

Adjoints of ideals

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Abstract. We characterize ideals whose adjoints are determined by their Rees valuations. We generalize the notion of a regular system of parameters, and prove that for ideals generated by monomials in such elements, the integral closure and adjoints are generated by monomials. We prove that the adjoints of such ideals and of all ideals in two-dimensional regular local rings are determined by their Rees valuations. We prove special cases of subadditivity of adjoints.

Adjoint ideals and multiplier ideals have recently emerged as a fundamental tool in commutative algebra and algebraic geometry. In characteristic 0 they may be defined using resolution of singularities. In positive prime characteristic p , Hara and Yoshida [4] introduced the analog of multiplier ideals as generalized test ideals for a tight closure theory. In all characteristics, even mixed, Lipman gave the following definition:

Definition 0.1: *Let R be a regular domain, I an ideal in R . The **adjoint** $\text{adj } I$ of I is defined as follows:*

$$\text{adj } I = \bigcap_v \{r \in R \mid v(r) \geq v(I) - v(J_{R_v/R})\},$$

where the intersection varies over all valuations v on the field of fractions K of R that are non-negative on R and for which the corresponding valuation ring R_v is a localization of a finitely generated R -algebra. The symbol $J_{R_v/R}$ denotes the Jacobian ideal of R_v over R .

By the assumption on v , each valuation in the definition of $\text{adj } I$ is Noetherian.

Many valuations v have the same valuation ring R_v ; any two such valuations are positive real multiples of each other, and are called equivalent. In the definition of $\text{adj } I$ above, one need only use one v from each equivalence class. In the sequel, we will always

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choose **normalized** valuations, that is, the integer-valued valuation v such that for all $r \in R$, $v(r)$ equals that non-negative integer n which satisfies that rR_v equals the n th power of the maximal ideal of R_v .

Lipman proved that for any ideal I in R and any $x \in R$, $\text{adj}(xI) = x \text{adj}(I)$. In particular, $\text{adj}(xR) = (x)$.

A crucial and very powerful property is the subadditivity of adjoints: $\text{adj}(IJ) \subseteq \text{adj}(I)\text{adj}(J)$. This was proved in characteristic zero by Demailly, Ein and Lazarsfeld [3], and for generalized test ideals in characteristic p by Hara and Yoshida [4, Theorem 6.10]. A simpler proof in characteristic p can be found in [1, Lemma 2.10]. A version of subadditivity formula on singular varieties was proved by Takagi in [21]. But subadditivity of adjoints is unknown in general. We prove it for generalized monomial ideals in Section 4, and for ideals in two-dimensional regular domains in Section 5. The case of subadditivity of adjoints for ordinary monomial ideals can be deduced from Howald's work [5] using toric resolutions, and the two-dimensional case has been proved by Takagi and Watanabe [22] using multiplier ideals. The case for generalized monomial ideals proved here is new.

An aspect of proving subadditivity and computability of adjoints is whether there are only finitely many valuations v_1, \dots, v_m such that for all n ,

$$\text{adj}(I^n) = \bigcap_{i=1}^m \{r \in R \mid v_i(r) \geq v_i(I^n) - v_i(J_{R_{v_i}/R})\}.$$

We prove in Sections 4 that Rees valuations suffice for the generalized monomial ideals. We also give an example (the first example in Section 5) showing that Rees valuations do not suffice in general. In Section 5 we give a general criterion for when the adjoint of an ideal is determined by its Rees valuations. A corollary is that Rees valuations suffice for ideals in two-dimensional regular domains. The first three sections develop the background on generalized monomial ideals.

We refer the reader to the article by Smith and Thompson [19] and to Järvilehto's thesis [8] for results on what divisors (i.e., valuations) are needed to compute the multiplier

ideals with rational coefficients. In general, Rees valuations do not suffice there, even in dimension two.

1. Generalized regular system of parameters

Definition 1.1: *Let R be a regular domain. Elements x_1, \dots, x_d in R are called a **generalized regular system of parameters** if x_1, \dots, x_d is a permutable regular sequence in R such that for every $i_1, \dots, i_s \in \{1, \dots, d\}$, $R/(x_{i_1}, \dots, x_{i_s})$ is a regular domain.*

Remark: Any part of a generalized regular system of parameters is again a generalized regular system of parameters.

For example, when R is regular local, an arbitrary regular system of parameters (or a part thereof) is a generalized regular system of parameters; or if R is a polynomial ring over a field, the variables are a generalized regular system of parameters.

Let \mathfrak{p} be any prime ideal containing the generalized regular system of parameters x_1, \dots, x_d . As $R/(x_1, \dots, x_d)$ is regular, so is $R_{\mathfrak{p}}/(x_1, \dots, x_d)_{\mathfrak{p}}$, whence x_1, \dots, x_d is part of a (usual) regular system of parameters in $R_{\mathfrak{p}}$.

Lemma 1.2: *Let R be a regular domain and x_1, \dots, x_d a generalized regular system of parameters. Then for any normalized valuation v as in the definition of adjoints, $v(J_{R_v/R}) \geq v(x_1 \cdots x_d) - 1$.*

Proof: By possibly taking a subset of the x_i , without loss of generality all $v(x_i)$ are positive. Let \mathfrak{p} be the contraction of the maximal ideal of R_v to R . After localizing at \mathfrak{p} , x_1, x_2, \dots, x_d are a part of a regular system of parameters (see comment above the lemma). We may possibly extend the x_i to a full regular system of parameters, so we may assume that $\mathfrak{p} = (x_1, \dots, x_d)$ is the unique maximal ideal in R . We may also assume that $v(x_1) \geq v(x_2) \geq \cdots \geq v(x_d) \geq 1$.

If $d = 0$, the lemma holds trivially. If $d = 1$, then v is the \mathfrak{p} -adic valuation, in which

case $R_v = R$, $J_{R_v/R} = R$. As v is normalized, $v(x_1) = 1$, and the lemma holds again. Now let $d > 1$ and let $S = R[\frac{x_1}{x_d}, \dots, \frac{x_{d-1}}{x_d}]$. Then S is a regular ring contained in R_v , and $\frac{x_1}{x_d}, \dots, \frac{x_{d-1}}{x_d}, x_d$ are a generalized regular system of parameters in S . Clearly $J_{S/R}$ equals $x_d^{d-1}S$. By induction on $\sum_i v(x_i)$ we conclude that $v(J_{R_v/S}) \geq v(\frac{x_1}{x_d} \cdots \frac{x_{d-1}}{x_d} x_d) - 1$, so that by Lipman and Sathaye [14, page 201], $v(J_{R_v/R}) = v(J_{S/R}) + v(J_{R_v/S}) \geq v(x_d^{d-1}) + v(\frac{x_1}{x_d} \cdots \frac{x_{d-1}}{x_d} x_d) - 1 = v(x_1 \cdots x_d) - 1$. \square

Though in general not as nicely behaved as variables in a polynomial or power series ring, generalized regular systems of parameters come close to them in many aspects. One interesting property is the following.

