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On the long–time asymptotics for the Korteweg–de Vries equation with steplike initial data associated with rarefaction waves

K. Andreiev, I. Egorova

B. Verkin Institute for Low Temperature Physics and Engineering of the National Academy of Sciences of Ukraine

E-mail: kyrylo.andreiev@gmail.com, iraegorova@gmail.com

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We discuss the long time asymptotics of the rarefaction wave for the KdV equation in the region behind the wave front. The first and the second term of the asymptotical expansion for such a solution were claimed without detailed analysis in [1]. In the present work we solve the corresponding parametrix problem and complete the asymptotic analysis. We also discuss an influence of the resonance on the asymptotical behavior of the solution.

Keywords: the KdV equation, rarefaction wave, paramerix

Mathematics Subject Classification 2010: 37K40, 35Q53 (primary); 35Q15 (secondary).

1 Introduction

This paper is a continuation of paper [1]. In [1] the long-time asymptotics is discussed of the Cauchy problem solution for the Korteweg-de Vries (KdV) equation

$$q_t(x,t) = 6q(x,t)q_x(x,t) - q_{xxx}(x,t), \quad (x,t) \in \mathbb{R} \times \mathbb{R}_+, \tag{1}$$

with steplike initial data $q(x,0) = q_0(x)$ of the following type:

$$\begin{cases} q_0(x) \to 0, & \text{as } x \to +\infty, \\ q_0(x) \to c^2, & \text{as } x \to -\infty. \end{cases} \tag{2}$$

Such an initial profile corresponds to the rarefaction wave. Its asymptotics is well understood on a physical level of rigor ([10], [15], [12]). In [1] the asymptotics of the solution for (1)-(2) is studied mathematically rigorously for the regions ahead of the back wave front by use of the nonlinear steepest descent method ([6]). As for the region behind the back wave front, in [1] a respective model Riemann-Hilbert (RH) problem is studied in the nonresonant case. This allows us to conjecture that the solution is asymptotically close to the respective background constant c^2, plus a decaying "radiation part" of order $O(t^{-1/2})$. Moreover, for this second term of the asymptotical expansion a formula was given which had the same form as for the decaying initial data ($q_0(x) \to 0$ as $x \to \pm\infty$).

The objects of the present paper are: (a) to justify the asymptotical expansion for the solution of (1)-(2) with respect to large t in the region $x < (-6c^2 - \epsilon)t$; (b) to check a possible influence of the resonance on the asymptotical expansion; (c) to clarify the formula for the second term.

We will assume that the initial profile (2) satisfies the following condition:

$$\int_0^{+\infty} \mathrm{e}^{(c+\kappa)x}(|q_0(x)|+|q_0(-x)-c^2|)dx < \infty, \ \ x^4 q^{(i)}(x) \in L_1(\mathbb{R}), \ i = 1, ..., 8, \quad (3)$$

where $\kappa > 0$ is a small number. Under this condition the solution of the Cauchy problem (1)-(2) exists in the classical sense and is unique in the domain $(x,t) \in \mathbb{R} \times [0,T]$ for any $T > 0$. Moreover, for each t it tends to the background constants $0, c^2$ with the first moment of perturbation finite at least (cf. [8]). Note that condition (3) is more restrictive than the decay condition from [1]. In fact, as we see later, namely (3) appears naturally in the domain behind the back wave front, especially in the resonant case. We prove the following

Theorem 1. *Let $q(x,t)$ be the solution of the Cauchy problem (1)-(3). Then for arbitrary small $\epsilon > 0$ in the domain $x < (-6c^2 - \epsilon)t$ the following asymptotics is valid as $t \to \infty$:*

$$q(x,t) = c^2 + \sqrt{\frac{4\nu(a)a}{3t}} \sin(16ta^3 - \nu(a)\log(192ta^3) + \Delta(a)) + o(t^{-\gamma}) \qquad (4)$$

for some $1/2 < \gamma < 1$. Here

$$a = \sqrt{-\frac{c^2}{2} - \frac{x}{12t}}, \quad \nu(a) = -\frac{1}{2\pi}\log\left(1 - |R(a)|^2\right), \qquad (5)$$

$$\Delta(a) = \frac{\pi}{4} + \arg(R(a)) + \arg(\Gamma(\mathrm{i}\nu(a))) + \frac{1}{\pi}\int_{\mathbb{R}\setminus[-a,a]} \log\left(\frac{1-|R(s)|^2}{1-|R(a)|^2}\right)\frac{ds}{s-a},$$

$\Gamma(z)$ is the Gamma-function and $R(k)$ is the left reflection coefficient of the initial data $q_0(x)$.

Remark 0.1. *The radiation part of formula (4) given by the* **left** *scattering data looks almost identical to that one which corresponds to the radiation part in the decaying case ([9]) up to two signs: in front of the left reflection coefficient and* *in front of the integral in $\Delta(a)$. However the investigation of Riemann-Hilbert problems associated with the steplike initial profile has distinctions respectively to the decaying case, especially in the resonant case.*

Note also that in order to simplify presentation we changed notations comparably with [1], omitting some indices.

2 Statement of the RH problem

Recall first briefly some facts from the scattering theory on steplike backgrounds ([4], [5], [7]). Let $q(x,t)$ be the solution of the Cauchy problem (1)-(3). Consider the underlying spectral problem for the operator $H(t) = -\frac{d^2}{dx^2} + q(x,t)$ on the whole axis:

$$(H(t)f)(x) := \lambda f(x), \qquad x \in \mathbb{R}, \tag{6}$$

where $\lambda \in \mathbb{C}$ is the spectral parameter. As is known, the spectrum of the operator $H(t)$ consists of an absolutely continuous part $\mathbb{R}_+$ and a finite number of negative eigenvalues $-\kappa_1^2 < \cdots < -\kappa_N^2 < 0$. The (absolutely) continuous spectrum consists of a part $[0, c^2]$ of multiplicity one and a part $[c^2, \infty)$ of multiplicity two. Instead of λ in equation (6) it is suitable for us to use another spectral parameter $k = \sqrt{\lambda - c^2}$. Here for the square root we choose the standard branch such that the function $k = k(\lambda)$ is a bijection between the domains $\mathbb{C}\backslash\mathbb{R}_+$ and $\mathfrak{D} := \mathbb{C}^+\backslash(0, \mathrm{i}c]$. The solutions of the equation (6) will be considered as functions of the parameter $k \in \overline{\mathfrak{D}} = \mathfrak{D} \cup \partial\mathfrak{D}$. In particular, the equation (6) has two Jost solutions $\phi(k,x,t)$ and $\phi_1(k,x,t)$, satisfying the conditions

$$\lim_{x\to+\infty} \mathrm{e}^{-\mathrm{i}\sqrt{k^2+c^2}x}\phi_1(k,x,t) = \lim_{x\to-\infty} \mathrm{e}^{\mathrm{i}kx}\phi(k,x,t) = 1, \quad k \in \overline{\mathfrak{D}}.$$

The Jost solutions satisfy the scattering relation

$$T(k,t)\phi_1(k,x,t) = \overline{\phi(k,x,t)} + R(k,t)\phi(k,x,t), \quad k \in \mathbb{R},$$

where $T(k,t)$, $R(k,t)$ are the left transmission and reflection coefficients. For the transmission coefficient the following formula is valid $T(k,t) = 2\mathrm{i}kW^{-1}(k,t)$, where

$$W(k,t) := \phi_{1,x}'(k,x,t)\phi(k,x,t) - \phi_1(k,x,t)\phi_x'(k,x,t)$$

is the Wronskian of the Jost solutions. The Wronskian is a holomorphic function in the domain $\mathfrak{D}$, it has continuous limit values at the boundary $\partial\mathfrak{D}$, and on the boundary $\partial\mathfrak{D}$ it never vanishes, except possibly at the point $k = \mathrm{i}c$. At this point there are two options:

(a) If $W(\mathrm{i}c, 0) \neq 0$, then $W(\mathrm{i}c, t) \neq 0$ for any t. In this case we say, that at the point $\mathrm{i}c$ there is no resonance. It is a general situation.

