Induced matching, ordered matching and Castelnuovo-Mumford regularity of bipartite graphs

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Abstract

Let $\boldsymbol{G}$ be a finite simple graph and let **ind-match**$(\boldsymbol{G})$ and **ord-match**$(\boldsymbol{G})$ denote the induced matching number and the ordered matching number of $\boldsymbol{G}$, respectively. We characterize all bipartite graphs $\boldsymbol{G}$ with **ind-match**$(\boldsymbol{G}) =$ **ord-match**$(\boldsymbol{G})$. We establish the Castelnuovo-Mumford regularity of powers of edge ideals and depth of powers of cover ideals for such graphs. We also give formulas for the count of connected non-isomorphic spanning subgraphs of $\boldsymbol{K_{m,n}}$ for which **ind-match**$(\boldsymbol{G}) =$ **ord-match**$(\boldsymbol{G}) = \boldsymbol{2}$, with an explicit expression for the count when $\boldsymbol{m \in \{2, 3, 4\}}$ and $\boldsymbol{m \leq n}$.

Keywords: Induced matching, ordered matching, Castelnuovo-Mumford regularity, depth, edge ideal, cover ideal

1 Introduction

Let G be a finite simple graph on the vertex set $V(G) = \{x_1, \ldots, x_d\}$ and edge set $E(G)$. We identify the vertices to variables and consider the polynomial ring $S = K[x_1, \ldots, x_d]$, where K is a field. The *edge ideal* of G is defined as $I(G) = \langle \{x_i x_j : x_i x_j \in E(G)\} \rangle \subset S$. Ever since the introduction of the edge ideal by Villarreal in [23], researchers have been trying to understand the interplay between the combinatorial properties of graphs and the algebraic properties of the associated edge ideals. One particular invariant, the Castelnuovo-Mumford regularity, has received much of the attention, compared to other invariants and properties. Several upper and lower bounds for the regularity of edge ideals were obtained by several researchers, see [1] and references therein. Whenever there is an upper and a lower bound for an invariant, it is natural to ask what are some necessary conditions and sufficient conditions for these two bounds to coincide, and structurally understand those objects for which these two bounds are equal. In this article, we address this question for the upper bound of ordered matching number and the lower bound of induced matching number.

Computing or bounding the Castelnuovo-Mumford regularity of the associated edge ideal and its powers, in terms of combinatorial data associated with G, has been a very active area of research for the past couple of decades. Bounds using several *matching numbers* have been obtained for the regularity. For graph G, let $\text{ind-match}(G), \text{ord-match}(G), \text{min-match}(G)$ and $\text{match}(G)$ denote induced matching number, ordered matching number, minimum matching number and matching number, respectively (see Section 2 for the definitions).

It is known that

$$\text{ind-match}(G) \leq \text{reg}(S/I(G)) \leq \{\text{ord-match}(G), \text{min-match}(G)\} \leq \text{match}(G),$$

where the first inequality was proved by Katzman, [15], the second inequality can be found in [24] (for min-match(G)) and [5] (for ord-match(G)) and the third inequality follows from the definition. A graph G is said to be a *Cameron-Walker graph* if ind-match(G) = match(G). This is a class of graphs which is well studied from both combinatorial and algebraic perspectives, [4, 11, 13]. In [12], Hibi et al. studied graphs with ind-match(G) = min-match(G). They gave a structural characterization of graphs satisfying ind-match(G) = min-match(G). In this article, we study graphs satisfying ind-match(G) = ord-match(G).

Besides the combinatorial reasons for understanding graphs G with ind-match(G) = ord-match(G), there is also an algebraic motivation to understand graphs with this property. It was proved by Cutkosky, Herzog and Trung, [6], and independently by Kodiyalam, [16], that for a homogeneous ideal I in a polynomial ring, reg(I^s) is a linear polynomial for $s \gg 0$. In the case of edge ideals, there have been extensive research in understanding this function and the polynomial in terms of combinatorial data associated with G, (see for example [1] and the references within). It was shown in [2, Theorem 4.5] and [20, Theorem 3.9] that for every integer $s \geq 1$,

$$2s + \text{ind-match}(G) - 2 \leq \text{reg}(S/I(G)^s) \leq 2s + \text{ord-match}(G) - 2.$$

If ind-match(G) = ord-match(G), then this gives an explicit expression for the regularity of powers of the edge ideal.

Classifying all graphs G with ind-match(G) = ord-match(G) would be an extremely hard problem in general, and in this paper we concentrate on classifying bipartite graphs satisfying this property. Another important reason for restricting our attention to the bipartite case is the behavior of the depth function of the cover ideal, see the end of Section 2.

For smaller values of induced and ordered matching, it is easier to handle the corresponding bipartite graphs. First we give graph theoretic characterization for graphs G with ind-match(G) = ord-match(G) = 1 (Theorem 3.4). We then move on to understand the structure of graphs in terms of the connectivity between the bipartitions. This gives us a classification of all bipartite graphs G with ind-match(G) = ord-match(G), (Theorem 3.8). To illustrate that this class of graphs is very different from the class of graphs G with ind-match(G) = min-match(G), we construct a class of graphs, $G_{r,m}$, $2 \leq r \leq m$, with reg($S/I(G_{r,m})$) = ind-match(G) = ord-match(G) = r and min-match(G) = m.

Our characterization of bipartite graphs with equal induced and ordered matching numbers allows us to count the number of graphs G satisfying ind-match(G) = ord-match(G) ≤ 2, see Theorem 3.4 and Theorem 5.4. The work for ind-match(G) = ord-match(G) = 2 leads us to some interesting connections between the presence of edges and certain number of integer sequences.

The paper is organized as follows. We collect some graph theory essentials in Section 2. Characterizing the equality of induced and ordered matching numbers is done in Section 3. In Section 4, we introduce some preliminaries to count the bipartite graphs with ind-match(G) = 2 = ord-match(G), namely we set up the notation for certain integer sequences and their counting. We count all graphs with ind-match(G) = 2 = ord-match(G) in Section 5. We provide the closed forms for certain elementary summations to count these graphs in Section 6 as an appendix.

2 Preliminaries

In this section, we provide the definitions and basic facts which will be used in the next sections.

Let $S = K[x_1, \dots, x_n]$ be the polynomial ring over a field K and let M be a finitely generated graded S-module. Suppose that the minimal graded free resolution of M is given by

$$0 \to \cdots \to \bigoplus_j S(-j)^{\beta_{1,j}(M)} \to \bigoplus_j S(-j)^{\beta_{0,j}(M)} \to M \to 0.$$

The Castelnuovo-Mumford regularity (or simply, regularity) of M, denoted by $\operatorname{reg}(M)$, is defined as

$$\operatorname{reg}(M) = \max\{j - i \mid \beta_{i,j}(M) \neq 0\}.$$

Also, the *projective dimension* of M is defined to be

$$\operatorname{pd}(M) = \max\{i \mid \beta_{i,j}(M) \neq 0 \text{ for some } j\}.$$

For a vertex $x_i \in V(G)$, the *neighbor set* of x_i is defined to be the set $N_G(x_i) = \{x_j \mid x_ix_j \in E(G)\}$. Moreover, the *closed neighborhood* of x_i is $N_G[x_i] = N_G(x_i) \cup \{x_i\}$. The cardinality of $N_G(x_i)$ is the *degree* of x_i and is denoted by $\deg_G(x_i)$. A vertex of degree one is called a *leaf* of G. The graph G is a *forest* if it does not have any cycle. The *distance* between x_i and x_j in G is defined to be the length of the shortest path between x_i and x_j in G. For a subset $W \subset V(G)$, $G \setminus W$ denotes the induced subgraph of G on the vertex set $V(G) \setminus W$. A subset A of $V(G)$ is said to be an *independent subset* of G if there are no edges among the vertices of A. The graph G is called *unmixed* (or *well-covered*) if all maximal independent sets of G have the same cardinality.

A *matching* in a graph is a subgraph consisting of pairwise disjoint edges. The cardinality of the largest matching in G is the *matching number* of G and is denoted by $\operatorname{match}(G)$. A matching in G is said to be a *maximal matching* if it is not properly contained in any matching of G. The *minimum matching number* of G, denoted by $\operatorname{min-match}(G)$, is the minimum cardinality of a maximal matching in G. A matching is said to be an *induced matching* if none the edges in the matching are joined by an edge in G. The largest size of an induced matching in G is called the *induced matching number* of G, denoted by $\operatorname{ind-match}(G)$. A graph G is called a *Cameron-Walker* graph if $\operatorname{ind-match}(G) = \operatorname{match}(G)$.