Proposition 1.3: *Let R be a regular domain and let x_1, \dots, x_d be a generalized regular system of parameters of R . Furthermore let $s \leq d$, $\mathfrak{p} = (x_1, \dots, x_s)$ and let f be a non-zero element of R . Then there exist monomials m_1, \dots, m_t in x_1, \dots, x_s and elements $h, g_1, \dots, g_t \in R \setminus \mathfrak{p}$ such that*

$$h \cdot f = \sum_{i=1}^t g_i \cdot m_i.$$

Proof: Clearly we may assume that R is local with maximal ideal \mathfrak{p} , and we then prove the proposition with $h = 1$. First we reduce to the case of complete local rings: Let \widehat{R} be the completion of R , and note that x_1, \dots, x_s is a regular system of parameters of \widehat{R} . Suppose we know the result for \widehat{R} and $x_1, \dots, x_s \in \widehat{R}$. Write $f = \sum_{i=1}^t h_i m_i$ for some $h_i \in \widehat{R}$, $h_i \notin \mathfrak{p}$. Clearly we may assume that none of the monomials is a multiple of another one. Let $I = (m_1, \dots, m_t) \subseteq R$. As $f \in I\widehat{R} \cap R = I$ by faithful flatness, we may write $f = \sum_{i=1}^t g_i m_i$ with $g_i \in R$, and in \widehat{R} we get $\sum_{i=1}^t (g_i - h_i) m_i = 0$ hence we conclude from [9], §5, that $g_i - h_i \in \mathfrak{p}\widehat{R}$, implying that $g_i \notin \mathfrak{p}$. Thus it suffices to prove the proposition in the case R is complete local with maximal ideal $\mathfrak{p} = (x_1, \dots, x_s)$.

Assume now that R is complete and let $f \in R$. Assume $f \in \mathfrak{p}^{n_1} \setminus \mathfrak{p}^{n_1+1}$. Then we may write

$$f = \sum_{i=1}^{t_1} a_{1i} m_{1i} + f_2$$

with some (unique) monomials m_{1i} of degree n_1 in x_1, \dots, x_s and some $a_{1i} \notin \mathfrak{p}$ (unique mod \mathfrak{p}) and with some $f_2 \in \mathfrak{p}^{n_1+1}$. Let $M_1 = (m_{11}, \dots, m_{1t_1})$. If $f_2 = 0$ we are done, otherwise we write

$$f_2 = \sum_{i=1}^{t_2} a_{2i} m_{2i} + f_3$$

with some $a_{2i} \notin \mathfrak{p}$, some monomials m_{2i} of degree n_2 in x_1, \dots, x_s , and some $f_3 \in \mathfrak{p}^{n_2+1}$. Set $M_2 = M_1 + (m_{21}, \dots, m_{2t_2})$ and continue. In this way we get an ascending chain $M_1 \subseteq M_2 \subseteq \dots$ of monomial ideals, which must stabilize eventually, $M_\rho = M_{\rho+1} = \dots =: M_\infty$. Let

$$M_\infty = (m_1, \dots, m_t),$$

with each m_i a monomial of degree d_i in x_1, \dots, x_s . We may assume that none of these monomials divides any of the other ones and that all m_i appear in a presentation of some f_j as above. Then in each step above we may write

$$f_l = \sum_{i=1}^t n_{li} m_i + f_{l+1}$$

with $n_{li} \in \mathfrak{p}^{n_l - d_i}$, and where furthermore if l is the smallest integer such that m_i appears with a non-trivial coefficient in the expansion of f_l , we have $n_{li} \notin \mathfrak{p}$. Hence

$$f = \sum c_i m_i + f_{l+1}$$

with some $c_i \notin \mathfrak{p}$ (or $c_i = 0$), and with $c_{l+1,i} - c_i \in \mathfrak{p}^{n_{l+1} - d_i}$ (and $f_{l+1} \in \mathfrak{p}^{n_{l+1}+1}$). As R is complete, this converges, and we get

$$f = \sum c_i m_i$$

with some $c_i \notin \mathfrak{p}$. □

2. Integral closures of (general) monomial ideals

Monomial ideals typically denote ideals in a polynomial ring or in a power series ring over a field that are generated by monomials in the variables. Such ideals have many good properties, and in particular, their integral closures and multiplier ideals are known to be monomial as well. The just stated result on multiplier ideals for the standard monomial ideals is due to Howald [5]. In this section we consider generalized monomial ideals and present their integral closures. For alternate proofs on the integral closure of generalized monomial ideals see Kiyek and Stückrad [10].

We define monomial ideals more generally:

Definition 2.1: *Let R be a regular domain, and let x_1, \dots, x_d in R be a generalized regular system of parameters. By a **monomial ideal (in x_1, \dots, x_d)** we mean an ideal in R generated by monomials in x_1, \dots, x_d .*

As in the usual monomial ideal case, we can define the Newton polyhedron:

Definition 2.2: *Let R, x_1, \dots, x_d be as above, and let I be an ideal generated by monomials $\underline{x}^{a_1}, \dots, \underline{x}^{a_s}$. Then the **Newton polyhedron of I** (relative to x_1, \dots, x_d) is the set*

$$NP(I) = \{\underline{e} \in \mathbb{Q}_{\geq 0}^d \mid \underline{e} \geq \sum_i c_i \underline{a}_i, \text{ for some } c_i \in \mathbb{Q}_{\geq 0}, \sum_i c_i = 1\}.$$

$NP(I)$ is an unbounded closed convex set in $\mathbb{Q}_{\geq 0}^d$. We denote the interior of $NP(I)$ as $NP^\circ(I)$.

Theorem 2.3: *Let R be a regular domain and x_1, \dots, x_d a generalized regular system of parameters. Let I be an ideal generated by monomials in x_1, \dots, x_d . The integral closure $\overline{I^n}$ of I^n equals*

$$\overline{I^n} = (\{\underline{x}^{\underline{e}} \mid \underline{e} \in n \cdot NP(I) \cap \mathbb{N}^d\}),$$

so it is generated by monomials.

