MATH 690 NOTES

1. Associated graded rings

Question 1.1. Let $G = k[G_1] = \bigoplus_{n\geq 0} G_n$ be a one-dimensional standard graded ring over the field $k = G_0$. What are necessary and sufficient conditions in order that G be the associated graded ring, $\mathrm{gr}_{\mathbf{m}}(R)$, of a one-dimensional Cohen-Macaulay local ring $(R, \mathbf{m})$?

The standard graded ring $G = k[G_1]$ is isomorphic to the associated graded ring of the local ring G_{G_+} and if G is Cohen-Macaulay, then G_{G_+} is Cohen-Macaulay. Hence every Cohen-Macaulay standard graded ring $k[G_1]$ is the associated graded ring of a Cohen-Macaulay local ring.

If $\dim_k G_1 \leq 2$, then $G = \mathrm{gr}_{\mathbf{m}}(R)$, where $(R, \mathbf{m})$ is Cohen-Macaulay, implies G is Cohen-Macaulay. For one may assume that the embedding dimension of R is at most two and that R is complete in the $\mathbf{m}$-adic topology. It follows that R is either a DVR, or of the form S/fS, where S is a two-dimensional RLR. If R is a DVR, then $\mathrm{gr}_{\mathbf{m}}(R)$ is a polynomial ring $k[t]$, while if $R = S/fS$, where $(S, \mathbf{n})$ is a two-dimensional RLR, then $\mathrm{gr}_{\mathbf{m}}(R) = \mathrm{gr}_{\mathbf{n}}(S)/(f^*)$, where $\mathrm{gr}_{\mathbf{n}}(S) = k[x, y]$ is a polynomial ring in two variables over the field k and f^* is the initial form of f and thus a nonzero homogeneous element of degree e, where e is the multiplicity of R and G.

There exist one-dimensional standard graded rings $G = k[G_1]$ with $\dim_k G_1 = 3$ that are not Cohen-Macaulay, but are of the form $\mathrm{gr}_{\mathbf{m}}(R)$, where $(R, \mathbf{m})$ is a one-dimensional Cohen-Macaulay local domain.

Example 1.2. Let $R = k[[t^4, t^5, t^{11}]]$, and define a k-algebra homomorphism of $S = k[[x, y, z]]$ onto R by $x \mapsto t^4$, $y \mapsto t^5$, and $z \mapsto t^{11}$. Then $R = S/I$, where $I = (xz - y^3,\, yz - x^4,\, z^2 - x^3y^2)S$. In an abuse of notation, we let x, y, z denote their own initial forms in the associated graded ring $\mathrm{gr}_{\mathbf{n}}(S)$. Thus $\mathrm{gr}_{\mathbf{n}}(S)$ is the polynomial ring $k[x, y, z]$ and the ideal I^* of initial forms of elements of I contains (xz, yz, z^2). Since these initial forms are all multiples of z, they do not generate I^*.

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One computes that

$$I^* = (xz, yz, z^2, y^4)k[x, y, z],$$

and therefore that

$$G = \mathrm{gr}_m(R) = \frac{k[x, y, z]}{(xz, yz, z^2, y^4)}.$$

Notice that $\mathrm{Ann}_G\, z = (x, y, z)G$, the graded maximal ideal of the one-dimensional standard graded ring G. Thus G is a one-dimensional standard graded ring that is not Cohen-Macaulay, but yet has the form $\mathrm{gr}_{\mathbf{m}}(R)$, where $(R, \mathbf{m})$ is Cohen-Macaulay. We have $e(R) = e(G) = 4$ and $t^4R = J$ is a principal reduction of $\mathbf{m}$. Since R is Cohen-Macaulay, we have $\ell(R/t^4R) = 4 = e(R)$. Notice that xG is a principal reduction of the maximal graded ideal G_+ of G. One computes that

$$\ell(\frac{G}{xG}) = \ell(\frac{k[y, z]}{((z^2, yz, y^4))}) = 5, \quad \text{and} \quad \ell(0 :_G x) = 1.$$