A set $A = \{x_{i1}x_{i2} \in E(G) \mid 1 \leq i \leq r\}$ is said to be an *ordered matching*, [5], if

1. A is a matching in G,
2. $\{x_{i1} \mid 1 \leq i \leq r\}$ is an independent set,
3. if $x_{i1}, x_{j2} \in E(G)$, then $i \leq j$.

The *ordered matching number* of G, denoted by $\operatorname{ord-match}(G)$, is defined to be

$$\operatorname{ord-match}(G) := \max\{|A| : \ A \text{ is an ordered matching of } G\}.$$

As already written in the introduction,

$$\operatorname{ind-match}(G) \leq \operatorname{reg}(S/I(G)) \leq \{\operatorname{ord-match}(G), \operatorname{min-match}(G)\} \leq \operatorname{match}(G),$$

There are several examples with inequality $\operatorname{min-match}(G) \leq \operatorname{ord-match}(G)$ and other examples with inequality $\operatorname{ord-match}(G) \leq \operatorname{min-match}(G)$.

A graph G is said to be *Cohen-Macaulay* (resp. *sequentially Cohen-Macaulay*) if $S/I(G)$ is Cohen-Macaulay (resp. sequentially Cohen-Macaulay).

A *bipartite graph* G is a graph with $V(G) = X \sqcup Y$ and $E(G) \subset X \times Y$. If $|X| = m$ and $|Y| = n$ and $E(G) = X \times Y$, then we say that G is a complete bipartite graph, and we denote G by $K_{m,n}$. If G is a bipartite graph, then G^{bc}, called the *bipartite complement*, is the bipartite graph with $V(G^{bc}) = V(G) = X \sqcup Y$ and $xy \in E(G^{bc})$ if and only if $xy \notin E(G)$. A subgraph H of a graph G is said to be a *spanning subgraph* if $V(H) = V(G)$. If $G = K_{m,n}$, then the set of connected spanning subgraphs of G is precisely the set of all connected bipartite graphs on $X \sqcup Y$, where $|X| = m$ and $|Y| = n$.

For the rest of the article, G always denotes a bipartite graph, without isolated vertices, on the finite vertex set $V(G) = X \sqcup Y$.

It was proved by Brodmann [3] that for any homogeneous ideal I in a graded ring R, $\operatorname{depth}(R/I^k)$ is a constant for $k \gg 0$. One consequence of ind-match and ord-match being equal is the constancy from the start of the depth function of powers of the *cover ideal* $J(G) = \bigcap_{x_ix_j \in E(G)}(x_i, x_j)$ of G, as we prove next.

Theorem 2.1. *Assume that G is a bipartite graph with $\operatorname{ind-match}(G) = \operatorname{ord-match}(G)$ and suppose $d = |V(G)|$. Let $J(G)$ denote the cover ideal of G. Then for every integer $k \geq 1$, we have*

$$\operatorname{depth}(S/J(G)^k) = d - \operatorname{ind-match}(G) - 1.$$

Proof. Since $\text{reg}(I(G)) = \text{ind-match}(G) + 1$ by inequalities in the Introduction, it follows from Terai's theorem [10, Proposition 8.1.10] that the projective dimension of $S/J(G)$ is equal to $\text{ind-match}(G) + 1$. Thus, Auslander-Buchsbaum formula implies that $\text{depth}(S/J(G)) = d - \text{ind-match}(G) - 1$. On the other hand, it follows from [8, Corollary 2.6] and [14, Theorem 3.2] that

$$d - \text{ind-match}(G) - 1 = \text{depth}(S/J(G)) \geq \text{depth}(S/J(G)^2) \geq \text{depth}(S/J(G)^3) \geq \cdots .$$

Moreover, we know from [14, Theorem 3.4] (see also [19, Theorem 3.1]) that $\text{depth}(S/J(G)^k) = d - \text{ord-match}(G) - 1$ for any $k \gg 0$. Since $\text{ord-match}(G) = \text{ind-match}(G)$, the assertion follows from the above inequalities. □

3 Equality of induced and ordered matching numbers

In this section, we characterize all bipartite graphs G with $\text{ord-match}(G) = \text{ind-match}(G)$. Before proving our general characterization (Theorem 3.8), we restrict ourselves to a special family of bipartite graphs for which the characterization has simpler formulation comparing with the general case. More precisely, we consider the class of sequentially Cohen-Macaulay bipartite graphs G. In [7, 21, 22], the authors studied the sequential Cohen-Macaulayness of $S/I(G)$ in terms of the combinatorial properties of G. Here, we study it in terms of the matching numbers. In the following theorem, we show that for a bipartite sequentially Cohen-Macaulay graph G, the equality $\text{ord-match}(G) = \text{match}(G)$ holds. As a consequence of this equality, we are able to characterize sequentially Cohen-Macaulay bipartite graphs G with $\text{ord-match}(G) = \text{ind-match}(G)$.

Theorem 3.1. *Let G be a bipartite graph. If G is sequentially Cohen-Macaulay, then* $\text{ord-match}(G) = \text{match}(G)$. *In particular, for a sequentially Cohen-Macaulay bipartite graph G, we have* $\text{ind-match}(G) = \text{ord-match}(G)$ *if and only if G is a Cameron-Walker graph.*

Proof. We prove the equality $\text{ord-match}(G) = \text{match}(G)$ by induction on $|V(G)|$. If $|V(G)| = 2$, then $\text{ord-match}(G) = \text{match}(G) = 1$. Assume by induction that if H is a sequentially Cohen-Macaulay bipartite graph with $|V(H)| < |V(G)|$, then $\text{ord-match}(H) = \text{match}(H)$. By [22, Corollary 3.11], there is a leaf $x \in V(G)$ such that $G \setminus N_G[x]$ is sequentially Cohen-Macaulay. Using [18, Lemma 2.1] and the induction hypothesis we have

$$\text{ord-match}(G) \geq \text{ord-match}(G \setminus N_G[x]) + 1 = \text{match}(G \setminus N_G[x]) + 1 = \text{match}(G)$$

where the last equality follows from the fact that x is a leaf of G. Thus, $\text{ord-match}(G) = \text{match}(G)$. The second assertion follows by observing that G is Cameron-Walker if and only if $\text{ind-match}(G) = \text{match}(G)$. □

The converse of the above Theorem does not hold. In fact, the following example shows that for each integer $k \geq 3$, there is a non-sequentially Cohen-Macaulay bipartite graph G with $\text{match}(G) = \text{ord-match}(G) = k$.

Example 3.2. *For any integer $k \geq 3$, let G_k be the graph obtained from a 4-cycle graph by attaching a path of length $2k - 3$ to exactly one of its vertices. Using induction on k, We show that G_k is not sequentially Cohen-Macaulay and* $\text{ord-match}(G_k) = \text{match}(G_k) = k$. *It is easy to see that* $\text{ord-match}(G_3) = \text{match}(G_3) = 3$. *Moreover, let x be the unique leaf of G_3 and let y be the unique neighbor of x. Then $G_3 \setminus N_{G_3}[y]$ is the 4-cycle graph that is not sequentially Cohen-Macaulay. Hence, by [22, Corollary 3.11], the graph G_3 is not sequentially Cohen-Macaulay. Now, suppose that $k \geq 4$. Let z be the be the unique leaf of G_k. Then $G_k \setminus N_{G_k}[z]$ is isomorphic to G_{k-1} that is not sequentially Cohen-Macaulay by induction hypothesis. Moreover, since z is a leaf of G_k, we have* $\text{match}(G_k) = \text{match}(G_{k_1}) + 1 = k$. *On the other hand, using [18, Lemma 2.1] and the induction hypothesis, we have*

$$\text{ord-match}(G_k) \geq \text{ord-match}(G_{k-1}) + 1 = k.$$

Thus, $\text{ord-match}(G_k) = k$.

In Example 3.2, we showed that for each integer $k \geq 3$, there is a non-sequentially Cohen-Macaulay bipartite graph G with $\text{match}(G) = \text{ord-match}(G) = k$. The following proposition shows that we cannot expect such an example when $k \leq 2$.