Proof: As $NP(I^n) = n \cdot NP(I)$, we may assume that $n = 1$. Write $I = (\underline{x}^{a_1}, \dots, \underline{x}^{a_s})$. Let $\alpha = \underline{x}^{\underline{e}}$ be such that $\underline{e} \in NP(I) \cap \mathbb{N}^d$. Then there exist $c_1, \dots, c_s \in \mathbb{Q}_{\geq 0}$ such that $\sum c_i = 1$

and $\underline{e} \geq \sum c_i \underline{a}_i$ (componentwise). Write $c_i = m_i/n$ for some $m_i \in \mathbb{N}$ and $n \in \mathbb{N}_{>0}$. Then

$$\alpha^n = x_1^{ne_1 - \sum m_i a_{i1}} \cdots x_d^{ne_d - \sum m_i a_{id}} (\underline{x}^{a_1})^{m_1} \cdots (\underline{x}^{a_s})^{m_s} \in I^{m_1 + \cdots + m_s} = I^n,$$

so that $\alpha \in \bar{I}$. It remains to prove the other inclusion.

Let S be the set of hyperplanes that bound $\text{NP}(I)$ and are not coordinate hyperplanes. For each $H \in S$, if an equation for H is $h_1 X_1 + \cdots + h_d X_d = h$ with $h_i \in \mathbb{N}$ and $h \in \mathbb{N}_{>0}$, define $I_H = (\underline{x}^e \mid e \in \mathbb{N}^d, \sum_i h_i e_i \geq h)$. Clearly $I \subseteq I_H$, $\text{NP}(I_H) \subseteq \{e \in \mathbb{Q}_{\geq 0}^d : \sum h_i e_i \geq h\}$, and $\text{NP}(I_H) \cap \mathbb{N}^d = \{e \in \mathbb{Q}_{\geq 0}^d : \sum h_i e_i \geq h\} \cap \mathbb{N}^d$. Suppose that the theorem is known for the (generalized) monomial ideals I_H . Then

$$\begin{aligned} \bar{I} &\subseteq \bigcap_{H \in S} \bar{I}_H \\ &\subseteq \bigcap_{H \in S} (\{\underline{x}^e \mid e \in \mathbb{N}^d, \sum_i h_i e_i \geq h, \text{ if } H \text{ is defined by } \sum_i h_i X_i = h\}) \\ &= (\{\underline{x}^e \mid e \in \text{NP}(I) \cap \mathbb{N}^d\}). \end{aligned}$$

Thus it suffices to prove the theorem for I_H . As before, let $\sum_i h_i X_i = h$ define H . By possibly reindexing, we may assume that h, h_1, \dots, h_t are positive integers and that $h_{t+1} = \cdots = h_d = 0$. As noted above it suffices to show

$$\bar{I}_H = (\{\underline{x}^e \mid e \in \mathbb{N}^d \text{ and } \sum h_i e_i \geq h\}).$$

Let Y_1, \dots, Y_t be variables over R and $R' = R[Y_1, \dots, Y_t]/(Y_1^{h_1} - x_1, \dots, Y_t^{h_t} - x_t)$. This is a free finitely generated R -module and Y_1, \dots, Y_t is a regular sequence in R' . Set $\mathfrak{p} = (Y_1, \dots, Y_t)R'$. Then $R'/\mathfrak{p} = R/(x_1, \dots, x_t)$ is a regular domain, so \mathfrak{p} is a prime ideal, and for any prime ideal \mathfrak{q} in R' containing \mathfrak{p} , $R'_\mathfrak{q}$ is a regular local ring. By construction, $I_H R'$ is contained in $(Y_1, \dots, Y_t)^h = \mathfrak{p}^h$. As $R'_\mathfrak{p}$ is a regular local ring, $\mathfrak{p}^h R'_\mathfrak{p}$ is integrally closed, and as R' is finitely generated over a locally formally equidimensional (regular) ring, $R'_\mathfrak{q}$ is locally formally equidimensional for every prime ideal \mathfrak{q} containing \mathfrak{p} . By a theorem of Ratliff, from [17], since \mathfrak{p} is generated by a regular sequence, the integral closure of $\mathfrak{p}^h R'_\mathfrak{q}$

has no embedded prime ideals. It follows that the integral closure of $\mathfrak{p}^h R'_q$ is $\mathfrak{p}^h R'_p \cap R'_q$. As R'_q is a regular domain and \mathfrak{p} is generated by a regular sequence, $\mathfrak{p}^h R'_p \cap R'_q = \mathfrak{p}^h R'_q$. It follows that $\mathfrak{p}^h R'_p \cap R' = \mathfrak{p}^h$ is the integral closure of \mathfrak{p}^h . Hence $\overline{I_H} \subseteq \overline{\mathfrak{p}^h} \cap R = \mathfrak{p}^h \cap R$, and by freeness of R' over R , the last ideal is exactly $(\underline{x}^\underline{e} \mid \underline{e} \in \mathbb{N}, \sum_i h_i e_i \geq h)$, which finishes the proof. \square

3. Rees valuations of (general) monomial ideals

Recall that the Rees valuations of a non-zero ideal in a Noetherian domain form a unique minimal set $\mathcal{RV}(I)$ of finitely many normalized valuations such that for all positive integers n , $\overline{I^n} = \{r \in R \mid v(r) \geq nv(I) \text{ for all } v \in \mathcal{RV}(I)\}$.

In an arbitrary Noetherian domain, for arbitrary ideals I and J , $\mathcal{RV}(I) \cup \mathcal{RV}(J) \subseteq \mathcal{RV}(IJ)$, and equality holds in two-dimensional regular domains. (This has appeared in the literature in several places, see for example Muhly-Sakuma [16], or the Rees valuations chapter in the upcoming book [20].)

We will prove that the Rees valuations of an ideal generated by monomials in a regular system of parameters are especially nice.

Definition 3.1: *Let R be a regular domain, and let x_1, \dots, x_d be a generalized regular system of parameters. A valuation v on the field of fractions of R is said to be **monomial** on x_1, \dots, x_d if for some $i_1, \dots, i_s \in \{1, \dots, d\}$, for any polynomial $f = \sum c_\nu x_{i_1}^{\nu_{i_1}} \cdots x_{i_s}^{\nu_{i_s}} \in R$ with all c_ν either 0 or not in $(x_{i_1}, \dots, x_{i_s})$, $v(f) = \min\{v(\underline{x}^\nu) \mid c_\nu \neq 0\}$. When the x_i are understood from the context, we say that v is **monomial**.*

Observe that $v(f) = 0$ for any $f \notin (x_{i_1}, \dots, x_{i_s})$. In particular $v(x_j) = v(1) = 0$ if $j \notin \{i_1, \dots, i_s\}$.

Proposition 3.2: *Let R be a regular domain, let x_1, \dots, x_d be a generalized regular system of parameters, and let a_1, \dots, a_d be non-negative rational numbers, not all of them zero. Then there exists a unique valuation v on the field of fractions $K = Q(R)$ of R that is*

monomial on x_1, \dots, x_d , with $v(x_i) = a_i$.

Proof: By reindexing we may assume that $a_1 > 0, \dots, a_s > 0$, $a_{s+1} = \dots = a_d = 0$ for some $s > 0$, and we also may assume that all a_i are integers.

