FROBENIUS NUMBERS OF NUMERICAL SEMIGROUPS GENERATED BY THREE CONSECUTIVE SQUARES OR CUBES

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Abstract. We establish and prove polynomial formulas for the Frobenius numbers of numerical semigroups generated by $n^2, (n+1)^2, (n+2)^2$ and by $n^3, (n+1)^3, (n+2)^3$. The formulas depend on the residue class of n modulo 4 and 18, respectively.

1. Introduction

A **numerical semigroup** is a subset of $\mathbb{N}_0$ that contains 0, is closed under addition, and is missing only finitely many positive integers. The largest integer not in the numerical semigroups is called its **Frobenius number**. In this paper we denote by $g(a_1, \ldots, a_k)$ the Frobenius number of the numerical semigroup generated by $\langle a_1, \ldots, a_k \rangle$. For example, $g(2,3) = 1$, $g(5,6,7) = 9$. The formula $g(a_1, a_2) = a_1a_2 - a_1 - a_2$ is usually attributed to Sylvester (see [10] or [6, p. 31]). By contrast, Curtis proved that there is no finite set of polynomials such that for each a_1, a_2, a_3 with common divisor 1, one of those polynomials gives $g(a_1, a_2, a_3)$ (see [1] or [6, p. 35]). Furthermore, Ramírez Alfonsin [5] proved that the computation of Frobenius number of a numerical semigroup generated by n integers is NP-hard for variable n. However, formulas for special types of numerical semigroups have been established, such as those for consecutive entries in an arithmetic sequence (see Roberts [7] or [6, pp. 60-61]), for some types of geometric sequences (see Ong, Ponomarenko [4]), or for certain entries in a Fibonacci sequence (see M. Marıin, Ramiirez Alfonsín, Revuelta [3]). A comprehensive account of results concerning the Frobenius number can be found in Ramírez Alfonsin's book [6]. A more recent paper by Dutch and Rickett [2] demonstrates that the growth rate of Frobenius number of the numerical semigroup generated by $n^2, (n+1)^2, \ldots,$ as a function of n, is $o(n^{2+\epsilon})$ for any positive ϵ.

In this paper, we prove formulas for the Frobenius numbers of numerical semigroups generated by three consecutive squares, and for those generated by three consecutive cubes. It is fairly elementary to compute any particular Frobenius number; the accomplishment of this paper is in giving general formulas.

The work on consecutive squares was done by the second author for his senior thesis at Reed College in 2009, and the work on consecutive cubes was done by the first author during the summers of 2011 and 2012, both under the supervision of the third author. In 2012 the first author was supported by a Fellowship for Faculty-Student Collaborative Research from Reed College. In the original version our proofs were hands-on using a mixture of linear algebra and number theory: we showed first of all for each case that the purported Frobenius number is not in the

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semigroup, and then showed that every strictly larger number is in the semigroup. We thank the anonymous referees for directing us to the paper [8] by Rødseth and the paper [9] by Ramírez Alfonsín and Rødseth; these significantly shortened our proofs. We also thank Alessio Sammartano for allerting us to [2].

Our work could not have been done without computer help, and we here describe the process. First, Frobenius numbers of specific numerical semigroups were computed by Mathematica. A plot of $g(n^2, (n+1)^2, (n+2)^2)$ as a function of n for thousands of n yielded three curves, and upon closer examination one of the three curves was thicker and actually consisted of two close curves. Further experimentation determined that the four curves depended on $n \mod 4$. With that information, we considered $G_{r,n} = g((4n+r)^2, (4n+r+1)^2, (4n+r+2)^2)$ as a function of n for each $r = 0, 1, 2, 3$. By Mathematica's computations, $G_{0,n}/n^3$ tends to 128 with n. This means that $G_{0,n}$ is roughly a polynomial in n of degree 3 with leading coefficient 128. With that, again with Mathematica's help, $(G_{0,n} - 128n^3)/n$ seemingly has limit 20, and $G_{0,n} - 128n^3 + 20n$ has limit 5, giving strong evidence that $g((4n)^2, (4n+1)^2, (4n+2)^2) = 128n^3 - 20n - 5$, at least for large n. This paper gives a proof of this for $n \geq 4$, as well as a proof of the other cases for consecutive squares and cubes. Indeed, we get four polynomial functions for squares and eighteen polynomial functions for cubes.

2. Rødseth's algorithm

Rødseth [8] introduced the following algorithm for computing the Frobenius number of a numerical semigroup generated by a, b, c. First one finds an integer $s_0 \in \{0, 1, 2, \ldots, a-1\}$ such that

$$bs_0 \equiv c \mod a.$$

Then, starting with $k = 1$ and using $s_{-1} = a$, one runs the Euclidean algorithm with negative remainders:

$$s_{k-2} = q_k s_{k-1} - s_k, \qquad 0 \leq s_k < s_{k-1},$$

until $s_{l+1} = 0$. (If $s_0 = 0$, which does not happen in our cases below, then l is set to 0.) With these q_i Rødseth defines integers P_i with $P_{-1} = 0$, $P_0 = 1$, and

$$P_k = q_k P_{k-1} - P_{k-2}, \qquad k = 1, 2, \ldots, l+1.$$

Since $q_k \geq 2$ and $P_k > P_{k-1}$, with the convention that $\frac{s_{-1}}{P_{-1}} = \infty$, we have that

$$0 = \frac{s_{l+1}}{P_{l+1}} < \frac{s_l}{P_l} < \cdots \frac{s_1}{P_1} < \frac{s_0}{P_0} < \frac{s_{-1}}{P_{-1}},$$

and so there exists a unique integer v such that

$$\frac{s_{v+1}}{P_{v+1}} \leq \frac{c}{b} < \frac{s_v}{P_v}. \tag{2.1}$$

The Frobenius number of the numerical semigroup generated by a, b, c is then

$$g(a, b, c) = -a + b(s_v - 1) + c(P_{v+1} - 1) - \min\{bs_{v+1}, cP_v\}. \tag{2.2}$$

Note that it is not necessary to run the whole Euclidean algorithm, but only through $(v+1)$ steps.

A modification of this procedure using the Euclidean algorithm with positive remainders is in the paper [9] by Ramírez Alfonsín and Rødseth.

In case that $a = n^e$, $b = (n+1)^e$, $c = (n+2)^e$, as $(n+1)(1-n+n^2-n^3+\cdots+(-1)^{e-1}n^{e-1}) = 1 \mod n^e$, it follows easily that

$$s_0 = (1-n+n^2-n^3+\cdots+(-1)^{e-1}n^{e-1})^e(n+2)^e \mod n^e.$$

In particular, when $e = 2$, $s_0 = n^2-4n+4$ for $n \geq 4$, and when $e = 3$, $s_0 = 18n^2-12n+8$ for $n \geq 18$. One may be lead to believe that the roles of 4 and 18 for these s_0 are the cause of why we need 4 and respectively 18 formulas for Frobenius numbers, however, by the same reasoning, for $e = 4$, with $s_0 = n^4-88n^3+56n^2-32n+16$ for $n \geq 88$, we would require 88 formulas, whereas experimental tests make us believe that we need 40 formulas.