Proposition 3.3. *Let G be a bipartite graph with* $\operatorname{match}(G) = \operatorname{ord-match}(G) \leq 2$*. Then G is a sequentially Cohen-Macaulay graph.*

Proof. By contradiction, suppose G is not sequentially Cohen-Macaulay. It is known that any forest is sequentially Cohen-Macaulay (see for instance, [22, Theorem 1.3]). As a consequence, G is not a forest and therefore, has a cycle C. Since $\operatorname{match}(G) \leq 2$, the length of C is equal to four. Suppose $V(C) = \{w, x, y, z\}$ and $E(C) = \{wx, xy, yz, wz\}$. Since $\operatorname{match}(C) = 2$ and $\operatorname{ord-match}(C) = 1$, we conclude that $G \neq C$. This implies that there is an edge say wv connected to C, where v is a vertex in $V(G) \setminus V(C)$. As G is a bipartite graph, v is not adjacent to the vertices x and z. If there is a vertex $u \in V(G) \setminus V(C)$ such that $ux \in E(G)$ or $uz \in E(G)$, then the matching number of G would be at least three, which is a contradiction. Thus, $\deg_G(x) = \deg_G(z) = 2$. If G has a vertex $t \in V(G) \setminus V(C)$ whose distance from w or y is at least two, then again, the matching number of G would be at least three, which is a contradiction. Hence, $V(G) = N_G[w] \cup N_G[y]$. Since G is a bipartite graph two distinct vertices belonging to $N_G(w) \cup N_G(y)$ can not be adjacent. Hence, $V(G) = \{w, y\} \sqcup (V(G) \setminus \{w, y\})$ is the bipartition for the vertex set of G. Consequently, G is a subgraph of $K_{2,m}$, for some integer $m \geq 3$. If $G = K_{2,m}$, then $\operatorname{ord-match}(G) = 1$, which is a contradiction. Thus, $G \neq K_{2,m}$. So, G has a leaf, say s. Then the unique neighbor of s is either w or y. Without loss of generality, we may assume that $ws \in E(G)$. Then the graphs $G \setminus N_G[w]$ and $G \setminus N_G[s]$ are forests (as they do not contain the vertex w and so, the cycle C). Therefore, using [22, Theorem 1.3], the graphs $G \setminus N_G[w]$ and $G \setminus N_G[s]$ are sequentially Cohen-Macaulay. Hence, [22, Corollary 3.11] implies that G is a sequentially Cohen-Macaulay graph, which is a contradiction. □

In the following result, we give a characterization of bipartite graphs with $\operatorname{ord-match}(G) = \operatorname{ind-match}(G) = 1$. A classification of bipartite graphs with ord-match and ind-match being equal to an arbitrary positive integer strictly bigger than 1 is in Theorem 3.8.

Theorem 3.4. *Let G be a bipartite graph. Then* $\operatorname{ord-match}(G) = \operatorname{ind-match}(G) = 1$ *if and only if G is a complete bipartite graph.*

Proof. If $G = K_{m,n}$ for some $m, n \geq 1$, then clearly $\operatorname{ind-match}(G) = 1 = \operatorname{ord-match}(G)$. Conversely suppose G is not a complete bipartite graph. Write $V(G) = \{x_1, \ldots, x_m\} \sqcup \{y_1, \ldots, y_n\}$. Since G is not complete bipartite, there exist i, j such that $x_i y_j \notin E(G)$. By permuting the vertices, we may assume that $x_j y_j \in E(G)$. Choose an r such that $x_i y_r \in E(G)$. Then $\{x_j y_j, x_i, y_r\}$ is an ordered matching in G. Hence $\operatorname{ord-match}(G) > 1$. □

Definition 3.5. *For every subset I of X, let C_I be the set of all vertices in Y that have an edge to all the vertices in I and to none in $X \setminus I$. Set $c_I = |C_I|$.*

Sometimes we shorten the notation and write in the subscripts not the set but its elements, without commas. For example, we abbreviate $c_{\{i,j\}}$ to c_{ij}. If I, J are distinct subsets of X, then C_I and C_J are disjoint. In fact, each $y \in Y$ belongs to exactly one of C_I. Since G does not have isolated vertices, we have $C_\emptyset = \emptyset$.

In order to characterize bipartite graphs G with $\operatorname{ind-match}(G) = \operatorname{ord-match}(G)$, we need the following two propositions.

Proposition 3.6. *Let G be a bipartite graph and let r be a positive integer. Then the induced matching number of G is at least r if and only if there exist subsets $J_1, J_2, \ldots, J_r$ of X such that none of the J_i is contained in the union of the others and such that $c_{J_1} c_{J_2} \cdots c_{J_r} > 0$.*

Proof. Suppose that $\operatorname{ind-match}(G)$ is at least r. Then there exist $a_1, \ldots, a_r \in X$ and $b_1, \ldots, b_r \in Y$ such that $a_1 b_1, a_2 b_2, \ldots, a_r b_r$ is an induced matching. For each $i \in [r]$, let K_i be the subset of Y consisting of all vertices with an edge to a_i and with no edge to any of the other a_j with $j \neq i$. So $b_i \in K_i$ and K_i is not empty. Let J_i be the set of all vertices in X with an edge to all elements of K_i. Then $a_i \in J_i$ and $a_j \notin J_i$ for all $j \neq i$. The conclusion follows for these $J_1, \ldots, J_r$.

Conversely, suppose that there exist subsets $J_1, J_2, \ldots, J_r$ of X such that none is contained in the union of the others and such that $c_{J_1} c_{J_2} \cdots c_{J_r} > 0$. For each $i \in r$, let $a_i \in J_i$ that is not in the union of the other J_j. Since $c_{J_i} > 0$, there exists $b_i \in C_{J_i}$. Then by the definition of these sets, $a_1 b_1, a_2 b_2, \ldots, a_r b_r$ is an induced matching, so that $\operatorname{ind-match}(G)$ is at least r. □

Proposition 3.7. *Let G be a bipartite graph and let r be a positive integer. Then the following are equivalent:*

1. ord-match$(G) \geq r$.
2. *There exist subsets* $J_1, J_2, \ldots, J_r$ *of* X *such that* $c_{J_1} c_{J_2} \cdots c_{J_r} > 0$ *and for all* $i \in [r-1]$, J_i *is not contained in* $J_1 \cup \cdots \cup J_{i-1}$.

Moreover, ord-match$(G) \leq r$ *if and only if for all possible sets* $J_1, \ldots, J_r$ *with the properties as in (2), their union equals* X.

Proof. (1) $\Rightarrow$ (2): Suppose that ord-match(G) is at least r. Then there exist $a_1, \ldots, a_r \in X$ and $b_1, \ldots, b_r \in Y$ such that $a_1b_1, a_2b_2, \ldots, a_rb_r$ is an ordered matching. For each $i \in [r]$, let K_i be the subset of Y consisting of all the vertices with an edge to a_i and with no edge to any of the a_j with $j > i$. So $b_i \in K_i$ and K_i is not empty. Let J_i be the set of all the vertices in X with an edge to all the elements of K_i. Then $a_i \in J_i$ and $a_j \notin J_i$ for all $j > i$. Thus (2) follows for these $J_1, \ldots, J_r$.

(2) $\Rightarrow$ (1): This is trivial for $r = 1$, so we may assume that $r > 1$. Let $J_1, J_2, \ldots, J_r$ be subsets of X such that for all $i \in [r-1]$, J_i is not contained in $J_1 \cup \cdots \cup J_{i-1}$ and such that $c_{J_1} c_{J_2} \cdots c_{J_r} > 0$. For each $i \in r$, let $a_i \in J_i \setminus J_1 \cup \cdots \cup J_{i-1}$. Since $c_{J_i} > 0$, there exists $b_i \in C_{J_i}$. By definition of these sets, $a_1b_1, a_2b_2, \ldots, a_rb_r$ is an ordered matching.

If there exists $a \in X \setminus J_1 \cup \cdots \cup J_r$, then since a is not an isolated vertex, there is a vertex $b \in Y$ which is adjacent to a. By definition of the sets C_I, there is no edge between a and $b_1, \ldots, b_r$. Thus $a_1b_1, a_2b_2, \ldots, a_rb_r, ab$ is an ordered matching. Thus ord-match(G) is strictly bigger than r.

If ord-match(G) is at least $r+1$, then by the equivalence of (1) and (2) the union of $J_1, \ldots, J_r$ cannot be X. This proves the last part. □

Using Propositions 3.6 and 3.7, we can classify all bipartite graphs for which ord-match and ind-match are equal.

Theorem 3.8. *(Classification) Let* G *be a bipartite graph and let* $r > 1$ *be a positive integer. Let* $J_1, \ldots, J_z$ *be all the subsets* I *of* X *for which* c_I *is positive. Then* G *has ind-match and ord-match equal to* r *if and only if the following conditions are satisfied:*

1. $z \geq r$.
2. *There exist distinct* $j_1, \ldots, j_r \in [z]$ *such that none of the* J_{j_i} *is contained in the union of the remaining* J_{j_k}.
3. *For all* $j_1, \ldots, j_r \in [z]$, *if for each* $i \in [r-1]$, J_{j_i} *is not contained in* $J_{j_1} \cup \cdots \cup J_{j_{i-1}}$, *then* $J_{j_1} \cup \cdots \cup J_{j_r} = X$.

