2×2 permanental ideals of hypermatrices

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Abstract. We study the structure of ideals generated by some classes of 2×2 permanents of hypermatrices. This generalizes [9] on 2×2 permanental ideals of generic matrices. We compare the obtained structure to that of the corresponding determinantal ideals in [11]: while the notion of t-switchability introduced in [11] plays a role for both permanental and determinantal ideals, the permanents require further restrictions, which in general increases the number of minimal primes. In the last two sections we examine a few related classes of permanental ideals.

1 Introduction

Determinants are ubiquitous in mathematics; if in the Laplace expansion of a determinant we replace all minus signs with plus signs, the result is the *permanent* of the matrix. The starting motivation for this work is the paper [11] on classes of 2×2 determinantal ideals of certain hypermatrices that arise in models of conditional independence; here we study the analogous 2×2 permanental ideals.

The computation of determinants can be done in polynomial time because of Gaussian elimination, whereas the computation of permanents is #-P-complete ([12]). In another comparison, the ideal generated by $r \times r$ determinants of a generic $m \times n$ matrix is a prime ideal with many nice properties ([2]), but the ideal generated by 2×2 permanents has $\binom{m}{2}\binom{n}{2} + m + n$ minimal components and one embedded component if $m, n \geq 3$ ([9]), and the structure of ideals generated by $r \times r$ permanents for $r \geq 3$ is even more complicated and not completely understood ([8]). This paper shows yet another aspect of the greater complexity of permanents over determinants.

Conditional independence ideals in algebraic statistics are ideals generated by some 2×2 determinants of generic hypermatrices. Their structures have been studied recently by Fink [4], Herzog–Hibi–Hreinsdottir–Kahle–Rauh [6], Ohtani [10], Swanson–Taylor [11], and Ah–Rauh [1]. In this paper we study the structure of analogous 2×2 permanental ideals of generic hypermatrices that depend on a parameter t and the size of the hypermatrix. The main result is Theorem 7.2 which gives a combinatorial description of all the prime ideals minimal over these permanental ideals. Not surprisingly, just as in [9], the number of minimal prime ideals is in general larger over these permanental ideals than over the corresponding determinantal ideals. We give an explicit set of generators of the minimal primes and give their Gröbner bases (Theorem 6.5). Section 5 shows many concrete examples. A consequence is that this paper puts the structure of the 2×2 permanental ideals (of generic matrices), as described in [9], into more general and new perspective (see Example 5.1).

In Sections 9 and 10 we present the structure of related permanental ideals, those generated by certain "diagonal" permanents, and by certan diagonal and slice permanents.

Many of the methods in this paper are similar to those in [11], but for permanental ideals one has to keep track of the signs, which adds not only notational difficulty but also changes many results. In particular, it increases the number of minimal primes significantly, as was already shown in the case of generic matrices (as opposed to hypermatrices) in [9].

This is an extension of the first author's senior thesis at Reed College, 2011, under the second author's supervision.

2 Set-up

We fix positive integers n and $r_1, \ldots, r_n$. Let R be the polynomial ring in $r_1 \cdots r_n$ variables over a field k. The variables will be written with lower case x with n-place subscripts, with the ith place in the subscript ranging from 1 through r_i. We arrange these variables into a generic $r_1 \times \cdots \times r_n$ hypermatrix that will be fixed throughout.

Throughout the paper $[r]$ denotes the set $\{1, 2, \ldots, r\}$. Thus, for example, the ring we use is $R = k[x_a : a \in [r_1] \times \cdots \times [r_n]]$. Throughout $N = [r_1] \times \cdots \times [r_n]$.

Let $L \subseteq [n]$. For $a, b \in N$ define the **switch** function $\mathrm{s}(L, a, b)$ that switches the L-entries of a into b: $\mathrm{s}(L, a, b)$ is an element of N whose ith component is

$$\mathrm{s}(L, a, b)_i = \begin{cases} b_i, & \text{if } i \in L; \\ a_i, & \text{otherwise.} \end{cases}$$

If $L = \{j\}$, we simply write $\mathrm{s}(L, a, b) = \mathrm{s}(j, a, b)$. For any a and b in N we define **the distance d** between them to be $\mathbf{d(a,b)} = \#\{i : a_i \neq b_i\}$. Note that $d(a, b) = d(\mathrm{s}(L, a, b), \mathrm{s}(L, b, a))$.

For any subsets $K \subseteq L \subseteq [n]$ and any $t \in [n]$ we define:

$$\begin{aligned} f_{K,a,b} &= x_a x_b - x_{\mathrm{s}(K,a,b)} x_{\mathrm{s}(K,b,a)}, \\ g_{K,a,b} &= x_a x_b + x_{\mathrm{s}(K,a,b)} x_{\mathrm{s}(K,b,a)}, \\ G_{L,K} &= \{g_{K,a,b} : a, b \in N, \{j : a_j \neq b_j\} = L\}. \end{aligned}$$

The elements $g_{K,a,b}$ above are (generalized) permanents, and the $f_{K,a,b}$ are (generalized) determinants. Whenever $K = \{i\}$, we write i instead of $\{i\}$, such as $f_{i,a,b}$, $g_{i,a,b}$, $G_{L,i}$. When $d(a, b) = 2$ and $a_i \neq b_i$, we call $g_{i,a,b}$ a **slice permanent** and $f_{i,a,b}$ a **slice determinant**. For any $t \in [n]$ we define:

$$\begin{aligned} I^{\langle t \rangle} &= (f_{i,a,b} : a, b \in N, d(a, b) = 2, i \in [t], a_i \neq b_i), \\ J^{\langle t \rangle} &= (g_{i,a,b} : a, b \in N, d(a, b) = 2, i \in [t], a_i \neq b_i). \end{aligned}$$

We examine the minimal primes over $J^{\langle t \rangle}$, whereas [11] did so for $I^{\langle t \rangle}$. The structure of ideals generated by classes of slice permanents turns out to be different from the structure of the corresponding ideals generated by slice determinants [11]. A special case of this difference was already demonstrated in [9] when $n = 2$. In order to have $J^{\langle t \rangle}$ different from $I^{\langle t \rangle}$, we assume throughout that the characteristic of the underlying field is not 2.

3 Lemmas for induction arguments

It is proved in [11] that for $a, a_1, b \in N$ and $i \in [n]$, $x_b f_{i,a,a_1} - x_{a_1} f_{i,a,b} = x_{\mathrm{s}(i,b,a)} f_{i,a_1,\mathrm{s}(i,a,b)} - x_{\mathrm{s}(i,a,a_1)} f_{i,\mathrm{s}(i,a_1,a),b}$. We prove the almost analogous result for permanental ideals:

Lemma 3.1 *For $a, a_1, b \in N$ and $i \in [n]$,*

$$x_b f_{i,a,a_1} - x_{a_1} f_{i,a,b} = x_{\mathrm{s}(i,b,a)} g_{i,a_1,\mathrm{s}(i,a,b)} - x_{\mathrm{s}(i,a,a_1)} g_{i,\mathrm{s}(i,a_1,a),b}.$$

In particular, if a and a_1 differ only in the jth component, and $j \neq i$, and $b_i \neq (a_1)_i$, then

$$x_a g_{i,a_1,b} - x_{a_1} f_{i,a,b} = x_{\mathrm{s}(i,b,a)} g_{i,a_1,\mathrm{s}(i,a,b)} \in (G_{\{i,j\},\{i\}}).$$

Proof. Just as in [11], the calculation of the first part is straightforward:

$$\begin{aligned}
x_b f_{i,a,a_1} - x_{a_1} f_{i,a,b} &+ x_{\mathrm{s}(i,a,a_1)} g_{i,\mathrm{s}(i,a_1,a),b} \\
&= x_b x_a x_{a_1} - x_b x_{\mathrm{s}(i,a,a_1)} x_{\mathrm{s}(i,a_1,a)} - x_{a_1} x_a x_b + x_{a_1} x_{\mathrm{s}(i,a,b)} x_{\mathrm{s}(i,b,a)} \\
&\qquad + x_{\mathrm{s}(i,a,a_1)} x_{\mathrm{s}(i,a_1,a)} x_b + x_{\mathrm{s}(i,a,a_1)} x_{\mathrm{s}(i,\mathrm{s}(i,a_1,a),b)} x_{\mathrm{s}(i,b,\mathrm{s}(i,a_1,a))} \\
&= x_{a_1} x_{\mathrm{s}(i,a,b)} x_{\mathrm{s}(i,b,a)} + x_{\mathrm{s}(i,a,a_1)} x_{\mathrm{s}(i,a_1,b)} x_{\mathrm{s}(i,b,a)} \\
&= x_{\mathrm{s}(i,b,a)} \big(x_{a_1} x_{\mathrm{s}(i,a,b)} + x_{\mathrm{s}(i,a,a_1)} x_{\mathrm{s}(i,a_1,b)} \big) \\
&= x_{\mathrm{s}(i,b,a)} g_{i,a_1,\mathrm{s}(i,a,b)}.
\end{aligned}$$

If a and a_1 differ only in the jth position with $j \neq i$, then $f_{i,a,a_1} = 0$ for all i, $\mathrm{s}(i, a_1, a) = a_1$, $\mathrm{s}(i, a, a_1) = a$, and a_1 and $\mathrm{s}(i, a, b)$ differ at most in the two components i and j, and the rest follows. □

The following lemma is via induction an immediate corollary of Lemma 3.1:

Lemma 3.2 *Let i be a positive integer, and let $a_0, a_1, \ldots, a_k, b \in N$. Suppose that the ith component of b differs from the ith components of $a_1, \ldots, a_k$, and that for all $j = 1, \ldots, k$, a_{j-1} and a_j differ exactly in component $l_j \neq i$. Then modulo $\sum_{j=1}^{k} (G_{\{i,l_j\},\{i\}})$,*

$$x_{a_1} x_{a_2} \cdots x_{a_k} f_{i,a_0,b} \equiv \begin{cases} x_{a_0} x_{a_1} \cdots x_{a_{k-1}} g_{i,a_k,b}, & \text{if } k \text{ is odd;} \\ x_{a_0} x_{a_1} \cdots x_{a_{k-1}} f_{i,a_k,b}, & \text{if } k \text{ is even;} \end{cases}$$

$$x_{a_1} x_{a_2} \cdots x_{a_k} g_{i,a_0,b} \equiv \begin{cases} x_{a_0} x_{a_1} \cdots x_{a_{k-1}} f_{i,a_k,b}, & \text{if } k \text{ is odd;} \\ x_{a_0} x_{a_1} \cdots x_{a_{k-1}} g_{i,a_k,b}, & \text{if } k \text{ is even.} \end{cases}$$

In particular, if $d(a_k, b) = 2$ and a_k and b differ in positions i and $l_0 \neq i$, then $\sum_{j=0}^{k} (G_{\{i,l_j\},\{i\}})$ contains $x_{a_1} x_{a_2} \cdots x_{a_k} f_{i,a_0,b}$ if k is odd, and it contains $x_{a_1} x_{a_2} \cdots x_{a_k} g_{i,a_0,b}$ if k is even. Also, if $d(a_k, b) = 1$, then $\sum_{j=0}^{k} (G_{\{i,l_j\},\{i\}})$ contains $x_{a_1} x_{a_2} \cdots x_{a_k} f_{i,a_0,b}$ if k is even, and it contains $x_{a_1} x_{a_2} \cdots x_{a_k} g_{i,a_0,b}$ if k is odd. □

4 Switchable and signed sets

Just as in [11], we need t-switchability and connectedness:

Definition 4.1 *A subset S of N is* **t-switchable** *if for all $a, b \in S$ with $d(a, b) = 2$, and all $i \in [t]$, we have that $\mathrm{s}(i, a, b), \mathrm{s}(i, b, a) \in S$.*

If a set S is t-switchable, define $a, b \in S$ to be **connected** *if there exist $c_0 = a, c_1, \ldots, c_{k-1}, c_k = b$ in S such that for all i, $d(c_i, c_{i+1}) = 1$. We call $c_0 = a, c_1, \ldots, c_{k-1}, c_k = b$ as above* **a path from a to b**. *We call k* **the length of the path**. *The smallest length of a path from a to b is called the* **path length from a to b** *and will be denoted as $\mathbf{pl_S(a, b)}$. We define $\overline{\mathbf{pl}}_{\mathbf{S}}(\mathbf{a}, \mathbf{b})$ to be $\mathrm{pl}_S(a, b) - 1$.*

Remark 4.2 Clearly connectedness is an equivalence relation. By Lemma 3.5 in [11], for any a, b that are connected in some t-admissible set and for any subset $K \subseteq [t]$, $\mathrm{s}(K, a, b), \mathrm{s}(K, b, a)$ are in S and connected to a and b in S.

