affine Varieties (Material from Chapter 1 of Hartshorne).**

Let k be a fixed algebraically closed field. Define *affine n-space over k* to be the set of all n-tuples of elements of k, denoted $\mathbb{A}^n_k = \mathbb{A}^n$.

Let $p \in \mathbb{A}^n$ be a point. Write $p = (a_1, \ldots, a_n)$, $a_i \in k$ the coordinates of p.

Let $A = k[x_1, \ldots, x_n]$. We can regard any $f \in A$ as a function $f : \mathbb{A}^n \to k$ by $f(p) = f(a_1, \ldots, a_n)$. Also for any $f \in A$ define the *zero set of f* to be

$$Z(f) = \{p \in \mathbb{A}^n \,|\, f(p) = 0\}.$$

More generally, if $T \subset A$ define

$$Z(T) = \{p \in \mathbb{A}^n \,|\, f(p) = 0 \text{ for all } f \in T\}$$

to be the zero set of T.

Definition 1.4. A subset Y of $\mathbb{A}^n$ is an algebraic set if there exists $T \subset A$ such that $Y = Z(T)$.

Proposition 1.5. *Let $\{Y_i\}_i$ be a collection of algebraic sets in $\mathbb{A}^n$. Then*

(1) $Y_i \cup Y_j$ is an algebraic set
(2) $\cap_i Y_i$ is an algebraic set
(3) $\mathbb{A}^n$ and $\emptyset$ are algebraic sets

We define a topology on $\mathbb{A}^n$ where the open sets are complements of algebraic sets, called the *Zariski topology.*

Definition 1.6. A nonempty subset Y of a topological space X is *irreducible* if it cannot be expressed as the union of two proper subsets, each of which is closed in Y.

Example 1.7. Any non-empty open subset Y of an irreducible space X is irreducible and dense in X.

Example 1.8. Consider the Zariski topology on $\mathbb{A}^1$. Let $Y \subset \mathbb{A}^1$ be an algebraic set. Then there exists a subset T of A such that $Y = Z(T) = Z(f)$ for some $f \in A$ since $A = k[x]$ is a PID. Since k is algebraically closed we can write $f = c(x - a_1) \cdots (x - a_t)$ such that $a_i \in k$. Hence $Y = Z(f) = \{a_1, \ldots, a_t\}$. Hence we can see that $\mathbb{A}^1$ is irreducible.

Definition 1.9. (1) An *affine algebraic variety* (or an affine variety) is an irreducible closed subset of $\mathbb{A}^n$.
(2) An open subset of an affine variety is a *quasi-affine variety.*

Definition 1.10. Let $Y \subset \mathbb{A}^n$ be any subset. Define the ideal of Y in A to be

$$\mathcal{I}(Y) = \{f \in A | f(p) = 0 \text{ for all } p \in Y\}.$$

Proposition 1.11. (a) *If $T_1 \subset T_2$ are subsets of A, then $Z(T_1) \supset Z(T_2)$.*
(b) *If $Y_1 \subset Y_2$ are subsets of $\mathbb{A}^n$, then $\mathcal{I}(Y_1) \supset \mathcal{I}(Y_2)$.*
(c) *Let Y_1 and Y_2 be subsets of $\mathbb{A}^n$, then $\mathcal{I}(Y_1 \cup Y_2) = \mathcal{I}(Y_1) \cap \mathcal{I}(Y_2)$.*
(d) *Let I be any ideal in A, then $\mathcal{I}(Z(I)) = \sqrt{I}$.*
(e) *Let Y be a subset of $\mathbb{A}^n$, then $Z(\mathcal{I}(Y)) = \overline{Y}$.*

Note that part (d) above is due to Hilbert's Nullstellensatz.

Corollary 1.12. *There is a one-to-one, inclusion reversing correspondence*

$$\{\textit{algebraic sets in } \mathbb{A}^n\} \leftrightarrow \{\textit{radical ideals in } A\}.$$

Furthermore, an algebraic set is irreducible if and only if its ideal is prime.

Example 1.13. $\mathbb{A}^n$ is irreducible.

Example 1.14. Let f_n be an irreducible polynomial in $A = k[x_1, \cdots, x_n]$. Then (f_n) is a prime ideal in A and hence $Z(f_n)$ is irreducible. We call $Z(f_n)$ an (affine) curve for $n = 2$, surface for $n = 3$, and hypersurface for $n > 3$.

Example 1.15. A maximal ideal m of $A = k[x_1, \cdots, x_n]$ corresponds to a minimal nonempty irreducible closed subset of $\mathbb{A}^n$, which must be a point.

Definition 1.16. If $Y \subset \mathbb{A}^n$ is an affine algebraic set, the *affine coordinate ring of* Y is $A(Y) = A/\mathcal{I}(Y)$.

Remark: If Y is an affine variety then $A(Y)$ is a domain and a finitely generated k-algebra. Conversely, any finitely generated k-algebra B which is a domain is the affine coordinate ring of some affine variety.

2. May 21, 2009

2.1. **Projective Varieties (Hartshorne, Section 1.2).** Let k be a fixed algebraically closed field. Define *projective n-space over k* to be the set of equivalence classes of $(n+1)$-tuples, $(a_0, a_1, \ldots, a_n)$, of elements in k with not all a_i zero under the equivalence relation

$$(a_0, a_1, \ldots, a_n) \sim \lambda(a_0, a_1, \ldots, a_n)$$

for $\lambda \in k - \{0\}$, denoted $\mathbb{P}^n_k = \mathbb{P}^n$.

Let $P \in \mathbb{P}^n$ be a point. Write

$$P = [a_0, a_1, \ldots, a_n] = \{(a_0, a_1, \ldots, a_n) | (a_0, a_1, \ldots, a_n) \sim \lambda(a_0, a_1, \ldots, a_n), \lambda \in k - \{0\}\}.$$

This set is called the *set of homogeneous coordinates of P*.

Example 2.1. (1) Consider $\mathbb{P}^0$. For all $a_0 \neq 0$, $[a_0] = a_0[1]$ so that $\mathbb{P}^0$ is a point.

(2) Consider $\mathbb{P}^1$. Let $[a_0, a_1] \in \mathbb{P}^1$. Then if $a_0 \neq 0$, we have $[a_0, a_1] = [1, a_1/a_0]$. Otherwise $a_0 = 0$ but then we must have $a_1 \neq 0$ so that $[a_0, a_1] = [0, a_1] = a_1[0, 1]$. We can think of projective 1-space as the collection of lines through the origin.

Let $S = k[x_0, x_1, \ldots, x_n]$ be a graded ring with $\deg x_i = 1$ for all i. Let $f \in S$ be any homogeneous polynomial of degree d. Then we can define the *zero set of f* to be

$$Z(f) = \{P \in \mathbb{P}^n \,|\, f(p) = 0\}.$$

Notice that this definition is well-defined since f is a homogeneous polynomial. Similarly for $T \subset S^h$ where S^h is the set of all homogeneous polynomials in S we define

$$Z(T) = \{P \in \mathbb{P}^n \,|\, f(P) = 0 \text{ for all } f \in T\}.$$

Definition 2.2. A subset Y of $\mathbb{P}^n$ is an *algebraic set* if there exists $T \subset S^h$ such that $Y = Z(T)$.

Proposition 2.3. *Let $\{Y_i\}_i$ be a collection of algebraic sets in $\mathbb{P}^n$. Then*

(1) $Y_i \cup Y_j$ is an algebraic set
(2) $\cap_i Y_i$ is an algebraic set
(3) $\emptyset$ and $\mathbb{P}^n$ are algebraic sets

Now we can define the *Zariski topology* on $\mathbb{P}^n$ by letting the open sets be the complements of the algebraic sets.

Notice that by exercise 2.2 on page 11 of Hartshorne for an ideal $a \subset S$ we have $Z(a) = \emptyset$ if and only if $\sqrt{a} =$ either S or the ideal $S_+ = \bigoplus_{d>0} S_d$ if and only if $a \supset S_d$ for some $d > 0$. Hence a proper ideal can have an empty zero set, unlike in affine n-space.

Definition 2.4. (1) Let $Y \subset \mathbb{P}^n$. Define the *homogeneous ideal* of Y in S to be

$$\mathcal{I}(Y) = \{f \in S | f \text{ is a homogeneous and } f(P) = 0 \text{ for all } P \in Y\}.$$

(2) If Y is an algebraic set in $\mathbb{P}^n$, define the *homogeneous coordiate ring* of Y to be $S(Y) = S/\mathcal{I}(Y)$.

If $f \in S$ is a linear homogeneous polynomial then $Z(f)$ is a hyperplane. In particular, if $H_i = Z(x_i)$ is the zero set of x_i then $U_i = \mathbb{P}^n - H_i$ is an open set for $i = 0, 1, \dots, n$. Then $\mathbb{P}^n$ is covered by the open sets U_i: Let $P \in \mathbb{P}^n$ and choose i so that $P = [a_0, a_1, \dots, a_n]$ with $a_i \neq 0$. Then $P \in U_i$. Now define a mapping $\varphi_i : U_i \to \mathbb{A}^n$ by $[a_0, a_1, \dots, a_n] \mapsto (\frac{a_0}{a_i}, \dots, \hat{\frac{a_i}{a_i}}, \dots, \frac{a_n}{a_i})$.

Proposition 2.5. *φ_i is a homeomorphism for all i.*

Now notice exercise 2.4 on page 11 of Hartshorne, which says that there is a 1-1 inclusion reversing correspondence between algebraic sets in $\mathbb{P}^n$ and homogeneous radical ideals of S not equal to S_+.

Example 2.6. Consider again $\mathbb{P}^1$ and the correspondence

$$\{\text{algebraic sets of } \mathbb{P}^n\} \leftrightarrow \{I \text{ homogeneous ideal in } S = k[x_0, x_1] | \sqrt{I} = I \neq S_+\}.$$

Under this correspondence we have

$$[1, 0] \leftrightarrow (x_1)$$

$$[0, 1] \leftrightarrow (x_0)$$

$$[1, 1] \leftrightarrow (x_0 - x_1)$$

$$[1,2] \leftrightarrow (2x_0 - x_1)$$

so for $f \in S^h$ of $\deg f = d$ then $f = g_1 \cdots g_d$ with g_i linear for all i.

2.2. **The Blowup is regular.**

Proposition 2.7. *Let $(R, \mathbf{m})$ be a two-dimensional normal local domain such that the associated graded ring $\operatorname{gr}_{\mathbf{m}}(R)$ a normal domain. Then $R[\frac{\mathbf{m}}{a}] := A$ is a regular ring for all $0 \neq a \in \mathbf{m}$. Moreover, if P is a maximal ideal of A that contains a, then A_P/aA_P is a DVR. Thus a is part of a minimal set of generators of PA_P.*

Proof. Since $\operatorname{gr}_{\mathbf{m}}(R)$ is a domain the $\mathbf{m}$-adic order function defines a valuation on R. Hence the powers of $\mathbf{m}$ are integrally closed so the Rees ring $R[\mathbf{m}\,t]$ and the extended Rees ring $R[\mathbf{m}\,t, t^{-1}]$ are normal domains. Since the graded ring $R[\mathbf{m}\,t][\frac{1}{at}] = R[\frac{\mathbf{m}}{a}][at, \frac{1}{at}]$ is a normal domain that is a Laurent polynomial ring over its degree zero component $R[\frac{\mathbf{m}}{a}] = A$, it follows that A is a normal domain. To show that A is regular, it remains to show that A_P is a RLR for every $P \in \operatorname{Spec} A$ with $\operatorname{ht} P = 2$. Notice that $\operatorname{ht} P = 2$ implies $P \cap R = \mathbf{m}$ and $a \in P$. Let

$$N = PR[\frac{\mathbf{m}}{a}][at, \frac{1}{at}] \cap R[\mathbf{m}\,t, t^{-1}].$$

Then

$$R[\mathbf{m}\,t, t^{-1}] \subset R[\frac{\mathbf{m}}{a}][at, \frac{1}{at}] = R[\mathbf{m}\,t, t^{-1}][\frac{1}{at}]$$

so $NR[\frac{\mathbf{m}}{a}][at, \frac{1}{at}] = PR[\frac{\mathbf{m}}{a}][at, \frac{1}{at}]$. Hence

$$R[\mathbf{m}\,t, t^{-1}]_N = A[at, \frac{1}{at}]_{PA[at, \frac{1}{at}]} = A_P(at).$$

Now A_P is a two-dimensional normal local domain, and A_P is a RLR if and only if $A_P(at)$ is a RLR. Since at is a unit of $A_P(at)$, the principal ideals generated by a and t^{-1} are equal in $A_P(at)$.