Proof. First suppose that ind-match(G) and ord-match(G) equal r. Then by Theorem 3.6, (1) and (2) hold. Theorem 3.7 implies (3).

Now suppose that the three conditions are satisfied. Then by Theorem 3.6, induced matching number of G is at least r and ord-match(G) is at most r. But ord-match(G) is greater than or equal to ind-match(G). So, ind-match(G) and ord-match(G) are both equal to r. □

We can say more in case ord-match(G) and ind-match(G) are both equal to 2:

Theorem 3.9. *(Classification for* $r = 2$*) Let* G *be a bipartite graph. Let* $J_1, \ldots, J_z$ *be all the subsets* I *of* X *for which* c_I *is positive. Then* G *has ind-match and ord-match equal to* 2 *if and only if the following conditions are satisfied:*

1. $z \geq 2$.
2. *There exist distinct* $i, j \in [z]$ *such that neither* J_i *nor* J_j *is contained in the other.*
3. *For any two distinct* $i, j \in [z]$, $J_i \cup J_j = X$.

Furthermore, if some J_i *and* J_j *are disjoint, then* z *equals* 2 *or* 3*; the later exactly when the graph is connected and in this case* $c_X > 0$.

Proof. Equivalence follows from Theorem 3.8. Note that part (3) here is equivalent to part (3) in Theorem 3.8 because J_i and J_j are distinct sets.

Let $k \in [z] \setminus \{i, j\}$. Then by condition (3), J_k contains the complement of J_i and J_j, so that if J_i and J_j are disjoint, then J_k contains X. This means that J_k has to equal X. Thus either $z = 3$ and G is connected, or else $z = 2$ and G is not connected. □

As a consequence of the above theorem, we obtain an explicit graph theoretic characterization of bipartite graphs G with $\text{ind-match}(G) = \text{ord-match}(G) = 2$.

Corollary 3.10. *Let G be a bipartite graph. Then* $\text{ind-match}(G) = \text{ord-match}(G) = 2$ *if and only if the bipartite complement G^{bc} of G is the disjoint union of complete bipartite graphs $H_1, \dots, H_s$ with $s \geq 2$ such that at least two of the H_i are not isolated vertices.*

Proof. Let $J_1, \dots, J_z$ be all the subsets I of X for which c_I is positive. Then for any pair of distinct integers $i, j \in [z]$, we have $J_i \cup J_j = X$ if and only if for any choice of vertices $y \in C_{J_i}$ and $y' \in C_{J_j}$, the equality $N_{G^{bc}}(y) \cap N_{G^{bc}}(y') = \emptyset$ holds. Thus, Condition (3) of Theorem 3.9 (and in fact, by Proposition 3.7, the inequality $\text{ord-match}(G) \leq 2$) is equivalent to say that G^{bc} is the disjoint union of complete bipartite graphs. Moreover, note that $\text{ind-match}(G) \geq 2$ if and only if $\text{ind-match}(G^{bc}) \geq 2$. Since G^{bc} is the disjoint union of complete bipartite graphs, we deduce at least two of these components are not isolated vertices. □

Let G be a bipartite graph. It is known that if G is either unmixed or sequentially Cohen-Macaulay, then $\text{reg}(S/I(G)) = \text{ind-match}(G)$. In the following theorem, for every pair of integers r, m with $2 \leq r \leq m$, we construct a bipartite graph $G_{r,m}$ which is neither sequentially Cohen-Macaulay nor unmixed, moreover, $\text{reg}(S/I(G_{r,m})) = \text{ind-match}(G_{r,m}) = \text{ord-match}(G_{r,m}) = r$ and $\text{min-match}(G_{r,m}) = m$. Hence, the class of graphs we study in this paper is not contained in two general classes of bipartite graphs for which the regularity of edge ideals is known. Furthermore, Theorem 3.11 shows that the family of graphs G with $\text{ind-match}(G) = \text{ord-match}(G)$ is far from the class of graphs considered in [12].

Theorem 3.11. *Let $2 \leq r \leq m$ be positive integers. Then there is a connected bipartite graph $G_{r,m}$ such that*

1. $\text{reg}(S/I(G_{r,m})) = \text{ind-match}(G_{r,m}) = \text{ord-match}(G_{r,m}) = r$ *and* $\text{min-match}(G_{r,m}) = m$.
2. *$G_{r,m}$ does not have any leaf (and hence $G_{r,m}$ is not a sequentially Cohen-Macaulay graph).*
3. *$G_{r,m}$ is not an unmixed graph.*

Proof. Set $X = \{x_1, \dots, x_{m+1}\}$ and $Y = \{y_1, \dots, y_{m+1}\}$. Let $G_{r,m}$ be the bipartite graph with vertex set $V(G_{r,m}) = X \sqcup Y$ and edge set

$$E(G_{r,m}) = \bigcup_{1 \leq j \leq r-1} \{x_1y_j, x_{j+1}y_j\} \cup \bigcup_{r \leq j \leq m} \{x_1y_j, x_iy_j \mid r+1 \leq i \leq m+1\} \cup \{x_iy_{m+1} \mid 1 \leq i \leq m+1\}.$$

For the convenience of the reader, we illustrate the graph $G_{r,m}$ with a sample picture below:

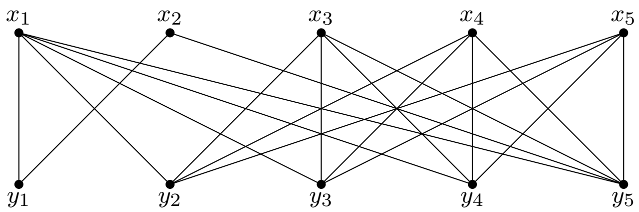

Fig. 1 $G_{2,4}$

It follows directly from the definition that

$$N_{G_{r,m}}(y_j) = \begin{cases} \{x_1, x_{j+1}\}, & 1 \leq j \leq r-1; \\ \{x_{r+1}, \dots, x_{m+1}\} \cup \{x_1\}, & r \leq j \leq m; \\ X, & j = m+1. \end{cases}$$

Clearly, $G_{r,m}$ is a connected bipartite graph that does not have any leaf. Therefore using [22, Corollary 2.10], it is not a sequentially Cohen-Macaulay graph. Moreover, it is easy to check that the set $\{x_2, \dots, x_r, y_r, \dots, y_m\}$ is a maximal independent set of $G_{r,m}$ with cardinality $m < |X|$. Thus, $G_{r,m}$ is not an unmixed graph. Hence, we only need to verify condition (i).

Set

$$J_1 = N_{G_{r,m}}(y_1), J_2 = N_{G_{r,m}}(y_2), \cdots, J_r = N_{G_{r,m}}(y_r), J_{r+1} = X.$$

Note that for a subset $J \subseteq X$, we have $c_J > 0$ if and only if $J \in \{J_1, \ldots, J_{r+1}\}$. Then it follows from Theorem 3.8 that

$$\text{reg}(S/I(G_{r,m})) = \text{ind-match}(G_{r,m}) = \text{ord-match}(G_{r,m}) = r.$$

Now, we compute min-match$(G_{r,m})$. It is obvious that

$$\{y_2x_3, y_3x_4, \ldots, y_{m-1}x_m, y_mx_1, y_{m+1}x_2\}$$

is a maximal matching in $G_{r,m}$ of size m. Hence, min-match$(G_{r,m}) \leq m$. Suppose M is a maximal matching in $G_{r,m}$ with $|M| \leq m-1$. Thus, there are two distinct vertices $y_i, y_j \in Y \setminus V(M)$. Without loss of generality, we may assume that $i < j$. We consider the following cases.

Case 1. Suppose $i, j \leq r-1$, then at least one of the vertices x_{i+1} and x_{j+1} does not belong to $V(M)$, as otherwise the edges $y_{m+1}x_{i+1}$ and $y_{m+1}x_{j+1}$ belong to M, which contradicts the definition of a matching. For example, suppose $x_{i+1} \notin V(M)$. Then $M \cup \{y_ix_{i+1}\}$ is a matching in $G_{r,m}$. Thus, M is not a maximal matching.