For a given a, b that are connected in S, it may happen that there are paths of different lengths between them. Furthermore, there can be paths of lengths of different parity between them, such as if $S = [3] \times [2] \times [2]$: $(1, 1, 1), (2, 1, 1), (3, 1, 1)$ and $(1, 1, 1), (3, 1, 1)$ are both paths from $(1, 1, 1)$ to $(3, 1, 1)$.

Proposition 4.3 *Let S be a t-switchable set, and let $a, b \in S$ be connected with $d(a, b) \geq 2$ and $a_i \neq b_i$ for some $i \in [t]$. Suppose that there are paths from a to b of different parity. Then $\sum_{j \neq i}(G_{\{i,j\},\{i\}})$ contains monomials all of whose subscripts are elements of S.*

Proof. Let $c_0 = a, c_1, \ldots, c_{k-1}, c_k = b$ be a path from a to b. Since $d(a, b) \geq 2$, it follows that $k \geq 2$.

Let j be least such that the ith entry of c_j is not a_i. Necessarily $j > 0$. Suppose that $j \leq k-2$. By assumption, c_{j-1} and c_j differ precisely in the ith component. Suppose that for all $l = j-1, \ldots, k-1$, c_l and c_{l+1} differ exactly in the ith component. Then we can cut an even number of elements $c_j, \ldots, c_{k-1}$ from the given path to still get a path from a to b with the same parity but such that the designated j is at least $k-1$. Now assume otherwise: then there exists an integer $l > j$ such that c_{l-1} and c_l differ in component $j \neq i$. We choose l to be the smallest such integer. Then the part $c_{j-1}, c_j, \ldots, c_l$ of the given path can by Remark 4.2 be replaced by $c_{j-1}, \mathrm{s}(i, c_l, c_{j-1}), \mathrm{s}(i, c_l, c_j), \mathrm{s}(i, c_l, c_{j+1}), \ldots, \mathrm{s}(i, c_l, c_{l-1}) = c_l$, which has the same length, and the designated j is increased by 1. By repeating these steps we may assume that the ith entries in $c_0, c_1, \ldots, c_{k-2}$ all equal a_i.

In particular, $1 \leq d(c_{k-2}, b) \leq 2$.

First suppose that $d(c_{k-2}, b) = 1$. Then c_{k-2}, c_{k-1}, b have distinct ith entries, and only differ in the ith entries. Since $d(a, b) \geq 2$, necessarily $k > 2$, and so c_{k-3} differs from c_{k-2}, c_{k-1}, b in the jth entry for some $j \neq i$, and differs from c_{k-1}, b also in the ith entry. Then

$$\begin{bmatrix} x_{c_{k-3}} & x_{\mathrm{s}(i,c_{k-3},c_{k-1})} & x_{\mathrm{s}(i,c_{k-3},b)} \\ x_{c_{k-2}} & x_{\mathrm{s}(i,c_{k-2},c_{k-1})} = x_{c_{k-1}} & x_{\mathrm{s}(i,c_{k-2},b)} = x_b \end{bmatrix}$$

is a submatrix of the hypermatrix, and all of its 2×2 permanents are in $(G_{\{i,j\},\{i\}})$. Let $p_{u,v}$ be the permanent of columns u, v in the matrix above. Then $(G_{\{i,j\},\{i\}})$ contains

$$x_{c_{k-3}} p_{2,3} - x_{\mathrm{s}(i,c_{k-3},c_{k-1})} p_{1,3} + x_{\mathrm{s}(i,c_{k-3},b)} p_{1,2} = 2 x_{c_{k-3}} x_{c_{k-1}} x_b,$$

and as the characteristic is not 2, $(G_{\{i,j\},\{i\}})$ contains a monomial all of whose subscripts are in S.

So we may assume that $d(c_{k-2}, b) = 2$ for all paths from a to b. Similarly, we may assume that if $d_0 = a, d_1, \ldots, d_{l-1}, d_l = b$ is a path from a to b with l of different parity from k, then the ith components of $d_0, d_1, \ldots, d_{l-2}$ are all a_i and $d(d_{l-2}, b) = 2$. Say k is even and l is odd. Then by Lemma 3.2, $x_{c_1} \cdots x_{c_{k-2}} g_{i,a,b}, \quad x_{d_1} \cdots x_{d_{l-2}} f_{i,a,b} \in \sum_{j \neq i} (G_{\{i,j\},\{i\}})$. In particular,

$$\mathrm{lcm}(x_{c_1} \cdots x_{c_{k-2}}, x_{d_1} \cdots x_{d_{l-2}})(g_{i,a,b}, f_{i,a,b}) \subseteq \sum_{j \neq i} (G_{\{i,j\},\{i\}}).$$

But $(g_{i,a,b}, f_{i,a,b}) = (x_a x_b, x_{\mathrm{s}(i,a,b)} x_{\mathrm{s}(i,b,a)})$ as ideals, so that $\sum (G_{\{i,j\},\{i\}})$ contains the monomials

$$\mathrm{lcm}(x_{c_1} \cdots x_{c_{k-2}}, x_{d_1} \cdots x_{d_{l-2}}) x_a x_b \text{ and } \mathrm{lcm}(x_{c_1} \cdots x_{c_{k-2}}, x_{d_1} \cdots x_{d_{l-2}}) x_{\mathrm{s}(i,a,b)} x_{\mathrm{s}(i,b,a)}.$$

□

For our analysis of prime ideals that are minimal over the permanental ideal $J^{\langle t \rangle}$, we deal with the parity question with the following further restriction on switchable sets:

Definition 4.4 *A t-switchable set S is called* **t-signed** *if for every equivalence class S_0 with respect to the connected property, one of the following conditions is satisfied:*
(1) All elements of S_0 have the same first t-coordinates;
(2) Any two elements of S_0 differ at most in one component;
(3) The parity of path length between any two elements in S_0 is independent of the path.

Remark 4.5 Let S be a t-signed set and let $a, b, c \in S$ be connected and in an equivalence class that satisfies property (3) in Definition 4.4. Then for any $K \subseteq [t]$, $\mathrm{pl}_S(a, b)$ and $\mathrm{pl}_S(\mathrm{s}(K, a, b), \mathrm{s}(K, b, a))$ have the same parity because we can make the path from $\mathrm{s}(K, a, b)$ to $\mathrm{s}(K, b, a)$ pass first from $\mathrm{s}(K, a, b)$ to a in $\#\{i \in K : a_i \neq b_i\}$ steps, then to b in $\mathrm{pl}_S(a, b)$ steps, and then to $\mathrm{s}(K, b, a)$ in $\#\{i \in K : a_i \neq b_i\}$ steps. Similarly, $\mathrm{pl}_S(a, b) + \mathrm{pl}_S(b, c)$ and $\mathrm{pl}_S(a, c)$ have the same parity.

Lemma 4.6 *Let S be a t-signed set. Let S_0 be an equivalence class in S with respect to connectedness. Then either for all $i \in [t]$, $\{a_i : a \in S_0\}$ contains at most two elements or else for all $a, b \in S_0$, $d(a, b) \leq 1$.*

Proof. Suppose for contradiction that there are $a, b, a', b', c' \in S_0$ such that for some $i \in [t]$, the ith components of a', b', c' are all distinct and such that $d(a, b) \geq 2$. Thus

S_0 fails conditions (1) and (2) in Definition 4.4, so S_0 must satisfy (3). But then $\mathrm{s}(i,a,a'), \mathrm{s}(i,a,b'), \mathrm{s}(i,a,c')$ and $\mathrm{s}(i,a,a'), \mathrm{s}(i,a,c')$ are both paths from $\mathrm{s}(i,a,a')$ to $\mathrm{s}(i,a,c')$, and they have different parities, which contradicts the condition in (3). $\square$

One might be tempted to think that for a t-signed equivalence class S_0, if there exists $i \in [t]$ such that $\{a_i : a \in S_0\}$ has two elements, then for all $j \in [n]$, $\{a_j : a \in S_0\}$ has at most 2 elements. This need not be the case, as demonstrated in Example 5.6 (4).

The following encapsulates and generalizes the $f_{K,a,b}$ and $g_{K,a,b}$, and will be used to describe generators of prime ideals minimal over $J^{\langle t\rangle}$ (details in Remark 4.8):

Definition 4.7 *Let S be t-signed. For any connected $a, b \in S$, and any $K \subseteq \{i \in [t] : a_i \neq b_i\}$, define*

$$h_{S,K,a,b} = x_a x_b - (-1)^{\#K \cdot \overline{pl}_S(a,b)} x_{\mathrm{s}(K,a,b)}, x_{\mathrm{s}(K,b,a)}.$$

Remark 4.8 Suppose that for some $L \subseteq \{i \in [t] : a_i \neq b_i\}$, $x_{\mathrm{s}(K,a,b)} x_{\mathrm{s}(K,b,a)} = x_{\mathrm{s}(L,a,b)} x_{\mathrm{s}(L,b,a)}$. We prove that then $h_{S,K,a,b} = h_{S,L,a,b}$. If $\overline{pl}_S(a,b)$ is even, this is certainly true. So we may assume that $\mathrm{pl}_S(a,b) = \overline{pl}_S(a,b) + 1$ is even. Without loss of generality $K \neq L$. The assumption $x_{\mathrm{s}(K,a,b)}, x_{\mathrm{s}(K,b,a)} = x_{\mathrm{s}(L,a,b)}, x_{\mathrm{s}(L,b,a)}$ is only possible if $(a_{t+1}, \ldots, a_n) = (b_{t+1}, \ldots, b_n)$ or if $t = n$. In either case, $L = \{i : a_i \neq b_i\} \setminus K$, and $\{i : a_i \neq b_i\} \subseteq [t]$, so that by Remark 4.2, $\mathrm{pl}_S(a,b) = d(a,b) = \{i : a_i \neq b_i\}$. But $\mathrm{pl}_S(a,b)$ is even, so that $\#L$ and $\#K$ must have the same parity. Thus $h_{S,K,a,b} = h_{S,L,a,b}$.

The following definitions will be applied mostly to t-signed S:

Definition 4.9 *For any subset S of N, define*

$$\begin{aligned}
\mathrm{Var}_S^{\langle t\rangle} &= (x_a : a \notin S),\\
\widetilde{J}_S^{\langle t\rangle} &= (h_{S,i,a,b} : a, b \text{ are connected in } S, i \in [t], a_i \neq b_i),\\
\mathrm{Q}_S^{\langle t\rangle} &= \mathrm{Var}_S^{\langle t\rangle} + \widetilde{J}_S^{\langle t\rangle}.
\end{aligned}$$

Note that $t = n$ and $t = n - 1$ give the same sets of ideals. Thus in the sequel, and especially in Section 6, we mostly talk about $t < n$. For t-switchable S, the analogous determinantal ideals $\widetilde{I}_S^{\langle t\rangle} = (f_{i,a,b} : i \in [t], a, b \text{ are connected in } S)$ and $\mathrm{P}_S^{\langle t\rangle} = \mathrm{Var}_S^{\langle t\rangle} + \widetilde{I}_S^{\langle t\rangle}$ were proved to be prime ideals in [11]. Much of what we prove in this paper mimics the proofs of [11], but with the added sign difficulty expressed through t-signedness.

Lemma 4.10 $\{h_{S,K,a,b} : a, b \text{ are connected in } S, K \subseteq \{i \in [t] : a_i \neq b_i\}\} \subseteq \widetilde{J}_S^{\langle t\rangle}$.

Proof. Let $a, b \in S$ and let $K \subseteq \{i \in [t] : a_i \neq b_i\}$. Write $K = \{k_1, \ldots, k_l\}$. By Remark 4.2, $\mathrm{s}(\{k_1, \ldots, k_{i-1}\}, a, b)$ and $\mathrm{s}(\{k_1, \ldots, k_{i-1}\}, b, a))$ are connected, and by Remark 4.5, $\mathrm{pl}_S(a,b) = \mathrm{pl}_S(\mathrm{s}(\{k_1, \ldots, k_{i-1}\}, a, b), \mathrm{s}(\{k_1, \ldots, k_{i-1}\}, b, a))$, so that

$$h_{S,K,a,b} = \sum_{i=1}^{l} (-1)^{(i-1)\cdot \overline{pl}_S(a,b)} h_{S,k_i,\mathrm{s}(\{k_1,\ldots,k_{i-1}\},a,b),\mathrm{s}(\{k_1,\ldots,k_{i-1}\},b,a)}.$$

$\square$

Proposition 4.11 *Let S be a t-switchable set (not necessarily t-signed). Then $J^{\langle t\rangle} \subseteq \mathrm{Q}_S^{\langle t\rangle}$.*

Proof. Let $a, b \in N$ with $d(a, b) = 2$ and $a_i \neq b_i$ for some $i \in [t]$. We need to prove that $h_{S,i,a,b} \in \mathrm{Q}_S^{\langle t\rangle}$. If $a, b \in S$, then by t-switchability of S, a and b are connected, so that $h_{S,i,a,b} \in \widetilde{J}_S^{\langle t\rangle} \subseteq \mathrm{Q}_S^{\langle t\rangle}$. Similarly, if $\mathrm{s}(i, a, b), \mathrm{s}(i, b, a) \in S$, then $h_{S,i,a,b} = \pm h_{S,i,\mathrm{s}(i,a,b),\mathrm{s}(i,b,a)} \in \mathrm{Q}_S^{\langle t\rangle}$, So we may assume that $a \notin S$ and that either $\mathrm{s}(i, a, b)$ or $\mathrm{s}(i, b, a)$ is not in S. But then $h_{S,i,a,b} \in \mathrm{Var}_S^{\langle t\rangle} \subseteq \mathrm{Q}_S^{\langle t\rangle}$. □

Definition 4.12 *Let S be t-signed. We say that S is* **maximal t-signed** *if for all t-signed subsets T of N containing S, $\mathrm{Q}_S^{\langle t\rangle} \subseteq \mathrm{Q}_T^{\langle t\rangle}$. (So $\mathrm{Var}_S^{\langle t\rangle}$ contains $\mathrm{Var}_T^{\langle t\rangle}$, that is why S is called maximal.)*

5 Some examples of switchable and signed sets

To bring the abstract notion of t-switchable sets down to earth, we now examine some concrete examples. In Section 7 we characterize more generally all minimal prime ideals over $J^{\langle t\rangle}$ (they are $\mathrm{Q}_S^{\langle t\rangle}$ for maximal t-signed S), but in examples below we simply state what the associated prime ideals are. We used Macaulay2 [5] together with Kahle's binomial package [7].