(b) If $W(\mathrm{i}c,0)=0$ (i.e. $W(\mathrm{i}c,t)=0$ for any t), then we deal with the resonance at point ic. Note (cf. [7]), that in this case

$$W(k,t)=C\sqrt{k-\mathrm{i}c}\,(1+o(1)),\quad C=C(t)\neq 0.$$

For the operator $H(t)$ the point ic is the only point where the resonance can happen, that is why we associate with the solution $q(x,t)$ the notion of the resonant or nonresonant cases.

Obviously, the transmission coefficient $T(k,t)$ has a meromorphic extension to the domain $\mathfrak{D}$ with simple poles at the points $\mathrm{i}\kappa_1,\dots,\mathrm{i}\kappa_N$. We set

$$\chi(k,t):=-\lim_{\varepsilon\to+0}\frac{\sqrt{(k+\epsilon)^2+c^2}}{k}|T(k+\varepsilon,t)|^2,\quad k\in[0,\mathrm{i}c].$$

This function is purely imaginary, moreover,

$$\chi(k,t)=\mathrm{i}\,|\chi(k,t)|,\quad k\in[0,\mathrm{i}c].$$

It is continuous on the set $[0,\mathrm{i}c)$ with $\chi(0,t)=0$ and in the nonresonant case

$$\chi(k,t)=C(t)\sqrt{k-\mathrm{i}c}(1+o(1)),\quad k\to\mathrm{i}c,\quad C(t)\neq 0. \tag{7}$$

In the resonant case the function $\chi(k,t)$ has a singularity

$$\chi(k,t)=\frac{C(t)}{\sqrt{k-\mathrm{i}c}}(1+o(1)),\quad k\to\mathrm{i}c,\quad C(t)\neq 0. \tag{8}$$

Next, it is evident that the Jost solutions $\phi(\mathrm{i}\kappa_j,x,t)$ are the eigenfunctions of the operator $H(t)$. Denote the inverse squares of the norms as

$$\gamma_j(t)=\left(\int_{\mathbb{R}}\phi^2(\mathrm{i}\kappa_j,x,t)dx\right)^{-1}.$$

The functions $R(k,t)$, $k\in\mathbb{R}$, and $\chi(k,t)$, $k\in[0,\mathrm{i}c]$, and also the quantities $-\kappa_j^2$, $\gamma_j(t)$, $j=1,\dots,N$ are the left scattering data of the operator $H(t)$. Their evolution due to the KdV flow is the following ([11]): $\gamma_j(t)=\gamma_j\mathrm{e}^{-8\kappa_j^3t+12c^2\kappa_jt}$,

$$\chi(\lambda,t)=\chi(k)\mathrm{e}^{-8\mathrm{i}tk^3-12\mathrm{i}tkc^2},\qquad R(\lambda,t)=R(k)\mathrm{e}^{-8\mathrm{i}tk^2-12\mathrm{i}tkc^2},$$

where we denoted $\chi(k)=\chi(k,0)$, $R(k)=R(k,0)$, and $\gamma_j=\gamma_j(0)$. By means of the Inverse Scattering Transform the solution $q(x,t)$ of the problem (1)–(3) can be uniquely recovered from the left initial scattering data (see [7])

$$\{R(k),\quad k\in\mathbb{R};\quad \chi(k),\ k\in[0,\mathrm{i}c];\quad -\kappa_j^2,\ \gamma_j>0,\quad j=1,\dots,N\}.$$

The properties of the scattering data listed above allow us to formulate the vector RH problem associated with the left scattering data. Namely, in $\mathfrak{D}$ we introduce a meromorphic vector function (variables x and t are treated as parameters)

$$\tilde{m}(k) = (\tilde{m}_1(k), \tilde{m}_2(k)) = \Big(T(k,t)\phi_1(k,x,t)\mathrm{e}^{-\mathrm{i}kx}, \quad \phi(k,x,t)\mathrm{e}^{\mathrm{i}kx}\Big). \tag{9}$$

This function has the following expansion as $k \to \infty$ (cf. [1])

$$\tilde{m}(k) = \begin{pmatrix}1 & 1\end{pmatrix} + \frac{1}{2\mathrm{i}k}\left(\int_{-\infty}^{x}(q(y,t)-c^2)dy\right)\begin{pmatrix}1 & -1\end{pmatrix} + O\left(\frac{1}{k^2}\right), \tag{10}$$

and therefore $\tilde{m}$ is bounded at infinity. The only singularities of this vector function in $\mathfrak{D}$ are the poles of its first component $\tilde{m}_1(k)$ at points $\mathrm{i}\kappa_j$. Out of these poles the function $\tilde{m}$ is continuous up to the boundary $\partial\mathfrak{D}$ except, probably, at the point $\mathrm{i}c$ in the resonant case. Let us extend $\tilde{m}$ to the domain $\mathfrak{D}^* = \{k : -k \in \mathfrak{D}\}$ by the symmetry condition $\tilde{m}(-k) = \tilde{m}(k)\sigma_1$, where $\sigma_1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ is the first Pauli matrix. After this extension the function $\tilde{m}(k)$ has poles in points $-\mathrm{i}\kappa_j$ (its second component) and jumps along the real axis and along the segment $[\mathrm{i}c, -\mathrm{i}c]$. Introduce cross-shaped contour $\Sigma := \mathbb{R} \cup [\mathrm{i}c, -\mathrm{i}c]$ with a natural orientation from minus to plus infinity on $\mathbb{R}$, and from up to down on $[\mathrm{i}c, -\mathrm{i}c]$. Denote by $\tilde{m}_+(k)$ (resp. $\tilde{m}_-(k)$) the limiting nontangential values of $\tilde{m}(k)$ from the right (resp. left) in the contour direction.

To simplify notations throughout of this paper along with the first Pauli matrix σ_1 we use also the third Pauli matrix $\sigma_3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ and three more matrices:

$$\mathbb{I} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \quad \mathbb{J} := \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}, \quad \mathbb{J}^\dagger := \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} = \sigma_1\mathbb{J}\sigma_1.$$

Let now $\mathbb{T}_j$ (resp., $\mathbb{T}_j^*$) be circles centred at $\mathrm{i}\kappa_j$ (resp., $-\mathrm{i}\kappa_j$) with radii $0 < \delta < \frac{1}{4}\min_{j=1}^N |\kappa_j - \kappa_{j-1}|$, $\kappa_0 = 0$. Choose $\delta > 0$ so small that the discs $|k - \mathrm{i}\kappa_j| < \delta$ lie inside the upper half-plane and do not intersect any of the other contours, moreover $\kappa_1 - \delta > \kappa + c$, where κ is the same as in estimate (3). The small circles $\mathbb{T}_j$ around $\mathrm{i}\kappa_j$ are oriented counterclockwise, and the circles $\mathbb{T}_j^*$ around $-\mathrm{i}\kappa_j$ are oriented clockwise.