Case 2. Suppose $1 \leq i \leq r-1$ and $r \leq j \leq m$. Since M is a maximal matching in $G_{r,m}$, we deduce that $N_{G_{r,m}}(y_i) \cup N_{G_{r,m}}(y_j) \subseteq V(M)$. This implies that

$$\{x_1, x_{i+1}, x_{r+1}, \ldots, x_{m+1}\} \subseteq V(M).$$

Since $x_{i+1} \in V(M)$ and $y_i \notin V(M)$, the edge $y_{m+1}x_{i+1}$ must belong to M. Then since $x_{r+1}, \ldots, x_{m+1} \in V(M)$, we conclude that $y_r, \ldots, y_m \in V(M)$. Moreover, according to Case 1, we may assume that $\{y_1, \ldots, y_{r-1}\} \setminus \{y_i\} \subseteq V(M)$. Therefore, $Y \setminus \{y_i\} \subseteq V(M)$. So, $|M| \geq m$, which is a contradiction.

Case 3. Suppose $r \leq i, j \leq m$. Since M is a maximal matching, we must have $N_{G_{r,m}}(y_i) \cup N_{G_{r,m}}(y_j) \subseteq V(M)$. In particular, $\{x_{r+1}, \ldots, x_{m+1}\} \subseteq V(M)$. So, in $G_{r,m}$, there are at least $m-r+1$ vertices other that y_i and y_j, which are adjacent to at least one of the vertices $x_{r+1}, \ldots, x_{m+1}$. But this is not the case according to the construction of $G_{r,m}$.

Case 4. Suppose $j = m+1$. Since M is a maximal matching, $N_{G_{r,m}}(y_j) \subseteq V(M)$. Therefore, $X \subseteq V(M)$. This implies that $|M| \geq m+1$, which is a contradiction. □

4 Set-up of k-sequences and J-sets

The notation in this section sets the stage for the counting of graphs in the next section.

Definition 4.1. *A* **k-sequence** *is a strictly decreasing sequence of integers* $\{k_0, k_1, \ldots, k_z\}$ *with* $z \geq 2$, $k_z = 0$, *and for all* $l = 2, 3, \ldots, z$, $k_{l+1} \geq 2k_l - k_{l-1}$.

Observe that if a k-sequence has at least four terms, then the truncation of the sequence by removing the first term is also a k-sequence. A truncated k-sequence is a truncation of infinitely many k-sequences.

Examples 4.2. When $k_0 = 2$, the only possible k-sequence is $\{2, 1, 0\}$.

The only possible k-sequences with $k_0 = 3$ are $\{3, 2, 1, 0\}$ and $\{3, 1, 0\}$.

When $k_0 = 4$, there are four k-sequences: $\{4, 3, 2, 1, 0\}$; $\{4, 2, 1, 0\}$; $\{4, 2, 0\}$ and $\{4, 1, 0\}$.

When $k_0 = 5$, there are six k-sequences: $\{5, 4, 3, 2, 1, 0\}$; $\{5, 3, 2, 1, 0\}$; $\{5, 3, 1, 0\}$; $\{5, 2, 1, 0\}$; $\{5, 2, 0\}$ and $\{5, 1, 0\}$.

Definition 4.3. *For any non-negative integer* n *and any* k*-sequence* $\{k_0, k_1, \ldots, k_z\}$ *with* $z \geq 3$, *set*

$$D(n; \{k_0, \ldots, k_z\}) = \begin{cases} \displaystyle\sum_{i=1}^{n-1} \sum_{j=1}^{n-i} C(n-i, j; k_1, \ldots, k_z), & \text{if } k_2 > 2k_1 - k_0; \\ \displaystyle\sum_{i=1}^{n-1} \sum_{j=1}^{\min\{n-i,i\}} C(n-i, j; k_1, \ldots, k_z), & \text{if } k_2 = 2k_1 - k_0, \end{cases}$$

where C is defined recursively as follows:

$$C(n,i;k_1,\ldots,k_z)=\begin{cases} n-i+1, & \text{if } z=3 \text{ and } k_3>2k_2-k_1;\\ \min\{n-i,i\}+1, & \text{if } z=3 \text{ and } k_3=2k_2-k_1;\\ \sum_{j=0}^{n-i} C(n-i,j;k_1,\ldots,k_z), & \text{if } z>3 \text{ and } k_3>2k_2-k_1;\\ \sum_{j=0}^{\min\{n-i,i\}} C(n-i,j;k_1,\ldots,k_z), & \text{if } z>3 \text{ and } k_3=2k_2-k_1. \end{cases}$$

Here is a full expansion of D:

$$D(n;\{k_0,\ldots,k_z\})=\sum_{i_1=1}^{n-1}\sum_{i_2=1}^{u_2}\sum_{i_3=0}^{u_3}\sum_{i_4=0}^{u_4}\cdots\sum_{i_z=0}^{u_z}1,$$

where the upper bounds u_l equal either $n-i_1-\cdots-i_{l-1}$ or $\min\{n-i_1-\cdots-i_{l-1},i_{l-1}\}$, depending on whether strict inequality or equality hold in $k_{l+1}\geq 2k_l-k_{l-1}$. The sum does not depend on the specific values in the k-sequence, but only on whether k_{l+1} is strictly greater than or equal to $2k_l-k_{l-1}$. Note that the counters i_1 and i_2 start with 1, but all other counters i_l start with 0.

Definition 4.4. *Let K be a finite set of k-sequences, each k-sequence in K having at least four terms (so $z\geq 3$ for each sequence in K). Let W be the ordered set from largest to smallest of all non-negative integers that appear in all sequences in K. So $0\in W$. For any positive integer $l\leq |W|$ and any $\underline{k}\in K$, let $o(l,\underline{k})$ be the subscript index of the lth element of W in $\underline{k}$. For any integer $n\geq 2$ we define $D_K(n)$ to be 0 if the first three terms in the sequences in K are not the same, and otherwise $D_K(n)$ is the sum of the common summands in the D-sums (from Theorem 4.3) coming from the W-terms for all sequences in K. Explicitly,*

$$D_K(n)=\sum_{i_1=1}^{n-1}\sum_{i_2=1}^{v_2}\sum_{i_3=0}^{v_3}\sum_{i_4=0}^{v_4}\cdots\sum_{i_{|W|}=0}^{v_{|W|}}1,$$

where for $l=2,\ldots,|W|$, v_l is the minimum of all the upper bounds in $D(n;\underline{k})$ in the $o(l,\underline{k})$th summation as $\underline{k}$ varies over all sequences in K. These upper bounds depend also on the summation indices not appearing in W, and those hidden indices are all set to their common hidden value 0.

For example, $D_K(n)=0$ for the set K consisting of two k-sequences $\{6,3,2,1,0\}$ and $\{6,3,1,0\}$ because their third terms are not the same.

For a more interesting example, let K consist of $\underline{k}=\{6,4,2,1,0\}$ and $\underline{l}=\{6,4,2,0\}$. Then $W=\{6,4,2,0\}$, $o(6,\underline{k})=o(6,\underline{l})=0$, $o(4,\underline{k})=o(4,\underline{l})=1$, $o(2,\underline{k})=o(2,\underline{l})=2$, $o(0,\underline{k})=4$, $o(0,\underline{l})=3$. We write out explicitly

$$D(n;\{6,4,2,1,0\})=\sum_{i_1=1}^{n-1}\sum_{i_2=1}^{\min\{n-i_1,i_1\}}\sum_{i_3=0}^{n-i_1-i_2}\sum_{i_4=0}^{\min\{n-i_1-i_2-i_3,i_3\}}1$$

and

$$D(n;\{6,4,2,0\})=\sum_{i_1=1}^{n-1}\sum_{i_2=1}^{\min\{n-i_1,i_1\}}\sum_{i_3=0}^{\min\{n-i_1-i_2,i_2\}}1=\sum_{i_1=1}^{n-1}\sum_{i_2=1}^{\min\{n-i_1,i_1\}}\sum_{i_3=0}^{0}\sum_{i_4=0}^{\min\{n-i_1-i_2,i_2\}}1,$$

where in the last line we inserted the trivial summation $\sum_{i_3=0}^{0}$ to correspond to the entry 1 in $\underline{k}$ that does not exist in $\underline{l}$, and we correspondingly renamed the old i_3 as new i_4. From these common rewritings of the two summations we read off

$$D_K(n)=\sum_{i_1=1}^{n-1}\sum_{i_2=1}^{\min\{n-i_1,i_1\}}\sum_{i_3=0}^{0}\sum_{i_4=0}^{\min\{n-i_1-i_2-i_3,i_3\}}1=\sum_{i_1=1}^{n-1}\sum_{i_2=1}^{\min\{n-i_1,i_1\}}1.$$

Incidentally, this sum simplifies to $\frac{1}{2}\lceil n/2\rceil(\lceil n/2\rceil - 1) + \frac{1}{2}\lfloor n/2\rfloor(\lfloor n/2\rfloor + 1)$.