The uniqueness of v is immediate by Proposition 1.3. To prove the existence we may replace R by $R_{\mathfrak{p}}$ (with $\mathfrak{p} = (x_1, \dots, x_s)$) and assume that R is local. Let R' be the regular local ring obtained by adjoining a a_i^{th} -root y_i of x_i to R ($i = 1, \dots, s$) and let \mathfrak{n} be the maximal ideal of R' . Then the \mathfrak{n} -adic valuation w on $L = Q(R')$ is monomial in y_1, \dots, y_s with $w(y_i) = 1$ for all i . The restriction $v := w|_K$ is a monomial valuation as desired. \square

Corollary 3.3: *Let R be a regular domain, and let x_1, \dots, x_d be a generalized regular system of parameters. Let I be an ideal generated by monomials in x_1, \dots, x_d . Then all the Rees valuations of I are monomial in x_1, \dots, x_d . Furthermore, if H_1, \dots, H_ρ are the non-coordinate hyperplanes bounding $\text{NP}(I)$, then the H_j are in one-to-one correspondence with the Rees valuations v_j of I .*

Proof: The Newton polyhedron $\text{NP}(I)$ of I is the intersection of finitely many half-spaces in \mathbb{Q}^d . Some of them are coordinate half-spaces $\{x_i \geq 0\}$, each of the others is determined by a hyperplane H of the form $h_1x_1 + \dots + h_dx_d = h$, with h_1, \dots, h_d, h non-negative integers, $h > 0$, and $\text{gcd}(h_1, \dots, h_d, h) = 1$. This hyperplane corresponds to a valuation v_H that is monomial on x_1, \dots, x_d and such that $v_H(x_i) = h_i$. By Theorem 2.3, the integral closure of I is determined by these v_H . Using $\text{NP}(I^n) = n \cdot \text{NP}(I)$, we see that the integral closure of I^n is also determined by these v_H . So each Rees valuation is one such v_H . Suppose that the set of Rees valuations is a proper subset of the set of all the v_H . Say one such v_H is not needed in the computation of the integral closures of powers of I . Since the hyperplanes H were chosen to be irredundant, by omitting any one of them, we get a point $(e_1, \dots, e_d) \in \mathbb{Q}_{\geq 0}^d$ which is on the unbounded side of all the hyperplanes bounding $\text{NP}(I)$ other than H , but is not on the unbounded side of H . There exists $m_1, \dots, m_d \in \mathbb{N}$ and $n > 0$ such that for each i , $e_i = m_i/n$. Then by assumption, $x_1^{m_1} \dots x_d^{m_d} \in \overline{I^n}$, but

$(m_1, \dots, m_n) \notin n \cdot \text{NP}(I)$, a contradiction. \square

The following is a local version of Howald [5, Lemma 1]. Howald's proof relies on the existence of a log resolution.

Lemma 3.4: *Let R be a regular domain, and let x_1, \dots, x_d be a generalized regular system of parameters. Let v be a discrete valuation that is monomial on x_1, \dots, x_d , non-negative on R , and has value group contained in \mathbb{Z} . Then*

$$v(J_{R_v}/R) = v(x_1 \cdots x_d) - \gcd(v(x_i)|i).$$

Proof: Since v is monomial in the x_i , the center of v on R is contained in $\mathfrak{m} = (x_1, \dots, x_d)$. By localizing, we may assume that \mathfrak{m} is the only maximal ideal in R . Let $a_i = v(x_i)$. Without loss of generality $a_1 \geq a_2 \geq \dots \geq a_d$, and let s be the largest integer such that $a_s > 0$. As v is monomial, if $s = 0$, then $v = 0$ and the lemma holds trivially. Thus we may assume that $s > 0$. If $s = 1$, necessarily $a_1 = \gcd(v(x_i)|i)$, and v is a_1 times the (x_1) -grading. Then $R_v = R_{(x_1)}$, $J_{R_v/R} = R_v$, $v(J_{R_v/R}) = 0 = v(x_1 \cdots x_d) - \gcd(v(x_i)|i)$. So the lemma holds in the case $s = 1$. We proceed by induction on $\sum_i a_i$. We may assume that $s > 1$. Let $S = R[\frac{x_1}{x_s}, \dots, \frac{x_{s-1}}{x_s}]$. Then S is a regular ring contained in R_v , $\frac{x_1}{x_s}, \dots, \frac{x_{s-1}}{x_s}, x_s, \dots, x_d$ is a generalized regular system of parameters. For these elements in S , v is still a monomial valuation, their v -values are non-negative integers, and the total sum of their v -values is strictly smaller than $\sum_i a_i$. Thus by induction,

$$\begin{aligned} v(J_{R_v/S}) &= v\left(\frac{x_1}{x_s} \cdots \frac{x_{s-1}}{x_s} x_s \cdots x_d\right) - \gcd(v(x_1/x_s), \dots, v(x_{s-1}/x_s), v(x_s), \dots, v(x_d)) \\ &= v(x_1 \cdots x_d) - (s-1)v(x_s) - \gcd(v(x_1), \dots, v(x_d)). \end{aligned}$$

As $R \subseteq S \subseteq R_v$ are all finitely generated algebras over R that are regular rings and have the same field of fractions, by Lipman and Sathaye [14, page 201], $J_{R_v/R} = J_{S/R}J_{R_v/S}$. Clearly $J_{S/R}$ equals x_s^{s-1} , whence

$$\begin{aligned} v(J_{R_v/R}) &= v(x_s^{s-1}) + v(x_1 \cdots x_d) - (s-1)v(x_s) - \gcd(v(x_1), \dots, v(x_d)) \\ &= v(x_1 \cdots x_m) - \gcd(v(x_1), \dots, v(x_d)). \end{aligned} \quad \square$$

4. Adjoints of (general) monomial ideals

A proof similar to the proof of Theorem 2.3 shows that the adjoint of a (general) monomial ideal is monomial. This generalizes Howald's result [5].

Theorem 4.1: *Let R be a regular domain, and let x_1, \dots, x_d be a generalized regular system of parameters. Let I be an ideal generated by monomials in x_1, \dots, x_d . Then for all $n \geq 1$,*

$$\begin{aligned} \text{adj}(I^n) &= \bigcap_v (\{\underline{x}^e \mid v(\underline{x}^e) \geq v(I^n) - v(x_1 \cdots x_d) + 1\}) \\ &= \bigcap_v (\{\underline{x}^e \mid v(\underline{x}^e) \geq v(I^n) - v(J_{R_v/R})\}) \\ &= (\{\underline{x}^e : \underline{e} \in \mathbb{N}^d \text{ and } e + (1, \dots, 1) \in NP^\circ(I^n)\}), \end{aligned}$$

as v varies over the (normalized) Rees valuations of I . In particular, the adjoint is also generated by monomials.