When the set S has a long name, we write $\mathrm{Q}^{\langle t\rangle}(S)$ instead of $\mathrm{Q}_S^{\langle t\rangle}$.

First we provide perspective in light of t-signed sets on the primary decomposition of 2×2 permanental ideals of generic matrices as first determined without t-signed notion in [9]. The main result of [9] was to give the description of the minimal primes over the ideal generated by 2×2 permanents of a generic $r_1 \times r_2$ matrix in terms of three types primes; but by the work in this paper, the three types of primes are all simply the prime ideals corresponding to 1-signed sets:

Example 5.1 Let $N = [r_1] \times [r_2]$, with $r_1, r_2 \geq 2$, which represents an ordinary matrix. For $t = 1, 2$, the t-signed subsets of N are identical, and they are: all 2×2, $1 \times r$ and $r \times 1$ submatrices of N. If $r_1 = r_2 = 2$, then the only maximal t-signed set is N. If $r_1 = 2 < r_2$, then the only maximal t-signed sets are 2×2 and $1 \times r_2$ submatrices of N, and if $r_1, r_2 > 2$, then the t-signed subsets of N are 2×2, $1 \times r_2$ and $r_1 \times 1$ submatrices of N. By Theorem 7.2, these maximal t-signed sets correspond precisely to the prime ideals minimal over $J^{\langle t\rangle}$, and this was first established in [9] without the vocabulary of signed sets. There it was also established that when $r_1, r_2 > 2$, there is exactly one embedded prime, corresponding to the (non-maximal) t-signed set $\emptyset$.

Example 5.2 Let $N = [2] \times [2] \times [2]$. Here we will think of N as a cube. The 1-signed subsets of N, together with their corresponding ideals, are as follows:

(1) N (the whole cube); $\mathrm{Q}_N^{\langle 1\rangle} = \widetilde{J}_N^{\langle t\rangle}$ is generated by four slice permanental and two diagonal determinantal generators.

(2) S is a face of the cube with nonconstant first component; $\mathrm{Q}^{\langle 1\rangle}(\text{face})$ is generated by the permanent of this face and the variables on the opposite face.
(3) S cannot be a face of the cube perpendicular to the x-axis as that is not even 1-switchable;
(4) S is an edge of the cube; $\mathrm{Q}^{\langle 1\rangle}(\text{edge})$ is generated by the variables not on this edge.
(5) two parallel edges of the cube with non-constant first component that are not on the same face, this is an example of a 1-switchable set with two equivalence classes with respect to connectedness; $\mathrm{Q}^{\langle 1\rangle}(\text{two edges})$ is generated by the variables not on these edges.
(6) a point; $\mathrm{Q}^{\langle 1\rangle}(\text{point})$ is generated by the other 7 variables.
(7) two points of distance three or two points of distance two with the same first component; $\mathrm{Q}^{\langle 1\rangle}(\text{two points})$ is generated by the variables not on these points.

Note that $\mathrm{Q}_N^{\langle 1\rangle}$ is contained in $\mathrm{Q}^{\langle 1\rangle}(\text{face})$, $\mathrm{Q}^{\langle 1\rangle}(\text{point})$, and $\mathrm{Q}^{\langle 1\rangle}(\text{two points})$, and that $\mathrm{Q}^{\langle 1\rangle}(\text{two edges})$ is contained in $\mathrm{Q}^{\langle 1\rangle}(\text{edge})$. From this we read that the maximal 1-signed sets are: N, and two edges parallel to the x-axis that are not on the same face. Note that there are two options for S being formed by the parallel edges. There are thus three maximal signed sets. Macaulay2 [5] says that these $\mathrm{Q}^{\langle t\rangle}(S)$ are all the minimal primes (and even all the components).

Example 5.3 Let $N = [2] \times [2] \times [2]$ as in the previous example. The 2-signed subsets of N (which are the same as 3-signed sets), together with their corresponding ideals, are as follows:
(1) N (the whole cube); $\mathrm{Q}_N^{\langle 2\rangle} = \widetilde{J}_N^{\langle 2\rangle}$ is (redundantly) generated by one slice permanent for each of the six faces, and three diagonal determinants for each of the opposite vertices.
(2) S is a face of the cube; $\mathrm{Q}^{\langle 2\rangle}(\text{face})$ is generated by the permanent of this face and the variables on the opposite face.
(3) an edge of the cube; $\mathrm{Q}^{\langle 2\rangle}(\text{edge})$ is generated by the variables not on this edge.
(4) a point; $\mathrm{Q}^{\langle 2\rangle}(\text{point})$ is generated by the remaining 7 variables.
(5) two points of distance three; $\mathrm{Q}^{\langle 2\rangle}(\text{two points})$ is generated by the variables not on these points.

Note that the union of two parallel edges that are not on the same face is not 2-signed. Here the maximal 2-signed sets are: N, and the two-opposite points. There are thus five maximal signed sets, and, indeed, Macaulay2 says that these are all the minimal components (and even all the associated primes).

Example 5.4 With $N = [3] \times [2] \times [2]$, the maximal 1-signed sets are:
(1) For each $k \in [3]$, S is the set of variables whose first coordinate is not k, so it looks like a $2 \times 2 \times 2$-block; $\mathrm{Q}_S^{\langle 1\rangle}$ is generated by the variables whose first coordinate is k, and by the four slice permanents and two diagonal determinants on the block. There are three such S.

(2) S consists of the variables on two lines parallel to the x-axis, the lines not lying on the same face of the cube; $\mathrm{Q}_S^{\langle 1\rangle}$ is generated by the variables not on these lines. There are two such S.

Indeed, Macaulay2 gives $3+2$ minimal primes, but it also gives one embedded prime ideal consisting of all variables. While the minimal components are all prime ideals themselves, the embedded component in this case may be taken to be $J^{\langle 1\rangle} + (x_a^3 : a \in N)$.

Example 5.5 With $N = [3] \times [2] \times [2]$ as in the previous example, the maximal 2-signed sets are:

(1) For each $i \in [3]$, S is the set of variables whose first coordinate is not i; $\mathrm{Q}_S^{\langle 2\rangle}$ is generated by the variables whose first coordinate is i, and by the slice permanents and diagonal determinants on the $2 \times 2 \times 2$-block. There are three such S.

(2) S is a line parallel to the x-axis; $\mathrm{Q}_S^{\langle 2\rangle}$ is generated by the 9 variables not in S. There are four such S.

(3) S is the disjoint union of a point and of two points on a line parallel to the x-axis, with all distances between the one point and the other two points being 3; $\mathrm{Q}_S^{\langle 2\rangle}$ is generated by the 9 variables not in S. There are $4 \cdot 3$ such S.

Indeed, Macaulay2 gives $3+4+12 = 19$ minimal primes, and it also gives one embedded prime ideal consisting of all variables. The minimal components are all prime ideals, and the embedded component in this case as well may be taken to be $J^{\langle 2\rangle} + (x_a^3 : a \in N)$.

Example 5.6 With $N = [2] \times [2] \times [3]$, by symmetry the maximal 2-signed sets and primary decomposition are as in Example 5.5. (In contrast, we do not have symmetry for 1-signed sets.) The maximal 1-signed sets are as follows:

(1) S consists of all variables that do not have a certain third coordinate (so it is a $2 \times 2 \times 2$ subhypermatrix); $\mathrm{Q}_S^{\langle 1\rangle}$ is generated by the variables not in S and by the diagonal determinants and the slice permanents of the block. There are 3 such S, one for each of the three third coordinates.

(2) S consists of variables on a face perpendicular to the x-axis; $\mathrm{Q}_S^{\langle 1\rangle}$ is generated by the variables not in S. There are two such S.

(3) S is a disjoint union of a line segment and of a 2×2 submatrix on distinct faces perpendicular to the y-axis; $\mathrm{Q}_S^{\langle 1\rangle}$ is generated by one permanent of the 2×2-matrix and by all variables not in S. There are $2 \cdot 3$ such S.

(4) S is the complement of the union of two lines parallel to the x-axis, the two lines having distinct second and third coordinates. This S is a non-disjoint union of three 2×2 submatrices, with one of the submatrices sharing edges with the other two. With this visualization, $\mathrm{Q}_S^{\langle 1\rangle}$ is generated by all variables not in S, by one permanent for each of the submatrices, by one diagonal determinant on the elements of S for each of the two pairs of adjacent 2×2 submatrices, and by one extra diagonal permanent of the two disjoint edges of the two 2×2 submatrices, There are $3 \cdot 2$ such S. (Note that this last S is one equivalence class under connectedness; there are two possible first

coordinates for each element, but the third coordinates can take on three different values for various first two coordinates.)

Indeed, Macaulay2 gives $17 = 3+2+6+6$ minimal primes, and no embedded primes. All the minimal components are prime.

6 Primes, Gröbner bases and similar reductions

The results here mimic those of [11]. The generators of $\widetilde{J}^{\langle t\rangle}$ are (monic) binomials that are, up to sign of the second coefficient in the binomial, the same as the generators of $\widetilde{I}^{\langle t\rangle}$. By Lemma 6.2 in [11], every monomial reduces modulo $\widetilde{I}^{\langle t\rangle}$ to a well-understood monomial; thus by the nature of the generators of $\widetilde{J}^{\langle t\rangle}$, every monomial reduces modulo $\widetilde{J}^{\langle t\rangle}$ to a well-understood monomial times a less well-understood sign. The bulk of this section is developing the machinery to keep track of this sign: Lemma 6.2 introduces an absolute measure for keeping track of what degree-three binomials are in $\widetilde{J}_S^{\langle t\rangle}$, and Lemma 6.3 proves that this measure behaves well under any partial switches in one of the two monomials in the binomial. The ultimate goal of this section is determining a Gröbner basis of $\widetilde{J}^{\langle t\rangle}$ and of proving that $\widetilde{J}^{\langle t\rangle}$ is a prime ideal.

Throughout this section we use the lexicographic order on the variables, with variables sorted in the lexicographic order on their indices. By **lt** we denote the leading term. Also, throughout S is a t-signed set, and for slight brevity we write $\mathrm{pl}_S(a,b), h_{S,K,a,b}$ without "S".

For the purposes of discussion below we introduce the following notion: we enumerate in the lexicographic order all possible elements of $[r_{t+1}] \times \cdots \times [r_n]$, with largest elements in the lexicographic order getting the largest numeral. Thus we can think of each element $a \in N$ as a $(t+1)$-tuple in $[r_1] \times \cdots \times [r_t] \times [r_{t+1} \cdots r_n]$, and in this notation without loss of generality $t = n-1 < n$.

Definition 6.1 *When we think of a as a $(t+1)$-tuple, we denote a as $\tilde{a}$.*

Lemma 6.2 *Let S be a t-signed set, and let $a,b,c,A,B,C \in S$ be mutually connected such that $\widetilde{a}_{t+1} = \widetilde{A}_{t+1}$, $\widetilde{b}_{t+1} = \widetilde{B}_{t+1}$, $\widetilde{c}_{t+1} = \widetilde{C}_{t+1}$, and for each $i = 1,\ldots,t$, the list a_i, b_i, c_i is up to order the same as the list A_i, B_i, C_i (with same multiplicities). Define*

$$\begin{aligned}
K_1 &= \{i \in [t] : A_i = b_i \neq a_i\},\\
K_2 &= \{i \in [t] : A_i = c_i \neq a_i\} \setminus K_1,\\
K_3 &= \{i \in K_1 : B_i \neq a_i\} \cup (\{i \in [t] : B_i \neq b_i\} \setminus K_1),\\
p &= \#K_1 \cdot \overline{pl}(a,b) + \#K_2 \cdot \overline{pl}(\mathrm{s}(K_1,a,b),c) + \#K_3 \cdot \overline{pl}(\mathrm{s}(K_1,b,a),\mathrm{s}(K_2,c,a)).
\end{aligned}$$

Then $x_a x_b x_c - (-1)^p x_A x_B x_C \in \widetilde{J}_S^{\langle t\rangle}$.

