Expanded lectures on binomial ideals

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Here is the gist of the Eisenbud–Sturmfels paper Binomial ideals, *Duke Math. J.* **84** (1996), 1–45. The main results are that the associated primes, the primary components, and the radical of a binomial ideal in a polynomial ring are binomial if the base ring is algebraically closed.

Throughout, $R = k[X_1, \ldots, X_n]$, where k is a field and $X_1, \ldots, X_n$ are variables over k. A **monomial** is an element of the form $\underline{X}^{\underline{a}}$ for some $a \in \mathbb{N}_0^n$, and a **term** is an element of k times a monomial. The words "monomial" and "term" are often confused. In particular, a **binomial** is defined as the difference of two terms, so it should better be called a "biterm", but this name is unlikely to stick. An ideal is **binomial** if it is generated by binomials.

Here are some easy facts:

(1) Every monomial is a binomial, hence every monomial ideal is a binomial ideal.
(2) The sum of two binomial ideals is a binomial ideal.
(3) The intersection of binomial ideals need not be binomial: $(t-1) \cap (t-2)$, even over a field of characteristic 0.
(4) Primary components of a binomial ideal need not be binomial: in $\mathbb{R}[t]$, the binomial ideal $(t^3 - 1)$ has exactly two primary components: $(t-1)$ and (t^2+t+1).
(5) The radical of a binomial ideal need not be binomial: Let $k = \mathbb{Z}/2\mathbb{Z}(t)$, $R = k[X, Y]$, $I = (X^2 + t, Y^2 + t + 1)$. Note that I is binomial (as $t + 1$ is in k), and $\sqrt{I} = (X^2 + t, X + Y + 1)$, and this cannot be rewritten as a binomial ideal as there is only one generator of degree 1 and it is not binomial.

Thus, we do need to make a further assumption, namely, **from now on**, all fields k are algebraically closed, and then the counterexamples to primary components and radicals do not occur. The ring is always $R = k[X_1, \ldots, X_n]$, and t is always a variable over R.

Comment: Can one repeat this for trinomial ideals (with obvious meanings)? The answer is that not really, because all ideals are trinomial – after adding variables and a change of variable. Namely, let $f = a_1 + a_2 + \cdots + a_m$ be a polynomial with m terms. Introduce new variables $t_3, \ldots, t_m$. Then $k[x_1, \ldots, x_n]/(f) = k[x_1, \ldots, x_n, t_3, \ldots, t_m]/(a_1 + a_2 - t_3, -t_3 + a_3 - t_4, -t_4 + a_4 - t_5, \ldots, -t_{m-2} + a_{m-2} - t_{m-1}, -t_{m-1} + a_{m-1} - t_m)$.

1 Commutative algebra facts

In this section I list some **commutative algebra facts** that I will refer to later in the paper, together with some easy propositions about binomial ideals.

(1) A Gröbner basis of a binomial ideal is binomial.
(2) In fact, an ideal is binomial if and only if it has a binomial Gröbner basis.

(3) For any ideals I, J in R,

$$I \cap J = (tI + (t-1)J)R[t] \cap R,$$

where t is a variable over R. More generally, for ideals $I_0, \dots, I_s$, let $t_1, \dots, t_s$ be variables over R, and then

$$I_0 \cap \cdots \cap I_s = \left((1 - \sum_i t_i)I_0 + t_1 t_1 + \cdots t_s I_s\right) R[t_1, \dots, t_s] \cap R.$$

(4) If we take a monomial ordering on $R[t]$ such that the leading term of f is not in R for all $f \in R[t] \setminus R$, then for any Gröbner basis G of an ideal K in $R[t]$,

$$(G \cap R) = K \cap R.$$

(Recall that G is a finite set, so $G \cap R$ is just a set intersection.)

(5) If K is a binomial ideal in $R[t]$, then $K \cap R$ is a binomial in R.

