

Minimal primes of ideals arising from conditional independence statements

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Abstract

We study ideals whose primary decomposition specifies the relevant structural zeros of certain conditional independence models. The ideals we study generalize the class of ideals considered by Fink [5] in a way distinct from the generalizations of Herzog-Hibi-Hreinsdottir-Kahle-Rauh [11] and Ay-Rauh [1]. We introduce switchable sets to give a combinatorial description of the minimal prime ideals, and for some classes we describe the minimal components. We discuss possible interpretations of the ideals we study, including as 2×2 minors of generic hypermatrices. We also introduce a definition of diagonal monomial orders on generic hypermatrices to compute some Gröbner bases.

Keywords: Primary Decomposition, Conditional Independence Ideals, Graphical Models, Gröbner Bases, Binomial Ideals

1. Introduction

Let X_1, \dots, X_n be n discrete random variables with r_i states each. A model is a set of joint probability distributions that satisfy some set of rules, and there are many options for these rules. Let A, B, C be disjoint subsets of $\{1, \dots, n\}$ and let X_A denote the random variable $\{X_{i_1}, \dots, X_{i_k} \mid i_j \in A, 1 \leq j \leq k\}$. We are interested in models specified by a set of conditional independence statements of the form $X_A \perp\!\!\!\perp X_B \mid X_C$ denoting that the state of the random variable X_A is independent of the state of the random variable X_B given any state of the random variable X_C . Such models arise in a wide variety of statistical contexts and there is a large literature discussing these models. Some references that are closely related to the work in this paper are [1, 11], which include some discussion of robustness theory, and [3, 6, 14], which discuss the connection to graphical models.

One reason conditional independence models are interesting from the perspective of commutative algebra arises from examining the *intersection axiom* [3, Proposition 3.1.3] which states that if all the probabilities in a joint distribution are positive then the pair of statements $X_A \perp\!\!\!\perp X_B \mid X_{C \cup D}$ and $X_A \perp\!\!\!\perp X_C \mid X_{B \cup D}$ together imply that $X_A \perp\!\!\!\perp \{X_B, X_C\} \mid X_D$.

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This axiom raises the question of what happens when there are *structural zeros*, that is what happens when a distribution is not strictly positive? Such zeros certainly happen in practice as there may be variables involved in a study for which some of the combinations of those variables are impossible. This problem can be looked at from the algebraic side as follows. Corresponding to the model there is a conditional independence ideal; see [3, 6, 11] for a general description, and see Remark 2.2 below for the algebraic formulation of the conditional independence ideals of this paper. The primary decomposition of these ideals identifies the structural zeros of concern in the context of the intersection axiom, including a component corresponding to the strictly positive distribution.

One component of the primary decomposition of conditional independence ideals is well understood, especially when the ideal is binomial, which is the case for the ideals we study. This component is an easy consequence of the Hammersley-Clifford Theorem [14], or Eisenbud and Sturmfels' work [4, Corollary 2.5], and is stated explicitly in Høsten and Shapiro's work [12, Theorem 2.1] and is "the graphical model" in Lauritzen [14]. One reason Lauritzen [14] refers to it as "the graphical model" is that it encodes the original conditional independence statements and any consequences of the intersection axiom, and is the only component of concern when the distribution is strictly positive. We summarize several ways of thinking about this ideal, particularly in our context, (via tensors, lattices, and hypermatrices) in Section 2 to set up a more in-depth discussion of its relationship to the ideals we study and we use this discussion in our arguments.

Our main result, Theorem 4.13, is a complete combinatorial description of the minimal prime ideals of the conditional independence ideal $I^{(t)}$ corresponding to the model

$$\{X_i \perp\!\!\!\perp X_j \mid X_T : \forall i \leq t, i < j \text{ and } T = \{1, \dots, n\} \setminus \{i, j\}\}, \quad (1)$$

where X_1, \dots, X_n have an arbitrary but finite number of states, and t is any positive integer at most n . This model is the model given by the set of pairwise Markov conditions (see [14, page 32]) on the graph with no edges on the first t vertices and a complete graph on the remaining $n - t$ vertices. We use $\tilde{I}^{(t)}$ to denote the primary component of $I^{(t)}$ which is the "graphical model" discussed above.

Computing the full primary decomposition of such ideals is computationally hard, even for small examples and with the use of the package Binomial.m2 [13] for binomial ideals. One of the first such computations was done by Garcia, Stillman, and Sturmfels in [6]: they computed the primary decomposition for a class of models arising from directed acyclic graphs. Fink [5] gave a combinatorial description of the primary decomposition of the model $\{X_1 \perp\!\!\!\perp X_2 \mid X_3, X_1 \perp\!\!\!\perp X_3 \mid X_2\}$, for any number of states. The description was conjectured by Cartwright and Engström [3, Page 146] at the workshop at Oberwolfach [3] run by Drton, Sturmfels and Sullivant.