It suffices to show that $R[\mathbf{m}\,t, t^{-1}]_N$ is a RLR, and that $R[\mathbf{m}\,t, t^{-1}]_N/t^{-1}R[\mathbf{m}\,t, t^{-1}]_N$ is a DVR. By hypothesis

$$\operatorname{gr}_{\mathbf{m}}(R) \cong R[\mathbf{m}\,t, t^{-1}]/(t^{-1}R[\mathbf{m}\,t, t^{-1}])$$

is a normal domain. Since $a \in P$, we have $t^{-1} \in NR[\mathbf{m}\,t, t^{-1}]$. Hence $N/(t^{-1})$ is a height one prime in $R[\mathbf{m}\,t, t^{-1}]/(t^{-1}R[\mathbf{m}\,t, t^{-1}])$. Therefore $R[\mathbf{m}\,t, t^{-1}]_N/(t^{-1}R[\mathbf{m}\,t, t^{-1}]_N)$ is a DVR. We conclude that R$[\mathbf{m}\,t, t^{-1}]_N$ is a RLR. $\square$

3. May 26, 2009

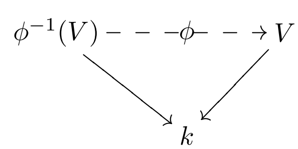
3.1. Morphisms (Section 1.3 of Hartshorne).

Definition 3.1.
(1) (Affine Case) Let Y be a quasi-affine variety of $\mathbb{A}^n$. A function $f : Y \to k$ is *regular at a point* $p \in Y$ if there exists an open neighborhood U of $p \in U \subset Y$ and there exist polynomials g and h in $A = k[x_1, \ldots, x_n]$ such that h is nowhere zero on U and $f = g/h$ on U. We say f is *regular on* Y if f is regular at every point of Y.
(2) (Projective Case) Let Y be a quasi-projective variety of $\mathbb{P}^n$. A function $f : Y \to k$ is *regular at a point* $p \in Y$ if there exists an open neighborhood U of $p \in U \subset Y$ and there exist homogeneous polynomials g and h of the same degree in $S = k[x_0, \ldots, x_n]$ such that h is nowhere zero on U and $f = g/h$ on U. We say f is *regular on* Y if f is regular at every point of Y.

Remark 3.2.
(1) Regular functions are continuous when we identify k with $\mathbb{A}^1$.
(2) If f and g are regular functions on a variety X and $f = g$ on some nonempty open subset $U \subset X$, then $f = g$ on X.

Proof: $f = g$ on X if and only if $(f - g)(Q) = 0$ for all $Q \in X$ if and only if $X = Z(f-g)$, but $U \subset Z(f-g) \subset X$ and U is dense in X so $\overline{U} = X \subset Z(f-g) \subset X$.

Definition 3.3. A variety over k is any affine, quasi-affine, projective, or quasi-projective variety. If X and Y are two varieties, a *morphism* is a continuous map $\phi : X \to Y$ such that for every open set $V \subset Y$ and for every regular function $f : V \to k$ we have that $f \circ \phi : \phi^{-1}(V) \to k$ is regular.



Remark 3.4. If ϕ and ψ are morphisms then $\phi \circ \psi$ is a morphism. Also a morphism $\phi : X \to Y$ is an isomorphism if there exists a morphism $\psi : Y \to X$ such that $\psi \circ \phi = id_X$ and $\phi \circ \psi = id_Y$. Also if a morphism is an isomorphism then it is bijective and bicontinuous, but the converse is not true in general as seen in the example below.

Example 3.5. Exercise 3.2(a): Let $\phi : \mathbb{A}^1 \to \mathbb{A}^2$ is defined by $t \mapsto (t^2, t^3)$. Then ϕ is bijective and bicontinuous, but ϕ is not an isomorphism since $A(X) \cong k[T]$ and $A(Y) \cong k[x, y]/(y^2 - x^3)$ by Corollary 3.7 in Hartshorne:

Corollary 3.6. *If X and Y are affine varieties, then X and Y are isomorphic if and only if $A(X)$ and $A(Y)$ are isomorphic as k-algebras.*

Definition 3.7. Let Y be a variety. Define $\mathcal{O}(Y)$ to be the ring of all regular functions on Y. For $p \in Y$ define $\mathcal{O}_{p,Y}$ or simply $\mathcal{O}_p$ to be the ring of germs of regular functions on Y near p, i.e. elements of $\mathcal{O}_p$ are a pair $< U, f >$ where U is an open subset of Y containing p and f is a regular function on U where we identify $< U, f >$ and $< V, g >$ if $f = g$ on $U \cap V$.

Note that $\mathcal{O}_p$ is a local ring with maximal ideal $\mathbf{m}_p = \{< U, f > \in \mathcal{O}_p : f(p) = 0\}$ as follows: If $< U, f > \in \mathcal{O}_p$ with $f(p) \neq 0$ then $< U, f >$ is a unit since $1/f$ is a regular function on $V := U - (U \cap Z(f))$.

Definition 3.8. Let Y be a variety. Define the function field of Y, denoted $K(Y)$, to be the set of all equivalence classes of pairs $< U, f >$ where U is a nonempty open set in Y, f is regular on Y, and we identify $< U, f >$ and $< V, g >$ if $f = g$ in $U \cap V$. Elements of $K(Y)$ are callled rational functions on Y.

Notice that $K(Y)$ is a field and that $\mathcal{O} \subset \mathcal{O}_p \subset K(Y)$ and if replace Y by an isomorphic variety then the corresponding rings are isomorphic so they are invariants up to isomorphism.

Theorem 3.9. *Let $Y \subset \mathbb{A}^n$ be an affine variety with affine coordinate ring $A(Y)$. Then*

(1) $\mathcal{O} \cong A(Y)$

(2) for $p \in Y$, let $\mathbf{m}_p \subset A(Y)$ be the ideal of functions vanishing at p. Then $p \mapsto \mathbf{m}_p$ gives a 1-1 correspondence between points of Y and maximal ideals of $A(Y)$.

(3) $K(Y) \cong Quot(A(Y))$ so $K(Y)$ is a finitely generated field extension of k and $trdeg_k K(Y) = \dim Y$.

Question: Let Y be any affine or quasi-affine variety. Is Y a quasi-projective variety?

Answer: See Hartshorne page 12 exercise 2.9 to see that every affine variety is a quasi-projective variety.

4. June 2, 2009

4.1. **An example where $(R, \mathbf{m})$ is a two-dimensional normal local domain and $\mathrm{gr}_{\mathbf{m}}(R)$ is a domain (but not normal) and the blowup of R at $\mathbf{m}$ is regular.**

Example 4.1. Let $S = k[X, Y, Z]_{(X,Y,Z)}$ and $f = X^3 - Y^2Z + Z^4 \in S$. Define $R = S/fS$ and let $\mathbf{n} = (X, Y, Z)S$ denote the maximal ideal of S and $\mathbf{m} = (x, y, z)$ denote the maximal ideal of R. We will show that the blowup of R at $\mathbf{m}$ is regular by showing that the following are regular rings

$$R\left[\frac{y}{x},\frac{z}{x}\right], R\left[\frac{x}{y},\frac{z}{y}\right], R\left[\frac{x}{z},\frac{y}{z}\right].$$

Proposition 4.2. *Let $(S,\mathbf{n})$ be a d-dimensional regular local ring. The blowup of S at $\mathbf{n}$ is regular.*

Proof: Let $\mathbf{n}=(x_1,\dots,x_d)$. We will show that $S[\frac{\mathbf{n}}{x_1}]$ is a regular ring then by symmetry we will have $S[\frac{\mathbf{n}}{x_i}]$ is regular for $1\le i\le d$ and since these rings cover the blowup we will be done.

Let $p\in \operatorname{Spec}(S[\frac{\mathbf{n}}{x_1}])$ and $q=p\cap S$. Suppose $q=\mathbf{n}$. Then $\mathbf{n}\,S[\frac{\mathbf{n}}{x_1}]=x_1S[\frac{\mathbf{n}}{x_1}]$. Now

$$S\left[\frac{\mathbf{n}}{x_1}\right]=S\left[\frac{x_2}{x_1},\frac{x_3}{x_1},\dots,\frac{x_d}{x_1}\right]\cong\frac{S[T_2,\dots,T_d]}{(x_1T_2-x_2,\dots,x_1T_d-x_d)}$$

so that

$$\frac{S\left[\frac{\mathbf{n}}{x_1}\right]}{\mathbf{n}\,S\left[\frac{\mathbf{n}}{x_1}\right]}\cong (S/\,\mathbf{n})[T_2,\dots,T_d]$$

which is clearly regular. Hence

$$\frac{S\left[\frac{\mathbf{n}}{x_1}\right]_p}{x_1S\left[\frac{\mathbf{n}}{x_1}\right]_p}$$

is regular which implies that $S[\frac{\mathbf{n}}{x_1}]_p$ is a regular ring since $x_1S\left[\frac{\mathbf{n}}{x_1}\right]_p$ is a height one principal ideal. Now suppose that $q\neq\mathbf{n}$. Then x_1 is not in p so $S[\frac{\mathbf{n}}{x_1}]_p=S[\frac{1}{x_1}]_{pS[\frac{\mathbf{n}}{x_1}]_p}$ and since $S[\frac{1}{x_1}]$ is regular $S[\frac{\mathbf{n}}{x_1}]_p$ is regular. Therefore we can conclude that $S[\frac{\mathbf{n}}{x_1}]$ is regular. ■

Now since we have that $S=k[X,Y,Z]_{(X,Y,Z)}$ is regular we will use the following version of the Jacobian criterion to show that the rings

$$R\left[\frac{y}{x},\frac{z}{x}\right], R\left[\frac{x}{y},\frac{z}{y}\right], R\left[\frac{x}{z},\frac{y}{z}\right]$$

are regular.

4.3 **Theorem 4.3** (Matsumura, Theorem 30.4(ii))**.** *Let R be a regular local ring and $P\in\operatorname{Spec}(R)$. Let $I\subset P$ be an ideal of R with $\operatorname{ht} IR_P=r$. If $D_1,\dots,D_r\in Der(R)$ and $f_1,\dots,f_r\in I$ are such that $\det(D_if_j)\notin P$ then*

$$IR_P=(f_1,\dots,f_r)R_P \text{ and } R_P/IR_P \text{ is regular.}$$

We have shown that

$$S\left[\frac{Y}{X},\frac{Z}{X}\right], S\left[\frac{X}{Y},\frac{Z}{Y}\right], S\left[\frac{X}{Z},\frac{Y}{Z}\right]$$

are regular rings. Then $S[\frac{Y}{X}, \frac{Z}{X}] \to R[\frac{y}{x}, \frac{z}{x}]$ is surjective with kernel K_1 of height 1. Let $p \in \operatorname{Spec}(R[\frac{y}{x}, \frac{z}{x}])$. Then there exists $P \in \operatorname{Spec}(S[\frac{Y}{X}, \frac{Z}{X}])$ such that $K_1 \subset P$ and $P/K_1 = p$ and

$$\frac{S[\frac{Y}{X}, \frac{Z}{X}]_P}{(K_1)_P} \cong R[\frac{y}{x}, \frac{z}{x}]_p.$$

We show that the left hand side above is a regular ring by using Theorem 4.3 ??. . Notice that $\operatorname{ht} K_1 S[\frac{Y}{X}, \frac{Z}{X}]_P = 1$. Then choose $f_1 \in I$ to be $f_1 = 1 - (\frac{Y}{X})^2(\frac{Z}{X}) + X(\frac{Z}{X})^4$. Also let $D : A \to A$ with $A = S[\frac{Y}{X}, \frac{Z}{X}]$ to be the derivation with $X \mapsto 1$, $Y/X \mapsto 0$, and $Z/X \mapsto 0$. Then $Df_1 = D(X(\frac{Z}{X})^4) = (\frac{Z}{X})^4$. If $(\frac{Z}{X})^4 \in P$ then $1 \in P$ since $f_1 \in P$ as well. Hence $Df_1 \notin P$ so by the theorem we have regularity. We similarly show that the other rings are regular, but I don't have time to type that up right now :-p

5. June 4, 2009

5.1. An example where $(R, \mathbf{m})$ is a two-dimensional normal local domain such that $\operatorname{gr}_{\mathbf{m}}(R)$ is a domain (not normal) and the blowup of R at $\mathbf{m}$ is NOT regular.