3. Frobenius number of three consecutive squares

Here are a few Frobenius numbers generated by three consecutive squares (omitting the void case containing 1): $g(4,9,16) = 23$, $g(9,16,25) = 119$, $g(16,25,36) = 167$, $g(25,36,49) = 312$, $g(64,81,100) = 1103$, and $g(144,169,196) = 3479$. The next theorem proves the general formulas for the Frobenius numbers of the numerical semigroups generated by all other instances of three consecutive squares:

Theorem 3.1. *The Frobenius number of the numerical semigroup generated by three consecutive squares $n^2, (n+1)^2, (n+2)^2$ is as follows:*

$$\begin{cases} 2n^3-5n-5, & \text{if } n \equiv 0 \mod 4, \text{ and } n \geq 16, \\ (5/2)n^3+(1/2)n^2-6n-6, & \text{if } n \equiv 1 \mod 4, \text{ and } n \geq 9, \\ 2n^3+2n^2-n-3, & \text{if } n \equiv 2 \mod 4, \text{ and } n \geq 6, \\ (17/4)n^3+(9/4)n^2-8n-8, & \text{if } n \equiv 3 \mod 4, \text{ and } n \geq 7. \end{cases}$$

In other words,

$$\begin{aligned} g\left((4m)^2,(4m+1)^2,(4m+2)^2\right) &= 128m^3-20m-5, \text{ if } m \geq 4, \\ g\left((4m+1)^2,(4m+2)^2,(4m+3)^2\right) &= 160m^3+128m^2+10m-9, \text{ if } m \geq 2, \\ g\left((4m+2)^2,(4m+3)^2,(4m+4)^2\right) &= 128m^3+224m^2+124m+19, \text{ if } m \geq 1, \\ g\left((4m+3)^2,(4m+4)^2,(4m+5)^2\right) &= 272m^3+648m^2+481m+103, \text{ if } m \geq 1. \end{aligned}$$

Thus the formula for $g(n^2,(n+1)^2,(n+2)^2)$ depends on $n \mod 4$, and is eventually periodically polynomial.

Proof. Let n be any positive integer. As noted in the previous section, $s_0 = n^2-4n+4$, so that the Euclidean algorithm in Rødseth's procedure starts as:

$$\begin{aligned} s_{-1} &= n^2, \\ s_0 &= n^2-4(n-1), \\ s_1 &= 2s_0-s_{-1} = n^2-8n+8 = n^2-2\cdot 4(n-1), \\ s_2 &= 2s_1-s_0 = n^2-3\cdot 4(n-1), \\ &\vdots \\ s_k &= 2s_{k-1}-s_{k-2} = n^2-(k+1)\cdot 4(n-1), \end{aligned}$$

as long as $n^2 - (k+1) \cdot 4(n-1) \geq 0$, i.e., as long as $\frac{n^2}{4(n-1)} \geq k+1$. Note that $s_{-1} > s_0 > s_1 > \cdots > s_k$ for $n \geq 4$, as needed for Rødseth's algorithm. Since all q_i are 2, it follows by induction that $P_i = 2P_{i-1} - P_{i-2} = i+1$ for $i = 1, \ldots, k$.

The simplest case is when $n = 4m+3$ for some integer $m \geq 1$. The (start of the) Euclidean algorithm in the first paragraph continues until $k = m$ as $s_m = (4m+3)^2 - (m+1) \cdot 4(4m+2) = 1$. Note that $s_{m-1} = s_m + 4(n-1) = 1 + 4(4m+2) = 16m+9$ and so that

$$\frac{s_m}{P_m} = \frac{1}{m+1} < \frac{(4m+5)^2}{(4m+4)^2} < 16 < \frac{16m+9}{m} = \frac{s_{m-1}}{P_{m-1}},$$

so that $v = m-1$, and by Equation 2.2,

$$\begin{aligned} g((4m+3)^2, (4m+4)^2, (4m+5)^2) \\ &= -(4m+3)^2 + (4m+4)^2(16m+9-1) + (4m+5)^2(m+1-1) \\ &\qquad - \min\{(4m+4)^2 \cdot 1, (4m+5)^2 \cdot m\} \\ &= -(4m+3)^2 + (4m+4)^2(16m+8) + (4m+5)^2 m - (4m+4)^2 \\ &= 272m^3 + 648m^2 + 481m + 103. \end{aligned}$$

Now let $n = 4m$ for some integer $m \geq 4$. The procedure in the first paragraph continues until $k = m-1$ as $s_{m-1} = (4m)^2 - m \cdot 4(4m-1) = 4m = n$ and as $n \not\geq 4(n-1)$. Observe that $s_{m-2} = s_{m-1} + 4(n-1) = 20m - 4$, so that by continuing the Euclidean algorithm one more step we get

$$s_m = 5s_{m-1} - s_{m-2} = 5(4m) - (20m-4) = 4,$$

and $P_m = 5P_{m-1} - P_{m-2} = 5m - (m-1) = 4m+1$. As

$$\frac{s_m}{P_m} = \frac{4}{4m+1} < \frac{(4m+2)^2}{(4m+1)^2} < 4 = \frac{4m}{m} = \frac{s_{m-1}}{P_{m-1}},$$

it follows that $v = m-1$, and by Equation 2.2,

$$\begin{aligned} g((4m)^2, (4m+1)^2, (4m+2)^2) \\ &= -(4m)^2 + (4m+1)^2(4m-1) + (4m+2)^2(4m+1-1) \\ &\qquad - \min\{(4m+1)^2 \cdot 4, (4m+2)^2 \cdot m\} \\ &= -(4m)^2 + (4m+1)^2(4m-1) + (4m+2)^2(4m) - (4m+1)^2 \cdot 4 \\ &= 128m^3 - 20m - 5. \end{aligned}$$

(Note that the "minimum" part above forces $m \geq 4$ for a clean formula.)

Now let $n = 4m+1$ for some integer $m \geq 2$. The procedure in the first paragraph continues until $k = m-1$ as $s_{m-1} = (4m+1)^2 - m \cdot 4(4m) = 8m+1$. Observe that $s_{m-2} = s_{m-1} + 4(n-1) = 24m+1$. By continuing the Euclidean algorithm one more step we get

$$s_m = 3s_{m-1} - s_{m-2} = 3(8m+1) - (24m+1) = 2,$$

and $P_m = 3P_{m-1} - P_{m-2} = 3m - (m-1) = 2m+1$. As

$$\frac{s_m}{P_m} = \frac{2}{2m+1} < \frac{(4m+3)^2}{(4m+2)^2} < 8 < \frac{8m+1}{m} = \frac{s_{m-1}}{P_{m-1}},$$

it follows that $v = m-1$, and by Equation 2.2,

$$g((4m+1)^2, (4m+2)^2, (4m+3)^2)$$

$$
\begin{aligned}
&= -(4m+1)^2 + (4m+2)^2(8m+1-1) + (4m+3)^2(2m+1-1)\\
&\qquad - \min\{(4m+2)^2 \cdot 2, (4m+3)^2 \cdot m\}\\
&= -(4m+1)^2 + (4m+2)^2(8m) + (4m+3)^2(2m) - (4m+2)^2 \cdot 2\\
&= 160m^3 + 128m^2 + 10m - 9.
\end{aligned}
$$

(Note that the “minimum” part forces $m \geq 2$ for a clean formula.)

Finally, let $n = 4m+2$ for some integer $m \geq 1$. The procedure in the first paragraph continues until $k = m-1$ as $s_{m-1} = (4m+2)^2 - m \cdot 4(4m+1) = 12m+4$. Observe that $s_{m-2} = s_{m-1} + 4(n-1) = 28m+8$. We continue the Euclidean algorithm three more steps to get

$$
\begin{aligned}
s_{m-2} &= 28m+8 = 3(12m+4) - (8m+4) = 3s_{m-1} - s_m, \textit{i.e.}, \quad s_m = 8m+4,\\
s_{m-1} &= 12m+4 = 2(8m+4) - (4m+4) = 2s_m - s_{m+1}, \textit{i.e.}, \quad s_{m+1} = 4m+4,\\
s_m &= 8m+4 = 2(4m+4) - 4 = 2s_{m+1} - s_{m+2}, \textit{i.e.}, \quad s_{m+2} = 4.
\end{aligned}
$$

This gives

$$
\begin{aligned}
P_m &= 3P_{m-1} - P_{m-2} = 3m - (m-1) = 2m+1,\\
P_{m+1} &= 2P_m - P_{m-1} = 2(2m+1) - m = 3m+2,\\
P_{m+2} &= 2P_{m+1} - P_m = 2(3m+2) - (2m+1) = 4m+3.
\end{aligned}
$$