(6) For any ideal I and any element m, $(I : m)m = I \cap (m)$. In particular, $I : m$ and $I \cap (m)$ is binomial. (Recall: false if m is not monomial.)

(7) For any Noetherian ring R, ideal I and $x \in R$, the following is a short exact sequence:

$$0 \longrightarrow \frac{R}{I : x} \longrightarrow \frac{R}{I} \longrightarrow \frac{R}{I + (x)} \longrightarrow 0,$$

where the first map is multiplication by x.

(8) If $0 \to M_1 \to M_2 \to M_3 \to 0$ is a short exact sequence of finitely generated modules over a Noetherian ring R, then $\mathrm{Ass}(M_2) \subseteq \mathrm{Ass}(M_1) \cup \mathrm{Ass}(M_3)$.

(9) If R is a Noetherian ring, then for any ideals I and J in R, the ascending chain $I : J \subseteq I : J^2 \subseteq I : J^3 \subseteq \cdots$ eventually stabilizes. The stabilized ideal is notated $I : J^\infty$ (without attaching any value to "J^∞").

(10) For any ideal I and any non-nilpotent element x, $I_x \cap R = I : (x)^\infty$.

(11) If $I : x^\infty = I : x^l$, then $I = (I : x^l) \cap (I + (x^l))$.

(12) With l as above, $\mathrm{Ass}(R/(I : x^l)) \subseteq \mathrm{Ass}(R/I) \subseteq \mathrm{Ass}(R/(I : x^l)) \cup \mathrm{Ass}(R/(I + (x^l)))$, and $\mathrm{Ass}(R/(I : x^l)) \cap \mathrm{Ass}(R/(I + (x^l))) = \emptyset$.

(13) Let $x_1, \dots, x_n \in R$. Then for any ideal I in R,

$$\begin{aligned}\sqrt{I} &= \sqrt{I + (x_1)} \cap \cdots \cap \sqrt{I + (x_n)} \cap \sqrt{I : (x_1 \cdots x_n)^\infty} \\ &= \sqrt{I + (x_1)} \cap \cdots \cap \sqrt{I + (x_n)} \cap \sqrt{I : x_1 \cdots x_n}.\end{aligned}$$

Proposition **1.1** *If I is a binomial ideal and J is a monomial ideal, then $I \cap J$ is binomial.*

Proof. Note that $(It + J(t-1))R[t]$ is a binomial ideal in $R[t]$. Let G be its Gröbner basis under an ordering as in commutative algebra fact (4). Then by commutative algebra fact (1), G is binomial, hence the set intersection $G \cap R$ is binomial, so that $(G \cap R)$ is a binomial ideal. Thus by commutative algebra fact (3), $I \cap J$ is binomial. □

Proposition **1.2** *If I is a binomial ideal and m is a monomial, then $I : m$ is binomial.*

Proof. By the previous proposition, $I \cap (m)$ is binomial. By commutative algebra fact (6), $(I : m)m$ is binomial, whence the division of each generator by its factor m still produces the binomial ideal $I : m$. □

Proposition **1.3** *Let I be a binomial ideal, and let $J_1, \ldots, J_l$ be monomial ideals. Then there exists a monomial ideal J such that $(I + J_1) \cap \cdots \cap (I + J_l) = I + J$.*

Proof. We can take a k-basis B of R/I to consist of monomials. By Gröbner bases of binomial ideals, $(I + J_k)/I$ is a subspace whose basis is a subset of B. Thus $\cap((I + J_k)/I)$ is a subspace whose basis is a subset of B, which proves the proposition. □

2 Binomial ideals in $S = k[X_1, \ldots, X_n, X_1^{-1}, \ldots, X_n^{-1}] = k[X_1, \ldots, X_n]_{X_1 \cdots X_n}$

Any binomial $\underline{X}^{\underline{a}} - c\underline{X}^{\underline{b}}$ can be written up to unit in S as $\underline{X}^{\underline{a}-\underline{b}} - c$.