Example 5.1. Let $S = k[X, Y, Z]_{(X,Y,Z)}$ and $f = X^3 - Y^2Z + Z^5 \in S$. Define $R = S/fS$ and let $\mathbf{n} = (X, Y, Z)S$ denote the maximal ideal of S and $\mathbf{m} = (x, y, z)$ denote the maximal ideal of R. This example is very similar to the one given on June 2, but we will show that in this case the blowup of R at $\mathbf{m}$ is not regular. It suffices to find an element $0 \neq a \in m$ and $p \in \operatorname{Spec}(R[\frac{\mathbf{m}}{a}])$ such that $R[\frac{\mathbf{m}}{a}]_p$ is not a regular local ring.

Recall as in the previous example that the natural map $S[\frac{X}{Z}, \frac{Y}{Z}] \mapsto R[\frac{x}{z}, \frac{y}{z}]$ is surjective with kernel K, a prime ideal of height one. Hence it is enough to find a $Q \in \operatorname{Spec}(S[\frac{\mathbf{n}}{Z}])$ such that $K \subset Q$ and $\dfrac{S[\frac{\mathbf{n}}{Z}]_Q}{K_Q}$ is not regular. Note that K is not a maximal ideal since $R[\frac{x}{z}, \frac{y}{z}]$ is not a field. Let $Q = (\frac{X}{Z}, \frac{Y}{Z}, Z)S[\frac{X}{Z}, \frac{Y}{Z}]$. Then Q is a height 3 maximal ideal that contains K and is generated by a regular sequence. Since $S[\frac{X}{Z}, \frac{Y}{Z}]_Q$ is a 3-dimensional regular local ring, it is a UFD. Hence K_Q, a height one prime ideal, is principal. In fact, letting $X_1 = \frac{X}{Z}$ and $Y_1 = \frac{Y}{Z}$ we can write $f = (X_1Z)^3 - (Y_1Z)^2Z + Z^5 = Z^3(X_1^3 - Y_1^2 + Z^2)$. Then $g = X_1^3 - Y_1^2 + Z^2$ is an irreducible element in K and hence $K_Q = gS[\frac{\mathbf{n}}{Z}]_Q$. Now

$$\frac{S[\frac{\mathbf{n}}{Z}]_Q}{K_Q} = \frac{S[\frac{\mathbf{n}}{Z}]_Q}{gS[\frac{\mathbf{n}}{Z}]_Q}$$

is a two-dimensional Noetherian local domain with maximal ideal $\frac{QS[\frac{\mathbf{n}}{Z}]_Q}{gS[\frac{\mathbf{n}}{Z}]_Q}$. Since $g \in Q^2$ we have the following isomorphism letting $A = S[\frac{\mathbf{n}}{Z}]$

$$\frac{QA_Q}{gA_Q} / \frac{Q^2A_Q}{gA_Q} \cong \frac{QA_Q}{Q^2A_Q}.$$

Hence by Nakayama we see that QA_Q is minimially generated by 3 elements and therefore A_Q is not a regular ring.

5.2. **Hartshorne, sections 1.3 and 1.4.**

Proposition 5.2. *Let X be any variety and Y an affine variety. Then there exists a natural bijective mapping of sets*

$$\alpha : \mathrm{Hom}(X, Y) \to \mathrm{Hom}(A(Y), \mathcal{O}(X)).$$

Corollary 5.3. *If X and Y are affine varieties, then $X \cong Y$ if and only if $A(X) \cong A(Y)$.*

Note that by exercise 3.9 we see that $S(X)$ is not invariant under isomorphism for a projective variety X.

Definition 5.4.

1. Let X and Y be varieties. A rational map $\phi : X \to Y$ is an equivalence class of pairs $< U, \phi_U >$ where U is a non-empty open subset of X and ϕ_U is a morphism of U to Y and where $< U, \phi_U > \sim < V, \phi_V >$ if $\phi_U = \phi_V$ on $U \cap V$.
2. The rational map ϕ is dominant if for some (and hence every) pair $< U, \phi_U >$ the image of ϕ_U is dense in Y.
3. A birational map $\phi : X \to Y$ is a rational map with an inverse rational map, i.e, there is a rational map $\psi : Y \to X$ such that $\psi \circ \phi = 1_X$ and $\phi \circ \psi = 1_Y$.

Note that $\mathbb{A}^n \times \mathbb{A}^m \cong \mathbb{A}^{n+m}$ but in general $\mathbb{P}^n \times \mathbb{P}^m \cong \mathbb{P}^{n+m}$. However by the Segre embedding given in exercise 3.14 $\mathbb{P}^n \times \mathbb{P}^m$ can be identified with a subvariety of $\mathbb{P}^N$ where $N = nm + n + m$. Also by exercise 3.16, if $X \subset \mathbb{P}^n$ and $Y \subset \mathbb{P}^m$ are respectively quasi-projective or projective varieties then so is $X \times Y$ in $\mathbb{P}^N$.

5.3. **The Blowup of $\mathbb{A}^n$ at the origin $\mathcal{O} = (0, \ldots, 0)$.** First consider $\mathbb{A}^n \times \mathbb{P}^{n-1}$ as a quasi-projective variety.

Now if $x_1, \ldots, x_n$ are the affine coordinates of $\mathbb{A}^n$ and $y_1, \ldots, y_n$ are the homogeneous coordinates of $\mathbb{P}^{n-1}$ we define the *blowup of* $\mathbb{A}^n$ *at* $\mathcal{O}$ as follows:

Let X be the closed subset of $\mathbb{A}^n \times \mathbb{P}^{n-1}$ defined by the equations $x_i y_j = x_j y_i$ for $i, j = 1, 2, \ldots, n$, i.e.,

$$X = \{(x_1, \ldots, x_n, y_1, \ldots, y_n) \in \mathbb{A}^n \times \mathbb{P}^{n-1} | x_i y_j = x_j y_i \text{ for } i, j = 1, 2, \ldots, n\}.$$

The blowup map ϕ is given as the composition of the embedding of X in $\mathbb{A}^n \times \mathbb{P}^{n-1}$ followed by the projection onto $\mathbb{A}^n$ as in the diagram below:

$$\begin{array}{ccc} X & \longrightarrow & \mathbb{A}^{\rm n} \times \mathbb{P}^{n-1} \\ & \searrow & \downarrow \\ & & \mathbb{A}^{\rm n} \end{array}$$

Properties of X:

(1) If $P \in \mathbb{A}^{\rm n}$ with $P \neq \mathcal{O}$ then we have the following:
   (a) $\phi^{-1}(P)$ consists of a single point.
   (b) $\phi : X - \phi^{-1}(\mathcal{O}) \to \mathbb{A}^{\rm n} - \mathcal{O}$ is an isomorphism.
   (c) The points of $\phi^{-1}(\mathcal{O})$ are in one-to-one correspondence with the set of lines through $\mathcal{O}$ in $\mathbb{A}^{\rm n}$.
   (d) X is irreducible.

(2) If $P \in \mathbb{A}^{\rm n}$ with $P = \mathcal{O}$ then $\phi^{-1}(P) \cong \mathbb{P}^{n-1}$.

6. June 16 and 18, 2009

6.1. **The degree function of Rees.** Let $(R, \mathbf{m})$ be a Noetherian local integral domain and let I be an $\mathbf{m}$-primary ideal. In [?], Rees introduced the interesting concept of the degree function $d_I(x)$ that is defined for each nonzero $x \in \mathbf{m}$ to be the multiplicity of the image of the ideal $I + xR$ in the ring R/xR. Rees proves that there exist a finite set $\{v_1, \ldots, v_k\}$ of discrete valuations centered on $\mathbf{m}$ such that for each nonzero $x \in \mathbf{m}$ one has

$$d_I(x) = \sum_{i=1}^{k} d_i(I) v_i(x),$$

where the $d_i(I)$ are positive integers called the degree coefficients of I. The discrete valuations v_i are now called the Rees valuations of the ideal I.

We would like to get some understanding of what these integers $d_i(I)$ are in various examples.

Example 6.1. If $(R, \mathbf{m})$ is a normal local domain of the form S/fS, where S is a 3-dimensional RLR with maximal ideal $(x, y, z)S$ and with residue field of characteristic zero, and if $f = x^n + y^n + z^n$, then the order valuation on R defined by the powers of $\mathbf{m}$ is the unique Rees valuation v of $\mathbf{m}$. Notice that $v(\mathbf{m}) = 1$ and the degree coefficient $d(\mathbf{m}, v) = n$. This gives examples where the degree coefficient $d(\mathbf{m}, v)$ can be any positive integer n that we want.

Example 6.2. Let $(R, \mathbf{m})$ be a 2-dimensional RLR with maximal ideal $\mathbf{m} = (x, y)R$ and for n a fixed positive integer, let $I = (x, y^n)R$. Then I is a simple complete $\mathbf{m}$-primary ideal with Rees valuation v having the property that $v(x) = n$ and $v(y) = 1$. Notice that

$v(I) = n$ and that $I + yR = \mathbf{m}$ and R/yR is a DVR. Therefore $d_I(y) = 1$. Hence we must have $d(I, v) = 1$. Notice also that $e(I)$, the multiplicity of I is n.

Remark 6.3. Van Lierde mentions that Rees and Sharp in RS [?] prove that if $(R, \mathbf{m})$ is a 2-dimensional local domain and I is an $\mathbf{m}$-primary ideal, then

$$e(I) = \sum_{i=1}^{k} d_i(I) v_i(I),$$

where $v_1, \ldots, v_k$ are the Rees valuations of I.

6.4 **Question 6.4.** Let $(R, \mathbf{m})$ be a 2-dimensional regular local ring and let I be a complete $\mathbf{m}$-primary ideal. Let $I = \prod_{i=1}^{r} I_i^{n_i}$ be the factorization of I as a product of distinct simple complete ideals, and let v_i denote the Rees valuation associated of I_i. Is the degree coefficient $d(I, v_i) = n_i$?

We have an afirmative answer of Question 6.4 ??, if the residue field $R/\,\mathbf{m}$ is an algebraically closed field.

6.41 **Proposition 6.5.** *Let $(R, \mathbf{m})$ be a 2-dimensional regular local ring with $\mathbf{m} := (x, y)R$. Assume that $R/\,\mathbf{m}$ is an algebraically closed field. Let $I \neq \mathbf{m}$ be a simple complete $\mathbf{m}$-primary ideal and let v denote the Rees valuation associated of I. Then*

(1) The degree coefficient $d(\mathbf{m}, \operatorname{ord}_R) = 1$.
(2) The degree coefficient $d(I, v) = 1$.