As

$$
\frac{s_{m+2}}{P_{m+2}} = \frac{4}{4m+3} < \frac{(4m+4)^2}{(4m+3)^2} < \frac{4m+4}{3m+2} = \frac{s_{m+1}}{P_{m+1}},
$$

it follows that $v = m+1$, and by Equation 2.2,

$$
\begin{aligned}
&g((4m+2)^2, (4m+3)^2, (4m+4)^2)\\
&\quad = -(4m+2)^2 + (4m+3)^2(4m+4-1) + (4m+4)^2(4m+3-1)\\
&\qquad - \min\{(4m+3)^2 \cdot 4, (4m+4)^2 \cdot (3m+2)\}\\
&\quad = -(4m+2)^2 + (4m+3)^2(4m+3) + (4m+4)^2(4m+2) - (4m+3)^2 \cdot 4\\
&\quad = 128m^3 + 224m^2 + 124m + 19.
\end{aligned}
$$

This proves the second formulation of Frobenius numbers in the statement of the theorem. The first set of formulas is now a straightforward derivation. □

4. Frobenius number of three consecutive cubes

The Frobenius numbers of numerical semigroups generated by three consecutive cubes are given by 18 polynomial formulas. We first define the candidate functions:

$$
\begin{aligned}
g_0(n) &= (2/3)n^5 + (55/18)n^4 + (23/3)n^3 + (43/3)n^2 + (77/9)n - 1,\\
g_1(n) &= (1/3)n^5 + (34/9)n^4 + (128/9)n^3 + (88/3)n^2 + (211/9)n + 35/9,\\
g_2(n) &= (2/3)n^5 + (55/18)n^4 + (53/9)n^3 + 9n^2 + (29/9)n - 25/9,\\
g_3(n) &= (1/3)n^5 + (34/9)n^4 + 16n^3 + (104/3)n^2 + (259/9)n + 17/3,\\
g_4(n) &= (2/3)n^5 + (55/18)n^4 + (61/9)n^3 + (35/3)n^2 + (53/9)n - 17/9,\\
g_5(n) &= (1/3)n^5 + (34/9)n^4 + (64/9)n^3 + 8n^2 + (19/9)n - 29/9,\\
g_6(n) &= (2/3)n^5 + (55/18)n^4 + (7/3)n^3 - (5/3)n^2 - (67/9)n - 19/3,\\
g_7(n) &= (1/3)n^5 + (34/9)n^4 + (32/9)n^3 - (8/3)n^2 - (77/9)n - 61/9,
\end{aligned}
$$

$$\begin{aligned}
g_8(n) =&(2/3)n^5 + (55/18)n^4 + (5/9)n^3 - 7n^2 - (115/9)n - 73/9,\\
g_9(n) =&(1/3)n^5 + (34/9)n^4 + (16/3)n^3 + (8/3)n^2 - (29/9)n - 5,\\
g_{10}(n) =&(2/3)n^5 + (55/18)n^4 + (13/9)n^3 - (13/3)n^2 - (91/9)n - 65/9,\\
g_{11}(n) =&(1/3)n^5 + (34/9)n^4 + (112/9)n^3 + 24n^2 + (163/9)n + 19/9,\\
g_{12}(n) =&(2/3)n^5 + (55/18)n^4 + 5n^3 + (19/3)n^2 + (5/9)n - 11/3,\\
g_{13}(n) =&(1/3)n^5 + (34/9)n^4 + (80/9)n^3 + (40/3)n^2 + (67/9)n - 13/9,\\
g_{14}(n) =&(2/3)n^5 + (55/18)n^4 + (29/9)n^3 + n^2 - (43/9)n - 49/9,\\
g_{15}(n) =&(1/3)n^5 + (34/9)n^4 + (32/3)n^3 + (56/3)n^2 + (115/9)n + 1/3,\\
g_{16}(n) =&(2/3)n^5 + (55/18)n^4 + (37/9)n^3 + (11/3)n^2 - (19/9)n - 41/9,\\
g_{17}(n) =&(1/3)n^5 + (34/9)n^4 + (16/9)n^3 - 8n^2 - (125/9)n - 77/9.
\end{aligned}$$

We find it easier to work with functions f_r below, where $f_r(m) = g_r(18m + r)$:

$$\begin{aligned}
f_0(m) &= 1259712m^5 + 320760m^4 + 44712m^3 + 4644m^2 + 154m - 1,\\
f_1(m) &= 629856m^5 + 571536m^4 + 190512m^3 + 31752m^2 + 2548m + 75,\\
f_2(m) &= 1259712m^5 + 1020600m^4 + 332424m^3 + 55404m^2 + 4698m + 157,\\
f_3(m) &= 629856m^5 + 921456m^4 + 532656m^3 + 153144m^2 + 21812m + 1223,\\
f_4(m) &= 1259712m^5 + 1720440m^4 + 946728m^3 + 263412m^2 + 37082m + 2107,\\
f_5(m) &= 629856m^5 + 1271376m^4 + 968112m^3 + 355752m^2 + 63828m + 4499,\\
f_6(m) &= 1259712m^5 + 2420280m^4 + 1840968m^3 + 693468m^2 + 129322m + 9537,\\
f_7(m) &= 629856m^5 + 1621296m^4 + 1590192m^3 + 753624m^2 + 173908m + 15695,\\
f_8(m) &= 1259712m^5 + 3120120m^4 + 3061800m^3 + 1488132m^2 + 358074m + 34087,\\
f_9(m) &= 629856m^5 + 1971216m^4 + 2398896m^3 + 1429704m^2 + 419252m + 48539,\\
f_{10}(m) &= 1259712m^5 + 3819960m^4 + 4609224m^3 + 2766636m^2 + 826058m + 98125,\\
f_{11}(m) &= 629856m^5 + 2321136m^4 + 3394224m^3 + 2466936m^2 + 892404m + 128663,\\
f_{12}(m) &= 1259712m^5 + 4519800m^4 + 6483240m^3 + 4648212m^2 + 1665946m + 238803,\\
f_{13}(m) &= 629856m^5 + 2671056m^4 + 4482864m^3 + 3730536m^2 + 1541908m + 253539,\\
f_{14}(m) &= 1259712m^5 + 5219640m^4 + 8637192m^3 + 7135452m^2 + 2943162m + 484897,\\
f_{15}(m) &= 629856m^5 + 3020976m^4 + 5758128m^3 + 5458968m^2 + 2576660m + 484767,\\
f_{16}(m) &= 1259712m^5 + 5919480m^4 + 11117736m^3 + 10433124m^2 + 4892186m + 917039,\\
f_{17}(m) &= 629856m^5 + 3370896m^4 + 7126704m^3 + 7455240m^2 + 3864564m + 794987.
\end{aligned}$$

This section is devoted to proving the following theorem:

Theorem 4.1. *The Frobenius number of the numerical semigroup generated by three consecutive cubes* $(18m + r)^3$, $(18m + r + 1)^3$ *and* $(18m + r + 2)^3$ *equals* $g_r(18m + r) = f_r(m)$, *where* $r = 0, 1, \ldots, 17$, *and* m *are as follows:*

(1) $m \geq 0$ *if* $r = 11, 12, 13, 14, 15, 16$.
(2) $m \geq 1$ *if* $r = 0, 1, 2, 3, 4, 5, 6, 9, 10$.
(3) $m \geq 2$ *if* $r = 7$.

(4) $m \geq 4$ *if* $r = 8$.
(5) $m \geq 15$ *if* $r = 17$.