Introduce also the phase function $\Phi(k) = \Phi(k,x,t)$:

$$\Phi(k) = -4\mathrm{i}k^3 - 6\mathrm{i}c^2k - 12\mathrm{i}\xi k, \quad \xi = \frac{x}{12t}.$$

This function is odd in $\mathbb{C}$. Its stationary points are $\pm a$, where $a := \sqrt{-\frac{c^2}{2} - \xi}$. The signature table for $\operatorname{Re}\Phi(k)$ when $\xi < -\frac{c^2}{2}$ is shown in Figure 1.

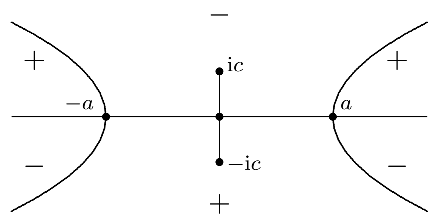

Fig. 1: The signature table of $\mathrm{Re}(\Phi(k))$

Redefine now $\tilde m(k)$ inside $\mathbb{T}_j, \mathbb{T}_j^*,\ j=1,...,N$ according to

$$m(k)=\begin{cases}\tilde m(k)A_j(k), & |k-\mathrm{i}\kappa_j|<\delta,\\ \tilde m(k)\sigma_1 A_j^{-1}(-k)\sigma_1, & |k+\mathrm{i}\kappa_j|<\delta,\\ \tilde m(k), & \text{else},\end{cases} \tag{11}$$

where

$$A_j(k)=\begin{pmatrix}1 & 0\\ -\frac{\mathrm{i}\gamma_j \mathrm{e}^{2t\Phi(\mathrm{i}\kappa_j)}}{k-\mathrm{i}\kappa_j} & 1\end{pmatrix}=\mathbb{I}-\frac{\mathrm{i}\gamma_j\mathrm{e}^{2t\Phi(\mathrm{i}\kappa_j)}}{k-\mathrm{i}\kappa_j}\mathbb{J}.$$

Thus $m(k)$ becomes holomorphic but with additional jumps along the circles $\mathbb{T}_j, \mathbb{T}_j^*,\ j=1,...,N$. Moreover, it preserves the asymptotics (10) of $\tilde m(k)$ as $k\to\infty$.

Theorem 2. *Let $\{R(k),\ k\in\mathbb{R};\ \chi(k),\ k\in[0,\mathrm{i}c];\ (\kappa_j,\gamma_j),\ 1\le j\le N\}$ be the left scattering data of the operator $H(0)$. Then the vector function $m(k)=m(k,x,t)$ defined by (9), (11) is the unique solution of the following vector Riemann–Hilbert problem:*

Find a vector function $m(k)$ which is holomorphic away from the contour $\Sigma=\bigcup_{j=1}^N(\mathbb{T}_j\cup\mathbb{T}_j^*)\cup\mathbb{R}\cup[-\mathrm{i}c,\mathrm{i}c]$, has continuous limiting values from both sides of the contour, except possibly of points $\pm\mathrm{i}c$, and satisfies:

A. *The jump condition $m_+(k)=m_-(k)v(k)$, where*

$$v(k)=\begin{cases}\begin{pmatrix}1-|R(k)|^2 & -\overline{R(k)}\mathrm{e}^{-2t\Phi(k)}\\ R(k)\mathrm{e}^{2t\Phi(k)} & 1\end{pmatrix}, & k\in\mathbb{R},\\ \begin{pmatrix}1 & 0\\ \chi(k)\mathrm{e}^{2t\Phi(k)} & 1\end{pmatrix}=\mathbb{I}+\chi(k)\mathrm{e}^{2t\Phi(k)}\mathbb{J}, & k\in[\mathrm{i}c,0],\\ A_j(k), & k\in\mathbb{T}_j,\quad k=1,...,N,\\ \sigma_1 v^{-1}(-k)\sigma_1, & k\in\cup_{j=1}^N\mathbb{T}_j^*\cup[0,-\mathrm{i}c];\end{cases}$$

B. *The symmetry condition*

$$m(-k) = m(k)\sigma_1; \tag{12}$$

C. *The normalization condition* $\lim_{\kappa\to\infty} m(\mathrm{i}\kappa) = \begin{pmatrix} 1 & 1 \end{pmatrix}$;

D. *In vicinities of points* $\pm\mathrm{i}c$*:*
 (a) *if* $\chi(k)$ *satisfies* (7)*, then* $m(k)$ *is continuous in the points* $\pm\mathrm{i}c$*;*
 (b) *under condition* (8)*,*

$$m(k) = \left(\frac{C_1}{\sqrt{k - \mathrm{i}c}}, \quad C_2 \right)(1 + o(1)), \quad k \to \mathrm{i}c, \quad C_1 \neq 0,$$

with a similar condition at $-\mathrm{i}c$ *due to* (12)*.*

P r o o f. This theorem follows from the union of the uniqueness result from [2] and a slight modification of the proof of Theorem 2.5 from [1] for the resonant case. ∎

3 Reduction to the model problem

In this section we describe some conjugation/deformation steps as $\xi < -c^2/2$ for the RH problem **A-D** which lead to an equivalent RH problem with the jump matrix close to the unitary matrix $\mathbb{I}$ for large time except of the small vicinities of points $\pm a$. A short description of these steps was proposed in Section 8 of [1]. We extend the respective analysis taking into account the resonant case.

According to the signature table of the phase function (see Figure 1), the matrix $v(k)$ is already exponentially close for large t to $\mathbb{I}$ on the segments $[-\mathrm{i}c, 0) \cup (0, \mathrm{i}c]$ and on the circles $\cup_{j=1}^{N}(\mathbb{T}_j \cup \mathbb{T}_j^*)$, but it is oscillatory with respect to t on the real axis. Besides one can have singularities of $v(k)$ at points $\pm\mathrm{i}c$. As a first step we apply the standard upper–lower and lower–upper factorization ([9], [6]) to the matrix $v(k)$ as $k \in \mathbb{R}$. To this end we construct an analytic in the domain $\mathbb{C} \setminus ((-\infty, -a) \cup (a, \infty))$ function $d(k)$ satisfying the jump condition

$$d_+(k) = d_-(k)(1 - |R(k)|^2) \text{ for } k \in \mathbb{R} \setminus [-a, a],$$

and such that $d(-k) = d^{-1}(k)$ and $d(k) \to 1$ as $k \to \infty$. By the Sokhotski–Plemelj formula this function is explicitly given by

$$d(k) = \exp\left(\frac{1}{2\pi\mathrm{i}} \int_{\mathbb{R}\setminus[-a,a]} \frac{\log(1 - |R(s)|^2)}{s - k} ds \right). \tag{13}$$

Since the domain of integration is even and the function $\log(1-|R(s)|^2)$ is also even, then $d(-k)=d^{-1}(k)$. For $k\to\infty$ we have

$$d(k)=1-\frac{1}{2\pi\mathrm{i}k}\int_{\mathbb{R}\setminus[-a,a]}\log(1-|R(s)|^2)ds+O\left(\frac{1}{k^2}\right). \tag{14}$$