Proof. (1) : We have $d_{\mathbf{m}}(\alpha) = d(\mathbf{m}, \operatorname{ord}_R) \operatorname{ord}_R(\alpha)$, for every $0 \neq \alpha \in \mathbf{m}$. Taking $\alpha := x$, then we have $d_{\mathbf{m}}(x) = e(\frac{\mathbf{m}+xR}{xR}) = e(R/xR) = 1$, since R/xR is an one dimensional RLR, and $\operatorname{ord}_R(x) = 1$. Hence $d(\mathbf{m}, \operatorname{ord}_R) = 1$.
(2) : By the result of Rees and Sharp RS [?], we have $d_I(\mathbf{m}) = d_{\mathbf{m}}(I)$. Hence we have

$$d(I, v)v(\mathbf{m}) = d_I(\mathbf{m}) = d_{\mathbf{m}}(I) = d(\mathbf{m}, \operatorname{ord}_R) \operatorname{ord}_R(I) = d(I, v)v(\mathbf{m}),$$

the last equality holds by Lipman's reciprocity theorem Lipman [?]. Since $v(\mathbf{m})$ is a positive integer, hence $d(I, v) = d(\mathbf{m}, \operatorname{ord}_R) = 1$. □

6.42 **Corollary 6.6.** *Let $(R, \mathbf{m})$ be a 2-dimensional regular local ring with $\mathbf{m} := (x, y)R$ and let I be a complete $\mathbf{m}$-primary ideal. Assume that $R/\,\mathbf{m}$ is an algebraically closed field. Let $I = \prod_{i=1}^{r} I_i^{n_i}$ be the factorization of I as a product of distinct simple complete ideals, and let v_i denote the Rees valuation associated of I_i. Then the degree coefficient $d(I, v_i) = n_i$, for all $i = 1, \ldots, r$.*

Proof. Notice that

$$d(I_j, v_i) = \begin{cases} 1, & \text{if} \quad j = i \quad \text{by Proposition 6.41 ??} \\ 0, & \text{if} \quad j \neq i, \quad \text{by R [?, Lemma 2.2].} \end{cases}$$

Hence we have

$$d(I, v_i) = d(\prod_{i=1}^{r} I_i^{n_i}, v_i) = \sum_{j=1}^{r} n_i d(I_j, v_i) = n_i d(I_i, v_i) = n_i,$$

the second equality holds by RS [?, lemma 5.1]. □

6.2. **The blowup of an ideal as a projective model.** Let I be a nonzero proper ideal of the integral domain R, and assume that $a_1, \ldots, a_n$ are nonzero elements of I that generate I. The blowup $\mathrm{Bl}_I(R)$ of R at I may be defined in terms of homogeneous localizations of the graded ring $R[It]$, denoted $\operatorname{Proj} R[It]$. Alternatively, one may consider the affine rings $R_i := R[I/a_i]$ over R and the union of $\operatorname{Spec} R_i, 1 \leq i \leq n$ over $\operatorname{Spec} R$. As in Zariski-Samuel ZS2 [?, page 116], we regard the blowup of I as a family of local rings. We identify each $P \in \operatorname{Spec} R_i$ with the local ring $R[I/a_i]_P := T$. Notice that the ideal IT is principal. We prove in Proposition 6.5 ?? other facts that hold in this situation, and use the following definition.

Definition 6.7. Let S be a subring of a local ring $(T, \mathbf{n})$.

(1) The prime ideal $P := S \cap \mathbf{n}$ is called the **center** of T on S.
(2) If S is also local and the center of T on S is the maximal ideal of S, then we say that T **dominates** S, or that S is **is dominated by** T.

6.5 **Proposition 6.8.** *Let I be a nonzero finitely generated ideal of the integral domain R, let a and b be nonzero elements of I and let $P \in \operatorname{Spec} R[I/a]$. Then*

(1) The ring $R[I/b] \subseteq R[I/a]_P \iff a/b \in R[I/a]_P \iff a/b$ is a unit of $R[I/a]_P$.
(2) If $R[I/b] \subseteq R[I/a]_P$, then $R[I/a]_P = R[I/b]_Q$, where $Q := PR[I/a]_P \cap R[I/b]$.

Proof. We clearly have $T := R[I/a]_P \supseteq R[I/b]_Q =: S$ and T dominates S. The ideal $IS = bS$. Since T dominates S, we have $aT = bT = IT$. Thus $a/b \in S$ is a unit of T. Since S and T are local rings with T dominating S, it follows that a/b is a unit of S. Therefore $R[I/a] \subseteq S$. Let $P' := QS \cap R[I/a]$. Then $R[I/a]_{P'}$ is dominated by S and S is dominated by T. Since $PT \cap R[I/a] = P$, we have $P' = P$, and thus $R[I/a]_P = R[I/b]_Q$. □

Corollary 6.9. *Let I be a nonzero finitely generated ideal of the integral domain R and let S be a local overring of R. Then we have:*

(1) S dominates a local ring on the blowup $\mathrm{Bl}_I(R)$ if and only if IS is a principal ideal of S.

(2) If IS is principal, then S dominates a unique local ring of $\mathrm{Bl}_I(R)$.

(3) Each valuation overring V of R dominates a unique local ring $S \in \mathrm{Bl}_I(R)$.

Proof. For item (1), if IS is principal, then $IS = aS$ for some nonzero $a \in I$. Hence $R[I/a] \subseteq S$. Let P denote the center of S on $R[I/a]$. Then $R[I/a]_P \in \mathrm{Bl}_I(R)$ is dominated by S. On the other hand, if S dominates $T \in \mathrm{Bl}_I(R)$, then IT is principal implies IS is principal. Item (2) follows from Proposition 6.5 ??.2, and item (3) is immediate from items (1) and (2). □

6.6 **Definition 6.10.** Let R be an integral domain.

(1) An extension domain S of R is said to be **birational over** R if S is contained in the field of fractions of R.

(2) Let I be a nonzero finitely generated ideal of R. Let Σ denote the family of birational local overrings S of R such that IS is principal. We define a partial order $\leq$ on Σ as follows:

$$\text{Let} \quad S, T \in \Sigma, \quad \text{then} \quad S \leq T \quad \Longleftrightarrow \quad T \quad \text{dominates} \quad S.$$

6.7 **Proposition 6.11.** *Let I be a nonzero finitely generated ideal of an integral domain R. Then*

(1) A birational local overring S of R is an element of $\mathrm{Bl}_I(R)$ if and only if IS is a principal ideal and S is a minimal element with respect to the partial order of Definition 6.6 ??.2.

(2) If T is a birational local overring of R and IT is principal, then there exists a unique local ring S such that T dominates S and S is a minimal element with respect to the partial order of Definition 6.6 ??.2.

Proof. For the proof of (1), if IS is principal, then $IS = aS$ for some nonzero $a \in I$. Hence $R[I/a] \subseteq S$. Let P denote the center of S on $R[I/a]$. Then S birationally dominates $R[I/a]_P$, and the minimality of S in Σ implies that $R[I/a]_P = S$. Hence $S \in \mathrm{Bl}_I(R)$. On the other hand, if $S \in \mathrm{Bl}_I(R)$, then $S = R[I/a]_P$ for some nonzero $a \in I$ and $P \in \operatorname{Spec} R[I/a]$. Suppose $T \in \Sigma$ with $T \leq S$. Then $IT = bT$ for some nonzero $b \in I$. Let Q denote the center of T on $R[I/b]$. Then $R[I/b]_Q \leq T \leq S$, and Proposition 6.5 ?? implies that $R[I/b]_Q = S$. Hence $S = T$ is minimal in Σ.

□

6.9 **Example 6.12.** Let $(R, \mathbf{m})$ be a 2-dimensional regular local ring and let $\mathbf{m} = (x, y)R$. The maximal ideals of $R[y/x]$ of height two have the form $(x, f(y/x))R[y/x]$, where $f(Z) \in R[Z]$ is a monic polynomial whose image in $(R/\mathbf{m})[Z]$ is irreducible. In particular $P = (x, \frac{y}{x} - a)R[y/x]$ is a maximal ideal of height two. If $a \notin \mathbf{m}$, then $R[x/y] \subseteq R[y/x]_P$. Let $Q = PR[y/x]_P \cap R[x/y]$. By Proposition 6.5 ??, $R[y/x]_P = R[x/y]_Q$. Notice that $Q = (y, \frac{x}{y} - a^{-1})R[x/y]$.

7. June 21 and 23, 2009

7.1. **The integrally closed m-primary ideals adjacent to m in a two-dimensional Muhly local domain $(R, \mathbf{m})$ of embedding dimension three.** According to Debremaeker in D1 [?], if $(R, \mathbf{m})$ is a two-dimensional Muhly local domain, then the integrally closed $\mathbf{m}$-primary ideals adjacent to $\mathbf{m}$ are in one-to-one correspondence with the set of immediate local quadratic transformations of R.

We give a method for finding these ideals in the case where R is the quotient of a 3-dimensional regular local ring, $(S, \mathbf{n})$, by an irreducible element, f. First consider Example 7.1 ??.

7.1 **Example 7.1.** Let $S = k[X, Y, Z]_{(X,Y,Z)}$ be the localized polynomial ring over an algebraically closed field k and let $\mathbf{n} = (X, Y, Z)S$. Let $f = XZ - Y^2 \in S$ and consider the two-dimensional Muhly local domain $R = S/fS$ with maximal ideal $\mathbf{m} = (x, y, z)R$. Now consider $R_1 = R[\frac{\mathbf{m}}{x}]$. We want to determine the maximal ideals of R_1 that contain $\mathbf{m} R_1 = xR_1$. Let

$$T_1 = \{p \in \operatorname{mSpec}(R_1) : \mathbf{m} R_1 \subset p\}.$$

We can write $R_1 = R[\frac{y}{x}, \frac{z}{x}]$ as a homomorphic image of $S_1 = S[\frac{\mathbf{n}}{x}]$. The kernel of the map, call it K, will be a prime ideal of height one. Notice that letting $y_1 = \frac{y}{x}$ and $z_1 = \frac{z}{x}$ in S_1 we have $f = xz - y^2 = x(z_1 x) - (y_1 x)^2 = x^2(z_1 - y_1^2)$ and $z_1 - y_1^2 \in K$ is irreducible.

Hence there is a one-to-one correspondence between T_1 and

$$T_2 = \{p \in \operatorname{mSpec}(S_1) : (\mathbf{n}, z_1 - y_1^2)S_1 \subset p\}.$$

Now since S is a regular local ring with maximal ideal $\mathbf{n} = (X, Y, Z)S$ we have that

$$S_1 \cong \frac{S[T_1, T_2]}{(xT_1 - y, xT_2 - z)}$$

by SH [?] so there is a one-to-one correspondence between T_2 and

$$T_3 = \{p \in \operatorname{mSpec}(S[T_1, T_2]) : (xT_1 - y, xT_2 - z, T_2 - T_1^2, \mathbf{n})S[T_1, T_2] \subset p\}.$$

Now if $p \in \operatorname{mSpec}(S[T_1, T_2])$ with $\mathbf{n} S[T_1, T_2] \subset p$ then this will correspond to a maximal ideal of $k[T_1, T_2]$, where $k = S/\mathbf{n}$ is algebraically closed. Hence we are reduced to finding the maximal ideals of $k[T_1, T_2]$ that contain $T_2 - T_1^2$. Since k is algebraically closed these maximal ideals of $k[T_1, T_2]$ are of the form $(T_1 - a, T_2 - a^2)$. This corresponds to ideals in R_1 of the form $n_a = (x, \frac{y}{x} - a, \frac{z}{x} - a^2)R_1$. Also the ideal in R that corresponds to n_a is $(x^2, y - xa, z - xa^2)R$ according to [?].

To find all the immediate quadratic transforms of R (or equivalently all the integrally closed $\mathbf{m}$-primary ideals adjacent to $\mathbf{m}$) we would also need to consider $R_2 = R[\frac{\mathbf{m}}{y}]$ and $R_3 = R[\frac{\mathbf{m}}{z}]$. However since $(x, z)R$ is a reduction of $\mathbf{m}\,R$ it suffices to consider R_3 along with R_1 as above. Since x and z play symmetric roles in R we know that the maximal ideals of R_3 that contain $\mathbf{m}\,R_3$ are of the form $(z, \frac{x}{z} - a^2, \frac{y}{z} - a)R_3$ which correspond to the ideals $(z^2, x - a^2z, y - az)R$ in R.