This theorem does not include all possible numerical semigroups generated by three consecutive cubes. The missing cases are routine to verify:

$$
\begin{aligned}
g(2^3, 3^3, 4^3) &= 181, & g(3^3, 4^3, 5^3) &= 1098,\\
g(4^3, 5^3, 6^3) &= 2107, & g(5^3, 6^3, 7^3) &= 5249,\\
g(6^3, 7^3, 8^3) &= 10745, & g(7^3, 8^3, 9^3) &= 21700,\\
g(8^3, 9^3, 10^3) &= 38919, & g(9^3, 10^3, 11^3) &= 55222,\\
g(10^3, 11^3, 12^3) &= 103589, & g(17^3, 18^3, 19^3) &= 881440,\\
g(25^3, 26^3, 27^3) &= 4868957, & g(26^3, 27^3, 28^3) &= 9413533,\\
g(35^3, 36^3, 37^3) &= 23887437, & g(44^3, 45^3, 46^3) &= 121672187,\\
g(53^3, 54^3, 55^3) &= 171468734, & g(62^3, 63^3, 64^3) &= 656175201,\\
g(71^3, 72^3, 73^3) &= 702420331, & g(89^3, 90^3, 91^3) &= 2107464204,\\
g(107^3, 108^3, 109^3) &= 5184832025, & g(125^3, 126^3, 127^3) &= 11115847882,\\
g(143^3, 144^3, 145^3) &= 21540510999, & g(161^3, 162^3, 163^3) &= 38633078456,\\
g(179^3, 180^3, 181^3) &= 65177647909, & g(197^3, 198^3, 199^3) &= 104643740310,\\
g(215^3, 216^3, 217^3) &= 161261882627, & g(233^3, 234^3, 235^3) &= 240099190564,\\
g(251^3, 252^3, 253^3) &= 347134951281, & g(269^3, 270^3, 271^3) &= 489336206114.
\end{aligned}
$$

Proof of Theorem 4.1. Let n be any positive integer. In Section 2 we derived $s_0 = 18n^2 - 12n + 8$ for all $n \geq 18$. The exceptional Frobenius numbers for $n = 2, 3, \ldots, 10, 17$ are given above, and it is straightforward to verify the stated formulas for cases $n = 11, 12, 13, 14, 15, 16$. Below we thus assume that $n \geq 18$. Set $n = 18m + r$ for some integers $m \geq 1$ and $r = 0, 1, \ldots, 17$. Then

$$
\begin{aligned}
s_{-1} &= n^3 = 5832m^3 + 972m^2r + 54mr^2 + r^3,\\
s_0 &= 18n^2 - 12n + 8 = 5832m^2 + 648mr + 18r^2 - 216m - 12r + 8,\\
s_1 &= (m+1)s_0 - s_{-1}\\
&= (5616 - 324r)m^2 + (636r - 36r^2 - 208)m - r^3 + 18r^2 - 12r + 8,
\end{aligned}
$$

and

$$P_{-1} = 0, \quad P_0 = 1, \quad P_1 = m + 1.$$

For the next step in the Euclidean algorithm, q_2 varies with r, so we analyze the eighteen cases of r separately below.

Case $\mathbf{n = 18m}$ for some integer $\mathbf{m \geq 1}$ (so $\mathbf{r = 0}$): Then Rødseth's algorithm started above proceeds as follows:

$$
\begin{aligned}
s_{-1} &= n^3 = 5832m^3,\\
s_0 &= 5832m^2 - 216m + 8,\\
s_1 &= (m+1)s_0 - s_{-1} = s_0 + (ms_0 - s_{-1}) = s_0 - (216m^2 - 8m),\\
s_2 &= 2s_1 - s_0 = s_0 - 2(216m^2 - 8m),
\end{aligned}
$$

$$s_3 = 2s_2 - s_1 = s_0 - 3(216m^2 - 8m),$$
$$\vdots$$
$$s_k = 2s_{k-1} - s_{k-2} = s_0 - k \cdot (216m^2 - 8m),$$

as long as $s_k = s_0 - k \cdot (216m^2 - 8m) \geq 0$, i.e., as long as

$$k \leq \frac{s_0}{216m^2 - 8m} = \frac{5832m^2 - 216m}{216m^2 - 8m} + \frac{8}{216m^2 - 8m} = 27 + \frac{8}{216m^2 - 8m},$$

i.e., for $k \leq 27$. With these by easy induction we have $P_k = km + 1$ for $k = 1, 2, \ldots, 27$. Then

$$\frac{s_{27}}{P_{27}} = \frac{5832m^2 - 216m + 8 - 27 \cdot (216m^2 - 8m)}{27m + 1} = \frac{8}{27m + 1}$$
$$< \frac{(18m+2)^3}{(18m+1)^3} < \frac{216m^2 - 8m + 8}{26m + 1} = \frac{s_{26}}{P_{26}},$$

so that $v = 26$. By Equation 2.2 then the Frobenius number in this case is

$$-(18m)^3 + (18m+1)^3(216m^2 - 8m + 8 - 1) + (18m+2)^3(27m + 1 - 1)$$
$$- \min\{(18m+1)^3 \cdot 8, (18m+2)^3(26m+1)\}$$
$$= -(18m)^3 + (18m+1)^3(216m^2 - 8m + 7) + (18m+2)^3(27m) - (18m+1)^3 \cdot 8$$
$$= 1259712m^5 + 320760m^4 + 44712m^3 + 4644m^2 + 154m - 1,$$

which equals $f_0(m)$ as desired.

Case $\mathbf{n = 18m + 1}$ for some integer $\mathbf{m \geq 1}$ (so $\mathbf{r = 1}$): Rødseth's algorithm proceeds as follows:

$$s_{-1} = n^3 = 5832m^3 + 972m^2 + 54m + 1,$$
$$s_0 = 18n^2 - 12n + 8 = 5832m^2 + 432m + 14,$$
$$s_1 = (m+1)s_0 - s_{-1} = s_0 + (ms_0 - s_{-1}) = s_0 - (540m^2 + 40m + 1),$$
$$s_2 = 2s_1 - s_0 = s_0 - 2(540m^2 + 40m + 1),$$
$$\vdots$$
$$s_9 = 2s_8 - s_7 = s_0 - 9 \cdot (540m^2 + 40m + 1) = 972m^2 + 72m + 5,$$
$$s_{10} = 2s_9 - s_8 = s_0 - 10 \cdot (540m^2 + 40m + 1) = 432m^2 + 32m + 4,$$

and $P_k = km + 1$ for $k = 1, 2, \ldots, 10$. This continues further as

$$s_{11} = 3s_{10} - s_9 = 324m^2 + 24m + 7, \quad P_{11} = 3P_{10} - P_9 = 3(10m+1) - (9m+1) = 21m + 2,$$
$$s_{12} = 2s_{11} - s_{10} = 216m^2 + 16m + 10, \quad P_{12} = 2P_{11} - P_{10} = 2(21m+2) - (10m+1) = 32m + 3,$$
$$s_{13} = 2s_{12} - s_{11} = 108m^2 + 8m + 13, \quad P_{13} = 2P_{12} - P_{11} = 2(32m+3) - (21m+2) = 43m + 4,$$
$$s_{14} = 2s_{13} - s_{12} = 16, \quad P_{14} = 2P_{13} - P_{12} = 2(43m+4) - (32m+3) = 54m + 5.$$

Then

$$\frac{s_{14}}{P_{14}} = \frac{16}{54m + 5} < \frac{(18m+3)^3}{(18m+2)^3} < \frac{108m^2 + 8m + 13}{43m + 4} = \frac{s_{13}}{P_{13}},$$

so that $v = 13$. By Equation 2.2 then the Frobenius number in this case is

$$-(18m+1)^3 + (18m+2)^3(108m^2 + 8m + 13 - 1) + (18m+3)^3(54m + 5 - 1)$$

$$
\begin{aligned}
&\quad - \min\{(18m+2)^3 \cdot 16, (18m+3)^3(43m+4)\} \\
&= -(18m+1)^3 + (18m+2)^3(108m^2+8m+12) + (18m+3)^3(54m+4) - (18m+2)^3 \cdot 16 \\
&= 629856m^5 + 571536m^4 + 190512m^3 + 31752m^2 + 2548m + 75,
\end{aligned}
$$

which equals $f_1(m)$ as desired.