Let I be a proper binomial ideal in S. Write $I = (\underline{X}^{\underline{e}} - c :\ \text{some } \underline{e} \in \mathbb{Z}^n, c_e \in k^*)$. (All c_e are non-zero since I is assumed to be proper.)

If e, e' occur in the definition of I, set $e'' = e - e', e''' = e + e'$. Then

$$\underline{X}^{\underline{e}} - c_e = \underline{X}^{\underline{e}'+\underline{e}''} - c_e \equiv c_{e'}\underline{X}^{\underline{e}''} - c_e \bmod I,$$
$$\underline{X}^{\underline{e}} - c_e = \underline{X}^{\underline{e}'''-\underline{e}'} - c_e \equiv c_{e'}^{-1}\underline{X}^{\underline{e}'''} - c_e \bmod I,$$

so that e'' is allowed with $c_{e''} = c_e c_{e'}^{-1}$, and e''' is allowed with $c_{e'''} = c_e c_{e'}$. In particular, the set of all allowed e forms a $\mathbb{Z}$-submodule of $\mathbb{Z}^n$. Say that it is generated by m vectors. Records these vectors into an $n \times m$ matrix A. We just performed some column reductions: neither these nor the rest of the standard column reductions over $\mathbb{Z}$ change the ideal I. But we can also perform column reductions! Namely, $S = k[X_1X_2^m, X_2, \ldots, X_n, (X_1X_2^m)^{-1}, (X_2)^{-1}, \ldots, (X_n)^{-1}]$, and we can rewrite any monomial $\underline{X}^{\underline{a}}$ as $(X_1X_2^m)^{a_1}X_2^{a_2-ma_1}X_3^{a_3} \cdots X_n^{a_n}$, which corresponds to the second row of the matrix becoming the old second row minus m times the old first row (and other rows remain unchanged). So this, and even all other, row reductions are allowed; whereas they do not change the ideal nor the constant coefficient in the binomial generating set, they do modify the variables. In any case, we can perform the standard row and column reductions on the occurring exponents e to get the $n \times n$ matrix into a standard form.

Example **2.1** Let $I = (x^3y - y^3z, xy - z^2)$ in $k[x, y, z]$. This yields the 3×2 matrix of occurring exponents:

$$A = \begin{bmatrix} 3 & 1 \\ -2 & 1 \\ -1 & -2 \end{bmatrix}.$$

We first perform some elementary column reductions (that possibly change the c_e to products of such, but our c_e are all 1, so there is no change):

$$A \to \begin{bmatrix} 1 & 3 \\ 1 & -2 \\ -2 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 1 & -5 \\ -2 & 5 \end{bmatrix}.$$

We next perform the row reductions, and for these we will keep track of the names of variables (in the obvious way):

$$\begin{matrix} x \\ y \\ z \end{matrix} \begin{bmatrix} 1 & 0 \\ 1 & -5 \\ -2 & 5 \end{bmatrix} \to \begin{matrix} xy \\ y \\ z \end{matrix} \begin{bmatrix} 1 & 0 \\ 0 & -5 \\ -2 & 5 \end{bmatrix} \to \begin{matrix} xyz^{-2} \\ y \\ z \end{matrix} \begin{bmatrix} 1 & 0 \\ 0 & -5 \\ 0 & 5 \end{bmatrix} \to \begin{matrix} xyz^{-2} \\ y \\ zy^{-1} \end{matrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 5 \end{bmatrix} \to \begin{matrix} xyz^{-2} \\ zy^{-1} \\ y \end{matrix} \begin{bmatrix} 1 & 0 \\ 0 & 5 \\ 0 & 0 \end{bmatrix}.$$

Thus, up to a monomial change of variables, once we bring the matrix of exponents into standard form, every proper binomial ideal in S is of the form $(X_1^{m_1} - c_1, \ldots, X_d^{m_d} - c_d)$ for some $d \le n$, some $m_i \in \mathbb{N}$, and some $c_i \in K^*$.