Example ?? indicates how we can find the immediate quadratic transforms of a two-dimensional Muhly local domain R with maximal ideal $\mathbf{m} = (x, y, z)$ that is the quotient of a 3-dimensional regular local ring S with maximal ideal $\mathbf{n} = (X, Y, Z)$ by an irreducible element $f(X, Y, Z)$. We summarize this method here: Let

$$R_1 = R[\frac{\mathbf{m}}{x}], R_2 = R[\frac{\mathbf{m}}{y}], R_3 = R[\frac{\mathbf{m}}{z}].$$

We seek to find the maximal ideals of these rings that contain $\mathbf{m}$. We consider R_1. Let $Y_1 = \frac{Y}{X}$ and $Z_1 = \frac{Z}{X}$. Then $f(X, Y, Z) = f(X, Y_1X, Z_1X) = X^d f_1(X, Y_1, Z_1)$. Then

$$R_1 \cong S[\frac{Y}{X}, \frac{Z}{X}]/(f_1(X, Y_1, Z_1))$$

and as in the agrument above the maximal ideals of R_1 that contain $\mathbf{m}\,R_1$ correspond to maximal ideals of $k[T_1, T_2]$ that contain $f_1(X, T_1, T_2)$ where $k = R/\,\mathbf{m}$. Similarly for R_2 and R_3.

Remark 7.2. With $(R, \mathbf{m})$ as in Example ??, the ideals $(x, y)R := P_1$ and $(y, z)R := P_2$ are height-one prime ideals of R such that R/P_i is a DVR. It follows that $(x^n, y, z)R$ and $(z^n, x, y)R$ are integrally closed ideals for every positive integer n. Debremaeker considers ideals having this form in [?].

7.3 **Question 7.3.** Let $(R, \mathbf{m})$ be a two-dimensional local domain. What are necessary and sufficient conditions for there to exist a prime ideal P of R such that R/P is a DVR?

Remark 7.4. If $(R, \mathbf{m})$ is a two-dimensional local UFD and if there exists $P \in \operatorname{Spec} R$ such that R/P is a DVR, then R is regular. Thus, with $S = k[X, Y, Z]_{(X,Y,Z)}$ as in Example ??, if $R = S/fS$, where $f = X^2 + Y^3 + Z^5$ or $f = X^2 + Y^3 + Z^7$, then R is a two-dimensional normal local domain that does not have any prime ideals P such that R/P is a DVR.

Remark 7.5. Let $S = k[X_1, \dots, X_n]$ be a polynomial ring in n variables over an algebraically closed field k. If I is a homogeneous ideal of S such that $\dim S/I \geq 1$, then there exists a prime ideal P of S such that $I \subseteq P$ and S/P is a polynomial ring in one variable over k. This follows because a graded integral domain of dimension one over k is a polynomial ring, hence S/P is a polynomial ring of dimension one over k for every homogeneous prime ideal P of S such that $\dim S/P = 1$. This provides lots of examples of two-dimensional local domains for which the answer to Question 7.3 ?? is affirmative. If I is a homogeneous prime ideal of S such that $\dim S/I = 2$ and R is the localization of S at its graded maximal ideal, then there exist infinitely many prime ideals Q of R such that R/Q is a DVR.

8. June 30 and July 2, 2009

Let $S = k[[X, Y, Z]]$ and $n = (X, Y, Z)S$ be a power series ring over a field k and its maximal ideal. Let $f = X^3 + Y^2Z + Z^4 + Y^5$ be in S and $R = S/fS$. We would like to understand the question 7.3 ??. To this end, we want to find maximal ideals of $R[\frac{m}{x}]$ that contain m. We have a surjective homomorpism,

$$R[\frac{y}{x}, \frac{z}{x}] \twoheadleftarrow S[\frac{Y}{X}, \frac{Z}{X}]$$

Set $Y_1 = \frac{Y}{X}$ and $Z_1 = \frac{Z}{X}$. Then $f = X^3 + Y^2X^2Z_1X + Z_1^4X^4 + Y_1^5X^5 = X^3(1 + Y_1^2Z_1 + Z_1^4X + Y_1^5X^2)$. Let $f_1 = 1 + Y_1^2Z_1 + Z_1^4X + Y_1^5X^2$. First, we look at those maximal ideals of $S[\frac{Y}{X}, \frac{Z}{X}]$ that contain (n, f_1). By the following isomorphisms

$$\begin{aligned}
\frac{S[T_1, T_2]}{(XT_1 - Y, XT_2 - Z)} &\cong S[\frac{Y}{X}, \frac{Z}{X}], \\
\frac{S[T_1, T_2]}{(n, XT_1 - Y, XT_2 - Z, f_1)} &= \frac{S[T_1, T_2]}{(X, Y, Z, XT_1 - Y, XT_2 - Z, 1 + Y_1^2Z_1 + Z_1^4X + Y_1^5X^2)} \\
&= \frac{S[T_1, T_2]}{(X, Y, Z, 1 + Y_1^2Z_1)} \\
&\cong \frac{k[T_1, T_2]}{(1 + Y_1^2Z_1)}
\end{aligned}$$

Those maximal ideals in $S[T_1, T_2]$ that contain (n, f_1) correspond to those maximal ideals in $k[T_1, T_2]$ that contain $f_1' = 1 + T_1^2T_2$. Since $T_1 = 1$ and $T_2 = -1$ is a root of f_1' the maximal ideal $(T_1 - 1, T_2 + 1)$ contains f_1' and this ideal correspoens to $(X, \frac{Y}{X} - 1, \frac{Z}{X} + 1)$ in S. Now let $(x^2, y - x, z + x) := I$ and $(y - x, z + x) := J$ in R.

Remark 8.1. $J = (y - x, x + z)$ is not a reduction of m. Suppose that it is. Because $J \subset I \subset m$, I is a reduction of m. This implies $\bar{I} = m$. But $\bar{I} = I \subsetneq m$. This is a contradiction.

8.1. **Minimal reduction J of I with certain conditions.** By the result of Debremaeker, $\mathrm{Rees}(I) = \{\mathrm{ord}_R, \mathrm{ord}_{R_1}\}$ where$R_1 := R[\frac{m}{x}]_M$, $M = (x, \frac{y}{x} - 1, \frac{z}{x} + 1)$ and $m_1 = (x, \frac{y}{x} - 1)$.

	x	y	z	x	$y - x$	$z + x$
ord_R	1	1	1	1	1	1
ord_{R_1}	1	1	1	1	2	≥ 2

If we change our f to $f = XY - Z^2$ then we ge the following table.

	x	z	y
$v_1 = \mathrm{ord}_R$	1	1	1
$v_2 = \mathrm{ord}_{R_1}$	1	2	3

Now $m_1 = (x, \frac{z}{x})$ since $\frac{xy}{x^2} = \frac{z^2}{x^2}$ implies $\frac{y}{x} = (\frac{z}{x})^2$. $I = (x^2, y, z)$. We want to find a two generated reduction $J = (a, b)$ of I such that $I = (a, b, c_i)$ for $i = 1, 2, v_1(c_1) > v_1(a) = v_1(b)$ and $v_2(c_2) > v_2(a) = v_2(b)$.

Example 8.2. We revisit Example 7.1 ??. Let $S = k[X, Y, Z]_{(X,Y,Z)}$ where k is a field. Let $f = XZ - Y^2$ and $R = S/fS$ and $m = (x, y, z)R$ where x, y, z denote the images of X, Y, Z in R. Further assume k is algebraically close and char $k = 0$. Then (R, m) is a 2-dimensional Muhly local domain. Then,

(1) Since (x, z) is a minimal reduction of m, $Bl_m(R) = Spec(R[\frac{m}{x}]) \cup Spec(R[\frac{m}{z}])$.

(2) Consider $R[\frac{m}{x}]$, the first quadratic transformation of R. Find all maximal ideals m_a of $R[\frac{m}{x}]$ that contain m.

$$m_a = (x, \frac{y}{x} - a, \frac{z}{x} - a^2)R[\frac{m}{x}] \qquad \forall a \in k$$

Let $R_{1_a} = R[\frac{m}{x}]_{m_a}$. This is a 2-dimensional regular local ring with maximal ideal $m_{1_a} := (x, \frac{y}{x} - a)R_{1_a}$.

(3) Set $I_{1_a} = xm_{1_a} \cap R$. We showed that $I_{1_a} = (x^2, y - ax, z - a^2x)R$.

(4) Consider the simplest case: Take $a = 0$. Then $I = (x^2, y, z)R$. Then $Rees(I) = \{ord_R, ord_{R_1}\}$ where $R_1 = R[\frac{m}{x}]_{(x, \frac{y}{x}, \frac{z}{x})}$.

Question 8.3. Find a minimal reduction $J = (a, b)$ of I and $c, d \in I$ such that

(i) $I = (a, b, c) = (a, b, d)$ and

(ii) $\mathrm{ord}_R(c) > \mathrm{ord}_R(a) = \mathrm{ord}_R(b)$, $ord_R(d) >$ $ord_{R_1}(a) =$ $ord_{R_1}(b)$.

Remark 8.4. Such a reduction cannot always be chosen

We start with the following table

	x	y	z
ord_R	1	1	1
ord_{R_1}	1	2	3

Consider $J = (x^2 - z, y - z)R$. Then (i) $(J, x^2) = (J, z) = (x^2, y, z) = I$ and (ii) + (iii)

	$x^2 - z$	$y - z$	x^2
ord_R	1	1	2
ord_{R_1}	2	2	3

Because $\mathrm{ord}_{R_1}(\frac{x^2-z}{x}) = \mathrm{ord}_{R_1}(x - \frac{z}{x}) = 1$ implies $\mathrm{ord}_{R_1}(x^2 - z) = 2$ and $\mathrm{ord}_{R_1}(\frac{y-z}{x}) = \mathrm{ord}_{R_1}(\frac{y}{x} - \frac{z}{x}) = 1$. Claim: J is a minimal reduction of I. Since $\lambda(R/J) < \infty$ J is generated by a SOP of R. Hence $e(J) = \lambda(R/J)$.

Remark 8.5. Let (A, n) be a Noetherian local ring. J, I n-primary ideals such that $J \subset I$ is a reduction. Then $e(J) = e(I)$. The converse is true if (A, n) is formally equidimentional i.e. $\dim(\hat{A}) = \dim(\frac{\hat{A}}{p})$. In particular when (A, n) is a Cohen-Macaulay local ring then the converse holds.

Show: $e(I) = e(J)$. We have $e(J) = \lambda(R/J) \geq \lambda(R/I)$. $2 = \lambda(R/I) \leq e(I)$ and $2 = e(m) < e(I)$ i.e $3 \leq e(I)$ because $I \subset m$ is not integral. It is enough to show $e(J) = 3$. Show: $\lambda(R/J) = 3$.

$$\begin{aligned}
\lambda(R/J) &= \lambda(S/(J, f)S) \\
&= \lambda(S/(x^2 - z, y - z, xz - y^2)) \\
&= \lambda(\frac{S/(y - z)}{(x^2 - z, xz - z^2)}, \qquad \text{mod } y - z \text{ and set } T = S/(y - z) \\
&= \lambda(\frac{T/(x^2 - z)}{x^3 - x^4}), \qquad \text{mod } x^2 - z \text{ and set } A = T/(x^2 - z) \\
&= \lambda(A/(x^3 - x^4)) \\
&= 3
\end{aligned}$$

9. July 9, 2009

9.1. **Generalize the previous section.** We would like to generalize the computation we did in the previous section to all such a. Let $S = k[X, Y, Z]_{(X,Y,Z)}$ be a polynomial ring over an algebraically closed field localized at a maximal ideal $n = (X, Y, Z)$ of S. Assume that char $k = 0$. Let $R = S/fS$ where $f = XY - Z^2$ in S. Let x, y, z denote the images of X, Y, Z in R. Then

(1) $M_a = (x, \frac{y}{x} - a^2, \frac{z}{x} - a), \quad \forall a \in k$

(2) $R_{1_a} = R[\frac{m}{x}]_{M_a}$ with $m_a = (x, \frac{z}{x} - a)$

(3) $I_a := (x^2, y - a^2x, z - ax)R$

(4) $Rees(I_a) = \{ord_R, ord_{R_{1_a}}\}$

We would like to find a minimal reduction $J_a = (u_a, v_a)$ of I_a that satisfies the following conditions.

(i) $I_a = (u_a, v_a, c_a) = (u_a, v_a, d_a)$ and

(ii) $ord_R(u_a) = ord_R(v_a) < ord_R(c_a)$ and $ord_{R_{1_a}}(u_a) = ord_{R_{1_a}}(v_a) < ord_{R_{1_a}}(c_a)$.