The non-trivial part of the last simplification involves exactly two non-trivial summations, and that is due to the two sequences having exactly two J-sets in common; J-sets are introduced next.

We will use J-sets in the counting of **connected bipartite** graphs with ind-match and ord-match both equal to 2.

Definition 4.5. *Let $z \geq 2$ and let $J_1, J_2, \ldots, J_z$ be distinct non-empty proper subsets of X with $|J_1| \leq |J_2| \leq \cdots \leq |J_z| < |X| = m$ such that $J_i \cup J_j = X$ for all distinct $i, j \in [z]$. (Unlike in Theorem 3.8 and Theorem 3.9, here the sets are proper.) All proper subsets I of X for which $c_I > 0$ for a given connected bipartite graph on $X \sqcup Y$ appear on this list (see Theorem 3.5), but this list of subsets may contain also some special subsets I for which $c_I = 0$. By possibly reindexing, we assume that $c_{J_1}c_{J_2} > 0$, that J_1 has the smallest cardinality among the J_i and that J_2 has the smallest possible cardinality among the remaining J_i.*

When J_1 and J_2 are disjoint, then by Theorem 3.9, $z = 2$ and c_X is positive, and otherwise c_X is only potentially positive.

For $l = 0, \ldots, z$ we set $K_l = J_1 \cap \cdots \cap J_l$ and $k_l = |K_l|$. So $k_0 = m$. Without loss of generality $J_1 = \{1, 2, \ldots, k_1\}$. Since J_2 contains exactly k_2 elements of J_1 and all elements of $X \setminus J_1$, without loss of generality $J_2 = \{1, 2, \ldots, k_2\} \cup \{k_1 + 1, k_1 + 2, \ldots, m\}$. Since J_2 does not contain J_1, necessarily $k_2 < k_1$. If $z \geq 3$, then J_3 must contain $\{k_2 + 1, k_2 + 2, \ldots, m\}$ and also exactly k_3 of the elements of $J_1 \cap J_2 = \{1, 2, \ldots, k_2\}$. So without loss of generality $J_3 = \{1, 2, \ldots, k_3\} \cup \{k_2+1, k_2+2, \ldots, m\}$. Since J_3 is a proper subset, necessarily $k_3 < k_2$. Similarly, for all $l \in [z]$, $J_l = \{1, 2, \ldots, k_l\} \cup \{k_{l-1}+1, k_{l-1}+2, \ldots, m\}$ and $k_l < k_{l-1}$. With this it is clear now that $k_0 = m > k_1 > k_2 > \cdots > k_l$. If $k_z > 0$, we can add to the sets also the one set $\{k_z + 1, k_z + 2, \ldots, m\}$, and then by increasing z by 1, we may assume that $k_z = 0$. With this we give our final requirement for the J-sets: we require $J_1 \cap \cdots \cap J_z = \emptyset$.

Note that $|J_l| = m - k_{l-1} + k_l$. So the condition $|J_1| \leq |J_2| \leq \cdots \leq |J_z| < |X| = m$ is equivalent to the condition $k_{l+1} \geq 2k_l - k_{l-1}$ for all $l = 1, 2, \ldots, z - 1$. Thus $k_0, \ldots, k_z = 0$ is a k-sequence (see Theorem 4.1). In addition, for any l, $|J_l| < |J_{l+1}|$ is equivalent to $k_{l+1} < 2k_l - k_{l-1}$.

Thus the possible unlabeled subsets $J_1, \ldots, J_z$ of X with $J_1 \cap \cdots \cap J_z = \emptyset$ determine the k-sequence $\{k_0, \ldots, k_z = 0\}$, and vice versa, a k-sequence defines up to relabeling the J-sets (with $J_1 \cap \cdots \cap J_z = \emptyset$).

The sets J_1 and J_2 in the definition contribute edges to G that will make ind-match and ord-match equal to 2, and the remaining $J_3, \ldots, J_z$ may only potentially contribute an edge.

Examples 4.6. The k-sequence $\{2, 1, 0\}$ corresponds to the sets $J_1 = \{1\}$, $J_2 = \{2\}$.

The k-sequence $\{3, 2, 1, 0\}$ corresponds to the sets $J_1 = \{1, 2\}$, $J_2 = \{1, 3\}$, $J_3 = \{2, 3\}$. The k-sequence $\{3, 1, 0\}$ corresponds to the sets $J_1 = \{1\}$, $J_2 = \{2, 3\}$.

The k-sequence $\{4, 3, 2, 1, 0\}$ corresponds to the sets $J_1 = \{1, 2, 3\}$, $J_2 = \{1, 2, 4\}$, $J_3 = \{1, 3, 4\}$, $J_4 = \{2, 3, 4\}$; the k-sequence $\{4, 2, 1, 0\}$ corresponds to the sets $J_1 = \{1, 2\}$, $J_2 = \{1, 3, 4\}$, $J_3 = \{2, 3, 4\}$; the k-sequence $\{4, 2, 0\}$ corresponds to the sets $J_1 = \{1, 2\}$, $J_2 = \{3, 4\}$; and the k-sequence $\{4, 1, 0\}$ corresponds to the sets $J_1 = \{1\}$, $J_2 = \{2, 3, 4\}$.

The k-sequence $\{6, 4, 2, 1, 0\}$, corresponds to the sets $J_1 = \{1, 2, 3, 4\}$, $J_2 = \{1, 2, 5, 6\}$, $J_3 = \{1, 3, 4, 5, 6\}$ and $J_4 = \{2, 3, 4, 5, 6\}$, and the k-sequence $\{6, 4, 2, 0\}$, corresponds to $J_1 = \{1, 2, 3, 4\}$, $J_2 = \{1, 2, 5, 6\}$, $J_3 = \{3, 4, 5, 6\}$. So these two k-sequences have exactly two J-sets in common, which explains why D_K computed earlier has only two summations, all other indices varying trivially.

5 Bipartite graphs with induced and ordered matching 2

The aim of this section is to count the number of connected bipartite graphs G whose ord-match and ind-match are equal to 2. For the rest of the section, let G be denote a connected bipartite graph. Suppose $X \sqcup Y$ is the bipartition for the vertex set of G. Set $m := |X|$ and $n := |Y|$. So G is a connected spanning subgraph of $K_{m,n}$. Without loss of generality we assume that $m \leq n$. Moreover, suppose that ind-match(G) = ord-match$(G) = 2$. As the main result of this section, in Theorem 5.4, we provide an inclusion-exclusion type formula for the number of such graphs. As a consequence, in Corollary 5.5, we obtain a closed formula for the number of these graphs when $m \leq 4$.

We first count a special type of connected bipartite graphs whose ind-match and ord-match are equal to 2.

Proposition 5.1. *Assume that I is a proper subset of X and that $n = |Y| \geq 3$. The number of connected spanning subgraphs of $K_{m,n}$ with* ind-match(G) = ord-match$(G) = 2$ *for which $c_I c_{X\setminus I} > 0$ equals $\binom{n-1}{2}$ if $|I| \neq n/2$ and $\lfloor \frac{n}{2} \rfloor \left(\lceil \frac{n}{2} \rceil - 1 \right)$ otherwise.*

Proof. By Theorem 3.9, I, $X \setminus I$ and X are the only subsets K of X for which c_K is positive.

When $|I| \neq |X \setminus I|$, then the number of such graphs equals

$$\sum_{c_I=1}^{n-2} \sum_{c_{X\setminus I}=1}^{n-c_I-1} 1 = \sum_{i=1}^{n-2} \sum_{j=1}^{n-i-1} 1 = \sum_{i=1}^{n-2} (n-i-1) = \binom{n-1}{2},$$

and if $|I| = |X \setminus I|$, then to not count unlabeled graphs twice, the number of such graphs equals

$$\sum_{c_I=1}^{n-2} \sum_{c_{X\setminus I}=1}^{\min\{c_I, n-c_I-1\}} 1 = \sum_{i=1}^{n-2} \sum_{j=1}^{\min\{i,n-i-1\}} 1,$$

which by Theorem 6.1 equals $\lfloor \frac{n}{2} \rfloor \left(\lceil \frac{n}{2} \rceil - 1 \right)$. □

The next proposition shows that the number of connected bipartite graphs G with ind-match(G) = ord-match$(G) = 2$ is closely related to the notion of k-sequences defined in Section 4.