The model we study is given in (1) and in that context Fink's [5] work covers the case $n = 3$ and $t = 1$. Recent work by Ohtani [15] and by Herzog et al. [11] on binomial edge ideals intersects our class of ideals when $t = 1$ and X_1 is binary. Ay and Rauh [1] generalized [11] to allow the number of states of X_1 to be arbitrary and thus their ideals intersect our class of ideals when $t = 1$.

Our main result, Theorem 4.13, is the description of the prime ideals minimal over $I^{(t)}$. We use a new combinatorial structure, t -switchable set, and a corresponding equivalence relation; these are defined in Section 3. We expect that these structures might be helpful in other contexts as well. We use a connection with Segre embeddings and the structure of t -switchable sets to prove in Theorem 4.4 that $\tilde{I}_S^{(t)}$ (see Definition 4.1) are prime ideals, thereby generalizing the fact that $\tilde{I}^{(t)}$ are prime ideals. By [4], a generating set of a minimal prime ideal of a binomial ideal consists of a set of variables and a set of binomials in the remaining variables. The content of this work is in giving an effective combinatorial description of the sets of variables and of the binomials, and we do so by using t -switchable sets and the associated equivalence relations: It is precisely such $\tilde{I}_S^{(t)}$, for restricted S , that are the binomial portion of the prime ideals minimal over $I^{(t)}$, and the variable portion is $(x_a : a \notin S)$ (proof is in Theorem 4.13).

Further, we prove in Theorem 5.5 that the minimal components of $I^{(n)}$ are prime ideals, while the work of Ay and Rauh [1] establishes that $I^{(1)}$ is radical and therefore all the primary components are minimal and prime. We give examples in Section 7 showing that ideals $I^{(t)}$ can have embedded primes for $n \geq 3$ and $t > 1$. Thus the ideals $I^{(t)}$ are not radical in general and therefore are not the same as the ideals in Herzog et al. [11], Ohtani [15], or Ay and Rauh [1]. Example 2.4 shows that the ideals $I^{(t)}$ are not lattice basis ideals.

Finally, we introduce a notion of diagonal monomial orders for generic hypermatrices and we use these orders to give Gröbner bases for $\tilde{I}_S^{(t)}$ in Section 6. This generalizes the well-known work of Caniglia, Guccione, and Guccione [2] for generic matrices and the work of Ha [10, Theorem 1.14] for $\tilde{I}^{(n)}$ in the reverse lexicographic monomial order.

2. Definitions and connections with tensors and Segre embeddings

After setting up the notation, we give an algebraic definition for the conditional independence ideal $I^{(t)}$ from the introduction. One of the prime ideals minimal over $I^{(t)}$ has many fruitful interpretations and in Discussion 2.3 we present these interpretations. In Example 2.4 we show that $I^{(t)}$ are not lattice basis ideals.

Throughout we fix positive integers n and r_1, \dots, r_n , the index set $N = [r_1] \times \dots \times [r_n]$, and the polynomial ring R over a field in variables x_a as a varies over N . Let M be the $r_1 \times \dots \times r_n$ hypermatrix whose a th entry is x_a . In this context, this paper is about the structure of certain determinantal ideals of this generic hypermatrix M .

Let $L \subseteq [n]$. For $a, b \in N$ define the **switch** function $s(L, a, b)$ that switches the L -entries of b into a : $s(L, a, b)$ is an element of N whose i th component is

$$s(L, a, b)_i = \begin{cases} b_i, & \text{if } i \in L; \\ a_i, & \text{otherwise.} \end{cases}$$

If $L = \{j\}$, we simply write $s(L, a, b) = s(j, a, b)$. For any two indices a and b in N we define the **distance** between them to be $d(a, b) = \#\{i : a_i \neq b_i\}$. This distance is the usual Hamming distance in coding theory. Note that $d(a, b) = d(s(L, a, b), s(L, b, a))$. For any $L \subseteq [n]$ and $i \in [n]$ we define:

$$f_{L,a,b} = x_a x_b - x_{s(L,a,b)} x_{s(L,b,a)},$$

$$f_{i,a,b} = x_a x_b - x_{s(i,a,b)} x_{s(i,b,a)}.$$

We call the $f_{i,a,b}$ the **2×2 minors** of the hypermatrix M . When $d(a,b) = 2$ and $a_i \neq b_i$, we call $f_{i,a,b}$ a **slice minor** of M . By a **slice submatrix** of M we refer to any submatrix of M consisting of all entries M_{i_1, \dots, i_n} with all but two of the indices identical. Thus a slice minor of M is simply a 2×2 minor of a slice submatrix of M . This notation provides flexibility over flattenings for discussing certain subsets of the minors of a hypermatrix such as the slice minors. We also think of $f_{i,a,b}$ as a minor of a flattening of the hypermatrix, using the i th component to index the rows. More generally, $f_{L,a,b}$ is a minor of a flattening of the hypermatrix, where the rows are indexed by the components in L (see also Discussion 2.3).