Notice that $ord_R(x) = ord_R(y) = ord_R(z) = 1$ and $ord_{R_{1_a}}(x) = 1, ord_{R_{1_a}}(y - a^2x) = 2, ord_{R_{1_a}}(z - ax) = 2$. Because $ord_{R_{1_a}}(\frac{y-a^2x}{x}) = ord_{R_{1_a}}(\frac{y}{x} - a^2) = ord_{R_{1_a}}((\frac{z}{x})^2 - a^2) = 1$.

Let $J_a = (x^2 - (z - ax), (y - a^2x) - (z - ax))$. Then

	$x^2 - (z - ax)$	$(y - a^2x) - (z - ax)$
ord_R	1	1
ord_{R_1}	2	2

Then, setting $I_a = (J_a, x^2)$ satisfies the first part of the condition (ii). We would like to find the element that satisfies the second part.

Compute $\lambda(R/J_a)$.

$$\begin{aligned}
\lambda(R/J_a) &= \lambda(S/(I_a, f)S) \\
&= \lambda(\frac{S}{(X^2 - (Z - aX), (Y - a^2X) - (Z - aX), XY - Z^2))}) \\
&= \lambda(\frac{T}{(X^2 - (Y - a^2X + aX), XY - (Y - a^2X + aX)^2)}) \quad \text{set } T = S/(Y - a^2X + aX - Z) \\
&= \lambda(\frac{A}{(X(X^2 + a^2X) - (X^2 + a^2X - a^2X + aX)^2)}) \quad \text{set } A = T/(X^2 + a^2X - Y) \\
&= \lambda(\frac{A}{(1 - 2a)X^3 - X^4)})
\end{aligned}$$

Hence $\lambda(R/J_a) = 3$ if $a \neq 1/2$ and $\lambda(R/J_a) = 4$ if $a = 1/2$.

10. July 16, 2009

Theorem 10.1 (Brodmann)**.** *Let (R, m) be a d-dimensional Cohen-Macaulay ring. Let I be complete intersection and generic complete intersection of height h. Then (1) $depth(R[It]) = min\{d, depth(R/I) + 2\}$ if $h = 0$ and $min\{d + 1, depth(R/I) + h + 2\}$ if $h > 0$.*
(2) $depth(gr_I(R)) = min\{d, depth(R/I) + h + 1\}$.

Theorem 10.2. *Let (R, m) be a 2-dimensional normal local domain. Let I be an m-primary ideal that is almost complete intersection. Suppose there exists a minimal reduction J of I with $\mu(I) = 2$ and $JI = I^2$. Then*
(1) $gr_I(R)$ is Cohen-Macaulay [VV].
(2) $R[It]$ is Cohen-Macaulay [GS].
(3) $F(I)$ is a Cohen-Macaulay ring having minimal multiplicity at its homogenous maximal ideal.
(4) If $Min(mR[It]) = \{Q_1, \dots, Q_n\}$. Then $n \leq 2$. (5) Suppose $Min(mR[It]) = \{Q_1, Q_2\}$. Then $R[It]/Q_i$ is regular for $i = 1, 2$.

Proof. (4) Let $N = (m, It)R[It]$ and $\bar{n}$ to be its image in $F(I)$. Then $\bar{N}$ is the unique homogeneous maximal ideal of $F(I)$. By (3),

$$edim(F(I)_{\bar{N}}) - dim(F(I)_{\bar{N}}) + 1 (= 3 - 2 + 1) = e(F(I)_{\bar{N}}) = 2$$

It is not hard to check that $dim(\frac{R[It]_N}{Q_{i_N}}) = 2$. By [Matsumura] 14.7,

$$\begin{aligned} e(F(I)_{\bar{N}}) &= \sum_{i=1}^{n} e((\frac{N}{Q_i})_{\bar{N}}, (\frac{R[It]}{Q_i})_{\bar{N}})\lambda(F(I)_{\bar{Q_i}}) \\ &= \sum_{i=1}^{n} e((\frac{R[It]}{Q_i})_{\bar{N}})\lambda((\frac{R[It]}{mR[It]})_{\bar{Q_i}}) \\ &\Rightarrow e(F(I)_{\bar{N}}) \geq n \\ &\Rightarrow n \leq 2 \end{aligned}$$

□

Example 10.3. This is an example of a 2-dimensional normal local domain such that the maximal ideal has two Rees valuations and $Bl_m(R)$ is regular. Let $S = k[X, Y, Z]_{(X,Y,Z)}$ where k is a field. Let $n = (X, Y, Z)S$ and $f = XY - Z^3$ in S. Let $R = S/fS$ and $m = (x, y, z)R$ where x, y, z denote the images of X, Y, Z in R. Since $gr_m(R) \cong \frac{k[x,y,z]}{(xy)}$ has two minimal primes, m has two Rees valuations. It remains to show that $Bl_m(R)$ is regular. When we blow up at m that is along the point $(0, 0, 0)$. Since the original curve is in C^3 the blow up is in $C^3 \times P^2$. Let the coordinates of P^2 be $[X' : Y' : Z']$. There exists three affine charts that cover the projective space P^2. Hence it is enough to check the regularity on them. Let $U_0 = \{X' \neq 0\}, U_1 = \{Y' \neq 0\}$ and $U_2 = \{Z' \neq 0\}$. Let's check on U_2 first. Since $Z' \neq 0$ we may assume that $Z' = 1$. Then $f = XY - Z^3$ becomes $f = XZ'YZ' - Z'^3 = Z'^2(XY - Z')$. The rank of the Jacobian matrix for XY - Z' at $(0, 0, 0)$ is one. This implies that it is regular on this chart (Hartshorne I.5). On U_0, we may set $X' = 1$. Then $f = X'YX' - (ZX')^3 = X'^2(Y - Z^3X')$. Similarly the rank of the Jacobian matrix for $(Y - Z^3X')$ is one. Hence it is regular on U_0. By symmetry it is regular on U_1. Hence the blow up is regular.

Remark 10.4. If we change $f = XY - Z^3$ to $f = XY - Z^m$ for $m \geq 4$ then $Bl_m(R)$ is not regular. This can be observed by looking at the chart U_2 above. The rank of Jacobian matrix at $(0, 0, 0)$ is zero. This tells us that $Bl_m(R)$ is not regular. Details can de found in I.5 of Hartshorne's Algebraic Geometry [?].

11. July 21, 2009

11.1 **Example 11.1.** Let $(R_0, \mathbf{m}_0)$ be a 2-dimensional RLR with $\mathbf{m}_0 = (x, y)R_0$. Consider the simple integrally closed ideal $I = (x, y^4)R_0$. The QDT-sequence determined by I is:

$$R_0 \subset R_1 = R_0[x/y, y]_{(x/y,y)} \subset R_1[x/y^2]_{(x/y^2,y)} = R_2 \subset R_2[x/y^3]_{(x/y^3,y)} = R_3$$

The proper transforms of I in these QDTs are

$$I^{R_1} = (x/y, y^3)R_1, \qquad I^{R_2} = (x/y^2, y^2)R_2, \qquad I^{R_3} = (x/y^3, y)R_3.$$

Definition 11.2. If $\alpha \subsetneq \beta$ is a birational extension of RLRs, then β is said to be **proximate** to α if $\beta \subset V_\alpha$, where V_α is the valuation ring of ord_α.

11.3 **Remark 11.3.** In Example [11.1] ??, we have R_1 is proximate to R_0 and R_2 is proximate to R_1, but R_2 is not proximate to R_0. It would be nice to cover a proof that if $\alpha_1 \subset \alpha_2 \subset \cdots \subset \alpha_r$ is a sequence of QDTs of 2-dimensional RLRs, then α_r is proximate to α_{r-1} and at most one other of the α_i.

Question 11.4. With the notation of Remark [11.3] ??, does there exist an example where $r = 4$ and α_4 is proximate to α_1 and α_3?

Example 11.5. For the RLRs R_i of Example [11.1] ??, let V_i denote the valuation ring of the ord valuation defined by the powers of the maximal ideal of R_i. Notice that x/y^{i+1} is residually transcendental for V_i.

With R_0 as in Example [11.1] ??, consider the affine components $R_0[y^2/x]$, $R_0[y^3/x]$ and $R_0[y^4/x]$ of the blowups of the integrally closed ideals $(x, y^2)R_0$, $(x, y^3)R_0$ and $(x, y^4)R_0$. These are all integrally closed domains of dimension two. Let

$$S_2 = R_0[y^2/x]_{(x,y,y^2/x)}, \quad S_3 = R_0[y^3/x]_{(x,y,y^3/x)}, \quad S_4 = R_0[y^4/x]_{(x,y,y^4/x)}.$$

Let $\mathbf{n}_i$ denote the maximal ideal of S_i. The associated graded ring $\text{gr}_{\mathbf{n}_2}(S_2)$ is a normal domain, so the blowup $\text{Bl}_{\mathbf{n}_2}(S_2)$ of the maximal ideal of S_2 is regular. Notice that the associated graded rings $\text{gr}_{\mathbf{n}_3}(S_3)$ and $\text{gr}_{\mathbf{n}_4}(S_4)$ are isomorphic rings having two minimal primes. The blowup $\text{Bl}_{\mathbf{n}_3}(S_3)$ is regular while the blowup $\text{Bl}_{\mathbf{n}_4}(S_4)$ is not regular. Notice that $\text{Rees}\,\mathbf{n}_2 = \{V_0\}$ and $\text{Rees}\,\mathbf{n}_3 = \{V_0, V_1\}$. To compute $\text{Rees}\,\mathbf{n}_4$, let $z = y^4/x$ and notice that $(x + z, y)S_4$ is a reduction of $\mathbf{n}_4$. The elements of $\text{Rees}\,\mathbf{n}_4$ are the DVRs V that birationally dominate S_4 and for which the element

$$\frac{x+z}{y} = \frac{x + y^4/x}{y} = \frac{x}{y} + \frac{y^3}{x}$$

is a unit of V and is residually transcendental for V. These DVRs are V_0 and V_2. The element x/y is residually transcendental for V_0 and y^3/x is in the maximal ideal of V_0, while y^3/x is residually transcendental for V_2 and x/y is in the maximal ideal of V_2. Thus $\operatorname{Rees} \mathbf{n}_4 = \{V_0, V_2\}$.

12. July 23, 2009

12.1. **An equation of integrality.** Let $R = k[x,y,z]_{(x,y,z)}$ where k is a field with $chak\ k \neq 3$. Let $f = x^3 + y^3 + z^3$. Let $S = R/(f)$. Let $I = (x^2, y + x, z)$ be an S-ideal. We would like to see that I^2 is not integrally closed. In particular the element $x(x+y) \in \bar{I^2} \setminus I^2$.

Claim: the following equation gives the integrality of $x(x+y)$ over I^2.

$$T^2 - (x+y)^2 T + xy(x+y)^2 = 0 \tag{1}$$

where $(x+y)^2 \in I^2$ and $xy(x+y)^2 \in I^4$. It is not hard to see that $x(x+y)$ satisfies this equation. Since $y + x \in I$, trivially $(y+x)^2 \in I^2$. It suffices to show that $xy(x+y)^2 \in I^4$. In particular, it is enough to show that $xy(x+y) \in I^3$.

$$x^3 + y^3 + z^3 = (x+y)(x^2 - xy + y^2) + z^3 \tag{2}$$

$$= (x+y)((x+y)^2 - 3xy) + z^3 \tag{3}$$

$$= (x+y)^3 - 3xy(x+y) + z^3 \tag{4}$$

Since $x^3 + y^3 + z^3 = 0$ in S. This implies

$$3xy(x+y) = (x+y)^3 + z^3 \tag{5}$$

$$= (x+y+z)((x+y)^2 - (x+y)z + z^2) \tag{6}$$

Since $x+y, z \in I$, $(x+y+z) \in I$. The same elements show that each term of $(x+y)^2 - (x+y)z + z^2$ is in I^2. Hence $xy(x+y) \in I^3$.

We can check that $x(x+y) \notin I^2$ by looking at the following equation

$$c_1(y+x)^2 + c_2(y+x)z + c_3 z^3 + c_4(x^3 + y^3 + z^3) = x(x+y) \tag{7}$$

and showing that there's no $c_i \in k$ that satisfy this equation over R.