Proposition 5.2. *Let $z \geq 3$ and $k_0 = m, k_1, \ldots, k_z = 0$ a k-sequence with corresponding sets $J_1, \ldots, J_z \subsetneq X = \{1, \ldots, m\}$ as in Theorem 4.5. Then $D(n; \{m, k_1, \ldots, k_z\})$ equals the number of connected bipartite graphs with edges from vertices in X to vertices in the set with n elements for which ord-match and ind-match are both equal to 2, and for which $c_{J_1} c_{J_2} > 0$ and $c_I = 0$ for all proper subsets I of X different from $J_1, \ldots, J_z$.*

Proof. The count is of unlabelled sets of vertices. With the set-up as in Theorem 4.5, we are counting the number of $(z+1)$-tuples $(c_{J_1}, \ldots, c_{J_z}, c_X)$ for which $c_{J_1}, c_{J_2} > 0$, $(\sum_{l=1}^{z} c_{J_l}) + c_X = |Y| = n$, and $c_I = 0$ for all proper subsets I of X other than $J_1, \ldots, J_z$. In order to not count identical unlabeled graphs twice, we need a further restriction: whenever $J_l, J_{l+1}, \ldots, J_{l+k}$ have the same number of elements, then we may assume that $c_{J_l} \geq c_{J_{l+1}} \geq \cdots \geq c_{l_k}$. If J_l and J_{l+1} have a different number of elements, then there is no restriction on the comparison of sizes of c_{J_l} and $c_{J_{l+1}}$. Thus the count equals precisely $D(n; \{m, k_1, \ldots, k_z\})$ from Theorem 4.3. □

In order to count all connected bipartite graphs with ord-match and ind-match both being equal to 2, we thus need a list of all k-sequences (or equivalently, of corresponding J-sets for one of the two sets of vertices), count the graphs for each of the k-sequences, and then use inclusion-exclusion to remove multiple counts of some of the graphs. This will be done in the next theorem.

We set some notation needed for the next theorem.

Notation 5.3. *For $m \leq n$, define*

$$N(m,n) = \left\lfloor \frac{m-1}{2} \right\rfloor \binom{n-1}{2} + (\delta_{m,even}) \left\lfloor \frac{n}{2} \right\rfloor \left(\left\lceil \frac{n}{2} \right\rceil - 1 \right) + \sum_K (-1)^{|K|-1} D_K(n),$$

where K varies over all non-empty subsets of the set of all k-sequences starting with $k_0 = m$ and with at least 4 terms, where D_K is from Theorem 4.4, and $\delta_{m,even}$ is 1 if m is even and 0 otherwise.

Theorem 5.4. *If $m < n$, then $N(m,n)$ equals the number of connected non-isomorphic spanning subgraphs of $K_{m,n}$ with both ord-match and ind-match being equal to 2.*

When $m = n$, then $N(m,n)$ counts the graphs as before with vertices unlabeled but treating the two sets of vertices as labeled.

Proof. Every graph under discussion corresponds to a unique collection of proper subsets $J_1, J_2, \ldots, J_z$ with non-decreasing cardinalities and their corresponding k-sequence $m = k_0, k_1, \ldots, k_z = 0$.

When $z = 2$, then J_1 and J_2 are disjoint, in which case Theorem 5.1 applies. Since $|J_1| \leq |J_2|$, necessarily $|J_1| \leq m/2$. The $\lfloor \frac{m-1}{2} \rfloor$ cases in which $|J_1| < |J_2|$ each give $\binom{n-1}{2}$ graphs, and when m is even, the case $|J_1| = |J_2|$ gives E graphs.

The cases without disjoint J-subsets are covered in Theorem 5.2. □

In the following corollary, we obtain a closed formula for the number of connected bipartite graphs with $\text{ind-match}(G) = \text{ord-match}(G) = 2$ when $\min\{|X|, |Y|\} \leq 4$.

Corollary 5.5. *For $2 \leq m \leq 4$, the number $N(m, n)$ of connected non-isomorphic spanning subgraphs of $K_{m,n}$ with ind-match and ord-match both equal to 2 is:*

$$\begin{cases} E, & \text{if } m = 2 \text{ and necessarily } n \geq 3; \\ 3, & \text{if } m = n = 3; \\ \binom{n-1}{2} + T, & \text{if } m = 3 < n; \\ 14, & \text{if } m = n = 4; \\ \binom{n-1}{2} + E + U + V, & \text{if } m = 4 < n, \end{cases}$$

where

$$\begin{aligned} E &= \left\lfloor \frac{n}{2} \right\rfloor \left(\left\lceil \frac{n}{2} \right\rceil - 1 \right), \\ T &= -\frac{25}{144} - \frac{n}{12} + \frac{7n^2}{24} + \frac{n^3}{36} + \frac{(-1)^n}{16} + \frac{3+\sqrt{3}i}{54} \left(\frac{-1-\sqrt{3}i}{2} \right)^n + \frac{3-\sqrt{3}i}{54} \left(\frac{-1+\sqrt{3}i}{2} \right)^n, \\ U &= \frac{-1}{16} + \frac{5}{12}n + \frac{5}{8}n^2 + \frac{1}{12}n^3 + \frac{1}{16}(-1)^n, \\ V &= \frac{-641 - 486n + 996n^2 + 132n^3 + 6n^4}{3456} + \frac{(11+2n)(-1)^n}{128} + \frac{(1-i)i^n + (1+i)(-i)^n}{32} \\ &\quad + \frac{1+\sqrt{3}i}{54} \left(\frac{-1-\sqrt{3}i}{2} \right)^n + \frac{1-\sqrt{3}i}{54} \left(\frac{-1+\sqrt{3}i}{2} \right)^n. \end{aligned}$$

Proof. The justification for $m = 2$ is that there is only one k-sequence with $z = 2$, and Theorem 5.1 applies.

The cases $m = n = 3$ and $m = n = 4$ are special because the two vertex sets can be switched, and the general formula does not accommodate such switching. These cases can be verified manually and we do not show the work here.

For $m = 3 < n$, the k-sequence $3, 1, 0$ is where then J_1 and J_2 are disjoint and Theorem 5.1 applies; and for the k-sequence $3, 2, 1, 0$, the number of graphs equals the sum

$$\sum_{i=1}^{n-1} \sum_{j=1}^{\min\{n-i,i\}} \sum_{k=0}^{\min\{n-i-j,j\}} 1,$$

which by Theorem 6.2 equals T.

For $m = 4 < n$, the k-sequence $4, 1, 0$ contributes $\binom{n-1}{2}$ graphs and the k-sequence $4, 2, 0$ contributes E graphs by Theorem 5.1. The contribution of the k-sequence $4, 2, 1, 0$ is

$$\sum_{i=1}^{n-1} \sum_{j=1}^{n-i} \sum_{k=0}^{\min\{n-i-j,j\}} 1,$$

which by Theorem 6.3 equals U. Finally, the k-sequence $4, 3, 2, 1, 0$ contributes

$$\sum_{i=1}^{n-1} \sum_{j=1}^{\min\{n-i,i\}} \sum_{k=0}^{\min\{n-i-j,j\}} \sum_{l=0}^{\min\{n-i-j-k,k\}} 1,$$

which by Theorem 6.4 equals V. □

6 Appendix: Some explicit summations

The closed forms of the sums in this appendix are used in the previous section.

Proposition 6.1. *For any integer $n \geq 3$,*

$$\sum_{i=1}^{n-2} \sum_{j=1}^{\min\{i,n-i-1\}} 1 = \left\lfloor \frac{n}{2} \right\rfloor \left(\left\lceil \frac{n}{2} \right\rceil - 1 \right).$$

Proof. Inequality $i \leq n-i-1$ holds if and only if $i \leq \lfloor \frac{n-1}{2} \rfloor$, so the count simplifies to:

$$\begin{aligned}
\sum_{i=1}^{n-2} \sum_{j=1}^{\min\{i,n-i-1\}} 1 &= \sum_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{j=1}^{i} 1 + \sum_{i=\lfloor \frac{n-1}{2} \rfloor+1}^{n-2} \sum_{j=1}^{n-i-1} 1 \\
&= \sum_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} i + (n-1) \left(n-2- \left\lfloor \frac{n-1}{2} \right\rfloor \right) - \sum_{i=\lfloor \frac{n-1}{2} \rfloor+1}^{n-2} i \\
&= \left\lfloor \frac{n-1}{2} \right\rfloor \left(\left\lfloor \frac{n-1}{2} \right\rfloor + 1 \right) + (n-1) \left(\frac{n}{2} - \left\lfloor \frac{n-1}{2} \right\rfloor - 1 \right) \\
&= \begin{cases} k^2-k, & \text{if } n=2k; \\ k^2, & \text{if } n=2k+1, \end{cases} \\
&= \left\lfloor \frac{n}{2} \right\rfloor^2 + \left\lfloor \frac{n}{2} \right\rfloor \left(\left\lceil \frac{n}{2} \right\rceil - \left\lfloor \frac{n}{2} \right\rfloor - 1 \right) \\
&= \left\lfloor \frac{n}{2} \right\rfloor \left(\left\lceil \frac{n}{2} \right\rceil - 1 \right).
\end{aligned}$$