Proof. K_1 and K_2 are disjoint by construction, so that $\mathrm{s}(K_2,c,a) = \mathrm{s}(K_2,c,\mathrm{s}(K_1,a,b))$.

Note that $\mathrm{s}(K_2, \mathrm{s}(K_1, a, b), c) = A$, that $\mathrm{s}(K_3, \mathrm{s}(K_1, b, a), c) = B$, and that $\mathrm{s}(K_3, \mathrm{s}(K_2, c, \mathrm{s}(K_1, a, b)), \mathrm{s}(K_1, b, a)) = C$. Thus by Lemma 4.10, the following elements are in $\widetilde{J}_S^{\langle t\rangle}$:

$$\begin{aligned}
&x_a x_b - (-1)^{\#K_1 \cdot \overline{pl}(a,b)} x_{\mathrm{s}(K_1,a,b)} x_{\mathrm{s}(K_1,b,a)},\\
&x_{\mathrm{s}(K_1,a,b)} x_c - (-1)^{\#K_2 \cdot \overline{pl}(\mathrm{s}(K_1,a,b),c)} x_A x_{\mathrm{s}(K_2,c,\mathrm{s}(K_1,a,b))},\\
&x_{\mathrm{s}(K_1,b,a)} x_{\mathrm{s}(K_2,c,\mathrm{s}(K_1,a,b))} - (-1)^{\#K_3 \cdot \overline{pl}(\mathrm{s}(K_1,b,a),\mathrm{s}(K_2,c,\mathrm{s}(K_1,a,b)))} x_B x_C.
\end{aligned}$$

It follows that

$$\begin{aligned}
&x_a x_b x_c - (-1)^{\#K_1 \cdot \overline{pl}(a,b) + \#K_2 \cdot \overline{pl}(\mathrm{s}(K_1,a,b),c) + \#K_3 \cdot \overline{pl}(\mathrm{s}(K_1,b,a),\mathrm{s}(K_2,c,\mathrm{s}(K_1,a,b)))} x_A x_B x_C\\
&\quad = \left(x_a x_b - (-1)^{\#K_1 \cdot \overline{pl}(a,b)} x_{\mathrm{s}(K_1,a,b)} x_{\mathrm{s}(K_1,b,a)}\right) x_c\\
&\qquad + (-1)^{\#K_1 \cdot \overline{pl}(a,b)} x_{\mathrm{s}(K_1,b,a)} \left(x_{\mathrm{s}(K_1,a,b)} x_c - (-1)^{\#K_2 \cdot \overline{pl}(\mathrm{s}(K_1,a,b),c)} x_A x_{\mathrm{s}(K_2,c,\mathrm{s}(K_1,a,b))}\right)\\
&\qquad + (-1)^{\#K_1 \cdot \overline{pl}(a,b) + \#K_2 \cdot \overline{pl}(\mathrm{s}(K_1,b,a),c)}\\
&\qquad\quad \cdot x_A \left(x_{\mathrm{s}(K_1,b,a)} x_{\mathrm{s}(K_2,c,\mathrm{s}(K_1,a,b))} - (-1)^{\#K_3 \cdot \overline{pl}(\mathrm{s}(K_1,b,a),\mathrm{s}(K_2,c,\mathrm{s}(K_1,a,b)))} x_B x_C\right)
\end{aligned}$$

is in $\widetilde{J}_S^{\langle t\rangle}$. □

Note that the order of the pairs (a, A), (b, B), (c, C) in the lemma above affects the number p. For this reason, the proof of the next result requires many subcases, but it leads to the eventual fact that the parity of p is independent of the order of the pairs.

Lemma 6.3 *With a, b, c, A, B, C as in Lemma 6.2, we fix A, B, C, and we allow a, b, c to vary. Thus p is now a function of a, b, c. Let $i \in [t]$, and assume one of the following:*
(1) $a_i \neq b_i$, *and* $a' = \mathrm{s}(i, a, b)$, $b' = \mathrm{s}(i, b, a)$, $c' = c$, $r = \overline{pl}(a, b)$.
(2) $a_i \neq c_i$, *and* $a' = \mathrm{s}(i, a, c)$, $b' = b$, $c' = \mathrm{s}(i, c, a)$, $r = \overline{pl}(a, c)$.
(3) $b_i \neq c_i$, *and* $a' = a$, $b' = \mathrm{s}(i, b, c)$, $c' = \mathrm{s}(i, c, b)$, $r = \overline{pl}(b, c)$.
Then $p(a', b', c') - p(a, b, c) + r$ is even.

Proof. By assumption and by Remark 4.2, $a, b, c, a', b', c', A, B, C$ are all in the same connected equivalence class.

To verify the evenness, we need to recall the construction of Lemma 6.2: to get from a, b, c to A, B, C, we first switch $\#K_1$ entries from b into a and then $\#K_2$ entries from c into a to then have the modified a equal to A; the procedure is to choose K_1 as large as possible, and then K_2 follows uniquely. In the third step using K_3, we switch the necessary entries of the new b and the new c to get the modified b equal B, and so necessarily the modified c equals C. Let K_1', K_2', K_3' be the corresponding sets when we start with a', b', c' in place of a, b, c.

Suppose that a, b, c all differ at most in the ith component. Then all $\overline{pl}$ in the definitions of $p(a, b, c)$ and $p(a', b', c')$ are 0, and $r = 0$, so that the lemma holds in this case. So we may assume that at least two of a, b, c differ not just in the ith component but also in some other jth component. By Lemma 4.6 a, b, c, the set $\{a_i, b_i, c_i\}$ contains exactly two elements.

Here are a few simplified expressions of $p(a, b, c)$ modulo 2:

$$\begin{aligned}
\overline{pl}(a,b) &= \mathrm{pl}(a,b) - 1,\\
\overline{pl}(\mathrm{s}(K_1,a,b),c) &= \mathrm{pl}(\mathrm{s}(K_1,a,b),c) - 1 \equiv \mathrm{pl}(\mathrm{s}(K_1,a,b),a) + \mathrm{pl}(a,c) - 1\\
&= \#K_1 + \mathrm{pl}(a,c) - 1,\\
\overline{pl}(\mathrm{s}(K_1,b,a),\mathrm{s}(K_2,c,a)) &= \mathrm{pl}(\mathrm{s}(K_1,b,a),\mathrm{s}(K_2,c,a)) - 1\\
&\equiv \mathrm{pl}(\mathrm{s}(K_1,b,a),b) + \mathrm{pl}(b,c) + \mathrm{pl}(c,\mathrm{s}(K_2,c,a)) - 1\\
&\equiv \#K_1 + \mathrm{pl}(b,c) + \#K_2 - 1,\\
p(a,b,c) &= \#K_1 \cdot (\mathrm{pl}(a,b) - 1) + \#K_2 \cdot (\#K_1 + \mathrm{pl}(a,c) - 1)\\
&\qquad + \#K_3 \cdot (\#K_1 + \mathrm{pl}(b,c) + \#K_2 - 1),
\end{aligned}$$

and there is a similar expressions for $p(a', b', c')$ in terms of K_1', K_2', K_3'.

We analyze the cases separately.

(1) As stated, by Lemma 4.6 we have two subcases: $A_i = a_i \neq b_i$, and $a_i \neq b_i = A_i$. In the first subcase, $i \in K_1' \setminus K_1$, the only change in the construction of A, B, C is that $K_1' = K_1 \cup \{i\}$ (but $\overline{pl}(a', b') = \overline{pl}(a, b)$), whence by Lemma 6.2 and the simplifications above, modulo 2,

$$\begin{aligned}
p(a',b',c') - p(a,b,c) + r &= \#K_1' \cdot (\mathrm{pl}(a',b') - 1) + \#K_2' \cdot (\#K_1' + \mathrm{pl}(a',c') - 1)\\
&\quad + \#K_3' \cdot (\#K_1' + \mathrm{pl}(b',c') + \#K_2' - 1)\\
&\quad - \#K_1 \cdot (\mathrm{pl}(a,b) - 1) - \#K_2 \cdot (\#K_1 + \mathrm{pl}(a,c) - 1)\\
&\quad - \#K_3 \cdot (\#K_1 + \mathrm{pl}(b,c) + \#K_2 - 1) + \overline{pl}(a,b)\\
&\equiv (\#K_1 + 1) \cdot (\mathrm{pl}(a,b) - 1) + \#K_2 \cdot (\#K_1 + 1 + \mathrm{pl}(a',c) - 1)\\
&\quad + \#K_3 \cdot (\#K_1 + 1 + \mathrm{pl}(b',c) + \#K_2 - 1)\\
&\quad - \#K_1 \cdot (\mathrm{pl}(a,b) - 1) - \#K_2 \cdot (\#K_1 + \mathrm{pl}(a,c) - 1)\\
&\quad - \#K_3 \cdot (\#K_1 + \mathrm{pl}(b,c) + \#K_2 - 1) + \mathrm{pl}(a,b) - 1\\
&\equiv (\#K_1 + 1) \cdot (\mathrm{pl}(a,b) - 1) + \#K_2 \cdot (\#K_1 + \mathrm{pl}(a',a) + \mathrm{pl}(a,c))\\
&\quad + \#K_3 \cdot (\#K_1 + \mathrm{pl}(b',b) + \mathrm{pl}(b,c) + \#K_2)\\
&\quad - \#K_1 \cdot (\mathrm{pl}(a,b) - 1) - \#K_2 \cdot (\#K_1 + \mathrm{pl}(a,c) - 1)\\
&\quad - \#K_3 \cdot (\#K_1 + \mathrm{pl}(b,c) + \#K_2 - 1) + \mathrm{pl}(a,b) - 1\\
&\equiv (\#K_1 + 1) \cdot (\mathrm{pl}(a,b) - 1) - \#K_1 \cdot (\mathrm{pl}(a,b) - 1) + \mathrm{pl}(a,b)\\
&\equiv \mathrm{pl}(a,b) - 1 + \mathrm{pl}(a,b) - 1,
\end{aligned}$$

which is even. In the second subcase ($a_i \neq b_i = A_i$), $K_1' = K_1 \setminus \{i\}$ and there are no other changes, so that the parity of $p(a', b', c') - p(a, b, c) + r$ in this case is the same as the parity in the previous case, namely even.

(2) We do some preliminary simplifications modulo 2, using Remark 4.2:

$$\begin{aligned}
p(a',b',c') &= \#K_1' \cdot (\mathrm{pl}(a',b') - 1) + \#K_2' \cdot (\#K_1' + \mathrm{pl}(a',c') - 1) \\
&\qquad + \#K_3' \cdot (\#K_1' + \mathrm{pl}(b',c') + \#K_2' - 1) \\
&\equiv \#K_1' \cdot (\mathrm{pl}(a',a) + \mathrm{pl}(a,b) - 1) + \#K_2' \cdot (\#K_1' + \mathrm{pl}(a,c) - 1) \\
&\qquad + \#K_3' \cdot (\#K_1' + \mathrm{pl}(b,c) + \mathrm{pl}(c,c') + \#K_2' - 1) \\
&\equiv \#K_1' \cdot \mathrm{pl}(a,b) + \#K_2' \cdot (\#K_1' + \mathrm{pl}(a,c) - 1) + \#K_3' \cdot (\#K_1' + \mathrm{pl}(b,c) + \#K_2').
\end{aligned}$$

Suppose that $i \in K_1$. Then $a_i \neq b_i = A_i$, and by Lemma 4.6, $a_i \neq b_i = A_i = c_i$. So $K_1' = K_1 \setminus \{i\}$. Then $i \notin K_2, K_2'$, and exactly one of K_3, K_3' contains i, so that $\#K_3' = \#K_3 \pm 1$. Thus

$$\begin{aligned}
p(a',b',c') - p(a,b,c) + r &= \#K_1' \cdot \mathrm{pl}(a,b) + \#K_2' \cdot (\#K_1' + \mathrm{pl}(a,c) - 1) \\
&\qquad + \#K_3' \cdot (\#K_1' + \mathrm{pl}(b,c) + \#K_2') \\
&\qquad - \#K_1 \cdot (\mathrm{pl}(a,b) - 1) - \#K_2 \cdot (\#K_1 + \mathrm{pl}(a,c) - 1) \\
&\qquad - \#K_3 \cdot (\#K_1 + \mathrm{pl}(b,c) + \#K_2 - 1) + \overline{pl}(a,c) \\
&\equiv (\#K_1 + 1) \cdot \mathrm{pl}(a,b) + \#K_2 \cdot (\#K_1 + 1 + \mathrm{pl}(a,c) - 1) \\
&\qquad + (\#K_3 + 1) \cdot (\#K_1 + 1 + \mathrm{pl}(b,c) + \#K_2) \\
&\qquad - \#K_1 \cdot (\mathrm{pl}(a,b) - 1) - \#K_2 \cdot (\#K_1 + \mathrm{pl}(a,c) - 1) \\
&\qquad - \#K_3 \cdot (\#K_1 + \mathrm{pl}(b,c) + \#K_2 - 1) + \mathrm{pl}(a,c) - 1 \\
&\equiv \mathrm{pl}(a,b) + \#K_2 + \#K_1 + 1 + \mathrm{pl}(b,c) + \#K_2 - \#K_1 + \mathrm{pl}(a,c) - 1,
\end{aligned}$$

which is even.