This confirms the equation

$$e(G) = \ell(G/xG) - \ell(0 :_G x).$$

Tom Marley [1] proved a result that implies as a special case that if $(R, \mathbf{m})$ is a one-dimensional Cohen-Macaulay local ring and $G = \mathrm{gr}_{\mathbf{m}}(R)$ is not Cohen-Macaulay, then $a_0(G) < a_1(G)$, where

$$a_0(G) = \max\{n \mid H^0_{G_+}(G)_n \neq 0\}, \qquad \text{where} \quad H^0_{G_+}(G) = \bigcup_{i \geq 0} (0) :_G G^i_+,$$

and

$$a_1(G) = \max\{n \mid H^1_{G_+}(G)_n \neq 0\}, \qquad \text{where} \quad H^1_{G_+}(G) = \frac{G[1/x]}{\varphi(G)}.$$

Here $x \in G_1$ generates a principal reduction of G_+ and $\varphi : G \to G[1/x]$ is the canonical homomorphism.

For R and G as in Example 1.2, one sees that $H^0_{G_+}(G) = zG = \langle z \rangle$ and $\deg z = 1$, so $a_0(G) = 1$. Also $G[1/x] = k[x, y][1/x]$, where $y^4 = 0$. Hence $y^3/x \in G[1/x]$ has a nonzero image in $\frac{G[1/x]}{\varphi(G)}$ and is a homogeneous element of maximal degree with this property. Since $\deg \frac{y^3}{x} = 2$, we have $a_1(G) = 2 > 1 = a_0(G)$, as is to be expected because of Marley's result.

Question 1.3. Does there exist a one-dimensional standard graded ring $G = k[G_1]$ as in Question 1.1 that also has $a_0(G) < a_1(G)$ and yet G is not an associated graded ring $\mathrm{gr}_{\mathbf{m}}(R)$, where $(R, \mathbf{m})$ is Cohen-Macaulay? Or said another way, could it be that the necessary condition $a_0(G) < a_1(G)$ is also a sufficient condition, and thus an answer to Question 1.1?

Example 1.4. Let $R = k[[t^6, t^7, t^{15}]]$, and define a k-algebra homomorphism of $S = k[[x, y, z]]$ onto R by $x \mapsto t^6$, $y \mapsto t^7$, and $z \mapsto t^{15}$. Then $R = S/I$, where $I = (xz - y^3,\ z^2 - x^5)S$. Thus R is a complete intersection and therefore is Gorenstein. As in Example 1.2, we let x, y, z denote their own initial forms in the associated graded ring $\mathrm{gr}_{\mathbf{n}}(S)$. Thus $\mathrm{gr}_{\mathbf{n}}(S)$ is the polynomial ring $k[x, y, z]$ and one computes that the ideal I^* of initial forms of elements of I is

$$I^* = (xz, z^2, zy^3, y^6)k[x, y, z],$$

and therefore that

$$G = \mathrm{gr}_m(R) = \frac{k[x, y, z]}{(xz, z^2, zy^3, y^6)}.$$

Notice that $\mathrm{Ann}_G\, z = (x, z, y^3)G$ is primary for the graded maximal ideal G_+ of the one-dimensional standard graded ring G. Thus G is not Cohen-Macaulay, but yet has the form $\mathrm{gr}_{\mathbf{m}}(R)$, where $(R, \mathbf{m})$ is Gorenstein. We have $e(R) = e(G) = 6$ and $t^6R = J$ is a principal reduction of $\mathbf{m}$. Since R is Cohen-Macaulay, we have $\ell(R/t^6R) = 6 = e(R)$. Notice that xG is a principal reduction of G_+. One computes that

$$\ell(\frac{G}{xG}) = \ell(\frac{k[y, z]}{((z^2, zy^3, y^6))}) = 9, \quad \text{and} \quad \ell(0 :_G x) = 3.$$

This confirms the equation

$$e(G) = \ell(G/xG) - \ell(0 :_G x).$$

One also computes that

$$(0) :_G x = \langle z, zy, zy^2\rangle G = H^0_{G_+}(G) \quad \text{and thus} \quad a_0(G) = 3,$$

while

$$G[\frac{1}{x}] = k[x, y][\frac{1}{x}], \quad \text{where} \quad y^6 = 0, \quad \text{so} \quad \frac{y^5}{x} \in G[\frac{1}{x}]$$

has a nonzero image in

$$H^1_{G_+}(G) = \frac{G[1/x]}{\varphi(G)}. \quad \text{Since } \deg \frac{y^5}{x} = 4, \quad a_1(G) \geq 4,$$

and since y^5/x has maximal degree among homogeneous elements of $G[1/x] \setminus \varphi(G)$, we have $a_1(G) = 4$.