Now the following are obvious: in characteristic 0,

$$I = \bigcap_{u_i^{m_i} = c_i} (X_1 - u_1, \ldots, X_d - u_d),$$

where all the primary components are distinct, binomial, and prime. Thus here all associated primes, all primary components, and the radical are all binomial ideals, and moreover all the associated primes have the same height and are thus all minimal over I.

In positive prime characteristic p, write each m_i as $p^{v_i} n_i$ for some positive v_i and non-negative n_i that is not a multiple of p. Then

$$I = \bigcap_{u_i^{m_i} = c_i} ((X_1 - u_1)^{p^{v_1}}, \ldots, (X_d - u_d)^{p^{v_d}}).$$

The listed generators of each component are primary. These primary components are binomial, as $(X_i - u_i)^{p_{v_i}} = X_i^{p_{v_i}} - u_i^{p_{v_i}}$. The radicals of these components are all the associated primes of I, and they are clearly the binomial ideals $(X_1 - u_1, \ldots, X_d - u_d)$. All of these prime ideals have the same height, thus they are all minimal over I. Furthermore,

$$\sqrt{I} = \bigcap_{u_i^{m_i} = c_i} (X_1 - u_1, \ldots, X_d - u_d) = (X_1^{n_1} - u_1^{n_1}, \ldots, X_d^{n_d} - u_d^{n_d}),$$

for any u_i with $u_i^{m_i} = c_i$. The last equality is in fact well-defined as if $(u_i')^{m_i} = c_i$, then $0 = c_i - c_i = u_i^{m_i} - (u_i')^{m_i} = (u_i^{n_i} - (u_i')^{n_i})^{p^{v_i}}$, so that $u_i^{n_i} = (u_i')^{n_i}$. In particular, $\sqrt{I}$ is binomial.

We summarize this section in the following theorem:

Theorem **2.2** *A proper binomial ideal in S has binomial associated primes, binomial primary components, and binomial radical. All associated primes are minimal. In characteristic* 0*, all components are prime ideals, so all binomial ideals in S are radical. In characteristic p, every binomial in the associated primes has a Frobenius power the corresponding primary component.* □

Example **2.3** (Continuation of Example 2.1.) In particular, if we analyze the ideal from Example 2.1, the already established row reduction shows that $I = (xyz^{-2}-1, (zy^{-1})^5-1)$. In characteristic 5, this is a primary ideal with radical $I = (xyz^{-2} - 1, zy^{-1} - 1) = (xy - z^2, z - y) = (x - z, z - y)$. In other characteristics, we get five associated primes $(xy - z^2, z - \alpha y) = (x - \alpha^2 y, z - \alpha y)$ as α varies over the roots of 1. All of these prime ideals are also the primary components of I.

Proposition **2.4** *Let I be an ideal in R such that IS is binomial. Then $IS \cap R$ is binomial.*

Proof. Let Q be a binomial ideal in R such that $QS = IS$. Then $IS \cap R = QS \cap R = Q : (X_1 \cdots X_n)^\infty$ is binomial by commutative algebra fact (10). □

3 Associated primes of binomial ideals are binomial

Theorem **3.1** *Let I be a binomial ideal. Then all associated primes of I are binomial ideals. (Recall that k is algebraically closed.)*

Proof. By factorization in polynomial rings in one variable, the theorem holds if $n \le 1$. So we may assume that $n \ge 2$. The theorem is clearly true if I is a maximal ideal. Now let I be arbitrary.