Proof: As I^n is monomial and as the Rees valuations of I^n are contained in the set of Rees valuations of I , it suffices to prove the theorem for $n = 1$. By Corollary 3.3 and by Lemma 3.4, the second and third equalities hold. So it suffices to prove that $\text{adj } I$ equals the other three expressions (when $n = 1$).

First we prove that $\underline{x}^e \in \text{adj}(I)$ whenever $\underline{e} \in \mathbb{N}^d$ with $\underline{e} + (1, \dots, 1) \in NP^\circ(I)$. Let v be a valuation as in the definition of $\text{adj}(I)$. As $(x_1 \cdots x_d \underline{x}^e)^n \in \overline{I^{n+1}}$ for some positive integer n , $v(x_1 \cdots x_d \underline{x}^e) > v(I)$. As v is normalized, $v(\underline{x}^e) \geq v(I) - v(x_1 \cdots x_d) + 1$. By Lemma 1.2, $v(J_{R_v/R}) \geq v(x_1 \cdots x_d) - 1$, so that $v(\underline{x}^e) \geq v(I) - v(J_{R_v/R})$. As v was arbitrary, this proves that $(\underline{x}^e \mid \underline{e} \in \mathbb{N}^d, \underline{e} + (1, \dots, 1) \in NP^\circ(I)) \subset \text{adj } I$. It remains to prove the other inclusion.

Let S be the set of bounding hyperplanes of $NP(I)$ that are not coordinate hyperplanes. For each $H \in S$, if an equation for H is $h_1 X_1 + \cdots + h_d X_d = h$ with $h_i \in \mathbb{N}$ and $h \in \mathbb{N}_{>0}$, define $I_H = (\underline{x}^e \mid \underline{e} \in \mathbb{N}^d, \sum_i h_i e_i \geq h)$. By the definition of Newton polyhedrons, $I \subseteq I_H$.

By possibly reindexing, without loss of generality $h_1, \dots, h_t > 0$ and $h_{t+1} = \dots = h_d = 0$. By Proposition 3.2 there exists a monomial valuation v_H on $Q(R)$ defined by $v_H(x_i) = h_i$. By construction, $v_H(I) \geq v_H(I_H) \geq h$ (even equalities hold), and $\text{adj}(I_H) \subseteq \{r \in R \mid v_H(r) \geq v_H(I_H) - v_H(J_{R_{v_H}/R})\}$. By the properties of v_H , the last ideal is generated by monomials in the x_i . By Lemma 3.4, $v_H(J_{R_{v_H}/R}) = v_H(x_1 \cdots x_d) - 1$, so that

$$\begin{aligned} \text{adj}(I_H) &\subseteq (\{\underline{x}^{\underline{e}} \mid \underline{e} \in \mathbb{N}^d, v_H(\underline{x}^{\underline{e}}) > v_H(I_H) - v_H(x_1 \cdots x_d)\}) \\ &\subseteq (\{\underline{x}^{\underline{e}} \mid \underline{e} \in \mathbb{N}^d, \sum_i h_i(e_i + 1) > v_H(I_H)\}) \\ &\subseteq (\{\underline{x}^{\underline{e}} \mid \underline{e} \in \mathbb{N}^d, \sum_i h_i(e_i + 1) > h\}), \end{aligned}$$

whence

$$\begin{aligned} \text{adj } I &\subseteq \bigcap_{H \in \mathcal{S}} \text{adj}(I_H) \\ &\subseteq \bigcap_{H \in \mathcal{S}} (\{\underline{x}^{\underline{e}} \mid \underline{e} \in \mathbb{N}^d, \sum_i h_i(e_i + 1) > h, \text{ if } H \text{ is defined by } \sum_i h_i X_i = h\}) \\ &= (\{\underline{x}^{\underline{e}} \mid \underline{e} \in \mathbb{N}^d, \underline{e} + (1, \dots, 1) \in \text{NP}^\circ(I)\}). \quad \square \end{aligned}$$

This theorem allows to address the subadditivity problem for monomial ideals:

Corollary 4.2: *Let $I, J \subseteq R$ be ideals generated by monomials in the generalized regular system of parameters x_1, \dots, x_d . Then*

$$\text{adj}(IJ) \subseteq \text{adj}(I) \cdot \text{adj}(J).$$

Proof: Let $\underline{x}^{\underline{a}} \in \text{adj}(IJ)$ be a monomial. By Theorem 4.1, $\underline{a} + (1, \dots, 1) \in \text{NP}^\circ(I \cdot J)$. As $\text{NP}^\circ(IJ) \subseteq \text{NP}^\circ(I) + \text{NP}^\circ(J)$, there exist $\underline{b} \in \text{NP}^\circ(I)$ and $\underline{c} \in \text{NP}^\circ(J)$ with $\underline{a} + (1, \dots, 1) = \underline{b} + \underline{c}$. Set $\underline{f} = (f_1, \dots, f_d)$ and $\underline{g} = (g_1, \dots, g_d)$ with $f_i = \lceil b_i \rceil - 1$ and $g_i = \lfloor c_i \rfloor$. Then $\underline{x}^{\underline{g}}$ and $\underline{x}^{\underline{f}}$ are monomials with $\underline{x}^{\underline{g}} \cdot \underline{x}^{\underline{f}} = \underline{x}^{\underline{a}}$, and furthermore

$$\begin{aligned} \underline{f} + (1, \dots, 1) &\in \underline{b} + \mathbb{Q}_{\geq 0}^d \subseteq \text{NP}^\circ(I), \\ \underline{g} + (1, \dots, 1) &\in \underline{c} + \mathbb{Q}_{\geq 0}^d \subseteq \text{NP}^\circ(J), \end{aligned}$$

implying by Theorem 4.1 that $\underline{x}^f \in \text{adj}(I)$ and $\underline{x}^g \in \text{adj}(J)$. \square

From the proof of Theorem 4.1, it is clear that the Rees valuations of the adjoint depend on the Rees valuations of the original ideal. The number of Rees valuations of I need not be an upper bound on the number of Rees valuations of $\text{adj}(I)$, and there is in general no overlap between the set of Rees valuations of I and the set of Rees valuations of $\text{adj} I$.

Example: Let R be a regular local ring with regular system of parameters x, y . Let I be the integral closure of (x^5, y^7) . Then by the structure theorem, I has only one Rees valuation, and $I = (x^5, x^4y^2, x^3y^3, x^2y^5, xy^6, y^7)$. By [7], by [5], or by Theorem 4.1, $\text{adj}(I) = (x^4, x^3y, x^2y^2, xy^4, y^5)$, which is not the integral closure of (x^4, y^5) . Thus $\text{adj}(I)$ has more than one Rees valuation. In fact, it has two Rees valuations, both of which are monomial and neither of which is equivalent to the Rees valuation of I : $v_1(x) = 1 = v_1(y)$, $v_1(\text{adj}(I)) = 4$, and $v_2(x) = 3, v_2(y) = 2, v_2(\text{adj}(I)) = 10$.