Put $m^{(1)}(k)=m(k)d(k)^{-\sigma_3}$. Evidently $m^{(1)}(-k)=m^{(1)}(k)\sigma_1$. One can check that (see e.g. [9]) $m^{(1)}(k)$ satisfies the jump condition $m_+^{(1)}(k)=m_-^{(1)}(k)v^{(1)}(k)$ with

$$v^{(1)}(k)=\begin{pmatrix}1-|R(k)|^2 & -\overline{R(k)}d^2(k)\mathrm{e}^{-2t\Phi(k)}\\ R(k)d^{-2}(k)\mathrm{e}^{2t\Phi(k)} & 1\end{pmatrix},\quad k\in[-a,a],$$

$$v^{(1)}(k)=\begin{pmatrix}(1-|R(k)|^2)d_+^{-1}(k)d_-(k) & -\overline{R}(k)d_+(k)d_-(k)\mathrm{e}^{-2t\Phi(k)}\\ R(k)d_+^{-1}(k)d_-^{-1}(k) & d_-^{-1}(k)d_+(k)\end{pmatrix},\ k\in\mathbb{R}\setminus[-a,a],$$

$$v^{(1)}(k)=d(k)^{\sigma_3}v(k)d(k)^{-\sigma_3},\quad k\in\cup_{j=1}^N(\mathbb{T}_j^U\cup\mathbb{T}_j^L)\cup[\mathrm{i}c,-\mathrm{i}c].$$

Recall that $\overline{R(k)}=R(-k)$ for $k\in\mathbb{R}$. Under condition (3) one can continue function $R(k)$ in a vicinity of of the contour Σ. Introduce the domains Ω_l^*, Ω_l, Ω_r^*, Ω_r, Ω^*, and Ω together with their boundaries $\mathcal{C}_l^*$, $\mathcal{C}_l$, $\mathcal{C}_r^*$, $\mathcal{C}_r$, $\mathcal{C}^*$, and $\mathcal{C}$, which are contained in the strip $\{k:|\operatorname{Im}k|<c+\epsilon\}$ as depicted in Figure 2. Introduce

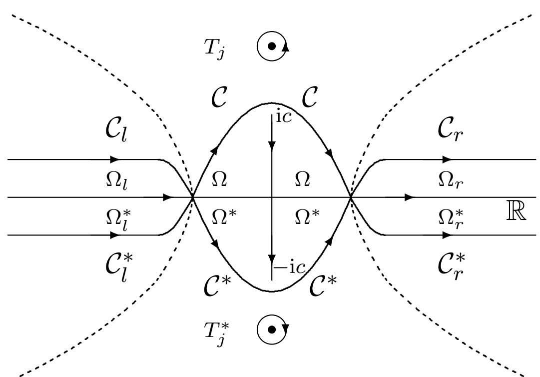

Fig. 2: Contour deformation in the domain $x<-6c^2t$

the following matrices

$$\begin{aligned} B(k) :=&\mathbb{I}+R(k)d^{-2}(k)\mathrm{e}^{2t\Phi(k)}\mathbb{J},\quad k\in\Omega,\\ B^*(k) :=&\mathbb{I}-\overline{R(k)}d^2(k)\mathrm{e}^{-2t\Phi(k)}\mathbb{J}^\dagger,\quad k\in\Omega^*,\\ A(k) :=&\mathbb{I}+\frac{\overline{R(k)}d^2(k)}{1-|R(k)|^2}\mathrm{e}^{-2t\Phi(k)}\mathbb{J}^\dagger,\quad k\in\Omega_r\cup\Omega_l,\\ A^*(k) :=&\mathbb{I}-\frac{R(k)d^{-2}(k)}{1-|R(k)|^2}\mathrm{e}^{2t\Phi(k)}\mathbb{J},\quad k\in\Omega_r^*\cup\Omega_l^*. \end{aligned}\tag{15}$$

Then

$$v^{(1)}(k) = \begin{cases} B_-^*(k)B_+(k), & k \in [-a,a], \\ A_-^*(k)A_+(k), & k \in \mathbb{R} \setminus [-a,a]. \end{cases}$$

Redefine $m^{(1)}(k)$ according to

$$m^{(2)}(k) = m^{(1)}(k) \begin{cases} B(k), & k \in \Omega, \\ B^*(k), & k \in \Omega^*, \\ A(k), & k \in \Omega_l \cup \Omega_r, \\ A^*(k), & k \in \Omega_l^* \cup \Omega_r^*, \\ \mathbb{I}, & \text{else}. \end{cases}$$

Lemma 1. *The following formulas are valid*

$$B_-(k)\, v^{(1)}(k)\, (B_+(k))^{-1} = \mathbb{I}, \quad k \in [\mathrm{i}c, 0];$$
$$(B_-^*(k))^{-1}\, v^{(1)}(k)\, B_+^*(k) = \mathbb{I}, \quad k \in [0, -\mathrm{i}c].$$

P r o o f. We observe that for $k \in [\mathrm{i}c, 0]$:

$$B_-(k)\, v^{(1)}(k)\, B_+(k)^{-1} = \begin{pmatrix} 1 & 0 \\ d(k)^{-2}(R_-(k) - R_+(k) + \chi(k))\mathrm{e}^{2t\Phi(k)} & 1 \end{pmatrix}.$$

As is known, under condition (3) the complex conjugated Jost solution $\overline{\phi}(k,x,t)$ can be continued analytically into a strip. Denote this continuation as $\breve{\phi}(k)$. It does not have a jump along the interval $[\mathrm{i}c, 0]$. Then the continuation of $R(k)$ can be represented via wronskians in a usual way ([7]). Let $\phi_1(k) := \lim_{\epsilon \to +0} \phi_1(k + \epsilon, x, t)$, then

$$R_-(k) = -\frac{\langle \overline{\phi}_1, \breve{\phi} \rangle}{\langle \overline{\phi}_1, \phi \rangle},\ R_+(k) = -\frac{\langle \phi_1, \breve{\phi} \rangle}{\langle \phi_1, \phi \rangle},\ \chi(k) = -\frac{\langle \phi, \breve{\phi} \rangle}{\langle \overline{\phi}_1, \breve{\phi} \rangle} \frac{\langle \phi_1, \overline{\phi}_1 \rangle}{\langle \phi_1, \phi \rangle},$$

where $\langle f, g \rangle$ is the usual Wronskian of two solutions of (6). Applying the Plücker identity (cf. [14]):

$$\langle f_1, f_2 \rangle \cdot \langle f_3, f_4 \rangle + \langle f_1, f_3 \rangle \cdot \langle f_4, f_2 \rangle + \langle f_1, f_4 \rangle \cdot \langle f_2, f_3 \rangle \equiv 0$$

to the functions $f_1 = \phi_1, f_2 = \phi, f_3 = \overline{\phi}_1, f_4 = \breve{\phi}$ we get

$$R_-(k) - R_+(k) + \chi(k) \equiv 0, \tag{16}$$

which proves the first identity of the Lemma. The proof of the second one is analogous. ■

Remark 0.2. *Equality* (16) *shows that independently of the resonant or nonresonant cases the jump of the vector function* $m^{(2)}(k)$ *along the interval* $[\mathrm{i}c, -\mathrm{i}c]$ *simply disappears, and therefore the final asymptotics will not depend on the resonance.*