Question 12.1. For the S-ideal $I = (x^2, x+y, z)S$, what is the reduction number of I with respect to a minimal reduction? I believe $J = (x+y+z, x^2 - z)S$ is a minimal reduction of I. Is JI properly contained in I^2?

12.2. **Proximate.** With regard to Remark 11.3 ??, we give the following argument for why α_r is always proximate to α_{r-1}:

Since α_r is a quadratic transformation of α_{r-1} there exists an $x \in \mathbf{m}_{r-1} - \mathbf{m}_{r-1}^2$ and a maximal ideal M of $\alpha_{r-1}[\frac{\mathbf{m}_{r-1}}{x}]$ such that $\alpha_r = \alpha_{r-1}[\frac{\mathbf{m}_{r-1}}{x}]_M$. Also, V_{r-1}, the valuation ring for the order valuation of α_{r-1}, is of the form $V_{r-1} = \alpha_{r-1}[\frac{\mathbf{m}_{r-1}}{x}]_{(x)}$, which is a further localization of α_r, so clearly $\alpha_r \subset V_{r-1}$. Note that the center of V_{r-1} on α_r is the regular prime $(x)\alpha_r$.

Now we want to consider any α_i such that α_r is proximate to α_i. Let $P_{r,i}$ be the center of V_i on α_r. Then $P_{r,i}$ is a height one prime and we claim that it is a regular prime, i.e. $\alpha_r/P_{r,i}$ is a DVR. As in the argument above, α_{i+1} is proximate to α_i and $\alpha_{i+1}/P_{i+1,i}$ is a DVR. Also since we have the chain of QDTs

$$\alpha_1 \subset \ldots \subset \alpha_i \subset \alpha_{i+1} \subset \ldots \subset \alpha_r$$

we have the birational extension $\alpha_{i+1}/P_{i+1,i} \subset \alpha_r/P_{r,i}$ where the smaller ring is a DVR so these two rings are equal and our claim is proven.

We would still like to understand why α_r is proximate to at most one other α_i and have an example where $r = 4$ and α_4 is proximate to α_3 and α_1 but not α_2.

Example 12.1 ?? is, I believe, an example where α_4 is proximate to α_1 and α_3, but is not proximate to α_2.

12.1 **Example 12.2.** Let $(R_0, \mathbf{m}_0)$ be a 2-dimensional RLR with $\mathbf{m}_0 = (x, y)R$. Define $R_1 = R_0[y/x]_{(x,y/x)}$. Let $y_1 = y/x$, and define $R_2 = R_1[x/y_1]_{(y_1,x/y_1)}$. Let $x_1 = x/y_1$ and define $R_3 = R_2[x_1/y_1]_(y_1, x_1/y_1)$. Then R_3 is proximate to R_2 and R_0. This can maybe better be described by regular parameters for the maximal ideals of R_0, R_1, R_2 and R_3 as

$$(x, y) \to (x, \frac{y}{x}) \to (\frac{x^2}{y}, \frac{y}{x}) \to (\frac{x^3}{y^2}, \frac{y}{x}).$$

Notice that the ord-valuation ring V_0 for R_0 contains R_3. Hence R_3 is proximate for R_0. Also notice that the ord valuation for V_1 does not contain R_3, for $v_1(x) = 1$ and $v_1(y) = 2$. Hence $v_1(x^3/y^2) = -1$ and R_3 is not proximate to R_1.

13. August 24, 2009

Example 13.1. Let $(R, \mathbf{m})$ be a 2-dim RLR with $\mathbf{m} = (x, y)R$. Then $\mathrm{Bl}_{\mathbf{m}}(R) = \mathrm{Proj}(R[\mathbf{m}\,t])$ is the set of homogeneous localizations of $R[\mathbf{m}\,t]$ at homogeneous prime ideals other than the unique homogeneous maximal ideal. We have

$R[\mathbf{m}\,t] = R[xt, yt] \subset R[t]$, a polynomial ring graded by $\mathbb{N}$

We can also think of the blowup of R as $\mathrm{Bl}_{\mathbf{m}}(R) = \mathrm{Spec}\, R[y/x] \cup \mathrm{Spec}\, R[x/y]$, where we regard $\mathrm{Spec}\, R[y/x]$ as the affine scheme that consists of the local rings

$$\{R[y/x]_P \mid P \text{ is a prime ideal of } R[y/x]\ \}.$$

It is often the case that $R[y/x]_P = R[x/y]_Q$ for some primes P and Q in the respective rings. Indeed, this is true for all prime ideals P of $R[y/x]$ other than the maximal ideal $(x, y/x)R[y/x]$ and all prime ideals Q of $R[x/y]$ other than the maximal ideal $(y, x/y)R[x/y]$.

Here is a more general example.

13.2 **Example 13.2.** Let R be a Noetherian domain with field of fractions K and let $I = (a_1, \dots, a_n)R$, where each a_i is a nonzero element of R. Then

$$\mathrm{Bl}_I(R) = \mathrm{Proj}\, R[It] = \bigcup_{i=1}^{n} \mathrm{Spec}\, R[I/a_i].$$

A local ring $(S, \mathbf{n})$ is in $\mathrm{Bl}_I(R)$ if and only if $S = R[I/a_i]_Q$ for some prime ideal Q of $R[I/a_i]$.

Question 13.3. Let I and J be ideals of the Noetherian domain R. When is $\mathrm{Bl}_I(R) = \mathrm{Bl}_J(R)$?

Remark 13.4. It is always true that $\mathrm{Bl}_I(R) = \mathrm{Bl}_{I^n}(R)$ for n a positive integer.

If V is a valuation domain with $R \subset V$ then V dominates a unique local ring of $\mathrm{Bl}_I(R)$. To see that V dominates some local ring of $\mathrm{Bl}_I(R)$ observe that $IV = aV$ for some $a \in I$. Hence $R[I/a] \subset V$. Let $\mathbf{n}$ be the maximal ideal of V and let $\mathbf{n} \cap R[I/a] = Q$. Then $R[I/a]_Q$ is dominated by V. This is said more clearly in Section 6.2.

Definition 13.5. Let S be a subring of a local ring $(T, \mathbf{n})$.

(1) The prime ideal $P := \mathbf{n} \cap S$ is called the **center** of T on S.
(2) If S is also local and $\mathbf{n} \cap S$ is the maximal ideal of S, then we say T **dominates** S.

With the notation of Example 13.2 ??, we have

$$\mathrm{Bl}_I(R) = \cup_{i=1}^{n} \mathrm{Spec}\, R[I/a_i] = \cup_{0 \neq a \in I} \mathrm{Spec}\, R[I/a].$$

Notice that $IR[I/a] = aR[I/a]$ is principal, so I extends to a principal ideal in each local ring of $\mathrm{Bl}_I(R)$. By Proposition 6.5 ?? as repeated below, if a and b are nonzero elements of I and $R[I/b] \subset R[I/a]_Q$, then $R[I/a]_Q = R[I/b]_{Q'}$, where $Q' = QR[I/a]_Q \cap R[I/b]$.

Proposition 13.6. *Let I be a nonzero finitely generated ideal of the integral domain R, let a and b be nonzero elements of I and let $P \in \mathrm{Spec}\, R[I/a]$. Then*

(1) The ring $R[I/b] \subseteq R[I/a]_P \iff a/b \in R[I/a]_P \iff a/b$ is a unit of $R[I/a]_P$.
(2) If $R[I/b] \subseteq R[I/a]_P$, then $R[I/a]_P = R[I/b]_Q$, where $Q := PR[I/a]_P \cap R[I/b]$.

Proof. We clearly have $T := R[I/a]_P \supseteq R[I/b]_Q =: S$ and T dominates S. The ideal $IS = bS$. Since T dominates S, we have $aT = bT = IT$. Thus $a/b \in S$ is a unit of T. Since S and T are local rings with T dominating S, it follows that a/b is a unit of S. Therefore $R[I/a] \subseteq S$. Let $P' := QS \cap R[I/a]$. Then $R[I/a]_{P'}$ is dominated by S and S is dominated by T. Since $PT \cap R[I/a] = P$, we have $P' = P$, and thus $R[I/a]_P = R[I/b]_Q$. □

Recall Definition 6.6 ??.

Definition 13.7. Let R be an integral domain.

(1) An extension domain S of R is said to be **birational over** R if S is contained in the field of fractions of R.
(2) Let I be a nonzero finitely generated ideal of R. Let Σ denote the family of birational local overrings S of R such that IS is principal. We define a partial order $\leq$ on Σ as follows:

$$\text{Let} \quad S, T \in \Sigma, \quad \text{then} \quad S \leq T \quad \Longleftrightarrow \quad T \quad \text{dominates} \quad S.$$

Question 13.8. Let I and J be integrally closed $\mathbf{m}$-primary ideals of a 2-dim RLR $(R, \mathbf{m})$. Then

(1) We have $\mathrm{Bl}_I(R) = \mathrm{Bl}_J(R)$ if and only if ... ???
(2) We have each $T \in \mathrm{Bl}_J(R)$ dominates some $S \in \mathrm{Bl}_I(R)$ if and only if ... ???

Definition 13.9. Let I and J be nonzero ideals of a Noetherian domain R. We say that the blowup $\mathrm{Bl}_J(R)$ **dominates** the blowup $\mathrm{Bl}_I(R)$ and write $\mathrm{Bl}_I(R) \leq \mathrm{Bl}_J(R)$ if each $T \in \mathrm{Bl}_J(R)$ dominates some $S \in \mathrm{Bl}_I(R)$.

14. August 26, 2009

Notes from the paper HL [?] of Heinzer and Lantz. Let $(R, \mathbf{m})$ be a 2-dimensional Noetherian normal local domain.

Definition 14.1. A **spot** over R is a local domain $(S, \mathbf{n})$ that is a localization of a finitely generated R-algebra (i.e., S is a local ring that is essentially finitely generated over R).

Consider the normal local domains $(S, \mathbf{n})$ that are birational over R and are spots over R. Other than the field of fractions K of R, there are three disjoint classes:

(1) $S = R_P$, where P is a height one prime of R, these are all DVRs, and are called the **essential prime divisors** of R. Denote this class by $\mathrm{epd}(R)$.

(2) $(S, \mathbf{n})$ is a DVR with $R \subset S \subset K$ and S dominating R, i.e., $\mathbf{n} \cap R = \mathbf{m}$ and S is a spot over R. These are called the **hidden prime divisors** of R, or the prime divisors of the second kind of R. Denote this class by $\operatorname{hpd}(R)$.

(3) The 2-dimensional normal local domains $(S, \mathbf{n})$ with $R \subset S \subset K$ and S is a spot over R. Denote this class by $\mathcal{S}_2(R)$.

Remark 14.2. Every $S \in \mathcal{S}_2(R)$ dominates R. More generally, if R is a Noetherian domain of dimension n, then any birational overring of R has dimension $\leq n$.

Remark 14.3. (1) $\operatorname{epd}(R) \cup \operatorname{hpd}(R) \cup \mathcal{S}_2(R) \cup \{K\}$ is the set of all normal local spots birational over R.

(2) If $(S, \mathbf{n}) \in \operatorname{hpd}(R)$, then $\operatorname{trdeg}_{R/\mathbf{m}} S/\mathbf{n} = 1$. This follows from the dimension formula: We have $S = R[\theta_1, \dots, \theta_n]]_{\mathbf{n} \cap R[\theta_1, \dots, \theta_n]]}$ and $\operatorname{trdeg} S/R = 0$ so $\dim R = \dim S + \operatorname{trdeg}_{R/m} S/n$. Let (V, N) be a DVR with $R \subset V \subset K = Q(R)$ and $N \cap R = \mathbf{m}$. If V is a spot over R, then $\operatorname{trdeg}_{R/\mathbf{m}} V/N = 1$. The dimension formula implies that $\operatorname{trdeg}_{R/\mathbf{m}} V/N > 0$ Choose $\theta \in V$ such that the image of θ in V/N is transcendental over R/M. Then $MR[\theta]$ is a nonmaximal prime ideal. It follows that $\operatorname{ht} MR[\theta] = 1$. Therefore S is a birational overring of the one-dimensional Noetherian local domain $R[\theta]_{MR[\theta]}$. The Krull-Akizuki Theorem implies that S is a localization of an integral extension of $R[\theta]_{MR[\theta]}$.