So we may suppose that $i \notin K_1$, and since $p(a',b',c') - p(a,b,c) + r$ is even if and only if $p(a,b,c) - p(a',b',c') + r$ is even, by symmetry of the construction we may assume that $i \notin K_1'$. Then $a_i = b_i$ or $a_i = A_i$, and $c_i = b_i$ or $c_i = A_i$. If $a_i = b_i$, then by Lemma 4.6, $a_i = b_i \neq c_i = A_i$, and necessarily $B_i = C_i = a_i$, so that $i \in K_2$, $K_2' = K_2 \setminus \{i\}$, $K_3' = K_3$, $K_1' = K_1$. In this case,

$$\begin{aligned}
p(a',b',c') - p(a,b,c) + r &= \#K_1' \cdot \mathrm{pl}(a,b) + \#K_2' \cdot (\#K_1' + \mathrm{pl}(a,c) - 1) \\
&\qquad + \#K_3' \cdot (\#K_1' + \mathrm{pl}(b,c) + \#K_2') \\
&\qquad - \#K_1 \cdot (\mathrm{pl}(a,b) - 1) - \#K_2 \cdot (\#K_1 + \mathrm{pl}(a,c) - 1) \\
&\qquad - \#K_3 \cdot (\#K_1 + \mathrm{pl}(b,c) + \#K_2 - 1) + \overline{pl}(a,c) \\
&\equiv \#K_1 \cdot \mathrm{pl}(a,b) + (\#K_2 - 1) \cdot (\#K_1 + \mathrm{pl}(a,c) - 1) \\
&\qquad + \#K_3 \cdot (\#K_1 + \mathrm{pl}(b,c) + \#K_2 - 1) \\
&\qquad - \#K_1 \cdot (\mathrm{pl}(a,b) - 1) - \#K_2 \cdot (\#K_1 + \mathrm{pl}(a,c) - 1) \\
&\qquad - \#K_3 \cdot (\#K_1 + \mathrm{pl}(b,c) + \#K_2 - 1) + \mathrm{pl}(a,c) - 1 \\
&\equiv -\#K_1 - \mathrm{pl}(a,c) + 1 - \#K_1 + \mathrm{pl}(a,c) - 1,
\end{aligned}$$

which is even. So we may suppose that $a_i \neq b_i$, so that $a_i = A_i$. Since $c_i \neq a_i$, by Lemma 4.6 then $A_i = a_i \neq b_i = c_i$. It follows that $K'_1 = K_1$, $i \notin K_2$, $K'_2 = K_2 \cup \{i\}$, $K'_3 = K_3$, so that the $p(a', b', c') - p(a, b, c) + r$ has the same parity as the one in the case $a_i = b_i$, which is even.

(3) We do some preliminary simplifications modulo 2, using Remark 4.2:

$$\begin{aligned} p(a', b', c') &= \#K'_1 \cdot (\mathrm{pl}(a', b') - 1) + \#K'_2 \cdot (\#K'_1 + \mathrm{pl}(a', c') - 1) \\ &\quad + \#K'_3 \cdot (\#K'_1 + \mathrm{pl}(b', c') + \#K'_2 - 1) \\ &\equiv \#K'_1 \cdot (\mathrm{pl}(a, b) + \mathrm{pl}(b, b') - 1) + \#K'_2 \cdot (\#K'_1 + \mathrm{pl}(a, c) + \mathrm{pl}(c, c') - 1) \\ &\quad + \#K'_3 \cdot (\#K'_1 + \mathrm{pl}(b, c) + \#K'_2 - 1) \\ &\equiv \#K'_1 \cdot \mathrm{pl}(a, b) + \#K'_2 \cdot (\#K'_1 + \mathrm{pl}(a, c)) \\ &\quad + \#K'_3 \cdot (\#K'_1 + \mathrm{pl}(b, c) + \#K'_2 - 1). \end{aligned}$$

If $i \in K_1$, then $a_i \neq b_i = A_i$. Since $b_i \neq c_i$, by Lemma 4.6 then $c_i = a_i \neq b_i = A_i$, so that $K'_1 = K_1 \setminus \{i\}$, $i \notin K_2, K_3$, $K'_2 = K_2 \cup \{i\}$ and $K'_3 = K_3$. Thus here

$$\begin{aligned} p(a', b', c') - p(a, b, c) + r &= \#K'_1 \cdot \mathrm{pl}(a, b) + \#K'_2 \cdot (\#K'_1 + \mathrm{pl}(a, c)) \\ &\quad + \#K'_3 \cdot (\#K'_1 + \mathrm{pl}(b, c) + \#K'_2 - 1) \\ &\quad - \#K_1 \cdot (\mathrm{pl}(a, b) - 1) - \#K_2 \cdot (\#K_1 + \mathrm{pl}(a, c) - 1) \\ &\quad - \#K_3 \cdot (\#K_1 + \mathrm{pl}(b, c) + \#K_2 - 1) + \overline{pl}(b, c) \\ &\equiv (\#K_1 - 1) \cdot \mathrm{pl}(a, b) + (\#K_2 + 1) \cdot (\#K_1 - 1 + \mathrm{pl}(a, c)) \\ &\quad + \#K_3 \cdot (\#K_1 - 1 + \mathrm{pl}(b, c) + \#K_2 + 1 - 1) \\ &\quad - \#K_1 \cdot (\mathrm{pl}(a, b) - 1) - \#K_2 \cdot (\#K_1 + \mathrm{pl}(a, c) - 1) \\ &\quad - \#K_3 \cdot (\#K_1 + \mathrm{pl}(b, c) + \#K_2 - 1) + \mathrm{pl}(b, c) - 1 \\ &\equiv -\mathrm{pl}(a, b) + \#K_1 - 1 + \mathrm{pl}(a, c)) - \#K_1 + \mathrm{pl}(b, c) - 1, \end{aligned}$$

which is even.

So we may assume that $i \notin K_1$, and since $p(a', b', c') - p(a, b, c) + r$ is even if and only if $p(a, b, c) - p(a', b', c') + r$ is even, by symmetry of the construction we may assume that $i \notin K'_1$. Then either $a_i = b_i$ or $a_i = A_i$, and either $a_i = c_i$ or $a_i = A_i$. If $a_i = A_i$, then $i \notin K'_2, K_2$, $K'_1 = K_1$, $K'_2 = K_2$, and exactly one of K_3, K'_3 has i, so that $\#K'_3 = \#K_3 \pm 1$. Thus modulo 2:

$$\begin{aligned} p(a', b', c') - p(a, b, c) + r &= \#K'_1 \cdot \mathrm{pl}(a, b) + \#K'_2 \cdot (\#K'_1 + \mathrm{pl}(a, c)) \\ &\quad + \#K'_3 \cdot (\#K'_1 + \mathrm{pl}(b, c) + \#K'_2 - 1) \\ &\quad - \#K_1 \cdot (\mathrm{pl}(a, b) - 1) - \#K_2 \cdot (\#K_1 + \mathrm{pl}(a, c) - 1) \\ &\quad - \#K_3 \cdot (\#K_1 + \mathrm{pl}(b, c) + \#K_2 - 1) + \overline{pl}(b, c) \\ &\equiv \#K_1 \cdot \mathrm{pl}(a, b) + \#K_2 \cdot (\#K_1 + \mathrm{pl}(a, c)) \end{aligned}$$

$$
\begin{aligned}
&+ (\#K_3+1)\cdot(\#K_1+\mathrm{pl}(b,c)+\#K_2-1)\\
&- \#K_1\cdot(\mathrm{pl}(a,b)-1) - \#K_2\cdot(\#K_1+\mathrm{pl}(a,c)-1)\\
&- \#K_3\cdot(\#K_1+\mathrm{pl}(b,c)+\#K_2-1)+\mathrm{pl}(b,c)-1\\
\equiv\ & \#K_1+\mathrm{pl}(b,c)+\#K_2-1+\#K_1+\#K_2+\mathrm{pl}(b,c)-1,
\end{aligned}
$$

which is even. So we may assume that $a_i \neq A_i$. Then $a_i = b_i = c_i$, which contradicts the assumption that $b_i \neq c_i$. ☐

Proposition 6.4 *Let S be a t-signed set, and let $a, b, c \in S$ be mutually connected. Then all reductions of $x_a x_b x_c$ in the lexicographic order with respect to $\{h_{S,K,a,b} : a, b$ are connected in $S, K \subseteq \{i \in [t] : a_i \neq b_i\}\}$ reduce to the same term, and the term is of the form $(-1)^u x_A x_B x_C$ for some integer u, unique up to parity.*

Proof. Set $G' = \{f_{K,d,e} : d, e \in S, K \subseteq \{i \in [t] : d_i \neq e_i\}\}$. This set is reminiscent of the set G in the statement of the proposition, the only difference is that each binomial in G' has the two coefficients $1, -1$, whereas some binomials in G have coefficients $1, 1$ and others have $1, -1$. By Theorem 4.5 of [11], $x_a x_b x_c$ and all of its reductions with respect to G' reduce to some minimal monomial $x_A x_B x_C$ with respect to G'. By the form of G' and G, then with respect to G, $x_a x_b x_c$ reduces to $(-1)^u x_A x_B x_C$ for some integer u, and we need to show that up to parity u is uniquely determined regardless of what the reduction steps are. In fact, we show that u has the same parity as $p(a, b, c)$ from Lemma 6.2. If $a = A$, $b = B$, and $c = C$, then $p(a, b, c) = 0$ and $x_a x_b x_c$ is in the reduced form and it does not reduce any further, so the conclusion holds trivially.

Now suppose that in the first step of the reduction, we reduce $x_a x_b x_c$ with respect to $h_{L,a,b}$ for some non-empty $L \subseteq \{i \in [t] : a_i \neq b_i\}$. Then $x_a x_b x_c$ is reduced to $(-1)^{\#L\cdot\overline{pl}(a,b)} x_{\mathrm{s}(L,a,b)} x_{\mathrm{s}(L,b,a)} x_c$. This is a proper reduction, so by induction on the order (in the lexicographic order), $x_{\mathrm{s}(L,a,b)} x_{\mathrm{s}(L,a,b)} x_c$ reduces to $(-1)^{p(\mathrm{s}(L,a,b),\mathrm{s}(L,b,e),c)} x_A x_B x_C$. Hence via this reduction we have $u = p(\mathrm{s}(L, a, b), \mathrm{s}(L, b, e), c) + \#L \cdot \overline{pl}(a, b)$. Write $L = \{l_1, \dots, l_k\}$. Then we have the following modulo 2:

$$
\begin{aligned}
u - p(a,b,c) &= p(\mathrm{s}(L,a,b),\mathrm{s}(L,b,a),c) - p(a,b,c) + \#L\cdot\overline{pl}(a,b)\\
&\equiv \sum_{i=1}^{k}\Big(p(\mathrm{s}(\{l_1,\dots,l_i\},a,b),\mathrm{s}(\{l_1,\dots,l_i\},b,a),c)\\
&\qquad - p(\mathrm{s}(\{l_1,\dots,l_{i-1}\},a,b),\mathrm{s}(\{l_1,\dots,l_{i-1}\},b,a),c)\\
&\qquad + \overline{pl}(l_i,\mathrm{s}(\{l_1,\dots,l_{i-1}\},a,b),\mathrm{s}(\{l_1,\dots,l_{i-1}\},b,a))\Big),
\end{aligned}
$$

which is even by Lemma 6.3. A very similar proof shows the same conclusion if we first reduce $x_a x_b x_c$ with respect to $h_{L,a,c}$ or $h_{L,b,c}$. ☐

Theorem 6.5 *Let S be a t-signed set. Then the set $\{h_{S,K,a,b} : a, b$ are connected in $S, K \subseteq \{i \in [t] : a_i \neq b_i\}\}$ is a (non-minimal) Gröbner basis for $\widetilde{J}_S^{\langle t\rangle}$ in the lexicographic order.*

Proof. A note: "S" in "S-polynomial" below is unrelated to the t-signed set S. Set $G = \{h_{K,a,b} : a, b \text{ connected in } S, K \subseteq \{i \in [t] : a_i \neq b_i\}\}, x_a x_b > x_{\mathrm{s}(K,a,b)} x_{\mathrm{s}(K,b,a)}\}$.