By use of Lemma 1 we conclude that the vector function $m^{(2)}(k)$ satisfies the jump $m_+^{(2)}(k) = m_-^{(2)}(k)v^{(2)}(k)$ with

$$v^{(2)}(k) = \begin{cases} B(k), & k \in \mathcal{C}, \\ B^*(k), & k \in \mathcal{C}^*, \\ A(k), & k \in \mathcal{C}_l \cup \mathcal{C}_r, \\ A^*(k), & k \in \mathcal{C}_l^* \cup \mathcal{C}_r^*, \\ v^{(1)}(k), & k \in \cup_{j=1}^N (\mathbb{T}_j \cup \mathbb{T}_j^*). \end{cases} \tag{17}$$

Thus matrix $v^{(2)}(k)$ has the structure

$$v^{(2)}(k) = \mathbb{I} + \begin{cases} F_1(k), & k \in \cup_{j=1}^N (\mathbb{T}_j \cup \mathbb{T}_j^*), \\ F_2(k), & k \in \mathcal{C}_l \cup \mathcal{C} \cup \mathcal{C}_r \cup \mathcal{C}_l^* \cup \mathcal{C}^* \cup \mathcal{C}_r^*, \end{cases}$$

with the matrices $F_{1,2}(k)$ admitting the estimates

$$||F_1(k)|| \le C\mathrm{e}^{-Ct}, \quad ||F_2(k)|| \le C(a)\mathrm{e}^{-t\mu(|k^2-a^2|)}, \tag{18}$$

where $\|\cdot\|$ is any norm of the matrix 2×2, $C > 0$, $C(a) > 0$ and $\mu(s)$, $s \in \mathbb{R}_+$, is a strictly increasing continuous function with $\mu(0) = 0$ and $\mu(s) = O(s^{3/4})$ as $s \to \infty$. Note that the vector function $m^{(2)}(k)$ has no jump along the contour Σ and therefore, the effect of resonance is not noticeable for $\xi < -c^2/2$. Due to (18) we can conclude that $m^{(2)}(k) \sim \begin{pmatrix} 1 & 1 \end{pmatrix}$ as $k \to \infty$. As it will be shown in the next section an error term has the structure $(1 \quad -1)O(k^{-1})O(t^{-1/2})$. Remind that for large imaginary k with $|k| > \kappa_1 + 1$ we have $\tilde m(k) = m^{(2)}(k)d(k)^{\sigma_3}$ with $d(k)$ defined by (13). By use of (14) one can expect that for $k \to \infty$

$$\tilde m_1(k) = m_1(k) \sim d(k) = 1 - \frac{\int_{\mathbb{R}\setminus[-a,a]} \log(1 - |R(s)|^2)ds}{2\pi \mathrm{i} k} + \frac{g(x,t)}{k} + O\left(\frac{1}{k^2}\right),$$

where $g(x,t) = o(1), g_x(x,t) = o(1)$ as $t \to \infty$ uniformly with respect to x. Function $g(x,t)$ appears from the effect of parametrix in small vicinities of points $\pm a$. A formula for this function will be obtained in the Section 4. Next, by (10)

$$q(x,t) = \frac{\partial}{\partial x} \lim_{k\to\infty} 2\mathrm{i}k\,(\tilde m_1(k) - 1). \tag{19}$$

Since $\frac{\partial}{\partial x} a(\xi) = O(t^{-1})$ then from (3) it follows that that after differentiation the integral from the r. h.s will be of order $O(t^{-1})$. Respectively,

$$q(x,t) = c^2 + o(1), \quad \text{as } t \to \infty,$$

so the leading term is equal to c^2 as expected. In the next section we show that the effect of the parametrix points implies in fact the term of order $O\left(t^{-1/2}\right)$.

4 The parametrix problem

We use the same approach as in [6] and [13], but for the vector RH problem as in [9]. Following these approaches we start with investigation in more details of the behavior of the jump matrix $v^{(2)}(k)$ near the point $-a$. Represent (13) as

$$\log d(k) = \frac{1}{2\pi\mathrm{i}}\int_{\mathbb{R}\setminus[-a,a]} \log\frac{1-|R(s)|^2}{1-|R(-a)|^2}\frac{ds}{s-k} + \frac{\log(1-|R(-a)|^2)}{2\pi\mathrm{i}}\int_{\mathbb{R}\setminus[-a,a]}\frac{ds}{s-k}.$$

Since $\int_{\mathbb{R}\setminus[-a,a]}\frac{ds}{s-k} = \log\frac{k+a}{a-k}$, then

$$d(k) = \left(\frac{k+a}{a-k}\right)^{\mathrm{i}\nu}\cdot \mathrm{e}^{\eta(k)},$$

where

$$\nu := \nu(a) = -\frac{1}{2\pi}\log(1-|R(-a)|^2),$$

$$\eta(k) := \eta(k,a) = \frac{1}{2\pi\mathrm{i}}\int_{\mathbb{R}\setminus[-a,a]} \log\frac{1-|R(s)|^2}{1-|R(-a)|^2}\frac{ds}{s-k}. \tag{20}$$

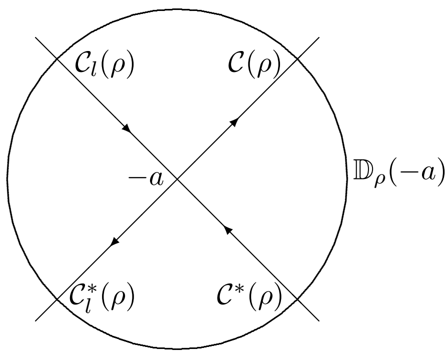

Fig. 3: The contour near the point $-a$

Let $\mathbb{D}_\rho(-a)$ be a circle of the radius $0<\rho<\inf\{\frac{1}{4},\frac{a}{4}\}$ centred at the point $-a$. Without loss generality one can assume that inside the domain $\mathbb{D}_\rho(-a)$ the contours $\mathcal{C}(\rho) := \mathcal{C}\cap\mathbb{D}_\rho(-a)$, $\mathcal{C}^*(\rho) := \mathcal{C}^*\cap\mathbb{D}_\rho(-a)$, $\mathcal{C}_l(\rho) := \mathcal{C}_l\cap\mathbb{D}_\rho(-a)$, $\mathcal{C}_l^*(\rho) := \mathcal{C}_l^*\cap\mathbb{D}_\rho(-a)$ are the parts of rays $\{-a+s\mathrm{e}^{\mathrm{i}(2n+1)\pi/4},\ \ s\in\mathbb{R}_+\}$ and they have orientations as depicted on Fig. 3. Denote by

$$\Gamma_\rho(-a) := \mathcal{C}(\rho)\cup\mathcal{C}^*(\rho)\cup\mathcal{C}_l(\rho)\cup\mathcal{C}_l^*(\rho).$$

Lemma 2. *The following inequalities hold for all $k \in \Gamma_\rho(-a)$ and $a > \epsilon$, where ϵ is a constant from Theorem 1 :*

$$\left| \mathrm{e}^{-2\eta(k)} - \mathrm{e}^{-2\eta(-a)} \right| \leq C|k+a|(1 + |\log|k+a||), \tag{21}$$

$$\left| 1 - \mathrm{e}^{-2\mathrm{i}\nu \log \frac{k-a}{2a}} \right| \leq C a^{-1} |k+a|, \tag{22}$$

where the constant $C = C(\epsilon)$ does not depend on ξ and k.