Let $j \in \{1, \ldots, n\}$. Note that $I + (x_j) = I_j + (x_j)$ for some binomial ideal I_j in $k[X_1, \ldots, X_{n-1}]$. By induction on n, all prime ideals in $\mathrm{Ass}(k[X_1, \ldots, X_{n-1}]/I_j)$ are binomial. But $\mathrm{Ass}(R/(I + (x_j))) = \{P + (x_j) : P \in \mathrm{Ass}(k[X_1, \ldots, X_{n-1}]/I_j)\}$, so that all prime ideals in $\mathrm{Ass}(R/(I + (x_j)))$ are binomial. By Proposition 1.2, $I : x_j$ is binomial. If x_j is a zerodivisor modulo I, then $I : x_j$ is strictly larger than I, so that by Noetherian induction, $\mathrm{Ass}(R/(I : x_j))$ contains only binomial ideals. By commutative algebra facts (7) and (8), $\mathrm{Ass}(R/I) \subseteq \mathrm{Ass}(R/(I+(x_j))) \cup \mathrm{Ass}(R/(I : x_j))$, whence all associated primes of I are binomial as long as some variable is a zerodivisor modulo I.

Now assume that all variables are non-zerodivisors modulo I. Let $P \in \mathrm{Ass}(R/I)$. Since $x_1 \cdots x_n$ is a non-zerodivisor modulo I, it follows that $P_{x_1 \cdots x_n} \in \mathrm{Ass}((R/I)_{x_1 \cdots x_n})$ $= \mathrm{Ass}(S/IS)$. By Theorem 2.2, $P_{x_1 \cdots x_n} = PS$ is binomial. By Proposition 2.4, P is binomial. □

Example **3.2** We first demonstrate this on a monomial ideal. Let $I = (y^3 z, z^2, x)$. Note that $I : y^3 = I : y^\infty = (z, x)$ is a prime ideal, and that $I + (y^3) = (y^3, z^2, x)$ is primary. Thus by commutative algebra fact (11),

$$I = (z, x) \cap (y^3, z^2, x)$$

is a primary decomposition, and it is an irredundant primary decomposition. Thus clearly $\operatorname{Ass}(R/I) = \{(x, z), (x, y, z)\}$. To get at the same thing via the methods in the proof of the theorem in this section, Observe that $I : z = (y^3, z, x)$ is primary with the only associated prime (x, y, z), and that $I + (z) = (z, x)$ is prime.

Comment: we were lucky that the method from the theorem produced exactly the set of associated primes and not a possibly larger list. In general, there is no such luck, and it is illustrated in the next example:

Example **3.3** (Continuation of Example 2.1, Example 2.3.) Let $I = (x^3y - y^3z, xy - z^2)$ in $k[x, y, z]$. We have already determined all associated prime ideals of I that do not contain any variables. So it suffices to find the associated primes of $I + (x^m)$, $I + (y^m)$ and of $I + (z^m)$, for some large m. But any prime ideal that contains I and x also contains z, so at least we have that (x, z) is minimal over I and thus associated to I. Similarly, (y, z) is minimal over I and thus associated to I. Also, any prime ideal that contains I and z contains in addition either x or y, so that at least we have determined $\operatorname{Min}(R/I)$. Any embedded prime ideal would have to contain of the the already determined primes. Since I is homogeneous, all associated primes are homogeneous, and in particular, the only embedded prime could be (x, y, z). It turns out that this prime ideal is not associated even if it came up in our construction, but we won't get to this until we have a primary decomposition.

4 Primary decomposition of binomial ideals

The main goal of this section is to prove that every binomial ideal has a binomial primary decomposition, if the underlying field is algebraically closed. See Theorem 4.4. We first need a lemma and more terms.

Definition **4.1** *An ideal I in a polynomial ring $k[X_1, \ldots, X_n]$ is* **cellular** *if for all $i = 1, \ldots, n$, X_i is either a non-zerodivisor or nilpotent modulo I.*

All primary monomial and binomial ideals are primary, as will be clear from constructions below.