□

Proposition 6.2. *For any non-negative integer n,*

$$\begin{aligned}
&\sum_{i=1}^{n-1} \sum_{j=1}^{\min\{n-i,i\}} \sum_{k=0}^{\min\{n-i-j,j\}} 1 \\
&= -\frac{25}{144} - \frac{n}{12} + \frac{7n^2}{24} + \frac{n^3}{36} + \frac{(-1)^n}{16} + \frac{3+\sqrt{3}i}{54} \left(\frac{-1-\sqrt{3}i}{2} \right)^n + \frac{3-\sqrt{3}i}{54} \left(\frac{-1+\sqrt{3}i}{2} \right)^n.
\end{aligned}$$

Proof. The summation on the left is the number a_n of quadruples of non-negative integers i, j, k, l that add to n with further restrictions that $i \geq j \geq 1$ and $j \geq k$. Set $u = i-j$, $v = j-k$. So we are looking for the number b_n of non-negative integer solutions u, v, k, l such that $u + 2v + 3k + l = n$ minus the number c_n of integer solutions $i \geq 0, j = 0 = k, l$ with $i + l = n$.

Let A be the set of non-negative integer solutions of $3x_1 + 2x_2 + x_3 + x_4 = n$. For each $i = 1, 2, 3, 4$, let A_i be the set of non-negative integer solutions of $3x_1 + 2x_2 + x_3 + x_4 = n$ with $x_i \geq 1$. Then

$$A = A_1 \cup A_2 \cup A_3 \cup A_3.$$

Using inclusion-exclusion formula

$$|A| = \sum_{i=1}^{4} |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - |A_1 \cap A_2 \cap A_3 \cap A_4|.$$

To compute the cardinality of A_1, set $y_1 = x_1 - 1$. Since for any solution in A_1, we have $x_1 \geq 1$, hence, for any such a solution, $y_1 \geq 0$. Moreover, it follows from $3x_1+2x_2+x_3+x_4 = n$ that $3y_1+2x_2+x_3+x_4 = n-3$.

Using induction hypothesis the number of solution of this equation is b_{n-3}. Therefore, $|A_1| = b_{n-3}$. Similarly, one an compute the cardinality of other A_i and their intersections. Hence,

$$b_n = 2b_{n-1} - b_{n-3} - b_{n-4} + 2b_{n-6} - b_{n-7}.$$

A similar argument shows that

$$c_n = 2c_{n-1} - c_{n-2}.$$

This yields that

$$\begin{aligned} c_n &= 2c_{n-1} - c_{n-2} \\ &= 2c_{n-1} - 2c_{n-3} + c_{n-4} \\ &= 2c_{n-1} - c_{n-3} - c_{n-3} + c_{n-4} \\ &= 2c_{n-1} - c_{n-3} - 2c_{n-4} + c_{n-5} + c_{n-4} \\ &= 2c_{n-1} - c_{n-3} - c_{n-4} + c_{n-5} \\ &= 2c_{n-1} - c_{n-3} - c_{n-4} + 2c_{n-6} - c_{n-7}. \end{aligned}$$

Therefore, c_n satisfies the same recursive formula as b_n. Hence, their subtraction $a_n = b_n - c_n$ also satisfies the same formula $a_n = 2a_{n-1} - a_{n-3} - a_{n-4} + 2a_{n-6} - a_{n-7}$.

The roots of the corresponding characteristic polynomial $x^7 - 2x^6 + x^4 + x^3 - 2x + 1$ are $1, 1, 1, 1, -1, -1/2 + \sqrt{3}i/2$ and $-1/2 - \sqrt{3}i/2$. Thus by standard theory of linear recursive sequences with constant coefficients (see for example Theorem 6.21 and Remark 6.23 in [17]), the closed form for a_n equals

$$a_n = c_0 + c_1 n + c_2 n^2 + c_3 n^3 + c_4(-1)^n + c_5\left((-1+\sqrt{3}\,i)/2\right)^n + c_6\left((-1-\sqrt{3}\,i)/2\right)^n$$

for some coefficients c_0 through c_6. With that we can set up a system of linear equations with manually computed $a_0 = 0$, $a_1 = 0$, $a_2 = 1$, $a_3 = 3$, $a_4 = 6$, $a_5 = 10$, $a_6 = 16$, and with the help of Macaulay2 [9] we computed the corresponding closed form for a_n to be

$$-\frac{25}{144} - \frac{n}{12} + \frac{7n^2}{24} + \frac{n^3}{36} + \frac{(-1)^n}{16} + \frac{3+\sqrt{3}i}{54}\left(\frac{-1-\sqrt{3}i}{2}\right)^n + \frac{3-\sqrt{3}i}{54}\left(\frac{-1+\sqrt{3}i}{2}\right)^n. \qquad \square$$

Proposition 6.3. *For any non-negative integer n,*

$$\sum_{i=1}^{n-1}\sum_{j=1}^{n-i}\sum_{k=0}^{\min\{n-i-j,j\}} 1 = \frac{-1}{16} + \frac{5}{12}n + \frac{5}{8}n^2 + \frac{1}{12}n^3 + \frac{1}{16}(-1)^n.$$

Proof. With methods as in the proof of Theorem 6.2, we get the recursive formula $a_n = 3a_{n-1} - 2a_{n-2} - 2a_{n-3} + 3a_{n-4} - a_{n-5}$. The corresponding polynomial is $x^5 - 3x^4 + 2x^3 + 2x^2 - 3x + 1$, whose roots are $1, 1, 1, 1, -1$. The initial conditions can easily be computed to be $a_1 = 0$, $a_2 = 1$, $a_3 = 4$, $a_4 = 9$, $a_5 = 17$, from which with linear algebra (or via `https://oeis.org/A005744`) we get the expression

$$a_n = \frac{-1}{16} + \frac{5}{12}n + \frac{5}{8}n^2 + \frac{1}{12}n^3 + \frac{1}{16}(-1)^n. \qquad \square$$

Proposition 6.4. *For any non-negative integer n,*

$$\begin{aligned} \sum_{i=1}^{n-1}\sum_{j=1}^{\min\{n-i,i\}}\sum_{k=0}^{\min\{n-i-j,j\}}\sum_{l=0}^{\min\{n-i-j-k,k\}} &= \frac{-641 - 486n + 996n^2 + 132n^3 + 6n^4}{3456} \\ &+ \frac{(11+2n)(-1)^n}{128} + \frac{(1-i)i^n + (1+i)(-i)^n}{32} \\ &- \frac{1}{27}\left(\frac{-1-\sqrt{3}i}{2}\right)^{n+1} - \frac{1}{27}\left(\frac{-1+\sqrt{3}i}{2}\right)^{n+1}. \end{aligned}$$

Proof. With the methods as in the proof of Theorem 6.2 we get the recursive formula

$$a_n = 2a_{n-1} - a_{n-3} - 2a_{n-5} + 2a_{n-6} + a_{n-8} - 2a_{n-10} + a_{n-11}.$$

The roots of the corresponding polynomial $x^{11} - 2x^{10} + x^8 + 2x^6 - 2x^5 - x^3 + 2x - 1$ are $1, 1, 1, 1, 1, -1, -1, i, -i, (-1 + \sqrt{3}\, i)/2, (-1 - \sqrt{3}\, i)/2$. The manually computed initial conditions are $a_1 = 0$, $a_2 = 1$, $a_3 = 3$, $a_4 = 7$, $a_5 = 12$, $a_6 = 20$, $a_7 = 30$, $a_8 = 44$, $a_9 = 61$, $a_{10} = 83$, $a_{11} = 109$. Thus with linear algebra we get the closed form

$$\begin{aligned} a_n = &\frac{-641 - 486n + 996n^2 + 132n^3 + 6n^4}{3456} + \frac{(11 + 2n)(-1)^n}{128} + \frac{(1 - i)i^n + (1 + i)(-i)^n}{32} \\ &+ \frac{1 + \sqrt{3}i}{54} \left(\frac{-1 - \sqrt{3}i}{2} \right)^n + \frac{1 - \sqrt{3}i}{54} \left(\frac{-1 + \sqrt{3}i}{2} \right)^n. \end{aligned}$$

□

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