Since $h_{K,a,b} = x_a x_b - (-1)^{\#K \cdot \overline{pl}(a,b)} x_{\mathrm{s}(K,a,b)} x_{\mathrm{s}(K,b,a)} = \pm h_{K,\mathrm{s}(K,a,b),\mathrm{s}(K,b,a)}$, it suffices to prove that G is a Gröbner basis. We have to prove that for any $h_{K,a,b}, h_{L,c,d} \in G$, the S-polynomial $S(h_{K,a,b}, h_{L,c,d})$ reduces to 0 in the Gröbner basis sense with respect to G.

If $x_a x_b$ and $x_c x_d$ have no variables in common, then by standard facts about Gröbner bases their S-polynomial reduces to 0. So we may assume that $a = d$. Then a, b, c, d are all connected in S.

Suppose first that in addition $b = c$. Then $S(h_{K,a,b}, h_{L,b,a})$ equals

$$-(-1)^{\#K \cdot \overline{pl}(a,b)} x_{\mathrm{s}(K,a,b)} x_{\mathrm{s}(K,b,a)} + (-1)^{\#L \cdot \overline{pl}(a,b)} x_{\mathrm{s}(L,a,b)} x_{\mathrm{s}(L,b,a)}.$$

If this is 0, we are done, otherwise, this equals

$$\begin{aligned}
-(-1)^{\#K \cdot \overline{pl}(a,b)} &\Big(x_{\mathrm{s}(K,a,b)} x_{\mathrm{s}(K,b,a)} + (-1)^{(\#L - \#K) \cdot \overline{pl}(a,b)} x_{\mathrm{s}(L,a,b)} x_{\mathrm{s}(L,b,a)} \Big) \\
&= \pm \big(x_{\mathrm{s}(K,a,b)} x_{\mathrm{s}(K,b,a)} \\
&\qquad + (-1)^{(\#(L \setminus K) + \#(K \setminus L)) \cdot \overline{pl}(a,b)} x_{\mathrm{s}((L \setminus K) \cup (K \setminus L), \mathrm{s}(K,a,b), \mathrm{s}(K,b,a))} \\
&\qquad\qquad \cdot x_{\mathrm{s}((L \setminus K) \cup (K \setminus L), \mathrm{s}(K,b,a), \mathrm{s}(K,a,b))} \big) \\
&= \pm h_{(L \setminus K) \cup (K \setminus L), \mathrm{s}(K,a,b), \mathrm{s}(K,b,a))},
\end{aligned}$$

which is a scalar multiple of an element in G, so it reduces to 0 with respect to G.

So we may assume that $a = d$ and $b \neq c$. Then

$$\begin{aligned}
&S(h_{K,a,b}, h_{L,a,c}) \\
&\quad = -(-1)^{\#K \cdot \overline{pl}(a,b)} x_{\mathrm{s}(K,a,b)} x_{\mathrm{s}(K,b,a)} x_c + (-1)^{\#L \cdot \overline{pl}(a,c)} x_{\mathrm{s}(L,a,c)} x_{\mathrm{s}(L,c,a)} x_b.
\end{aligned}$$

Both $(-1)^{\#K \cdot \overline{pl}(a,b)} x_{\mathrm{s}(K,a,b)} x_{\mathrm{s}(K,b,a)} x_c$ and $(-1)^{\#L \cdot \overline{pl}(a,c)} x_{\mathrm{s}(L,a,c)} x_{\mathrm{s}(L,c,a)} x_b$ are reductions of $x_a x_b x_c$, so that by Proposition 6.4, S-polynomial reduces to 0 with respect to G. □

Theorem 6.6 *If S is a t-signed set, then the ideals $\widetilde{J}_S^{\langle t \rangle}$ and $\mathrm{Q}_S^{\langle t \rangle}$ are prime.*

Proof. This proof mimics that of Theorem 6.3 in [11]. By faithfully flatness we may assume without loss of generality that the underlying field is algebraically closed.

Let $S = S_1 \cup \cdots \cup S_k$ be a partition of S into equivalence classes with respect to connectedness. Each S_i is t-signed. Then $\widetilde{J}_S^{\langle t \rangle} = \cup_i \widetilde{J}_{S_i}^{\langle t \rangle}$ and $\mathrm{Q}_S^{\langle t \rangle} = \sum_i \widetilde{J}_{S_i}^{\langle t \rangle} + \mathrm{Var}_S^{\langle t \rangle}$, and the generators of $\widetilde{J}_{S_1}^{\langle t \rangle}, \ldots, \widetilde{J}_{S_k}^{\langle t \rangle}$, and $\mathrm{Var}_S^{\langle t \rangle}$ use disjoint variables. By a well-known fact, it suffices to prove that each $\widetilde{J}_{S_i}^{\langle t \rangle}$ is a prime ideal. By renaming we now assume that t-signed $S = S_i$ is one equivalence class under connectedness.

Let $G = \{h_{S,K,a,b} : a, b \text{ are connected in } S, K \subseteq \{i \in [t] : a_i \neq b_i\}\}$. By Theorem 6.5, G is a Gröbner basis for $\widetilde{J}_S^{\langle t \rangle}$.

Here we quote Lemma 6.2 from [11]: Suppose that $a_1, \ldots, a_r, b_1, \ldots, b_r \in S$ have the property that for all $i = 1, \ldots, t$, up to order, the multiset $\{a_{1i}, a_{2i}, \ldots, a_{ri}\}$ is the same as the multiset $\{b_{1i}, b_{2i}, \ldots, b_{ri}\}$, and such that, up to order, the multiset $\{(a_{1,t+1}, a_{1,t+2}, \ldots, a_{1,n}), (a_{2,t+1}, a_{2,t+2}, \ldots, a_{2,n}), \ldots, (a_{r,t+1}, a_{r,t+2}, \ldots, a_{r,n})\}$ is the same as the multiset $\{(b_{1,t+1}, b_{1,t+2}, \ldots, b_{1,n}), (b_{2,t+1}, b_{2,t+2}, \ldots, b_{2,n}), \ldots, (b_{r,t+1}, b_{r,t+2}, \ldots, b_{r,n})\}$. Then in the lexicographic order, $x_{a_1} x_{a_2} \cdots x_{a_r} - x_{b_1} x_{b_2} \cdots x_{b_r}$ reduces with respect to $\{f_{K,a,b} : K \subseteq [t], a, b \in S\}$ to 0.

A consequence is that under the conditions in the previous paragraph, either $x_{a_1} \cdots x_{a_r} - x_{b_1} \cdots x_{b_r}$ or $x_{a_1} \cdots x_{a_r} + x_{b_1} \cdots x_{b_r}$ reduces to 0 with respect to G.

Suppose that $\widetilde{J}_S^{\langle t \rangle}$ is not a prime ideal. Since this is a binomial ideal, by Eisenbud–Sturmfels [3] there exists a zerodivisor modulo $\widetilde{J}_S^{\langle t \rangle}$ of the form $\alpha - c\beta$ for some monomials α and β and some possibly zero coefficient c in the base field. Then there exists f not in $\widetilde{J}_S^{\langle t \rangle}$ such that $(\alpha - c\beta) \cdot f \in \widetilde{J}_S^{\langle t \rangle}$. Without loss of generality we may assume that α, β, and each monomial in f is reduced with respect to C. Write $f = c_1 m_1 + c_2 m_2 + \cdots + c_k m_k$, where the c_i are non-zero elements of the underlying field and the m_i are monomials. We may assume that $m_1 > m_i$ for all i and $\alpha > \beta$ in the lexicographic order. Since $(\alpha - c\beta) \cdot f$ reduces to 0 with respect to G, αm_1 must reduce with respect to G up to sign to the same monomial as some other monomial in $(\alpha - c\beta) \cdot f$. If it reduces to the same monomial as αm_i for some $i > 1$, then by the previous paragraph, m_1 and m_i reduce up to sign to the same monomial, which contradicts the assumption on the monomials in f being reduced. Hence αm_1 reduces with respect to G to the same monomial as some βm_{i_2}, and by the reduced assumption on $\alpha - c\beta$ necessarily $i_2 > 1$ and c is not zero. Similarly, αm_{i_2} reduces to the same monomial as βm_{i_3}, for some $i_3 \neq i_2$, and more generally for all s, αm_{i_s} reduces to the same monomial as $\beta m_{i_{s+1}}$ for some $i_s \neq i_{s+1}$. Necessarily some i_s must equal some i_j for $s < j$. Hence $\alpha^{j-s+1} m_{i_s} \cdots m_{i_j}$ reduces to the same monomial as $\beta^{j-s+1} m_{i_s} \cdots m_{i_j}$, whence again by the previous paragraph, α^{j-s+1} reduces to the same monomial as β^{j-s+1}, and even α reduces to the same monomial as β, which is a contradiction. Thus $\widetilde{J}_S^{\langle t \rangle}$ is a prime ideal. □

In particular, $\widetilde{J}_N^{\langle t \rangle}$ is prime in case $N = [2] \times [2] \times [2]$. We give here an easier proof in this special case based on results of [11], but this easier proof does not generalize to arbitrary N. Namely, $\widetilde{J}_N^{\langle t \rangle} = (f_{i,a,b} : d(a,b) = 3, i \in [t]) + (g_{i,a,b} : d(a,b) = 2, i \in [t], a_i \neq b_i)$. Let $\varphi : R \to R$ be the ring isomorphism that restricts to the identity map on the underlying field and maps x_{ijk} to $-x_{ijk}$ if $i = j = k$ and to x_{ijk} otherwise. It is straightforward to see that φ takes $\widetilde{J}_N^{\langle t \rangle}$ onto $\widetilde{I}_N^{\langle t \rangle}$. But by [11], $\widetilde{I}_N^{\langle t \rangle}$ is a prime ideal (for all sizes of N), and since isomorphisms map prime ideals to prime ideals, the conclusion follows.

7 Prime ideals minimal over $\widetilde{J}_S^{\langle t \rangle}$

We prove in this section that the maximal t-signed sets correspond precisely to prime ideals minimal over $J^{\langle t \rangle}$. In [11] it was proved that the maximal t-switchable sets correspond precisely to prime ideals minimal over $I^{\langle t\rangle}$. The flow of the proofs resembles those in [11], but again, here in addition parity has to be checked and controlled.

Theorem 7.1 *If P is a prime ideal minimal over $J^{\langle t\rangle}$, then $P = \mathrm{Q}_S^{\langle t\rangle}$ for some maximal t-signed set S.*

Proof. Let S be the set of all $a \in [r_1] \times \cdots \times [r_n]$ such that $x_a \notin P$.

Let $a, b \in S$ have $d(a, b) = 2$ and $a_i \neq b_i$ for some $i \in [t]$. Since P contains $J^{\langle t\rangle}$ and $i \in [t]$, P contains $h_{S,i,a,b} = x_a x_b - (-1)^{\overline{pl}_S(a,b)} x_{\mathrm{s}(i,b,a)} x_{\mathrm{s}(i,a,b)}$. Since $a, b \in S$, then $x_a x_b \notin P$, so that necessarily $x_{\mathrm{s}(i,b,a)} x_{\mathrm{s}(i,a,b)} \notin P$, and hence $\mathrm{s}(i, b, a), \mathrm{s}(i, a, b) \in S$. This proves that S is t-switchable.

We next prove that S is t-signed. It suffices to prove that every connected component S_0 of S is t-signed. If all elements of S_0 share the same first t coordinates, or if all elements in S_0 only differ in the ith component for some $i \in [t]$, then S_0 is t-signed. So we may suppose that the distance between some elements a, b of S_0 is at least 2 and that for some $i \in [t]$, $\{c_i : c \in S_0\}$ has more than one element. By Remark 4.2, by possibly replacing a by $\mathrm{s}(i, a, c)$ for some $c \in S_0$, we may assume that a and b differ in the ith component. Now let $c, e \in S_0$ be arbitrary. If there are paths from c to e whose lengths have different parities, then there are paths from a to b via paths from c to e whose lengths have different parities. Then by Proposition 4.3, $J^{\langle t\rangle}$ and hence P contain monomials whose variables have subscripts in S. But P is a prime ideal, so P contains a factor x_c for some $c \in S$, which is a contradiction. This proves that S is t-signed.

By Proposition 4.11, $J^{\langle t\rangle} \subseteq \mathrm{Q}_S^{\langle t\rangle}$, and by Theorem 6.6, $\mathrm{Q}_S^{\langle t\rangle}$ is a prime ideal.