(3) If $(S, N) \in \mathcal{S}_2(R)$ then the field S/N is finite algebraic over R/M.

If (R, M) is assumed to be excellent (or more generally, if the M-adic completion of R is a domain, i.e. R is analytically irreducible), then every 2-dimensional normal Noetherian local domain S such that $R \subset S \subset K$ is in $\mathcal{S}_2(R)$.

Discussion 14.4. Fix (R, M) a 2-dimemsional normal Noetherian local domain and $S \in \mathcal{S}_2(R)$. Then we can consider $\operatorname{epd}(R)$, $\operatorname{hpd}(R)$, $\mathcal{S}_2(R)$ and $\operatorname{epd}(S)$, $\operatorname{hpd}(S)$, $\mathcal{S}_2(S)$.

(1) Clearly $\mathcal{S}_2(S) \subset \mathcal{S}_2(R)$ and $hpd(S) \subset hpd(R)$.

(2) Also $V \in \operatorname{epd}(R)$ and $S \subset V$ implies $V \in \operatorname{epd}(S)$. For if $S \subset V_p$, let $q = pR_p \cap S$, then $V = R_p \subset S_q = V$.

(3) If $V \in \operatorname{epd}(S)$ and $MV = V$, then $V \in \operatorname{epd}(R)$. If $S_Q = V$, then either $Q \cap R = M$ or $Q \cap R = P$, where $\operatorname{ht} P = 1$. If $Q \cap R = P$ then $R_P = S_Q = V$ and $V \in \operatorname{epd}(R)$. Since MS is a nonzero ideal of S, MS is contained in at most finitely many height one primes $Q_1, \dots, Q_n$ of S. Then $V_1 = S_{Q_1}, \dots, V_n = S_{Q_n}$ are precisely the elements of $\operatorname{epd}(S)$ that are not in $\operatorname{epd}(R)$. We say that $V_1, \dots, V_n$ are the **exceptional prime divisors** of S/R and write $\operatorname{xpd}(S/R) = \operatorname{epd}(S) \cap \operatorname{hpd}(R)$.

(4) If $S \in \mathcal{S}_2(R)$ and $R \neq S$ then $\operatorname{epd}(S) \cap \operatorname{hpd}(R)$ is a finite nonempty set.

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(R, M) a 2-dimensional normal local domains with

$$\operatorname{epd}(R) = \{R_P | P \text{ is a height one prime}\}$$

$$hpd(R) = \{V | R \subset V \subset Q(R) \text{ and } V \text{ is a DVR that dominates } R \text{ and } V \text{ is a spot over } R\}$$

$$\mathcal{S}_2 = \{S | S \text{ is a 2-dimensional normal local domain with } R \subset S \subset Q(R) \text{ and } S \text{ is a spot over } R\}$$

For $S \in \mathcal{S}_2$, we have $\operatorname{xpd}(S/R) = \operatorname{epd}(S) \cap \operatorname{hpd}(R)$ where

$$\operatorname{xpd}(S/R) = \{S_Q | Q \text{ is a height one prime of } S \text{ and } MS \subset Q\}.$$

Proposition 15.1. *Assuming R is excellent, let S_1 and S_2 and T be in $\mathcal{S}_2(R)$.*

(1) If $S_1 \subset T$, then $\operatorname{xpd}(T/S_1) = \emptyset \iff S_1 = T$.

(2) Assume T is a localization of the integral closure of $S_1[S_2]$. Then we have

(a) $\operatorname{xpd}(T/R) \subset \operatorname{xpd}(S_1/R) \cup \operatorname{xpd}(S_2/R)$,

(b) $\operatorname{xpd}(T/S_2) \subset \operatorname{xpd}(S_1/R)$, and $\operatorname{xpd}(T/S_1) \subset \operatorname{xpd}(S_2/R)$.

(3) If $\operatorname{xpd}(S_1/R) = \operatorname{xpd}(S_2/R)$, then either $S_1 = S_2$ or no element of $\mathcal{S}_2(R)$ dominates both S_1 and S_2, i.e. $S_1[S_2]$ has dimension less than or equal to 1.

Proof: (1)$\Leftarrow$ is clear

$\Rightarrow$: Let N_1 be the maximal ideal of S_1 and N the maximal ideal of T. We need to see that if N_1T is N-primary then $S_1 = T$. This follows from a form of Zariski's Main Theorem - see Nagata (37.4).

(2) Suppose $\operatorname{xpd}(T/R)$ is not a subset of $\operatorname{xpd}(S_1/R) \cup \operatorname{xpd}(S_2/R)$. Then there exists a height one prime P of T such that $T_P \in \operatorname{xpd}(T/R) \setminus (\operatorname{xpd}(S_1/R) \cup \operatorname{xpd}(S_2/R))$. Hence $P \cap S_1 = N_1$ and $P \cap S_2 = N_2$. Let $Q = P \cap S_1[S_2]$. Notice that $S_1[S_2]/Q$ is generated over R/M by the images of S_1/N_1 and S_2/N_2 in $S_1[S_2]/Q$. Since S_1, S_2 are in $\mathcal{S}_2(R)$, the fields S_1/N_1 and S_2/N_2 are finite algebraic over R/M. Hence the integral domain $S_1[S_2]/Q$ is a field, finite algebraic over R/M. Let $P' = P \cap \overline{S_1[S_2]}$ then $T_P = \overline{S_1[S_2]}_{P'}$. Hence P' is a height one non-maximal prime of $\overline{S_1[S_2]}$. But $P \cap S_1[S_2] = Q = P' \cap S_1[S_2]$ and Q is maximal. By the going up theorem P' is maximal. This is a contradicition. Hence $\operatorname{xpd}(T/R) \subset \operatorname{xpd}(S_1/R) \cup \operatorname{xpd}(S_2/R)$.

(3) If $V \in \operatorname{xpd}(T/S_2)$ then $V \in \operatorname{xpd}(T/R) \subset \operatorname{xpd}(S_1/R) \cup \operatorname{xpd}(S_2/R)$. Then $V \in \operatorname{xpd}(S_1/R)$ and (3) now follows, for if $S_1 \neq S_2$ and $\dim S_1[S_2] = 2$ then there exists a maximal ideal N of $S_[S_2]$ of height two. We then have $N \cap S_i = N_i$, $i = 1, 2$. This implies

the existence of $T \in \mathcal{S}_2(R)$ such that T dominates both S_1 and S_2. But item (2) then implies that $\operatorname{xpd}(T/S_1) = \emptyset$. ■

Remark 15.2. Given (S_1, N_1) and (S_2, N_2) dominating R, let $S = S_1 \cap S_2$.

(1) The ring S has at most 2 maximal ideals $N_1 \cap S$ and $N_2 \cap S$, for each nonunit of S is contained in $(N_1 \cap S) \cup (N_2 \cap S)$.
(2) If S has two maximal ideals then $\dim S_1 \cap S_2 \leq 1$. For if there exists a maximal ideal N of $S_1[S_2]$ with $\operatorname{ht} N = 2$, then $N_1 \cap S \subset N \cap S$ and $N_2 \cap S \subset N \cap S$. Hence $N \cap S$ is the unique maximal ideal of S.

Example 15.3. Let (R, M) be a 2-dimensional RLR with $M = (x, y)R$. Let $S_1 = R[y^2/x]_{(x,y,y^2/x)}$ and $S_2 = R[x/y]_{(y,x/y)}$. We claim that $R = S_1 \cap S_2$. It suffices to show that every valuation domain dominating R contains either S_1 or S_2 for

$$R = \bigcap\{V \mid V \text{ is a valuation domain birationally dominating } R\ \}.$$

Now $(x, y)V$ is principal generated either by xV or yV. If $(x, y)V = xV$, then $v(y) \geq v(x) > 0$. Hence $v(y^2) > v(x)$, so $v(y^2/x) > 1$ so V dominates S_1. If $(x, y)V \neq xV$ then $v(x) > v(y)$ and $v(x/y) > 0$ and V dominates S_2.

Question 15.4. Let (R, M) be a 2-dimensional normal local domain and let S_1 and $S_2 \in \mathcal{S}_2(R)$. If $R = S_1 \cap S_2$ does it follow that every valuation domain V that birationally dominates R contains either S_1 or S_2?

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Let (R, M) be a 2-dimensional RLR with $M = (x, y)R$. Assume R/M is algebraically closed. Then $R_1 = R[y/x]_{(x,y/x)}$ is a "first" local QDT of R. We have $R \subsetneq R_1$ and there are no RLRs S with $R \subsetneq S \subsetneq R_1$. We have $\operatorname{xpd}(R_1/R) = \{\operatorname{ord}_R\}$. Let $V = \operatorname{ord} R = (\mathrm{R}_1)_{M\,\mathrm{R}_1} = (\mathrm{R}_1)_{x\,\mathrm{R}_1}$.

Question 16.1. Which of the local QDTs of R_1 are contained in V?

Let $M_1 = (x, y/x)R_1$ and $y_1 = y/x$ then $y = xy_1$. Some local QDTs of R_1 are $R_1[y_1/x]_{(x,y_1/x)}$ and $R_1[y_1/x]_{(x,y_1/x-a)}$ where $0 \neq \overline{a} \in R/M_1$. None of these are contained in V since $v(y_1/x) = -1 = v(y_1/x - a)$. But $R_1[x/y_1]_{(y_1,x/y_1)} \subset V$. Hence there is exactly one local first QDT of R_1 that is contained in V, namely $R_1[x/y_1]_{(y_1,x/y_1)}$.

To check this statement: how do we see that $R_1[x/y_1]_{(y_1,x/y_1-a)}$, with $0 \neq \overline{a} \in R/M_1$, is not contained in V? Notice that x/y_1 is a unit of $R_1[x/y_1]_{(y_1,x/y_1-a)}$, but $v(x/y_1) = 1$ so $v(y_1/x) = -1$.

Let $R_2 = R_1[x/y_1]_{(y_1, x/y_1)}$ and let $x_1 = x/y_1$ so $x = y_1 x_1$. Then $V = \operatorname{ord} R$ is centered on a height one prime P_2 of R_2, $P_2 = \frac{x}{y_1} R_2$. Notice that R_2/P_2 is a DVR. Let $V_1 = \operatorname{ord} R_1$, then $v_1(x) = v_1(y_1) = 1$ and x/y_1 is residually transcendental in the residue field of V_1.

$R_2 \subset V_1$ and V_1 is centered on a height one prime Q_2 of R_2, $Q_2 = y_1 R_2$. Note that R_2/Q_2 is a DVR and $M_2 = P_2 + Q_2$.

Question 16.2. Which of the first local QDTs of R_2 are contained in V and which are contained in V_1?

Let $M_2 = (y_1, x_1)R_2$ and $v_1(y_1) = 1$, $v_1(x_1) = 0$ and $v(x_1) = 1$, $v(y_1) = 0$. Then

(1) $R_2[y_1/x_1]_{(x_1, y_1/x_1)} \subset V_1$, but no other first local QDT of R_2 is contained in V_1.
(2) $R_2[x_1/y_1]_{(y_1, x_1/y_1)} \subset V$, and no other first local QDE of R_2 is contained in V.

Remark 16.3. Let $R = R_0 \subset R_1 \subset R_2 \subset \cdots \subset R_n$ be a sequence of local QDTs. Let V_i be the ord-valuation of R_i. Then $R_{i+1} \subset V_i$ for $i = 0, 1, \ldots n-1$ and $V_i \in \operatorname{epd}(R_{i+1})$. Let P_{i+1} denote the center of V_i on R_{i+1}, then R_{i+1}/P_{i+1} is a DVR. There exists at most one j with $j < i$ such that $R_{i+1} \subset V_j$. If $R_{i+1} \subset V_j$ with $j < i$ then $V_j \in \operatorname{epd}(R_{i+1})$ and if Q_{i+1} is the center of V_j on R_{i+1} then R_{i+1}/Q_{i+1} is a DVR and $M_{i+1} = P_{i+1} + Q_{i+1}$.

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