Definition **4.2** *For any binomial $g = \underline{X}^{\underline{a}} - c\underline{X}^{\underline{b}}$ and for any non-negative integer d, define*

$$g^{[d]} = \underline{X}^{d\underline{a}} - c^d \underline{X}^{d\underline{b}}.$$

The following is a crucial lemma:

Lemma **4.3** *Let I be a binomial ideal, let $g = \underline{X}^{\underline{a}} - c\underline{X}^{\underline{b}}$ be a non-monomial binomial in R such that $\underline{X}^{\underline{a}}$ and $\underline{X}^{\underline{b}}$ are non-zerodivisors modulo I. Then there exists a monomial ideal I_0 such that for all large d, $I : g^{[d!]} = I : (g^{[d!]})^2 = I + I_0$.*

Proof. For all integers d and e, $g^{[d]}$ is a factor of $g^{[de]}$, so that $I : g^{[d]} \subseteq I : g^{[de]}$. Thus there exists d such that for all $e \geq d$, $I : g^{[d!]} = I : g^{[e!]}$.

Let $f \in I : g^{[d!]}$. Write $f = f_1 + f_2 + \cdots + f_s$ for some terms (coefficient times monomial) $f_1 > f_2 > \cdots > f_s$. Without loss of generality $\underline{X}^{\underline{a}} > \underline{X}^{\underline{b}}$. We have that

$$f_1\underline{X}^{\underline{a}} + f_2\underline{X}^{\underline{a}} + \cdots + f_s\underline{X}^{\underline{a}} + f_1\underline{X}^{\underline{b}} + f_2\underline{X}^{\underline{b}} + \cdots + f_s\underline{X}^{\underline{b}} \in I.$$

In the Gröbner basis sense, each $f_i\underline{X}^{\underline{a}}$, $f_i\underline{X}^{\underline{b}}$ reduces to some unique term (coefficient times monomial) modulo I. Since $\underline{X}^{\underline{a}}$ and is a non-zerodivisor modulo I, $f_i\underline{X}^{\underline{a}}$ and $f_j\underline{X}^{\underline{a}}$ cannot reduce to a scalar multiple of the same monomial, and similarly $f_i\underline{X}^{\underline{b}}$ and $f_j\underline{X}^{\underline{b}}$ cannot reduce to a scalar multiple of the same monomial. Thus for each $j = 1, \ldots, s$ there exists $\pi(j) \in \{1, \ldots, s\}$ such that $f_j\underline{x}^{d!\underline{a}} - c^{d!}f_{\pi(j)}\underline{x}^{d!\underline{b}} \in I$. The function $\pi : \{1, \ldots, s\} \to \{1, \ldots, s\}$ is injective. By easy induction, for all i, $f_j(\underline{x}^{d!\underline{a}})^i - c^{d!i}f_{\pi^i(j)}(\underline{x}^{d!\underline{b}})^i \in I$. By elementary group theory, $\pi^{s!}(j) = j$, so that for all j, $f_jg^{[d!][s!]} \in I$. Then $f_jg^{[((d!)(s!))!]} \in I$, and by the choice of d, $f_jg^{[d!]} \in I$. Thus $I : g^{[d!]}$ contains monomials $f_1, \ldots, f_s$. Thus set I_0 to be the monomial ideal generated by all the monomials appearing in the generators of $I : g^{[d!]}$.

Let $f \in I : (g^{[d!]})^2$. We wish to prove that $f \in I : g^{[d!]}$. By possibly enlarging I_0 we may assume that I_0 contains all monomials in $I : g^{[d!]} = I + I_0$. This in particular means that any Gröbner basis G of $I : g^{[d!]}$ consists of monomials in I_0 and binomial non-monomials in I. Write $f = f_1 + f_2 + \cdots + f_s$ for some terms $f_1 > f_2 > \cdots > f_s$. As in the previous paragraph, for each j, either $f_j\underline{x}^{d!\underline{a}} \in I_0$ or else $f_j\underline{x}^{d!\underline{a}} - c^{d!}f_{\pi(j)}\underline{x}^{d!\underline{b}} \in I$. If $f_j\underline{x}^{d!\underline{a}} \in I_0 \subseteq I : g^{[d]}$, then by the non-zerodivisor assumption, $f_j \in I : g^{[d]}$, which contradicts the assumption. So necessarily we get the injective function $\pi : \{1, \ldots, s\} \to \{1, \ldots, s\}$. As in the previous paragraph we then get that each $f_j \in I : g^{[d]}$. □

Without loss of generality assume that no f_i is in $I : g^{[d!]}$. Note that $fg^{[d!]} \in I : g^{[d!]}$. Consider the case that $f_j\underline{x}^{d!\underline{a}} \in I_0$ and get a contradiction. Now repeat the π argument as in a previous part to make the conclusion.