We next prove that $\mathrm{Q}_S^{\langle t\rangle} \subseteq P$. By the definition of S, $\mathrm{Var}_S^{\langle t\rangle} \subseteq P$. Let $h_{S,i,a,b} \in \widetilde{J}_S^{\langle t\rangle}$, with $i \in [t]$ and a and b connected in S such that $a_i \neq b_i$. If $d(a, b) = 1$, then $h_{S,i,a,b} = 0$, so it is an element of P. So we may assume that $d(a, b) \geq 2$. By the definition of connectedness, there exist elements $c_0 = a, c_1, \ldots, c_k, c_{k+1}, b \in S$ such that for all $j = 1, \ldots, k$, c_{j-1} and c_j differ only in one position. Without loss of generality $d(c_k, b) = 2$. After omitting any repetitions in $c_0, \mathrm{s}(i, c_1, a), \ldots, \mathrm{s}(i, c_{k+1}, a)$, all the ith components on the list equal a_i. By Remark 4.2, this list is in S. Note that $2 \leq d(\mathrm{s}(i, c_k, a), b) \leq 3$. If $d(\mathrm{s}(i, c_k, a), b) = 3$, then we take the list $c_0, \mathrm{s}(i, c_1, a), \ldots, \mathrm{s}(i, c_{k+1}, a)$ with redundancies removed, and the last element on the list differs from b in 2 entries, and if $d(\mathrm{s}(i, c_k, a), b) = 2$, then we take the list $c_0, \mathrm{s}(i, c_1, a), \ldots, \mathrm{s}(i, c_k, a)$ with redundancies removed. In either case, after renaming we have a path $a = c_0, c_1, \ldots, c_k$ with all ith components being a_i and $d(c_k, b) = 2$. Then by Lemma 3.2, $x_{c_1} \cdots x_{c_k} h_{S,i,a,b} \in J^{\langle t\rangle} \subseteq P$, and since $x_{c_j} \notin P$, it follows that $h_{S,i,a,b} \in P$, as desired. Thus $\widetilde{J}_S^{\langle t\rangle} \subseteq \mathrm{Q}_S^{\langle t\rangle} \subseteq P$. Since $\mathrm{Q}_S^{\langle t\rangle}$ is a prime ideal, by minimality of P, $\mathrm{Q}_S^{\langle t\rangle} = P$.

Finally, let T be t-signed and properly containining S. Then $\mathrm{Var}_T^{\langle t\rangle} \subsetneq \mathrm{Var}_S^{\langle t\rangle}$, and so $\mathrm{Q}_T^{\langle t\rangle} \neq \mathrm{Q}_S^{\langle t\rangle}$. By Proposition 4.11, $\mathrm{Q}_T^{\langle t\rangle}$ contains $J^{\langle t\rangle}$, and by Theorem 6.6, $\mathrm{Q}_T^{\langle t\rangle}$ is a prime ideal. This combined with the fact that $P = \mathrm{Q}_S^{\langle t\rangle}$ is minimal over $J^{\langle t\rangle}$, implies that $\mathrm{Q}_T^{\langle t\rangle} \not\subseteq \mathrm{Q}_S^{\langle t\rangle}$. Therefore $\mathrm{Q}_S^{\langle t\rangle}$ and $\mathrm{Q}_T^{\langle t\rangle}$ are incomparable. Thus S is a maximal t-signed set.

Theorem 7.2 *The set of prime ideals minimal over $J^{\langle t\rangle}$ equals the set of ideals of the form $\mathrm{Q}_S^{\langle t\rangle}$ as S varies over maximal t-signed sets.*

Proof. Let S be a maximal t-signed set. Then by Proposition 4.11 and Theorem 6.6, $\mathrm{Q}_S^{\langle t\rangle}$ is a prime ideal that contains $J^{\langle t\rangle}$. Suppose that P is a prime ideal that is minimal over $J^{\langle t\rangle}$ and is contained in $\mathrm{Q}_S^{\langle t\rangle}$. By Theorem 7.1, $P = \mathrm{Q}_T^{\langle t\rangle}$ for some maximal t-signed set T. Since $\mathrm{Q}_T^{\langle t\rangle} \subseteq \mathrm{Q}_S^{\langle t\rangle}$, necessarily $\mathrm{Var}_T^{\langle t\rangle} \subseteq \mathrm{Var}_S^{\langle t\rangle}$, so that $S \subseteq T$. But then maximality of S forces $\mathrm{Q}_T^{\langle t\rangle} = \mathrm{Q}_S^{\langle t\rangle}$. Thus $\mathrm{Q}_S^{\langle t\rangle}$ is a minimal prime ideal of $J^{\langle t\rangle}$. Theorem 7.1 proves the other direction. □

8 Structure of t-signed sets and the radical of $J^{\langle t\rangle}$

The aim of this section is to establish the structure of t-signed sets in more detail, with the bigger goal of then determining the radical of $J^{\langle t\rangle}$. However, our description of this radical is only indirect.

Lemma 8.1 *For $i = 1, \ldots, n$, let $S_i \subseteq [r_i]$. Suppose that one of the following conditions holds:*
(1) $|S_1| = \cdots = |S_t| = 1$.
(2) There exists i such that for all $j \neq i$, $|S_j| = 1$.
(3) For all i, $|S_i| \leq 2$.
Then $S = S_1 \times \cdots \times S_n$ is a t-signed set consisting of one equivalence class only.

Proof. It is clear that S forms one equivalence class. Conditions (1) and (2) above correspond precisely to conditions (1) and (2) in Definition 4.4.

Now assume that condition (3) above holds. If $a, b \in S$ such that $d(a, b) = 2$ and $i \in [t]$ with $a_i \neq b_i$, then $\mathrm{s}(i, a, b), \mathrm{s}(i, b, a) \in S$ as well, so that S is t-switchable. Any path from an arbitrary a to an arbitrary b in S may involve several switches in each of the entries, and the parity of the number of switches in the kth entry is 1 if $a_k \neq b_k$ and 0 otherwise. So the parity of each path, namely the parity of the sum of the switches in all the components, is uniquely determined. Thus S is t-signed. □

By Example 5.6 (4), the t-switchable sets can have a form different from the forms given in Lemma 8.1 above.

Lemma 8.2 *Let U be a subset of N that is contained in a 2×3 submatrix of N with entries varying in the i and jth components only. Suppose that U is not contained in a 2×2 submatrix or in a 1×3 submatrix, and that either i or j is in $[t]$. Then U is not a subset of any t-signed set.*

Conversely, if U is not a subset of any t-signed S, then the smallest t-switchable set containing U contains a 2×3 submatrix of N with entries varying in the i and jth components with either i or j in $[t]$.

Proof. Suppose that U is a subset of a t-signed set S. Then since S is t-switchable, by Remark 4.2, the whole 2×3 matrix is in S, and even all its entries are in the same connected equivalence class S_0. So, let $a, b, c \in S_0$ be in a 1×3 submatrix. Then a, b, c and a, c are both paths from a to b in S_0, but then S is not t-signed, which is a contradiction. The proof of the converse is similar. □

Definition 8.3 *For any finite multiset M of elements of N, define $x_M = \prod_{a\in M} x_a$. (When M is a multiset, x_M allows for products of powers of variables, but when M is a set, then x_M is square-free.)*

Lemma 8.4 *Let $\mathbf{M}$ be the set of all subsets of N that are not contained in any t-signed sets. A monomial x_M is in $\sqrt{J^{\langle t\rangle}}$ if and only if $M \in \mathbf{M}$.*

Proof. By Theorem 7.2, $x_M \in \sqrt{J^{\langle t\rangle}}$ if and only if for all maximal t-signed S, $x_M \in \mathrm{Q}_S^{\langle t\rangle}$, which by the structure of these prime ideals holds if and only if for all S, M is not a subset of S. This is the same as saying that $M \in \mathbf{M}$. □

Corollary 8.5 *Let $\mathbf{M}_i$ be the set of all sets in $\mathbf{M}$ with i elements. Then $(x_M : M \in \mathbf{M}) + J^{\langle t\rangle} = (x_M : M \in \mathbf{M}_3) + J^{\langle t\rangle}$.*

Proof. For any $a, b \in N$, by Lemma 8.1, $\{a_1, b_1\} \times \cdots \times \{a_n, b_n\}$ is t-signed, so that $\mathbf{M}_0 = \mathbf{M}_1 = \mathbf{M}_2 = \emptyset$. It suffices to prove that $(x_M : M \in \mathbf{M}) \subseteq (x_M : M \in \mathbf{M}_3) + J^{\langle t\rangle}$. Let $M \in \mathbf{M}$. By Lemma 8.2, the smallest t-switchable set containing M contains a 2×3 submatrix with fixed coordinates i, j such that $\{i, j\} \cap [t] \neq \emptyset$. It need not be the case that this 2×3 submatrix contains 3 elements of M that do not lie in a 1×3 or 2×2 submatrix. However, let S be the smallest subset of N containing S and such that whenever $a, b \in S$ with $d(a, b) = 2$, then $\mathrm{s}(i, a, b) \in S$. The assumption is that elements of S fill that 2×3 submatrix. But generating $\mathrm{s}(i, a, b), \mathrm{s}(i, b, a)$ from a, b is on the algebraic side the same as subtracting multiples of generators of $J^{\langle t\rangle}$, which says that after repeatedly subtracting from x_M specific elements of $J^{\langle t\rangle}$, we get a monomial $x_{M'}$ that has a factor $x_a x_b x_c$ such that the smallest t-switchable set containing a, b, c contains that 2×3 submatrix. This proves the corollary. □

It is not true that $(x_M : M \in \mathbf{M}) = (x_M : M \in \mathbf{M}_3)$. For example, if $n = t = 3$, then $M = \{(1, 1, 1), (2, 1, 1), (3, 1, 1), (1, 2, 2)\}$ is in $\mathbf{M}$. By Lemma 8.1, the proper subsets $\{(1, 1, 1), (3, 1, 1), (1, 2, 2)\}$ $\{(1, 1, 1), (2, 1, 1), (1, 2, 2)\}$ $\{(1, 1, 1), (2, 1, 1), (3, 1, 1)\}$ are not in $\mathbf{M}$, and $\{(2, 1, 1), (3, 1, 1), (1, 2, 2)\}$ is t-signed with exactly two equivalence classes under connectedness ($\{(2, 1, 1), (3, 1, 1)\}$ and $\{(1, 2, 2)\}$), and so this last subset of M is also not in $\mathbf{M}$. Thus $x_M \notin (x_{M'} : M' \in \mathbf{M}_3)$.

Theorem 8.6 *Let $\mathbf{T}$ be the set of all pairs (M, M') of finite lists of elements of N for which there exists an integer $v(M, M')$ such that for all t-signed S, $x_M - (-1)^{v(M,M')} x_{M'} \in \mathrm{Q}_S^{\langle t\rangle}$. The radical of $J^{\langle t\rangle}$ is generated by all elements $x_M - (-1)^{v(M,M')} x_{M'}$ for $(M, M') \in \mathbf{T}$, and by all x_M for $M \in \mathbf{M}_3$.*

Proof. Let $J' = (x_M - (-1)^{v(M,M')} x_{M'} : (M, M') \in \mathbf{T})$ and $J'' = (x_M : M \in \mathbf{M}_3)$. Clearly $J^{\langle t \rangle} \subseteq J'$.

By Theorem 7.2, $J' \subseteq \sqrt{J^{\langle t \rangle}}$, and by Lemma 8.4, $J'' \subseteq \sqrt{J^{\langle t \rangle}}$. Thus $J' + J'' \subseteq \sqrt{J^{\langle t \rangle}}$.

For the other inclusion, first assume that the underlying field is algebraically closed. By [3], $\sqrt{J^{\langle t \rangle}}$ is generated by binomials and monomials. Let $f = x_M - c x_{M'} \in \sqrt{J^{\langle t \rangle}}$ for some finite lists M, M' and some possibly zero scalar c.

Suppose that $x_M \in \sqrt{J^{\langle t \rangle}}$. Then by Lemma 8.4, the set M is in $\mathbf{M}$, so that by Corollary 8.5, $x_M \in J^{\langle t \rangle} + J''$, and either $c = 0$ or similarly $x_{M'} \in J^{\langle t \rangle} + J''$, whence $f \in J^{\langle t \rangle} + J''$.

Next assume that $x_M \notin \sqrt{J^{\langle t \rangle}}$. Thus c is non-zero. By Theorem 7.2, $f \in \mathrm{Q}_S^{\langle t \rangle}$ for all t-signed S, and for at least one such S, $x_M \notin \mathrm{Q}_S^{\langle t \rangle}$. By the structure of these prime ideals, $x_M - (-1)^{p_S} x_{M'} \in \mathrm{Q}_S^{\langle t \rangle}$ for some integer p_S depending on S. But also $x_M - c x_{M'} \in \sqrt{J^{\langle t \rangle}} \subseteq \mathrm{Q}_S^{\langle t \rangle}$, so that $c = (-1)^{p_S}$. Thus by assumption for any t-signed set T, $x_M - (-1)^{p_S} x_{M'} \in \sqrt{J^{\langle t \rangle}} \subseteq \mathrm{Q}_T^{\langle t \rangle}$, hence $f \in J'$.