P r o o f. We give the proof for $k \in \mathcal{C}(\rho)$. The other cases are similar. First we show that

$$|\eta(k) - \eta(-a)| \leq C|k+a|(1 + |\log|k+a||), \quad a \in I, \quad k \in \mathcal{C}(\rho). \tag{23}$$

Divide the domain of integration in (20) into three parts $[-\infty, -2a]$, $[-2a, -a]$, $[a, \infty]$, and denote by $I_1(k)$, $I_2(k)$, $I_3(k)$ the respective integrals. For $k \in \mathcal{C}(\rho)$ the following estimates are straightforward:

$$|I_1(k) - I_1(-a)| \leq C|k+a|, \quad |I_3(k) - I_3(-a)| \leq C|k+a|. \tag{24}$$

Integrating $2\pi\mathrm{i} I_2(\xi, k)$ by parts:

$$\int_{-2a}^{a} \log \frac{1 - |R(s)|^2}{1 - |R(-a)|^2} \frac{ds}{s-k} =$$
$$- \log \frac{1 - |R(-2a)|^2}{1 - |R(-a)|^2} \log(-2a - k) - \int_{-2a}^{a} \log(s-k) d \log(1 - |R(s)|^2),$$

we get

$$\begin{aligned} |I_2(k) - I_2(-a)| = & \frac{1}{2\pi} \left| \log \frac{k+2a}{a} \log \frac{1 - |R(-2a)|^2}{1 - |R(-a)|^2} \right. \\ & \left. + \int_{-2a}^{-a} \log \frac{s-k}{s+a} d \log(1 - |R(s)|^2) \right|. \end{aligned}$$

Since $|R(s)| \leq C(\epsilon) < 1$ as $|s| > \epsilon$ then

$$|I_2(k) - I_2(-a)| \leq C(\epsilon) \left(\left| \int_{-2a}^{-a} \left| \log \frac{s-k}{s+a} \right| ds \right| + |k+a| \right).$$

The change of variables $v = -|k+a|/(s+a)$ gives

$$\left| \int_{-2a}^{-a} \log \left| \frac{s-k}{s+a} \right| ds \right| = |k+a| \left| \int_{\frac{|k+a|}{a}}^{\infty} \log(1 + v \mathrm{e}^{\mathrm{i}\frac{\pi}{4}}) \frac{dv}{v^2} \right|,$$

where we took into account that $k \in \mathcal{C}(\rho)$. Combining this estimate with estimate

$$|\log|1 + v\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}|| \leq C \begin{cases} v, & 0 \leq v \leq 2, \\ \log v, & 2 \leq v \leq \infty, \end{cases}$$

and with (24) we get (23). Next, by Lemma 23.2 in [3]:

$$\sup_{\xi<(-c^2/2-\epsilon)} \sup_{k\in\mathbb{C}\setminus\mathbb{R}} |\eta(k)| < \infty.$$

Using this, (23) and inequality $|\mathrm{e}^{w} - 1| \leq |w| \max(1, \mathrm{e}^{\mathrm{Re}\, w})$, $\quad w \in \mathbb{C}$, we get (21) and also

$$\begin{aligned} \left|1 - \mathrm{e}^{-2\nu \log \frac{k+a}{2a}}\right| &\leq \left|2\nu \log \frac{k+a}{2a}\right| \mathrm{e}^{|\,\mathrm{Re}(2\mathrm{i}\nu \log \frac{k+a}{2a})|} \\ &\leq C \left|\log\left(1 + \frac{k-a}{2a}\right)\right| \leq C a^{-1}|k+a|. \end{aligned}$$

This proves (22). ■

Introduce a local parameter $z = \sqrt{48a}(k+a)$. Then $z \in \mathbb{D}_{\rho_1}$, where $\mathbb{D}_{\rho_1}$ is the circle of the radius $\rho_1 = \sqrt{48a}\,\rho$ centred at 0. The contour $\Gamma_\rho(-a)$ in terms of the variable z will have notation Γ_{ρ_1}, and for the constituents of this contour we will keep notations $\mathcal{C}$ and so on. Taking into account (5) put $\varphi(z) := -8\mathrm{i}a^3 + \frac{\mathrm{i}}{4}z^2$,

$$\begin{aligned} r_1(z) :=& \tilde{R}(z)\mathrm{e}^{-2\tilde{\eta}(z)}\mathrm{e}^{2\mathrm{i}\nu\log(2a\sqrt{48a}-z)}, \\ r_2(z) :=& \frac{\overline{\tilde{R}(z)}}{1-|\tilde{R}(z)|^2}\mathrm{e}^{2\tilde{\eta}(z)}\mathrm{e}^{-2\mathrm{i}\nu\log(2a\sqrt{48a}-z)}, \\ r_3(z) :=& \frac{\tilde{R}(z)}{1-|\tilde{R}(z)|^2}\mathrm{e}^{-2\tilde{\eta}(z)}\mathrm{e}^{2\mathrm{i}\nu\log(2a\sqrt{48a}-z)}, \\ r_4(z) :=& \overline{\tilde{R}(z)}\mathrm{e}^{2\tilde{\eta}(z)}\mathrm{e}^{-2\mathrm{i}\nu\log(2a\sqrt{48a}-z)}. \end{aligned}$$

where $\tilde{R}(z) := R(k(z))$, $\tilde{\eta}(z) := \eta(k(z))$. The phase function is represented as

$$\tilde{\Phi}(z) := \Phi(k(z)) = \varphi(z) - \frac{\mathrm{i}z^3}{12a\sqrt{48a}}.$$

From (15) and (25) it follows that the jump matrix $v^{(2)}(k)$ as a function of the variable $z \in \Gamma_{\rho_1}$ has the form

$$\tilde{v}^{(2)}(z) = \mathbb{I} + \begin{cases} r_1(z)z^{-2\mathrm{i}\nu}\mathrm{e}^{2t\tilde{\Phi}(z)}\mathbb{J}, & z \in \mathcal{C}, \\ -r_2(z)z^{2\mathrm{i}\nu}\mathrm{e}^{-2t\tilde{\Phi}(z)}\mathbb{J}^\dagger, & z \in \mathcal{C}_l, \\ -r_3(z)z^{-2\mathrm{i}\nu}\mathrm{e}^{2t\tilde{\Phi}(z)}\mathbb{J}, & z \in \mathcal{C}_l^*, \\ r_4(z)z^{2\mathrm{i}\nu}\mathrm{e}^{-2t\tilde{\Phi}(z)}\mathbb{J}^\dagger, & z \in \mathcal{C}^*. \end{cases} \tag{25}$$

Put now

$$f := f(a) = R(-a)\mathrm{e}^{-2\eta(-a)}\mathrm{e}^{2\mathrm{i}\nu(a)\log(2a\sqrt{48a})}. \tag{26}$$