It is interesting to also consider the principal reduction $x - y$ of G_+. We have

$$\ell(\frac{G}{(x - y)G}) = \ell(\frac{k[y, z]}{((z^2, zy, y^6))}) = 7, \quad \text{and} \quad \ell(0 :_G x - y) = 1.$$

Since G is not Cohen-Macaulay, for each $w \in G_1$ that generates a reduction of G_+, we must have $\ell(G/wG) > e(G) = 6$, but the length varies depending on the length of $(0) :_G w$.

Remark 1.5. Let $G = k[G_1] = \bigoplus_{n \geq 0} G_n$ be a one-dimensional standard graded ring over the field $k = G_0$. If G is not Cohen-Macaulay and $\dim_k G_1 = 2$, then $a_0(G) > a_1(G)$. For $G = k[x, y] = S/I$, where $S = k[X, Y]$ is a polynomial ring and I is a homogeneous ideal with $\operatorname{ht} I = 1$. Since the polynomial ring S is a UFD, $I = fL$, where f is a homogeneous polynomial of degree $d \geq 1$ and either $L = S$, or L is a homogeneous ideal that is primary for the maximal ideal $(X, Y)S$. Since G is not Cohen-Macaulay, $L \neq S$. We may assume that the field k is infinite and that f is monic as a polynomial in Y. Then xG is a principal reduction of G_+ and

$$H^1_{G_+}(G) = \frac{G[1/x]}{\varphi(G)}, \quad \text{where} \quad \varphi(G) = G/fG = S/fS \quad \text{and} \quad a_1(G) = d - 2$$

since $\frac{y^{d-1}}{x} \in G[1/x] \setminus \varphi(G)$ is a homogeneous element of degree $d-2$ and this is the maximal degree of a homogeneous element of $G[1/x] \setminus \varphi(G)$. On the other hand, $H^0_{G_+}(G) = J/I$, where J is the saturation in $S = k[X, Y]$ of $I = fL$, i.e.,

$$J = \bigcup_{n \geq 0} (fL :_S (X, Y)^n = fS, \quad \text{and thus} \quad H^0_{G_+}(G) = fS/fL.$$

It follows that $a_0(G) = \deg f + \operatorname{socdeg}(L)$, where $\operatorname{socdeg}(L)$ is the maximal degree of a homogeneous element in $(L :_S (X, Y)) \setminus L$. Since L is $(X, Y)S$-primary, $\operatorname{socdeg}(L)$ is a well-defined nonnegative integer.

We conclude that $a_1(G) + 2 \leq a_0(G)$ if $\dim_k G_1 = 2$ and G is not Cohen-Macaulay.

Question 1.6. Let $S = k[x, y, z]$ be the graded polynomial ring in the variables x, y, z over the field k. Can one describe or classify in some way the homogeneous ideals I of S such that

(1) the radical of I is the prime ideal $(y, z)S$,
(2) I has the maximal ideal $(x, y, z)S$ as an associated prime,
(3) $G = S/I$ has the property that $a_0(G) < a_1(G)$?

Example 1.2 shows that the ideal $(xz, yz, z^2, y^4)S$ has the properties enumerated in Question 1.6, but one sees that the ideals $(xz, yz, z^2, y^2)S$ and $(xz, yz, z^2, y^3)S$ fail to satisfy the condition $a_0(G) < a_1(G)$. Example 1.4 shows that the ideal $(xz, z^2, yz^3, y^6)S$ has the properties enumerated in Question 1.6, but one sees that the ideals $(xz, z^2, yz^3, y^4)S$ and $(xz, z^2, yz^3, y^5)S$ fail to satisfy the condition $a_0(G) < a_1(G)$.

2. The associativity formula for multiplicities

References

[1] T. Marley, The reduction number of an ideal and the local cohomology of the associated graded ring, *Proc. Amer. Math. Soc.* **117** (1993), 335-341.

Department of Mathematics, Purdue University, West Lafayette, IN 47907
E-mail address: `heinzer@math.purdue.edu`