Nevertheless, the one Rees valuation of I still determines the adjoints of all the powers of I .

5. Adjoint of ideals and Rees valuations

In this section we characterize those ideals I for which $\text{adj}(I^n)$ is determined by the Rees valuations of I for all n . In the last section we have seen that this is the case for monomial ideals. That the Rees valuations of an ideal I should play a crucial role in determining the adjoint of I in general is also implied by the following result:

Proposition 5.1: *Let I be an ideal in a regular domain R , and let \mathcal{V} be a finite set of valuations on the field of fractions of R such that for all $n \in \mathbb{N}$,*

$$\text{adj} I = \bigcap_{v \in \mathcal{V}} \{r \in R \mid v(r) \geq v(I) - v(J_{R_v/R})\}.$$

Then \mathcal{V} contains the Rees valuations of I .

Proof: Assume that there exist some Rees valuations of I not contained in \mathcal{V} . By the defining property of Rees valuations there exist a non-negative integer n and an element $r \in R$ with

- (1) $v(r) \geq n \cdot v(I)$ for all $v \in \mathcal{V}$.
- (2) $r \notin \overline{I^n}$.

Let w be a Rees valuation of I with $w(r) \leq n \cdot w(I) - 1$. Assume that I is l -generated and let $t \geq l \cdot w(I)$. Then

$$w(r^t) = t \cdot w(r) < (nt - l + 1)w(I),$$

hence

$$r^t \notin \overline{I^{nt-l+1}}.$$

On the other hand,

$$v(r^t) \geq nt \cdot v(I) \geq nt \cdot v(I) - v(J_{R_v/R}) \quad \text{for all } v \in \mathcal{V},$$

implying that

$$r^t \in \text{adj}(I^{nt}) \subseteq \overline{I^{nt-l+1}}$$

by [12], (1.4.1), a contradiction. □

It is not true in general that the set of Rees valuations determines the adjoint of an arbitrary ideal:

Example: Let (R, \mathfrak{m}) be a d -dimensional regular local ring, with $d > 2$, and let \mathfrak{p} be a prime ideal in R of height $h \in \{2, \dots, d-1\}$ generated by a regular sequence. Then the \mathfrak{p} -adic valuation $v_{\mathfrak{p}}$ is the only Rees valuation of \mathfrak{p} . If $v_{\mathfrak{p}}$ defined $\text{adj}(\mathfrak{p}^n)$ in the sense that

$$\text{adj}(\mathfrak{p}^n) = \{r \in R \mid v_{\mathfrak{p}}(r) \geq nv_{\mathfrak{p}}(\mathfrak{p}) - v_{\mathfrak{p}}(J_{R_{v_{\mathfrak{p}}}/R})\} \quad \text{for all } n,$$

then, as $v_{\mathfrak{p}}(\mathfrak{p}) = 1$ and $J_{R_{v_{\mathfrak{p}}}/R} = \mathfrak{p}^{h-1}R_{v_{\mathfrak{p}}}$, it follows that

$$\text{adj}(\mathfrak{p}^{h-1}) = \{r \in R \mid v_{\mathfrak{p}}(r) \geq 0\} = R.$$

However, if \mathfrak{p} is generated by elements in \mathfrak{m}^e , where $e \geq d/(h-1)$, and if v denotes the \mathfrak{m} -adic valuation, then

$$\text{adj}(\mathfrak{p}^{h-1}) \subseteq \{r \in R \mid v(r) \geq v(\mathfrak{p}^{h-1}) - v(J_{R_v/R})\} \subseteq \{r \in R \mid v(r) \geq d - (d-1)\} \subseteq \mathfrak{m},$$

which is a contradiction. A concrete example of this is $R = k[[X, Y, Z]]$ with the prime ideal $\mathfrak{p} = (X^4 - Z^3, Y^3 - X^2Z)$, which defines the monomial curve (t^9, t^{10}, t^{12}) .

The following is a geometric reformulation of [18], see also [2], 2.3 or [13], 1.4:

Remark: Let R be a regular domain and let $I \subseteq R$ be an ideal of R . Let $Y = \text{Spec}(R)$, $P = R[IT]$, the Rees ring of I and let \bar{P} be its normalization and $\varphi : X = \text{Proj}(\bar{P}) \rightarrow Y$ the induced scheme. Then X/Y is essentially of finite type by [14], p. 200 (see also [20, 9.2.3], for details). Thus φ is a projective, birational morphism, X is a normal, Noetherian scheme and $I\mathcal{O}_X$ is an invertible ideal. Let $\mathfrak{P}_1, \dots, \mathfrak{P}_r$ be the irreducible components of the vanishing set $\mathfrak{V}(I\mathcal{O}_X)$ of $I\mathcal{O}_X$ (i.e. those points x of X of codimension 1 such that $I\mathcal{O}_{X,x}$ is a proper ideal of $\mathcal{O}_{X,x}$). Then $\mathcal{O}_{X,\mathfrak{P}_i}$ is a discrete valuation ring (with field of fractions $K = Q(R)$) and the corresponding valuations v_1, \dots, v_r are exactly the Rees valuations of I .

If (R, \mathfrak{m}) is local and I is \mathfrak{m} -primary, the Rees valuations correspond to the irreducible components of the closed fibre $\varphi^{-1}(\mathfrak{m})$ which in this case is a $(\dim(R) - 1)$ -dimensional projective scheme (in general neither reduced nor irreducible).

Let $f : Z \rightarrow Y$ be birational and of finite type. Then the Jacobian ideal $\mathcal{J}_{Z/Y} \subseteq \mathcal{O}_Z$ is well-defined (being locally the 0th-Fitting ideal of the relative Kähler differentials). If in addition Z is normal, then

$$\omega_{Z/Y} := \mathcal{O}_Z : \mathcal{J}_{Z/Y} = \mathcal{H}om_Z(\mathcal{J}_{Z/Y}, \mathcal{O}_Z)$$

is a canonical dualizing sheaf for f with

$$\mathcal{O}_Z \subseteq \omega_{Z/Y} \subseteq \mathcal{M}_Z,$$

where \mathcal{M}_Z denotes the constant sheaf of meromorphic functions on Z . If

$$g : Z' \longrightarrow Z$$

is another birational morphism and if g is proper and Z' is normal as well, then

$$g_*\omega_{Z'/Y} \subseteq \omega_{Z/Y}$$

(cf. [14, 2.3] and [15, §4]).