Theorem **4.4** *If k is algebraically closed, then any binomial ideal has a binomial primary decomposition.*

Proof. Let I be a binomial ideal. For each variable X_j by commutative algebra fact (11) there exists l such that $I = (I : X_j^l) \cap (I + (X_j)^l)$, so it suffices to find the primary decompositions of the two ideals $I : X_j^l$ and $I + (X_j)^l$. These two ideals are binomial, the former by Proposition 1.2. By repeating this for another X_i on the two ideals, and then repeat for X_k on the four new ideals, et cetera, with even some j repeated, we may assume that each of the intersectands is cellular. It suffices to prove that each cellular binomial ideal has a binomial primary decomposition.

So let I be cellular and binomial. By possibly reindexing, we may assume that $X_1, \ldots, X_d$ are non-zerodivisors modulo I, and $X_{d+1}, \ldots, X_n$ are nilpotent modulo I. Let $P \in \text{Ass}(R/I)$. By Theorem 3.1, P is a binomial prime ideal. Since I is contained in P,

P must contain $X_{d+1}, \ldots, X_n$, and since the other variables are non-zerodivisors modulo I, these are the only variables in P. Thus $P = P_0 + (X_{d+1}, \ldots, X_n)$, where P_0 is a binomial prime ideal whose generators are binomials in $k[X_1, \ldots, X_d]$, and $X_1, \ldots, X_d$ are non-zerodivisors modulo I.

So far we have I "cellular with respect to variables". (For example, we could have $I = (X_3(X_1^2 - X_2^2), X_3^2)$ and $P = (X_1 - X_2, X_3)$.) Now we will make it more "cellular with respect to binomials in the subring". Namely, let g be a non-zero binomial in P_0. (In the parenthetical example, we could have $g = X_1 - X_2$.) By Lemma 4.3, there exists $d \in \mathbb{N}$ such that $I : g^{[d]} = I : (g^{[d]})^2 = I +$ (monomial ideal). This in particular implies that P is not associated to $I : g^{[d]}$, and by commutative algebra fact (11), P is associated to $I + (g^{[d]})$. Furthermore, the P-primary component of I is the P-primary component of the binomial ideal $I + (g^{[d]})$. We replace the old I by the binomial ideal $I + (g^{[d]})$. We repeat this to each g a binomial generator of P_0, so that we may assume that P is minimal over I. (In the parenthetical example above, we would now have with $d = 6$ that $I = (X_1^6 - X_2^6, X_3(X_1^2 - X_2^2), X_3^2)$.) Now $X_{d+1}, \ldots, X_n$ are still nilpotent modulo I. The P-primary component of I is the same as the P-primary component of binomial ideal $I : (X_1 \cdots X_d)^\infty$, so by replacing I with $I : (X_1 \cdots X_d)^\infty$ we may assume that I is still cellular.