Case $\mathbf{n = 18m + 2}$ for some integer $\mathbf{m \geq 1}$:

$$
\begin{aligned}
s_{-1} &= n^3 = 5832m^3 + 1944m^2 + 216m + 8, \\
s_0 &= 18n^2 - 12n + 8 = 5832m^2 + 1080m + 56, \\
s_1 &= (m+1)s_0 - s_{-1} = 4968m^2 + 920m + 48 = s_0 - (864m^2 + 160m + 8), \\
s_2 &= 2s_1 - s_0 = s_0 - 2(864m^2 + 160m + 8) \\
&\vdots \\
s_5 &= 2s_4 - s_3 = s_0 - 5 \cdot (864m^2 + 160m + 8) = 1512m^2 + 280m + 16, \\
s_6 &= 2s_5 - s_4 = s_0 - 6 \cdot (864m^2 + 160m + 8) = 648m^2 + 120m + 8, \\
s_7 &= 3s_6 - s_5 = 432m^2 + 80m + 8, \\
s_8 &= 2s_7 - s_6 = 216m^2 + 40m + 8, \\
s_9 &= 2s_8 - s_7 = 8,
\end{aligned}
$$

$P_k = km + 1$ for $k = 1, 2, \ldots, 6$, $P_7 = 3P_6 - P_5 = 3(6m+1) - (5m+1) = 13m + 2$, $P_8 = 2P_7 - P_6 = 2(13m+2) - (6m+1) = 20m + 3$, $P_9 = 2P_8 - P_7 = 2(20m+3) - (13m+2) = 27m + 4$. Then

$$\frac{s_9}{P_9} = \frac{8}{27m+4} < \frac{(18m+4)^3}{(18m+3)^3} < \frac{216m^2+40m+8}{20m+3} = \frac{s_8}{P_8},$$

so that $v = 8$. By Equation 2.2 the Frobenius number in this case is

$$
\begin{aligned}
&-(18m+2)^3 + (18m+3)^3(216m^2 + 40m + 8 - 1) + (18m+4)^3(27m + 4 - 1) \\
&\quad - \min\{(18m+3)^3 \cdot 8, (18m+4)^3(20m+3)\} \\
&= -(18m+2)^3 + (18m+3)^3(216m^2+40m+7) + (18m+4)^3(27m+3) - (18m+3)^3 \cdot 8 \\
&= 1259712m^5 + 1020600m^4 + 332424m^3 + 55404m^2 + 4698m + 157,
\end{aligned}
$$

which equals $f_2(m)$ as desired.

Case $\mathbf{n = 18m + 3}$ for some integer $\mathbf{m \geq 1}$:

$$
\begin{aligned}
s_{-1} &= n^3 = 5832m^3 + 2916m^2 + 486m + 27, \\
s_0 &= 18n^2 - 12n + 8 = 5832m^2 + 1728m + 134, \\
s_1 &= (m+1)s_0 - s_{-1} = 4644m^2 + 1376m + 107 = s_0 - (1188m^2 + 352m + 27), \\
s_2 &= 2s_1 - s_0 = s_0 - 2(1188m^2 + 352m + 27) \\
s_3 &= 2s_2 - s_1 = s_0 - 3 \cdot (1188m^2 + 352m + 27) = 2268m^2 + 672m + 53, \\
s_4 &= 2s_3 - s_2 = s_0 - 4 \cdot (1188m^2 + 352m + 27) = 1080m^2 + 320m + 26, \\
s_5 &= 3s_4 - s_3 = 972m^2 + 288m + 25 = s_4 - (108m^2 + 32m + 1), \\
s_6 &= 2s_5 - s_4 = 864m^2 + 256m + 24 = s_4 - 2(108m^2 + 32m + 1),
\end{aligned}
$$

$$\begin{aligned}
&\vdots\\
s_{13} &= 2s_{12} - s_{11} = s_4 - 9(108m^2+32m+1) = 108m^2+32m+17,\\
s_{14} &= 2s_{13} - s_{12} = s_4 - 10(108m^2+32m+1) = 16,
\end{aligned}$$

so that by easy induction arguments, $P_k = km+1$ for $k = 1,2,3,4$, $P_k = (5k-16)m + k - 3$ for $k = 5,6,\ldots,14$. Then

$$\frac{s_{14}}{P_{14}} = \frac{16}{54m+11} < \frac{(18m+5)^3}{(18m+4)^3} < \frac{108m^2+32m+17}{49m+10} = \frac{s_{13}}{P_{13}},$$

so that $v = 13$. By Equation 2.2 the Frobenius number in this case is

$$\begin{aligned}
&-(18m+3)^3 + (18m+4)^3(108m^2+32m+17-1) + (18m+5)^3(54m+11-1)\\
&\qquad - \min\{(18m+4)^3\cdot 16, (18m+5)^3(49m+10)\}\\
&\quad = -(18m+3)^3 + (18m+4)^3(108m^2+32m+16) + (18m+5)^3(54m+10) - (18m+4)^3\cdot 16\\
&\quad = 629856m^5 + 921456m^4 + 532656m^3 + 153144m^2 + 21812m + 1223,\\
&\quad = f_3(m).
\end{aligned}$$

Case $\mathbf{n = 18m + 4}$ for some integer $\mathbf{m \geq 1}$:

$$\begin{aligned}
s_{-1} &= n^3 = 5832m^3 + 3888m^2 + 864m + 64,\\
s_0 &= 18n^2 - 12n + 8 = 5832m^2 + 2376m + 248,\\
s_1 &= (m+1)s_0 - s_{-1} = 4320m^2 + 1760m + 184,\\
s_2 &= 2s_1 - s_0 = 2808m^2 + 1144m + 120,\\
s_3 &= 2s_2 - s_1 = 1296m^2 + 528m + 56,\\
s_4 &= 3s_3 - s_2 = 1080m^2 + 440m + 48,\\
s_5 &= 2s_4 - s_3 = 864m^2 + 352m + 40,\\
s_6 &= 2s_5 - s_4 = 648m^2 + 264m + 32,\\
s_7 &= 2s_6 - s_5 = 432m^2 + 176m + 24,\\
s_8 &= 2s_7 - s_6 = 216m^2 + 88m + 16,\\
s_9 &= 2s_8 - s_7 = 8,
\end{aligned}$$

so that $P_k = km+1$ for $k = 1,2,3$, $P_k = (4k-9)m + k - 2$ for $k = 4,5,\ldots,9$. Then

$$\frac{s_9}{P_9} = \frac{8}{27m+7} < \frac{(18m+6)^3}{(18m+5)^3} < \frac{216m^2+88m+16}{23m+6} = \frac{s_8}{P_8},$$

so that $v = 8$. By Equation 2.2 the Frobenius number in this case is

$$\begin{aligned}
&-(18m+4)^3 + (18m+5)^3(216m^2+88m+16-1) + (18m+6)^3(27m+7-1)\\
&\qquad - \min\{(18m+5)^3\cdot 8, (18m+6)^3(23m+6)\}\\
&\quad = -(18m+4)^3 + (18m+5)^3(216m^2+88m+15) + (18m+6)^3(27m+6) - (18m+5)^3\cdot 8\\
&\quad = 1259712m^5 + 1720440m^4 + 946728m^3 + 263412m^2 + 37082m + 2107\\
&\quad = f_4(m).
\end{aligned}$$