Theorem 5.2: *Let R be a regular domain and let $I \subseteq R$ be a non-trivial ideal. Furthermore let $Y = \text{Spec}(R)$ and $\varphi : X \rightarrow Y$ be the normalized blow-up of I . Then the following are equivalent:*

- (1) $\text{adj}(I^n) = \bigcap_{v \in \mathcal{RV}(I)} \{r \in R \mid v(r) \geq n \cdot v(I) - v(J_{R_v/R})\}$ for all positive integers n .
(2) If Z is a normal scheme and $\pi : Z \rightarrow X$ is proper and birational, then

$$\pi_*\omega_{Z/Y} = \omega_{X/Y}.$$

Remark: If in the situation of 5.2(2) the scheme X is Cohen–Macaulay as well, then X has pseudo-rational singularities only ([15, §4]).

Remark: In the situation of 5.2(1) the set $\mathcal{RV}(I)$ is the unique smallest set of valuations defining $\text{adj}(I^n)$ in view of Proposition 5.1.

Proof of Theorem 5.2: If $f : Z \rightarrow Y$ is proper and birational with Z normal and $I\mathcal{O}_Z$ invertible, we set

$$\text{adj}_Z(I^n) = H^0(Z, I^n\omega_{Z/Y}) \quad (\subseteq R).$$

Then $\text{adj}(I^n) = \bigcap \text{adj}_Z(I^n)$ by [12], where $f : Z \rightarrow Y$ varies over all such morphisms. By the universal properties of blow-up and normalization, any such f factors as

$$Z \xrightarrow{\pi} X \xrightarrow{\varphi} Y,$$

As $\pi_*\omega_{Z/Y} \subseteq \omega_{X/Y}$, and as $I\mathcal{O}_Z$ is invertible, this implies by the projection formula

$$\begin{aligned}
H^0(Z, I^n \omega_{Z/Y}) &= H^0(X, \pi_* I^n \omega_{Z/Y}) \\
&= H^0(X, I^n \pi_* \omega_{Z/Y}) \\
&\subseteq H^0(X, I^n \omega_{X/Y}),
\end{aligned}$$

and therefore

$$(*) \quad \text{adj}_Z(I^n) \subseteq \text{adj}_X(I^n) \quad \text{for all positive integers } n$$

for any such $f : Z \rightarrow Y$.

As $\omega_{X/Y}$ is reflexive by [14, p. 203], and as $I\mathcal{O}_X$ is invertible, $I^n \omega_{X/Y}$ is a reflexive coherent subsheaf of the sheaf of meromorphic functions of X , and therefore we have

$$H^0(X, I^n \omega_{X/Y}) = \bigcap_{x \in X: \text{ht}(x)=1} (I^n \omega_{X/Y})_x.$$

For $x \in X$ with $\varphi(x) \notin \mathfrak{B}(I)$, the set of primes containing I , we have

$$I\mathcal{O}_{X,x} = \mathcal{O}_{X,x},$$

$$\omega_{X/Y,x} = \mathcal{O}_{X,x},$$

as φ is an isomorphism away from $\mathfrak{B}(I)$. Those $x \in X$ with $\text{ht}(x) = 1$ and $\varphi(x) \in \mathfrak{B}(I)$ correspond to the Rees valuations of I , and thus

$$\begin{aligned}
\text{adj}_X(I^n) &= H^0(X, I^n \omega_{X/Y}) \\
&= \bigcap_{x \in X: \text{ht}(x)=1} (I^n \omega_{X/Y})_x \\
&= \bigcap_{x \in X: \text{ht}(x)=1, \varphi(x) \in \mathfrak{B}(I)} (I^n \omega_{X/Y})_x \cap \bigcap_{x \in X: \text{ht}(x)=1, \varphi(x) \notin \mathfrak{B}(I)} \mathcal{O}_{X,x} \\
&\supseteq \bigcap_{v \in \mathcal{RV}(I)} \{r \in R : v(r) \geq n \cdot v(I) - v(J_{R_v/R})\},
\end{aligned}$$

where we also use, that $\omega_{R_v/R}$ is an invertible fractional ideal with inverse $J_{R_v/R}$. As $\pi_* \omega_{X/Y} = \mathcal{O}_X$ by [15, §4], hence $H^0(X, \omega_{X/Y}) = R$, the converse inclusion is obvious:

$$\text{adj}_X(I^n) = \bigcap_v \{r \in K : v(r) \geq n \cdot v(I) - v(J_{R_v/R})\} \cap R$$

$$\subseteq \bigcap_{v \in \mathcal{RV}(I)} \{r \in R : v(r) \geq n \cdot v(I) - v(J_{R_v/R})\},$$

and we conclude that

$$\text{adj}_X(I^n) = \bigcap_{v \in \mathcal{RV}(I)} \{r \in R \mid v(r) \geq n \cdot v(I) - v(J_{R_v/R})\}.$$

Thus (1) is equivalent to

$$\text{adj}_X(I^n) = \text{adj}_Z(I^n) \quad \text{for all } n \in \mathbb{N}$$

for all $f : Z \rightarrow Y$ as above.

First assume (2). This direction is implicit in [12], cf. [12], 1.3.2(b). Let $f : Z \rightarrow Y$ be as above. By the assumptions we have trivially

$$H^0(X, I^n \pi_* \omega_{Z/Y}) = H^0(X, I^n \omega_{X/Y})$$

implying by the calculations preceding (*) that $\text{adj}_X(I^n) = \text{adj}_Z(I^n)$ for all positive integers n . Thus (1) follows.

Conversely suppose that (1) holds, i.e., that $\text{adj}(I^n) = \text{adj}_X(I^n)$ for all positive integers n . Then by (*) we must have that the canonical inclusions

$$H^0(X, I^n \pi_* \omega_{Z/Y}) = \text{adj}_Z(I^n) \hookrightarrow \text{adj}_X(I^n) = H^0(X, I^n \omega_{X/Y})$$

are isomorphisms for all positive integers n . If X' denotes the blow-up of I on Y , then $I\mathcal{O}_{X'}$ is a very ample invertible sheaf on X' . As X/X' is finite, $I\mathcal{O}_X$ is an ample invertible sheaf on X , and thus the above isomorphisms imply that the canonical inclusion

$$\pi_* \omega_{Z/Y} \hookrightarrow \omega_{X/Y}$$

is an isomorphism, i.e., that (2) holds. □

For both conditions in the theorem some examples are known, as we show below.

Recall that two ideals $I, J \subseteq R$ are called projectively equivalent if there exist positive integers i, j with $\overline{I^i} = \overline{J^j}$, cf. [2].