This proves that $\sqrt{J^{\langle t \rangle}} = J' + J''$ whenever the underlying field is algebraically closed.

For an arbitrary underlying field, let R' be R tensored with the algebraic closure of the field. Then by above, $\sqrt{J^{\langle t \rangle} R'} = (J' + J'')R'$, which has generators in R, whence by faithful flatness of R' over R, also $\sqrt{J^{\langle t \rangle}} = J' + J''$. □

If $N = [r_1] \times [r_2]$, by [9], $\sqrt{J^{\langle t \rangle}} = J^{\langle t \rangle} + (x_M : M \in \mathbf{M}_3)$. However, for $N = [2] \times [2] \times [2] \times [2]$, $\sqrt{J^{\langle t \rangle}}$ properly contains $J^{\langle t \rangle} + (x_M : M \in \mathbf{M}_3)$.

9 Related ideals I: generated by $g_{i,a,b}$ when $d(a, b) = 3$

In this section we introduce another notion of distance

$$d_t(a, b) = \#\{i \in [t] : a_i \neq b_i\},$$

and the set $\{i \in [t] : a_i \neq b_i\}$ will be denoted as $D_t(a, b)$.

Definition 9.1 *Let $t \leq n$ and let $\widehat{J}^{\langle t \rangle}$ be the ideal generated by $g_{i,a,b}$ with $a, b \in N$ such that $d(a, b) = d_t(a, b) = 3$ and $i \in D_t(a, b)$.*

Lemma 9.2 *$\widehat{J}^{\langle t \rangle}$ is generated by monomials $x_a x_b$ such that $d(a, b) = d_t(a, b) = 3$ and $i \in D_t(a, b)$. Thus $\widehat{J}^{\langle t \rangle}$ is a radical ideal.*

Proof. If $t \leq 2$, $\widehat{J}^{\langle t \rangle}$ is the zero ideal, and so is the ideal generated by the non-empty set of specified monomials. So we may assume that $t \geq 3$.

Let $a, b \in N$ satisfy $d(a, b) = d_t(a, b) = 3$. Let $D_t(a, b) = \{i, j, k\}$. Then $g_{i,a,b}, g_{j,a,b}, g_{k,a,b}$ are among the generators of $\widehat{J}^{\langle t \rangle}$, but $g_{i,a,b} - g_{j,a,b} = x_{\mathrm{s}(i,a,b)} x_{\mathrm{s}(i,b,a)} - x_{\mathrm{s}(j,a,b)} x_{\mathrm{s}(j,b,a)}$, and $g_{k,a,b} = x_{\mathrm{s}(i,a,b)} x_{\mathrm{s}(i,b,a)} + x_{\mathrm{s}(j,a,b)} x_{\mathrm{s}(j,b,a)}$, so that as the characteristic of the underlying field is not 2, $\widehat{J}^{\langle t \rangle}$ contains $x_{\mathrm{s}(i,a,b)} x_{\mathrm{s}(i,b,a)}$ and $x_{\mathrm{s}(j,a,b)} x_{\mathrm{s}(j,b,a)}$, whence it also contains $x_a x_b$. This proves one inclusion, and the other is trivial. So $\widehat{J}^{\langle t\rangle}$ is generated by square-free monomials, so it is a radical ideal. $\square$

The following is now immediate:

Theorem 9.3 *Let $\widehat{\mathbf{S}}$ be the collection of all subsets S of N such that for any $a, b \in N$ with $d(a,b) = d_t(a,b) = 3$, at least one of a and b is not in S. Then the prime ideals that are minimal over $\widehat{J}^{\langle t\rangle}$ are the minimal ideals in $\{\mathrm{Var}_S^{\langle t\rangle} : S \in \widehat{\mathbf{S}}\}$ (see Definition 4.9).* $\square$

It is clear that when $t = 3$, the set $\widehat{\mathbf{S}}$ consists of sets S containing points no two of which have distance 3. In order to get the minimal primes, we need $\mathrm{Var}_S^{\langle 3\rangle}$ minimal possible, so we want S maximal possible.

Example 9.4 If $N = [r_1] \times [r_2] \times [r_3]$ and $t = 3$, the minimal prime ideals over $\widehat{J}^{\langle 3\rangle}$ correspond to the sets S, whose geometric descriptions are as followis:

(1) For $i = 1, 2, 3$, S consists of all elements with a fixed ith coordinate; there are $r_1 + r_2 + r_3$ such sets.

(2) S consists of four elements in a $2 \times 2 \times 2$ subhypermatrix consisting of one of the corners plus its adjacent neighbors; there are $\binom{r_1}{2}\binom{r_2}{2}\binom{r_3}{2}$ such subhypermatrices, and each one has eight such sets.

(3) S consists of four elements in a $2 \times 2 \times 2$ subhypermatrix all of whose pairs have distance 2; there are $\binom{r_1}{2}\binom{r_2}{2}\binom{r_3}{2}$ such subhypermatrices, and each one has two such sets.

Thus in total there are $r_1 + r_2 + r_3 + 10\binom{r_1}{2}\binom{r_2}{2}\binom{r_3}{2}$ such minimal primes. Since $\widehat{J}^{\langle t\rangle}$ is a radical monomial ideal, it has no embedded primes.

10 Related ideals II: generated by $g_{i,a,b}$ when $d(a,b) = 2, 3$

We still use $d_t(a,b) = \#\{i \in [t] : a_i \neq b_i\}$ and $D_t(a,b) = \{i \in [t] : a_i \neq b_i\}$.

Definition 10.1 *Let $t \leq n$ and let $\check{J}^{\langle t\rangle}$ be the ideal generated by $g_{i,a,b}$ with $a, b \in N$ and $i \in D_t(a,b)$ such that either $d(a,b) = d_t(a,b) = 3$ or $d(a,b) = 2$.*

Theorem 10.2 *Let $\check{\mathbf{S}}$ be the collection of all t-signed subsets S of N (see Definition 4.1) such that for any $a, b \in N$, if $d(a,b) = d_t(a,b) = 3$, then at least one of a and b is not in S. Then the prime ideals that are minimal over $\check{J}^{\langle t\rangle}$ are the minimal ideals in $\{\mathrm{Q}_S^{\langle t\rangle} : S \in \check{\mathbf{S}}\}$ (see Definition 4.9).*

Proof. Let $\mathrm{Q}_S^{\langle t\rangle}$ be in $\check{\mathbf{S}}$. By Proposition 4.11, $J^{\langle t\rangle} \subseteq \mathrm{Q}_S^{\langle t\rangle}$, and by Lemma 9.2 and by the d_t, d-conditions on S, $\widehat{J}^{\langle t\rangle} \subseteq \mathrm{Q}_S^{\langle t\rangle}$. Thus $\check{J}^{\langle t\rangle} \subseteq \mathrm{Q}_S^{\langle t\rangle}$. By Theorem 6.6, $\mathrm{Q}_S^{\langle t\rangle}$ is a prime ideal.

Note that $\check{J}^{\langle t\rangle} = J^{\langle t\rangle} + \widehat{J}^{\langle t\rangle}$. Let P be a prime ideal minimal over $\check{J}^{\langle t\rangle}$. Let S be the set of all $a \in N$ such that $x_a \notin P$. As in the proof of Theorem 7.1, S is t-switchable. Thus by Theorem 6.6, $\mathrm{Q}_S^{\langle t\rangle}$ is a prime ideal, and by Proposition 4.11, $\mathrm{Q}_S^{\langle t\rangle}$ contains $J^{\langle t\rangle}$. Since P contains $\widehat{J}^{\langle t\rangle}$, then at least one of a and b is not in S whenever $d(a,b) = d_t(a,b) = 3$.

Again as in the proof of Theorem 7.2, S is t-signed, and $\check{J}^{\langle t\rangle} \subseteq \mathrm{Q}_S^{\langle t\rangle} \subseteq P$. Thus $S \in \check{\mathbf{S}}$, and $\mathrm{Q}_S^{\langle t\rangle} = P$. If $\mathrm{Q}_T^{\langle t\rangle} \subseteq \mathrm{Q}_S^{\langle t\rangle}$ for some $T \in \check{\mathbf{S}}$, then by the first paragraph and minimality of P, $\mathrm{Q}_T^{\langle t\rangle} = \mathrm{Q}_S^{\langle t\rangle} = P$ is minimal in $\check{\mathbf{S}}$.

Now assume that $\mathrm{Q}_S^{\langle t\rangle}$ is minimal in $\check{\mathbf{S}}$. By the first paragraph, $\check{J}^{\langle t\rangle} \subseteq \mathrm{Q}_S^{\langle t\rangle}$, and $\mathrm{Q}_S^{\langle t\rangle}$ is a prime ideal. Let P be a prime ideal that is minimal over $\check{J}^{\langle t\rangle}$ and is contained in $\mathrm{Q}_S^{\langle t\rangle}$. By the already established part, $P = \mathrm{Q}_T^{\langle t\rangle}$ for some $T \in \check{\mathbf{S}}$. Since $\mathrm{Q}_T^{\langle t\rangle} \subseteq \mathrm{Q}_S^{\langle t\rangle}$, then by the minimality, $\mathrm{Q}_T^{\langle t\rangle} = \mathrm{Q}_S^{\langle t\rangle}$. □

Remark 10.3 We can give a more precise combinatorial description of the sets S in the theorem above. Let S_0 be a connected component of S. Then S_0 is also in $\check{\mathbf{S}}$. Suppose that for all $a, b \in S_0$, $d_t(a,b) \leq 2$. We claim that S_0 is contained in a subset of N in which all except some two coordinates in $[t]$ are fixed. If not, there exist $a, b, c \in S_0$ such that a and b differ in coordinates i and j in $[t]$, a and c differ in coordinates k and l in $[t]$, and $\{i,j,k,l\}$ has at least three elements. Since $d_t(b,c) \leq 2$, necessarily $\#\{i,j,k,l\} = 3$. Say $i = l$, so that b and c differ in positions j and k. But then $b' = \mathrm{s}(i,b,a)$ differs from c in positions i, j and k. By Remark 4.2, $b', c' = \mathrm{s}(\{i,j,k\}, c, b') \in S_0$, but $d_t(c,c') = d(c,c') = 3$, which contradicts that $S_0 \in \check{\mathbf{S}}$. Thus indeed S_0 is contained in a subset of N in which all except some two coordinates in $[t]$ are fixed.

Example 10.4 Let $t = n = 3$ and let $S \in \check{\mathbf{S}}$ be such that $\mathrm{Q}_S^{\langle t\rangle}$ is minimal over $\check{J}^{\langle t\rangle}$. Since $t = n = 3$, $d_t(a,b) = d(a,b) \leq 3$ for all $a, b \in S$. But since $S \in \check{\mathbf{S}}$, $d(a,b) \neq 3$. Thus for all $a, b \in S$, $d(a,b) \leq 2$. Since $t = 3$, by switchability, all elements of S are connected. Thus by the remark, there exists k such that all elements of S have the same kth coordinate. Let $\{1,2,3\} = \{i,j,k\}$.

(1) If also the jth coordinate of all elements of S is fixed, then the minimal $\mathrm{Q}_S^{\langle t\rangle}$ can only be achieved if S is the whole line segment with those fixed jth and kth coordinates. There are $r_1 \cdot r_2 + r_1 \cdot r_3 + r_2 \cdot r_3$ such sets.

(2) Now suppose that S is not contained in a line segment. Since S is t-signed, by Lemma 4.6, S is contained in a 2×2 submatrix parallel to a coordinate plane. To achieve a minimal $\mathrm{Q}_S^{\langle t\rangle}$, S then consists of all four points of that submatrix. There are $r_1\binom{r_2}{2}\binom{r_3}{2} + r_2\binom{r_1}{2}\binom{r_3}{2} + r_3\binom{r_1}{2}\binom{r_2}{2}$ such primes.

(3) Suppose that $r_i = 2$. Then a prime ideal $\mathrm{Q}_S^{\langle t\rangle}$ of the type as in part (1) strictly contains a prime ideal of part (2), so not all S in parts (1) and (2) correspond to minimal primes.

Note that there are no further containments among these prime ideals, so that in total there are $(r_1 \cdot r_2)\delta_{r_3>2} + (r_1 \cdot r_3)\delta_{r_2>2} + (r_2 \cdot r_3)\delta_{r_1>2} + r_1\binom{r_2}{2}\binom{r_3}{2} + r_2\binom{r_1}{2}\binom{r_3}{2} + r_3\binom{r_1}{2}\binom{r_2}{2}$ minimal primes, where δ_C is 1 if condition C is true and is 0 otherwise.

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