Case $\mathbf{n = 18m + 5}$ for some integer $\mathbf{m \geq 1}$:

$$\begin{aligned}
s_{-1} &= n^3 = 5832m^3 + 4860m^2 + 1350m + 125,\\
s_0 &= 18n^2 - 12n + 8 = 5832m^2 + 3024m + 398,\\
s_1 &= (m+1)s_0 - s_{-1} = 3996m^2 + 2072m + 273,\\
s_2 &= 2s_1 - s_0 = 2160m^2 + 1120m + 148,\\
s_3 &= 2s_2 - s_1 = 324m^2 + 168m + 23,\\
s_4 &= 7s_3 - s_2 = 108m^2 + 56m + 13,\\
s_5 &= 3s_4 - s_3 = 16,
\end{aligned}$$

so that $P_k = km + 1$ for $k = 1, 2, 3$, $P_4 = 7P_3 - P_2 = 19m + 6$, $P_5 = 3P_4 - P_3 = 54m + 17$. Then

$$\frac{s_5}{P_5} = \frac{16}{54m + 17} < \frac{(18m+7)^3}{(18m+6)^3} < \frac{108m^2 + 56m + 13}{19m + 6} = \frac{s_4}{P_4},$$

so that $v = 4$. By Equation 2.2 the Frobenius number in this case is

$$\begin{aligned}
&-(18m+5)^3 + (18m+6)^3(108m^2 + 56m + 13 - 1) + (18m+7)^3(54m + 17 - 1)\\
&\qquad - \min\{(18m+6)^3 \cdot 16, (18m+7)^3(19m+6)\}\\
&= -(18m+5)^3 + (18m+6)^3(108m^2 + 56m + 12) + (18m+7)^3(54m + 16) - (18m+6)^3 \cdot 16\\
&= 629856m^5 + 1271376m^4 + 968112m^3 + 355752m^2 + 63828m + 4499\\
&= f_5(m).
\end{aligned}$$

Case $\mathbf{n = 18m + 6}$ for some integer $\mathbf{m \geq 1}$:

$$\begin{aligned}
s_{-1} &= n^3 = 5832m^3 + 5832m^2 + 1944m + 216,\\
s_0 &= 18n^2 - 12n + 8 = 5832m^2 + 3672m + 584,\\
s_1 &= (m+1)s_0 - s_{-1} = 3672m^2 + 2312m + 368,\\
s_2 &= 2s_1 - s_0 = 1512m^2 + 952m + 152,\\
s_3 &= 3s_2 - s_1 = 864m^2 + 544m + 88,\\
s_4 &= 2s_3 - s_2 = 216m^2 + 136m + 24,\\
s_5 &= 4s_4 - s_3 = 8,
\end{aligned}$$

so that $P_k = km + 1$ for $k = 1, 2$, $P_3 = 3P_2 - P_1 = 5m + 2$, $P_4 = 2P_3 - P_2 = 8m + 3$, $P_5 = 4P_4 - P_3 = 27m + 10$. Then

$$\frac{s_5}{P_5} = \frac{8}{27m + 10} < \frac{(18m+8)^3}{(18m+7)^3} < \frac{216m^2 + 136m + 24}{8m + 3} = \frac{s_4}{P_4},$$

so that $v = 4$. By Equation 2.2 the Frobenius number in this case is

$$\begin{aligned}
&-(18m+6)^3 + (18m+7)^3(216m^2 + 136m + 24 - 1) + (18m+8)^3(27m + 10 - 1)\\
&\qquad - \min\{(18m+7)^3 \cdot 8, (18m+8)^3(8m+3)\}\\
&= -(18m+6)^3 + (18m+7)^3(216m^2 + 136m + 23) + (18m+8)^3(27m + 9) - (18m+7)^3 \cdot 8\\
&= 1259712m^5 + 2420280m^4 + 1840968m^3 + 693468m^2 + 129322m + 9537\\
&= f_6(m).
\end{aligned}$$

Case $\mathbf{n = 18m + 7}$ for some integer $\mathbf{m \geq 2}$:

$$\begin{aligned}
s_{-1} &= n^3 = 5832m^3 + 6804m^2 + 2646m + 343,\\
s_0 &= 18n^2 - 12n + 8 = 5832m^2 + 4320m + 806,\\
s_1 &= (m+1)s_0 - s_{-1} = 3348m^2 + 2480m + 463,\\
s_2 &= 2s_1 - s_0 = 864m^2 + 640m + 120,\\
s_3 &= 4s_2 - s_1 = 108m^2 + 80m + 17,\\
s_4 &= 8s_3 - s_2 = 16,
\end{aligned}$$

so that $P_k = km+1$ for $k = 1, 2$, $P_3 = 4P_2 - P_1 = 7m+3$, $P_4 = 8P_3 - P_2 = 54m+23$. Then

$$\frac{s_4}{P_4} = \frac{16}{54m+23} < \frac{(18m+9)^3}{(18m+8)^3} < \frac{108m^2+80m+17}{8m+3} = \frac{s_3}{P_3},$$

so that $v = 3$. By Equation 2.2 the Frobenius number in this case is

$$\begin{aligned}
&-(18m+7)^3 + (18m+8)^3(108m^2+80m+17-1) + (18m+9)^3(54m+23-1)\\
&\qquad - \min\{(18m+8)^3 \cdot 16, (18m+9)^3(8m+3)\}\\
&= -(18m+7)^3 + (18m+8)^3(108m^2+80m+16) + (18m+9)^3(54m+22) - (18m+8)^3 \cdot 16\\
&= 629856m^5 + 1621296m^4 + 1590192m^3 + 753624m^2 + 173908m + 15695\\
&= f_7(m).
\end{aligned}$$

Note that the "minimum" part forces $m \geq 2$ for a clean formula.

Case $\mathbf{n = 18m + 8}$ for some integer $\mathbf{m \geq 4}$:

$$\begin{aligned}
s_{-1} &= n^3 = 5832m^3 + 7776m^2 + 3456m + 512,\\
s_0 &= 18n^2 - 12n + 8 = 5832m^2 + 4968m + 1064,\\
s_1 &= (m+1)s_0 - s_{-1} = 3024m^2 + 2576m + 552,\\
s_2 &= 2s_1 - s_0 = 216m^2 + 184m + 40,\\
s_3 &= 14s_2 - s_1 = 8,
\end{aligned}$$

so that $P_k = km + 1$ for $k = 1, 2$, $P_3 = 14P_2 - P_1 = 27m + 13$. Then

$$\frac{s_3}{P_3} = \frac{8}{27m+13} < \frac{(18m+10)^3}{(18m+9)^3} < \frac{216m^2+184m+40}{2m+1} = \frac{s_2}{P_2},$$

so that $v = 2$. By Equation 2.2 the Frobenius number in this case is

$$\begin{aligned}
&-(18m+8)^3 + (18m+9)^3(216m^2+184m+40-1) + (18m+10)^3(27m+13-1)\\
&\qquad - \min\{(18m+9)^3 \cdot 8, (18m+10)^3(2m+1)\}\\
&= -(18m+8)^3 + (18m+9)^3(216m^2+184m+39) + (18m+10)^3(27m+12) - (18m+9)^3 \cdot 8\\
&= 1259712m^5 + 3120120m^4 + 3061800m^3 + 1488132m^2 + 358074m + 34087\\
&= f_8(m).
\end{aligned}$$

Note that the "minimum" part forces $m \geq 4$ for a clean formula.

Case $\mathbf{n = 18m + 9}$ for some integer $\mathbf{m \geq 1}$:

$$\begin{aligned}
s_{-1} &= n^3 = 5832m^3 + 8748m^2 + 4374m + 729,\\
s_0 &= 18n^2 - 12n + 8 = 5832m^2 + 5616m + 1358,
\end{aligned}$$

$$
\begin{aligned}
s_1 &= (m+1)s_0 - s_{-1} = 2700m^2 + 2600m + 629,\\
s_2 &= 3s_1 - s_0 = 2268m^2 + 2184m + 529 = s_1 - (432m^2 + 416m + 100),\\
s_3 &= 2s_2 - s_1 = s_1 - 2(432m^2 + 416m + 100),\\
&\ \vdots\\
s_6 &= 2s_5 - s_4 = s_1 - 5(432m^2 + 416m + 100) = 540m^2 + 520m + 129,\\
s_7 &= 2s_6 - s_5 = s_1 - 6(432m^2 + 416m + 100) = 108m^2 + 104m + 29,\\
s_8 &= 5s_7 - s_6 = 16,
\end{aligned}
$$

so that $P_1 = m+1$, $P_2 = 3P_1 - P_0 = 3m+2$, $P_k = 2P_{k-1} - P_{k-2} = (2k-1)m + k$ for $k = 3, \dots, 7$, $P_8 = 5P_7 - P_6 = 5(13m+7) - (11m+6) = 54m+29$. As

$$\frac{s_8}{P_8} = \frac{16}{54m+29} < \frac{(18m+11)^3}{(18m+10)^3} < \frac{108m^2+104m+29}{13m+7} = \frac{s_7}{P_7},$$

so that $v = 7$. By Equation 2.2 the Frobenius number in this case is

$$
\begin{aligned}
-&(18m+9)^3 + (18m+10)^3(108m^2+104m+29-1) + (18m+11)^3(54m+29-1)\\
&\qquad - \min\{(18m+10)^3 \cdot 16, (18m+11)^3(13m+7)\}\\
&= -(18m+9)^3 + (18m+10)^3(108m^2+104m+28) + (18m+11)^3(54m+28) - (18m+10)^3 \cdot 16\\
&= 629856m^5 + 1971216m^4 + 2398896m^3 + 1429704m^2 + 419252m + 48539\\
&= f_9(m).
\end{aligned}
$$