Since $\nu \in \mathbb{R}$ and $\eta(-a) \in \mathrm{i}\mathbb{R}$, then $|f| = |R(-a)|$. From Lemma 2 it follows that for $z \in \mathbb{D}_{\rho_1}$ the functions $\{r_j(z)\}_1^4$ satisfy inequalities:

$$\begin{aligned} |r_1(z) - f| &\le C(\epsilon)|z|^\alpha, \quad z \in \mathcal{C}, \\ \left| r_2(z) - \frac{\overline{f}}{1-|f|^2} \right| &\le C(\epsilon)|z|^\alpha, \quad z \in \mathcal{C}_l, \\ \left| r_3(z) - \frac{f}{1-|f|^2} \right| &\le C(\epsilon)|z|^\alpha, \quad z \in \mathcal{C}_l^*, \\ \left| r_4(z) - \overline{f} \right| &\le C(\epsilon)|z|^\alpha, \quad z \in \mathcal{C}^*, \end{aligned} \tag{27}$$

where $\alpha < 1$ can be choose arbitrary close to 1. Now we are ready to formulate an auxiliary RH problem in the domain $\mathbb{D}_{\rho_1}$, which is called the parametrix problem. We are looking for a holomorphic in $\mathbb{D}_{\rho_1} \setminus \Gamma_{\rho_1}$ matrix function $M^{par}(z)$ satisfying the jump condition

$$M_+^{par}(z) = M_-^{par}(z)v^{par}(z), \quad z \in \Gamma_{\rho_1}, \quad \text{with} \tag{28}$$

$$v^{par}(z) := \mathbb{I} + \begin{cases} f z^{-2\mathrm{i}\nu}\mathrm{e}^{2t\varphi(z)}\mathbb{J}, & z \in \mathcal{C}, \\ \overline{f} z^{2\mathrm{i}\nu}\mathrm{e}^{-2t\varphi(z)}\mathbb{J}^\dagger, & z \in \mathcal{C}^*, \\ -\frac{\overline{f}}{1-|f|^2} z^{2\mathrm{i}\nu}\mathrm{e}^{-2t\varphi(z)}\mathbb{J}^\dagger, & z \in \mathcal{C}_l, \\ -\frac{f}{1-|f|^2} z^{-2\mathrm{i}\nu}\mathrm{e}^{2t\varphi(z)}\mathbb{J}, & z \in \mathcal{C}_l^*. \end{cases} \tag{29}$$

and the boundary condition $M^{par}(z) \sim \mathbb{I}$,as $z \in \partial\mathbb{D}_{\rho_1}$.

This problem is solved in [9] and [13]. We recall briefly the main steps in the construction of its solution. Denote $\zeta = \sqrt{t}\, z$. We study the parametrix problem solution for large t. Consider first another auxiliary matrix RH-problem in the domain $\mathbb{C}\backslash Y$, where $Y = Y_1 \cup Y_2 \cup Y_3 \cup Y_4$ and $Y_i = \{s\mathrm{e}^{\mathrm{i}(2n+1)\pi/4}, \ \ s \in \mathbb{R}_+\}$ are the contours as depicted in Fig. 4. Let $M^Y(\zeta)$ solve the following problem:

$$M^Y(\zeta) \to \mathbb{I}, \quad \zeta \to \infty, \tag{30}$$

$$M_+^Y(\zeta) = M_-^Y(\zeta)v^Y(\zeta), \quad \zeta \in Y, \tag{31}$$

where the jump matrix $v^Y(\zeta)$ is defined by

$$v^Y(\zeta) := \mathbb{I} + \begin{cases} f\zeta^{-2\mathrm{i}\nu}\mathrm{e}^{\frac{\mathrm{i}\zeta^2}{2}}\mathbb{J}, & \zeta \in Y_1, \\ -\frac{\overline{f}}{1-|f|^2}\zeta^{2\mathrm{i}\nu}\mathrm{e}^{\frac{-\mathrm{i}\zeta^2}{2}}\mathbb{J}^\dagger, & \zeta \in Y_2, \\ -\frac{f}{1-|f|^2}\zeta^{-2\mathrm{i}\nu}\mathrm{e}^{\frac{\mathrm{i}\zeta^2}{2}}\mathbb{J}, & \zeta \in Y_3, \\ \overline{f}\zeta^{2\mathrm{i}\nu}\mathrm{e}^{-\frac{\mathrm{i}\zeta^2}{2}}\mathbb{J}^\dagger, & \zeta \in Y_4. \end{cases} \tag{32}$$

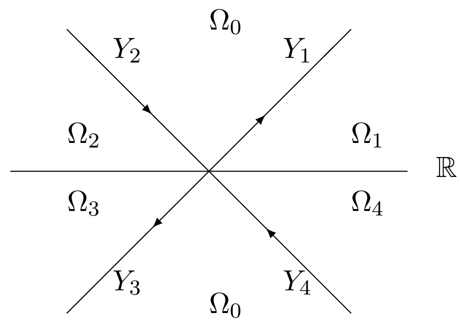

Fig. 4: The sets Ω_j and the rays Y_i, $j = 1, ..., 4$.

Following ([13]) define a sectionally analytic function $\tilde{M}^Y(\zeta)$ by

$$\tilde{M}^Y(\zeta) := \begin{pmatrix} \psi_{11}(\zeta) & \frac{(\frac{d}{d\zeta} - \frac{\mathrm{i}\zeta}{2})\psi_{22}(\zeta)}{\overline{\beta}} \\ \frac{(\frac{d}{d\zeta} + \frac{\mathrm{i}\zeta}{2})\psi_{11}(\zeta)}{\beta} & \psi_{22}(\zeta) \end{pmatrix}, \quad \zeta \in \mathbb{C}\backslash\mathbb{R},$$

where $\beta = \beta(a)$ is given by

$$\beta := \sqrt{\nu(a)} \mathrm{e}^{\mathrm{i}(\frac{\pi}{4} - \arg f(a) + \arg \Gamma(\mathrm{i}\nu(a)))}, \tag{33}$$

and the functions ψ_{11}, ψ_{22} are defined by

$$\psi_{11}(\zeta) = \begin{cases} \mathrm{e}^{-\frac{3\pi\nu}{4}} D_{\mathrm{i}\nu}(\mathrm{e}^{-\frac{3\mathrm{i}\pi}{4}}\zeta), & \operatorname{Im}\zeta > 0, \\ \mathrm{e}^{\frac{\pi\nu}{4}} D_{\mathrm{i}\nu}(\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}\zeta), & \operatorname{Im}\zeta < 0, \end{cases}$$

$$\psi_{22}(\zeta) = \begin{cases} \mathrm{e}^{\frac{\pi\nu}{4}} D_{-\mathrm{i}\nu}(\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}}\zeta), & \operatorname{Im}\zeta > 0, \\ \mathrm{e}^{-\frac{3\pi\nu}{4}} D_{-\mathrm{i}\nu}(\mathrm{e}^{\frac{3\mathrm{i}\pi}{4}}\zeta), & \operatorname{Im}\zeta < 0. \end{cases}$$

Here $D_s(z)$ denotes the parabolic cylinder function. Then ([13]) the solution $M^Y(\zeta)$ of matrix the RH-problem (30)-(32) is the following

$$M^Y(\zeta) = \tilde{M}^Y(\zeta) D_j(\zeta), \quad \zeta \in \Omega_j, \quad j = 0, ..., 4,$$

where $D_0(\zeta) = \zeta^{-\mathrm{i}\nu\sigma_3} \mathrm{e}^{\frac{\mathrm{i}\zeta^2}{4}\sigma_3}$ and

$$D_1(\zeta) = (\mathbb{I} - f\mathbb{J})D_0(\zeta), \quad D_2(\zeta) = (\mathbb{I} + \frac{\overline{f}}{1 - |f|^2}\mathbb{J}^\dagger)D_0(\zeta),$$
$$D_3(\zeta) = (\mathbb{I} + \frac{f}{1 - |f|^2}\mathbb{J})D_0(\zeta), \quad D_4(\zeta) = (\mathbb{I} - \overline{f}\mathbb{J}^\dagger)D_0(\zeta).$$