If $\mathrm{Ass}(R/I) = \{P\}$, then I is P-primary, and we are done. So we may assume that there exists an associated prime ideal Q of I different from P. Since P is minimal over I and different from Q, necessarily there exists an irreducible binomial $g = \underline{X}^{\underline{a}} - c\underline{X}^{\underline{b}} \in Q \setminus P$. Necessarily $g \notin (X_{d+1}, \ldots, X_n)R$. Thus Lemma 4.3 applies, so there exists $d \in \mathbb{N}$ such that $I : g^{[d]} = I : (g^{[d]})^2 = I +$ (monomial ideal). Note that Q is not associated to this ideal but Q is associated to I, so that the binomial ideal $I : g^{[d]}$ is strictly larger than I. If $g^{[d]} \notin P$, then the P-primary component of I equals the P-primary component of $I : g^{[d]}$, and so by Noetherian induction (if we have proved it for all larger ideals, we can then prove it for one of the smaller ideals) we have that the P-primary component of I is binomial. So without loss of generality we may assume that $g^{[d]} \in P$. Then $g^{[d]}$ contains a factor in P of the form $g_0 = \underline{X}^{\underline{a}} - c'\underline{X}^{\underline{b}}$ for some $c' \in k$. If the characteristic of R is p, $g_0^{p^m}$ is a binomial for all m, we choose the largest m such that p^m divides d, and set $h = g^{[d]}/g_0$, $b = g_0^{p^m}$. In characteristic 0, we set $h = g^{[d]}/g_0$ and $b = g_0$. In either case, b is a binomial, $b \in I : h$ and $h \notin P$. Thus the P-primary component of I is the same as the P-primary component of $I : h$, and in particular, since $I \subseteq I + (b) \subseteq I : h$, it follows that the P-primary component of I is the same as the P-primary component of the binomial ideal $I + (b)$. If $b \in Q$, then $g_0 = \underline{X}^{\underline{a}} - c'\underline{X}^{\underline{b}}$ and $g = \underline{X}^{\underline{a}} - c\underline{X}^{\underline{b}}$ are both in Q. Necessarily $c \neq c'$, so that $\underline{X}^{\underline{a}}, \underline{X}^{\underline{b}} \in Q$, and since $g \notin (X_{d+1}, \ldots, X_n)R$, it follows that Q contains one of the variables $X_1, \ldots, X_d$. But these variables are non-zerodivisor modulo I, so that Q cannot be associated to I, which proves that $b \notin Q$. But then I is strictly contained in $I + (b)$, and by Noetherian induction, the P-primary component is binomial. □

5 The radical of a binomial ideal is binomial

Theorem **5.1** *If the underlying field is algebraically closed, then the radical of any binomial ideal in a polynomial ring is binomial.*

Proof. This is clear if $n = 0$. So assume that $n > 0$. By commutative algebra fact (13),

$$\sqrt{I} = \sqrt{I + (X_1)} \cap \cdots \cap \sqrt{I + (X_n)} \cap \sqrt{I : (X_1 \cdots X_n)^\infty}.$$

Let $I_0 = \sqrt{I : (X_1 \cdots X_n)^\infty}$. We have established in Theorem 2.2 that $\sqrt{I_0}S = \sqrt{IS}$ is binomial in S. By Proposition 2.4, $\sqrt{I_0}$ is binomial.

Note that $I + (X_1) = I \cap k[X_2, \ldots, X_n] + (X_1) +$ (monomial ideal). By commutative algebra fact (5), $I_1 = I \cap k[X_2, \ldots, X_n]$ is binomial, and so by induction on n, the radical of I_1 is binomial. This radical is contained in $\sqrt{I}$, and by possibly adding the binomial generators of $\sqrt{I_1}$ to I, we may assume that $\sqrt{I_1} \subseteq I$, and subsequently that $\sqrt{I_1} = I_1$. A minimal prime ideal P over $I_1 + (X_1) +$ (monomial ideal) is of the form $I_1 + (X_1, X_{j_1}, \ldots, X_{j_s})$ for some $j_1, \ldots, j_s$. Then by Proposition 1.3, $\sqrt{I_1 + (X_1) + \text{ (monomial ideal)}}$ equals $I_1 + J_1$ for some monomial ideal J_1. But this is precisely the radical of $I + (X_1)$. Similarly, $\sqrt{I + (X_j)} = I + J_j$ for some monomial ideals $J_1, \ldots, J_l$. By Proposition 1.3, $\sqrt{I} = (I + J) \cap I_0$ for some monomial ideal J. But $I \subseteq I_0$, so that $\sqrt{I} = I + J \cap I_0$, and this is a binomial ideal by Proposition 1.1. □