Case $\mathbf{n = 18m + 10}$ for some integer $\mathbf{m \geq 1}$:

$$
\begin{aligned}
s_{-1} &= n^3 = 5832m^3 + 9720m^2 + 5400m + 1000,\\
s_0 &= 18n^2 - 12n + 8 = 5832m^2 + 6264m + 1688,\\
s_1 &= (m+1)s_0 - s_{-1} = 2376m^2 + 2552m + 688,\\
s_2 &= 3s_1 - s_0 = 1296m^2 + 1392m + 376,\\
s_3 &= 2s_2 - s_1 = 216m^2 + 232m + 64,\\
s_4 &= 6s_3 - s_2 = 8,
\end{aligned}
$$

so that $P_1 = m+1$, $P_2 = 3m+2$, $P_3 = 2P_2 - P_1 = 5m+3$. $P_4 = 6P_3 - P_2 = 27m+16$. Then

$$\frac{s_4}{P_4} = \frac{8}{27m+16} < \frac{(18m+12)^3}{(18m+11)^3} < \frac{216m^2+232m+64}{5m+3} = \frac{s_3}{P_3},$$

so that $v = 3$. By Equation 2.2 the Frobenius number in this case is

$$
\begin{aligned}
-&(18m+10)^3 + (18m+11)^3(216m^2+232m+64-1) + (18m+12)^3(27m+16-1)\\
&\qquad - \min\{(18m+11)^3 \cdot 8, (18m+12)^3(5m+3)\}\\
&= -(18m+10)^3 + (18m+11)^3(216m^2+232m+63) + (18m+12)^3(27m+15) - (18m+11)^3 \cdot 8\\
&= 1259712m^5 + 3819960m^4 + 4609224m^3 + 2766636m^2 + 826058m + 98125\\
&= f_{10}(m).
\end{aligned}
$$

Case $\mathbf{n = 18m + 11}$ for some integer $\mathbf{m \geq 1}$: Then Rødseth's algorithm from the first paragraph continues as:

$$\begin{aligned} s_{-1} &= n^3 = 5832m^3 + 10692m^2 + 6534m + 1331, \\ s_0 &= 18n^2 - 12n + 8 = 5832m^2 + 6912m + 2054, \\ s_1 &= (m+1)s_0 - s_{-1} = 2052m^2 + 2432m + 723, \\ s_2 &= 3s_1 - s_0 = 324m^2 + 384m + 115, \\ s_3 &= 7s_2 - s_1 = 216m^2 + 256m + 82, \\ s_4 &= 2s_3 - s_2 = 108m^2 + 128m + 49, \\ s_5 &= 2s_4 - s_3 = 16, \end{aligned}$$

so that $P_1 = m + 1$, $P_2 = 3P_1 - P_0 = 3m + 2$, $P_3 = 7P_2 - P_1 = 20m + 13$, $P_4 = 2P_3 - P_2 = 37m + 24$, $P_5 = 2P_4 - P_3 = 54m + 35$, and

$$\frac{s_5}{P_5} = \frac{16}{54m+35} < \frac{(18m+13)^3}{(18m+12)^3} < \frac{108m^2+128m+49}{37m+24} = \frac{s_4}{P_4},$$

so that $v = 4$. By Equation 2.2 the Frobenius number in this case is

$$\begin{aligned} &-(18m+11)^3 + (18m+12)^3(108m^2+128m+49-1) + (18m+13)^3(54m+35-1) \\ &\qquad - \min\{(18m+12)^3 \cdot 16, (18m+13)^3(37m+24)\} \\ &= -(18m+11)^3 + (18m+12)^3(108m^2+128m+48) + (18m+13)^3(54m+34) - (18m+12)^3 \cdot 16 \\ &= 629856m^5 + 2321136m^4 + 3394224m^3 + 2466936m^2 + 892404m + 128663 \\ &= f_{11}(m). \end{aligned}$$

Case $\mathbf{n = 18m + 12}$ for some integer $\mathbf{m \geq 1}$:

$$\begin{aligned} s_{-1} &= n^3 = 5832m^3 + 11664m^2 + 7776m + 1728, \\ s_0 &= 18n^2 - 12n + 8 = 5832m^2 + 7560m + 2456, \\ s_1 &= (m+1)s_0 - s_{-1} = 1728m^2 + 2240m + 728, \\ s_2 &= 4s_1 - s_0 = 1080m^2 + 1400m + 456, \\ s_3 &= 2s_2 - s_1 = 432m^2 + 560m + 184, \\ s_4 &= 3s_3 - s_2 = 216m^2 + 280m + 96, \\ s_5 &= 2s_4 - s_3 = 8, \end{aligned}$$

so that $P_1 = m + 1$, $P_2 = 4P_1 - P_0 = 4m + 3$, $P_3 = 2P_2 - P_1 = 7m + 5$, $P_4 = 3P_3 - P_2 = 17m + 12$. $P_5 = 2P_4 - P_3 = 27m + 19$. Then

$$\frac{s_5}{P_5} = \frac{8}{27m+19} < \frac{(18m+14)^3}{(18m+13)^3} < \frac{216m^2+280m+96}{17m+12} = \frac{s_4}{P_4},$$

so that $v = 4$. By Equation 2.2 the Frobenius number in this case is

$$\begin{aligned} &-(18m+12)^3 + (18m+13)^3(216m^2+280m+96-1) + (18m+14)^3(27m+19-1) \\ &\qquad - \min\{(18m+13)^3 \cdot 8, (18m+14)^3(17m+12)\} \\ &= -(18m+12)^3 + (18m+13)^3(216m^2+280m+95) + (18m+14)^3(27m+18) - (18m+13)^3 \cdot 8 \\ &= 1259712m^5 + 4519800m^4 + 6483240m^3 + 4648212m^2 + 1665946m + 238803 \\ &= f_{12}(m). \end{aligned}$$

Case $\mathbf{n = 18m + 13}$ for some integer $\mathbf{m \geq 1}$:

$$\begin{aligned}
s_{-1} &= n^3 = 5832m^3 + 12636m^2 + 9126m + 2197,\\
s_0 &= 18n^2 - 12n + 8 = 5832m^2 + 8208m + 2894,\\
s_1 &= (m+1)s_0 - s_{-1} = 1404m^2 + 1976m + 697,\\
s_2 &= 5s_1 - s_0 = 1188m^2 + 1672m + 591,\\
s_3 &= 2s_2 - s_1 = 972m^2 + 1368m + 485 = s_2 - (216m^2 + 304m + 106),\\
&\ \vdots\\
s_6 &= 2s_5 - s_4 = s_2 - 4(216m^2 + 304m + 106) = 324m^2 + 456m + 167,\\
s_7 &= 2s_6 - s_5 = s_2 - 5(216m^2 + 304m + 106) = 108m^2 + 152m + 61,\\
s_8 &= 3s_7 - s_6 = 16,
\end{aligned}$$

so that $P_1 = m + 1$, $P_k = (4k-3)m + 3k - 2$ for $k = 2, \ldots, 7$, $P_8 = 3P_7 - P_6 = 3(25m + 19) - (21m + 16) = 54m + 41$, and

$$\frac{s_8}{P_8} = \frac{16}{54m + 41} < \frac{(18m+15)^3}{(18m+14)^3} < \frac{108m^2 + 152m + 61}{25m + 19} = \frac{s_7}{P_7},$$

so that $v = 7$. By Equation 2.2 the Frobenius number in this case is

$$\begin{aligned}
&-(18m+13)^3 + (18m+14)^3(108m^2 + 152m + 61 - 1) + (18m+15)^3(54m + 41 - 1)\\
&\qquad - \min\{(18m+14)^3 \cdot 16, (18m+15)^3(25m+19)\}\\
&= -(18m+13)^3 + (18m+14)^3(108m^2 + 152m + 60) + (18m+15)^3(54m+40) - (18m+14)^3 \cdot 16\\
&= 629856m^5 + 2671056m^4 + 4482864m^3 + 3730536m^2 + 1541908m + 253539\\
&= f_{13}(m).
\end{aligned}$$