The matrix $M^Y(\zeta)$ is analytic for $\zeta \in \mathbb{C}\backslash Y$ and satisfies the jump condition $M^Y_+(\zeta) = M^Y_-(\zeta)v^Y(\zeta)$, where $v^Y(\zeta)$ defined by (32). Also $M^Y(\zeta)$ satisfies the asymptotic formula

$$M^Y(\zeta) = \mathbb{I} + \frac{\mathrm{i}}{\zeta}\begin{pmatrix} 0 & -\beta \\ \overline{\beta} & 0 \end{pmatrix} + O\left(\frac{1}{\zeta^2}\right), \quad \zeta \to \infty, \tag{34}$$

where $\beta = \beta(a)$ is defined by (33). Put $D(t) := \mathrm{e}^{8\mathrm{i}a^3 t\sigma_3} t^{-\mathrm{i}\nu\sigma_3/2}$ and introduce the matrix $M^{par}(z)$ by formula

$$M^{par}(z) := D(t) M^Y(\sqrt{t}z) D(t).$$

It is straightforward to check that $M^{par}(z)$ satisfies (28)-(29). Due to (34) it is close as $t \to \infty$ to the identity matrix on $\partial\mathbb{D}_{\rho_1}$.

Put now $M_{-a}(k) = M^{par}(\sqrt{48a}(k+a))$. This function is holomorphic in $\mathbb{D}_\rho(-a) \setminus (\mathcal{C} \cup \mathcal{C}^* \cup \mathcal{C}_l \cup \mathcal{C}_l^*)$ has the jump with the matrix $v^{par}(\sqrt{48a}(k+a))$. it is easy to see that the matrix $M_a(k) := \sigma_1 M_{-a}(k)\sigma_1$ solves the corresponding parametrix problem in the domain $\mathbb{D}_\rho(a) \setminus (\mathcal{C} \cup \mathcal{C}^* \cup \mathcal{C}_r \cup \mathcal{C}_r^*)$. Moreover, due to (34):

$$\begin{aligned} M_{-a}(k) &= \mathbb{I} + \frac{\mathrm{i}}{\sqrt{48at}(k+a)}\begin{pmatrix} 0 & -\beta \mathrm{e}^{16\mathrm{i}a^3 t} t^{-\mathrm{i}\nu} \\ \overline{\beta}\mathrm{e}^{-16\mathrm{i}a^3 t} t^{\mathrm{i}\nu} & 0 \end{pmatrix} + O\left(\frac{1}{t}\right), \ k \in \partial\mathbb{D}_\rho(-a), \\ M_a(k) &= \mathbb{I} - \frac{\mathrm{i}}{\sqrt{48at}(k-a)}\begin{pmatrix} 0 & \overline{\beta}\mathrm{e}^{-16\mathrm{i}a^3 t} t^{\mathrm{i}\nu} \\ -\beta \mathrm{e}^{16\mathrm{i}a^3 t} t^{-\mathrm{i}\nu} & 0 \end{pmatrix} + O\left(\frac{1}{t}\right), \ k \in \partial\mathbb{D}_\rho(a). \end{aligned} \tag{35}$$

The completion of the asymptotical analysis repeats now almost literally the same considerations as in [13], Theorem 2.1. Namely, denote

$$\tilde{\Gamma} := \mathcal{C}_l \cup \mathcal{C}_l^* \cup \mathcal{C} \cup \mathcal{C}^* \cup \mathcal{C}_r \cup \mathcal{C}_r^* \cup \partial\mathbb{D}_\rho(-a) \cup \partial\mathbb{D}_\rho(a).$$

For the vector $m^{(2)}(k)$ corresponding to the jump matrix (25) put

$$\hat{m}(k) = \begin{cases} m^{(2)}(k)(M_{\mp a}(k))^{-1}, & |k \pm a| < \rho, \\ m^{(2)}(k), & \text{otherwise}. \end{cases}$$

Then the vector function $\hat{m}(k)$ is holomorphic in $\mathbb{C} \setminus \tilde{\Gamma}$, satisfies the standard symmetry and the normalization conditions, that is $\hat{m}(k) \to (1 \ \ 1)$ as $k \to \infty$ and $\hat{m}(-k) = \hat{m}(k)\sigma_1$. Moreover, it has a jump on $\tilde{\Gamma}$ with the jump matrix

$$\hat{v}(k) = \begin{cases} (M_{\mp a}(k))_- \, v^{(2)}(k)(M_{\mp a}(k))_+^{-1}, & k \in \Gamma_\rho(\mp a), \\ (M_{\mp a}(k))^{-1}, & |k \pm a| = \rho, \\ v^{(2)}(k), & \text{otherwise}. \end{cases}$$

Now from Theorem 2.1 of [13], estimates (27), (35) and a trivial equality

$$\frac{1}{2\pi \mathrm{i}}\int_{|k\pm a|=\rho}\frac{dk}{k\pm a}=1,$$

(the integration is counterclockwise) it follows that

$$\begin{aligned}
&\lim_{k\to \mathrm{i}\infty} 2\mathrm{i}k\left(\hat m(k)-\begin{pmatrix}1 & 1\end{pmatrix}\right) = -\frac{1}{\pi}\begin{pmatrix}1 & 1\end{pmatrix}\left(\int_{|k+a|=\rho}(M_{-a}(k)-\mathbb{I})\,dk\right.\\
&\left.+\int_{|k-a|=\rho}(M_a(k)-\mathbb{I})\,dk\right)+O(t^{-\frac{1+\alpha}{2}})\\
&=\frac{2}{\sqrt{48at}}(1,\,-1)\left(\beta \mathrm{e}^{16\mathrm{i}a^3t-\mathrm{i}\nu\log t}+\overline{\beta}\mathrm{e}^{-16\mathrm{i}a^3t+\mathrm{i}\nu\log t}\right)+O(t^{-\frac{1+\alpha}{2}})\\
&=\frac{\sqrt{\nu(a)}}{\sqrt{3at}}\cos\left(16a^3t-\nu(a)\log t-\mathrm{i}\log\frac{\beta(a)}{\sqrt{\nu(a)}}\right)+O(t^{-\frac{1+\alpha}{2}}),
\end{aligned}\tag{36}$$

where the term $O(t^{-\frac{1+\alpha}{2}})$ can be differentiated with respect to x, and the derivative has the same order $O(t^{-\frac{1+\alpha}{2}})$ as $t\to\infty$ uniformly with respect to ξ in the domain $\xi<-\frac{c^2}{2}-\epsilon$ (cf. [13], Theorem 2.1). Next, by (5) we have $\frac{\partial a}{\partial x}=-\frac{1}{24at}$. Combining this with (19), (26),(33) and (36) we get

$$q(x,t)=c^2+\sqrt{\frac{4\nu(a)a}{3t}}\sin\left(16ta^3-\nu(a)\log t+\frac{\pi}{4}-\arg f(a)+\arg\Gamma(\mathrm{i}\nu)\right)$$

with

$$\arg f(a)=\nu(a)\log(192a^3)+\arg R(-a)+\frac{1}{\pi}\int_{\mathbb{R}\setminus[-a,a]}\log\left(\frac{1-|R(s)|^2}{1-|R(a)|^2}\right)\frac{ds}{s+a}.$$

The result of Theorem 1 is now immediate from $\arg R(-a)=-\arg R(a)$ and the oddness of the last integral with respect to a.

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