Corollary 5.3: *Let R be a regular domain, let x_1, \dots, x_d be a generalized regular system of parameters, and let I be an ideal projectively equivalent to an ideal generated by monomials $\underline{x}^{a_1}, \dots, \underline{x}^{a_s}$ in x_1, \dots, x_d . Then $\text{adj}(I)$ is a monomial ideal in x_1, \dots, x_d , determined by the Rees valuations of I , and the normalized blow-up of I satisfies condition (2) of 5.2.*

Proof: It remains to note that

$$\text{Proj}(\overline{R[It]}) = \text{Proj}(\overline{R[I^i t]}) = \text{Proj}(\overline{R[\overline{I^i t}]}).$$

Then the corollary follows from Theorem 4.1. \square

By the work of Lipman and Teissier we also know (2) in some cases.

Corollary 5.4: *Let (R, \mathfrak{m}) be a two-dimensional regular domain. Then for any non-zero ideal I ,*

$$\text{adj}(I) = \bigcap_{v \in \mathcal{RV}(I)} \{r \in R \mid v(r) \geq v(I) - v(J_{R_v/R})\}.$$

Proof: It remains to note that in the two-dimensional case the normalized blow-up of I has pseudo-rational singularities only by [15], p. 103 and [11], 1.4. Thus condition (2) of 5.2 is satisfied. \square

Remark: In the case of two-dimensional regular rings an elementary direct proof of 5.4 can be given as well: We may assume that (R, \mathfrak{m}) is local with infinite residue field and that I is \mathfrak{m} -primary. Then it follows from [6] and [7] (see also [12]) that for a generic $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ the ideals I and $\text{adj}(I)$ are contracted from $S := R[\frac{\mathfrak{m}}{x}]$ and that $\text{adj}(I)S = \frac{1}{x} \text{adj}(IS)$. From this 5.4 follows by an easy induction on the multiplicity $\text{mult}(I)$ of I .

With this line of argument we can also give an easy proof of subadditivity of adjoint ideals in the two-dimensional case. Again we may assume that (R, \mathfrak{m}) is local with infinite

residue field, and that I and J are \mathfrak{m} -primary. For a generic $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ we will have that I , J , IJ , $\text{adj}(I)$, $\text{adj}(J)$, $\text{adj}(IJ)$ and $\text{adj}(I)\text{adj}(J)$ are contracted from $S = R[\frac{\mathfrak{m}}{x}]$. Denoting by I' , resp. J' , the strict transforms of I , resp. J , we conclude by the above and by induction on $\text{mult}(I) + \text{mult}(J)$:

$$\begin{aligned} \text{adj}(IJ) &= \text{adj}(IJ)S \cap R \\ &= \frac{1}{x} \text{adj}(IJS) \cap R \\ &= x^{\text{ord}(I)+\text{ord}(J)-1} \text{adj}(I'J') \cap R \\ &\subseteq x^{\text{ord}(I)-1} \text{adj}(I') \cdot x^{\text{ord}(J)-1} \text{adj}(J') \cap R \\ &= \text{adj}(I) \text{adj}(J)S \cap R \\ &= \text{adj}(I) \text{adj}(J). \end{aligned}$$

Alternatively, the subadditivity result may be deduced from [13] and [12]. We note that Tagaki and Watanabe [22] proved subadditivity of adjoint ideals more generally, for two-dimensional log-terminal singularities. The argument given here does not extend to their situation.

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References

1. M. Blickle, M. Mustața and K. E. Smith, Discreteness and rationality of F-thresholds, [arXiv:math.AG/0607660](https://arxiv.org/abs/math/0607660).
2. C. Ciuperca, W. Heinzer, L. Ratliff and D. Rush, Projectively equivalent ideals and Rees valuations, *J. Algebra* **282** (2004), 140–156.
3. J.-P. Demailly, L. Ein and R. Lazarsfeld, A subadditivity property of multiplier ideals, *Mich. Math. J.* **48** (2000), 137–156.
4. N. Hara and K. I. Yoshida, A generalization of tight closure and multiplier ideals, *Trans. Am. Math. Soc.* **355** (2003), 3143–3174.
5. J. A. Howald, Multiplier ideals of monomial ideals, *Trans. Amer. Math. Soc.* **353** (2001), 2665–2671.
6. C. Huneke, Complete ideals in two-dimensional regular local rings, *Commutative algebra (Berkeley, CA, 1987)* (1989), Math. Sci. Res. Inst. Publ., **15**, Springer, New York, 325–338.
7. C. Huneke and I. Swanson, Cores of ideals in two dimensional regular local rings, *Michigan Math. J.* **42** (1995), 193–208.

8. T. Järvilehto, Jumping numbers of a simple complete ideal in a two dimensional-regular local ring, Ph. D. thesis, University of Helsinki, 2007. [arXiv:math.AC/0611587](https://arxiv.org/abs/math/0611587).
9. I. Kaplansky, R -sequences and homological dimension, *Nagoya Math. J.* **20** (1962), 195–199.
10. K. Kiyek and J. Stückrad, Integral closure of monomial ideals on regular sequences, *Rev. Math. Iberoamericana* **19** (2003), 483–508.
11. J. Lipman, Desingularization of two-dimensional schemes, *Ann. of Math.* **107** (1978), 151–207.
12. J. Lipman, Adjointsof ideals in regular local rings, with an appendix by S. D. Cutkosky, *Math. Research Letters* **1** (1994), 739–755.
13. J. Lipman, Proximity inequalities for complete ideals in two-dimensional regular local rings, *Contemp. Math.* **159** (1994), 293–306.
14. J. Lipman and A. Sathaye, Jacobian ideals and a theorem of Briançon-Skoda, *Michigan Math. J.* **28** (1981), 199–222.
15. J. Lipman and B. Teissier, Pseudo-local rational rings and a theorem of Briançon-Skoda about integral closures of ideals, *Michigan Math. J.* **28** (1981), 97–112.
16. H. Muhly and M. Sakuma, Asymptotic factorization of ideals, *J. London Math. Soc.* **38** (1963), 341–350.
17. L. J. Ratliff, Jr., Locally quasi-unmixed Noetherian rings and ideals of the principal class, *Pacific J. Math.* **52** (1974), 185–205.
18. D. Rees, Valuations associated with ideals (II), *J. London Math. Soc.* **36** (1956), 221–228.
19. K. E. Smith and H. M. Thompson, Irrelevant exceptional divisor for curves on a smooth surface, [arXiv:math.AG/0611765](https://arxiv.org/abs/math/0611765).
20. I. Swanson and C. Huneke, *Integral Closure of Ideals, Rings, and Modules*, book, to be published by Cambridge University Press, 2006.
21. S. Takagi, Formulas for multiplier ideals on singular varieties, *Amer. J. Math.* **128** (2006), 1345–1362.
22. S. Takagi and K.-I. Watanabe, When does subadditivity theorem for multiplier ideals hold? *Trans. Amer. Math. Soc.* **356** (2004), 3951–3961.

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