Case $\mathbf{n = 18m + 14}$ for some integer $\mathbf{m \geq 1}$:

$$\begin{aligned}
s_{-1} &= n^3 = 5832m^3 + 13608m^2 + 10584m + 2744,\\
s_0 &= 18n^2 - 12n + 8 = 5832m^2 + 8856m + 3368,\\
s_1 &= (m+1)s_0 - s_{-1} = 1080m^2 + 1640m + 624,\\
s_2 &= 6s_1 - s_0 = 648m^2 + 984m + 376,\\
s_3 &= 2s_2 - s_1 = 216m^2 + 328m + 128,\\
s_4 &= 3s_3 - s_2 = 8,
\end{aligned}$$

so that $P_1 = m + 1$, $P_2 = 6P_1 - P_0 = 6m + 5$, $P_3 = 2P_2 - P_1 = 11m + 9$, $P_4 = 3P_3 - P_2 = 27m + 22$. Then

$$\frac{s_4}{P_4} = \frac{8}{27m + 22} < \frac{(18m+16)^3}{(18m+15)^3} < \frac{216m^2 + 328m + 128}{11m + 9} = \frac{s_3}{P_3},$$

so that $v = 3$. By Equation 2.2 the Frobenius number in this case is

$$\begin{aligned}
&-(18m+14)^3 + (18m+15)^3(216m^2 + 328m + 128 - 1) + (18m+16)^3(27m + 22 - 1)\\
&\qquad - \min\{(18m+15)^3 \cdot 8, (18m+16)^3(11m+9)\}\\
&= -(18m+14)^3 + (18m+15)^3(216m^2 + 328m + 127) + (18m+16)^3(27m+21) - (18m+15)^3 \cdot 8\\
&= 1259712m^5 + 5219640m^4 + 8637192m^3 + 7135452m^2 + 2943162m + 484897
\end{aligned}$$

$= f_{14}(m).$

Case $\mathbf{n = 18m + 15}$ for some integer $\mathbf{m \geq 1}$:

$$\begin{aligned}
s_{-1} &= n^3 = 5832m^3 + 14580m^2 + 12150m + 3375,\\
s_0 &= 18n^2 - 12n + 8 = 5832m^2 + 9504m + 3878,\\
s_1 &= (m+1)s_0 - s_{-1} = 756m^2 + 1232m + 503,\\
s_2 &= 8s_1 - s_0 = 216m^2 + 352m + 146,\\
s_3 &= 4s_2 - s_1 = 108m^2 + 176m + 81,\\
s_4 &= 2s_3 - s_2 = 16,
\end{aligned}$$

so that $P_1 = m + 1$, $P_2 = 8P_1 - P_0 = 8m + 7$, $P_3 = 4P_2 - P_1 = 31m + 27$, $P_4 = 2P_3 - P_2 = 54m + 47$, and

$$\frac{s_4}{P_4} = \frac{16}{54m + 47} < \frac{(18m+17)^3}{(18m+16)^3} < \frac{108m^2 + 176m + 81}{31m + 27} = \frac{s_3}{P_3},$$

so that $v = 3$. By Equation 2.2 the Frobenius number in this case is

$$\begin{aligned}
&-(18m+15)^3 + (18m+16)^3(108m^2 + 176m + 81 - 1) + (18m+17)^3(54m + 47 - 1)\\
&\qquad - \min\{(18m+16)^3 \cdot 16, (18m+17)^3(31m+27)\}\\
&= -(18m+15)^3 + (18m+16)^3(108m^2 + 176m + 80) + (18m+17)^3(54m+46) - (18m+16)^3 \cdot 16\\
&= 629856m^5 + 3020976m^4 + 5758128m^3 + 5458968m^2 + 2576660m + 484767\\
&= f_{15}(m).
\end{aligned}$$

Case $\mathbf{n = 18m + 16}$ for some integer $\mathbf{m \geq 1}$:

$$\begin{aligned}
s_{-1} &= n^3 = 5832m^3 + 15552m^2 + 13824m + 4096,\\
s_0 &= 18n^2 - 12n + 8 = 5832m^2 + 10152m + 4424,\\
s_1 &= (m+1)s_0 - s_{-1} = 432m^2 + 752m + 328,\\
s_2 &= 14s_1 - s_0 = 216m^2 + 376m + 168,\\
s_3 &= 2s_2 - s_1 = 8,
\end{aligned}$$

so that $P_1 = m+1$, $P_2 = 14P_1 - P_0 = 14m + 13$, $P_3 = 2P_2 - P_1 = 27m + 25$. Then

$$\frac{s_3}{P_3} = \frac{8}{27m + 25} < \frac{(18m+18)^3}{(18m+17)^3} < \frac{216m^2 + 376m + 168}{14m + 13} = \frac{s_2}{P_2},$$

so that $v = 2$. By Equation 2.2 the Frobenius number in this case is

$$\begin{aligned}
&-(18m+16)^3 + (18m+17)^3(216m^2 + 376m + 168 - 1) + (18m+18)^3(27m + 25 - 1)\\
&\qquad - \min\{(18m+17)^3 \cdot 8, (18m+18)^3(14m+13)\}\\
&= -(18m+16)^3 + (18m+17)^3(216m^2 + 376m + 167) + (18m+18)^3(27m+24) - (18m+17)^3 \cdot 8\\
&= 1259712m^5 + 5919480m^4 + 11117736m^3 + 10433124m^2 + 4892186m + 917039\\
&= f_{16}(m).
\end{aligned}$$

Case $\mathbf{n = 18m + 17}$ for some integer $\mathbf{m \geq 15}$:

$$s_{-1} = n^3 = 5832m^3 + 16524m^2 + 15606m + 4913,$$

$$s_0 = 18n^2 - 12n + 8 = 5832m^2 + 10800m + 5006,$$
$$s_1 = (m+1)s_0 - s_{-1} = 108m^2 + 200m + 93,$$
$$s_2 = 54s_1 - s_0 = 16,$$

so that $P_1 = m+1$, $P_2 = 54P_1 - P_0 = 54m + 53$, and

$$\frac{s_2}{P_2} = \frac{16}{54m+53} < \frac{(18m+19)^3}{(18m+18)^3} < \frac{108m^2+200m+93}{m+1} = \frac{s_1}{P_1},$$

so that $v = 1$. By Equation 2.2 the Frobenius number in this case is

$$\begin{aligned}
&-(18m+17)^3 + (18m+18)^3(108m^2+200m+93-1) + (18m+19)^3(54m+53-1) \\
&\qquad - \min\{(18m+18)^3 \cdot 16, (18m+19)^3(m+1)\} \\
&\quad = -(18m+17)^3 + (18m+18)^3(108m^2+200m+92) + (18m+19)^3(54m+52) - (18m+18)^3 \cdot 16 \\
&\qquad\qquad (\text{because } m \geq 15) \\
&\quad = 629856m^5 + 3370896m^4 + 7126704m^3 + 7455240m^2 + 3864564m + 794987 \\
&\quad = f_{17}(m).
\end{aligned}$$

This finishes the proof of Theorem 4.1. □

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