Definition 2.1. For any $t \in [n]$, let

$$\begin{aligned} I^{(t)} &= (f_{i,a,b} : a, b \in N, d(a,b) = 2, i \in [t]), \\ \tilde{I}^{(t)} &= (f_{i,a,b} : a, b \in N, i \in [t]). \end{aligned}$$

Note that the generators of $I^{(t)}$ (resp. $\tilde{I}^{(t)}$) are those slice (resp. all) minors of M for which one of the two components that varies is in $[t]$. Alternatively, the generators of $I^{(t)}$ (resp. $\tilde{I}^{(t)}$) are the slice (resp. all) minors of the generic $r_1 \times r_2 \times \dots \times r_t \times (r_{t+1} \dots r_n)$ hypermatrix.

Remark 2.2. By standard construction of conditional independence ideals, $I^{(t)}$ corresponds to the conditional independence model given in (1) (see [3, Proposition 3.1.4]).

Discussion 2.3. Now we connect the ideal $\tilde{I}^{(t)}$, generated by the 2×2 minors of M in which one of the entries being switched is at most t , to other ideals in the literature to facilitate later arguments. The following describe the same ideal:

- (1) The ideal cutting out the rank-one tensors in the flattenings of $V_1 \otimes \dots \otimes V_n$ of the form $V_i \otimes (\otimes_{j \neq i} V_j)$ as i varies over $[t]$.
- (2) The defining ideal for the Segre embedding of $\mathbb{P}(V_1) \times \dots \times \mathbb{P}(V_t) \times \mathbb{P}(V_{t+1} \otimes \dots \otimes V_n) \rightarrow \mathbb{P}(V_1 \otimes \dots \otimes V_n)$.
- (3) The ideal generated by all the 2×2 minors of the generic $r_1 \times r_2 \times \dots \times r_t \times (r_{t+1} \dots r_n)$ hypermatrix.
- (4) The lattice ideal, where the lattice is the kernel of the matrix which computes two sets of marginals. The first set of marginals are the 1-marginals for each $i \in [t]$ so that for each possible i th state we marginalize over the remaining variables. The second set are the marginals for each possible state of the remaining $n - t$ variables.

The fact that these all define the same ideal is scattered through the literature, but most of the key ideas are in [7]. For example, the connection between (2) and (4) follows from [7] since the model is given as distributions in the image of a monomial parameterization given by the marginals matrix, and that monomial parameterization is exactly the Segre map. One argument that (2) is equivalent to (3) is in Ha [10]. Finally, it is well known that a

matrix (hypermatrix) has rank one if and only if its 2×2 minors vanish and that such matrices (hypermatrices) represent rank-one tensors in the corresponding tensor product of vector spaces which shows that (1) is equivalent to (3). The statement of (4) uses the language of marginals and as such suggests that the underlying field is \mathbb{R} which is the case in [7]. However, lattice ideals and the corresponding monomial map are not field-dependent. Similarly [10] assumes the underlying field is of characteristic zero, but the fact that (3) is a prime ideal and corresponds to (2) does not depend on the underlying field.

The many interpretations of $\tilde{I}^{(t)}$ have several useful consequences:

- (1) $\tilde{I}^{(t)}$ is prime say by [10] (and we generalize this fact in Lemma 4.3);
- (2) $I^{(t)} : \mathbf{x}^\infty = \tilde{I}^{(t)}$, where \mathbf{x} is the product of the ring variables (by the Hammersley-Clifford Theorem [14], [4], or [12]);
- (3) therefore $\tilde{I}^{(t)}$ is a minimal prime component of $I^{(t)}$, and
- (4) $\tilde{I}^{(t)}$ is the unique smallest binomial prime ideal containing $I^{(t)}$ and no variables.

Since $I^{(t)} \subseteq \tilde{I}^{(t)}$ and $\tilde{I}^{(t)}$ is a lattice ideal for a saturated lattice, it is natural to ask if $I^{(t)}$ is a lattice basis ideal (as defined in [12]) for some basis for the same lattice. The following example illustrates that this is not the case, and that the lattice basis ideal is properly contained in $I^{(t)}$.

Example 2.4. For a simple illustration of how $I^{(t)}$ and $\tilde{I}^{(t)}$ relate to lattice ideals, consider the example of three random variables, each with two states, and $t = 3$. The $2 \times 2 \times 2$ hypermatrix has 6 faces and the determinants of these faces give six minimal generators of $I^{(t)}$. There are six non-slice minors which we add in to generate $\tilde{I}^{(t)}$ (only three of these are needed to get a minimal generating set). Finally, the lattice basis ideal is minimally generated by four binomials, which correspond to four of the six faces (depending on which basis one chooses) and is strictly contained in $I^{(t)}$.

Therefore the primary decomposition of $I^{(t)}$ does not follow from [12]. By [4], describing the minimal prime ideals of $I^{(t)}$ consists of establishing sets of variables and binomials. We use *t-switchable sets* S and ideals $\tilde{I}_S^{(t)}$ for this purpose (defined in Section 3). In Section 6 we place these ideals in the wider theory and prove that they are prime ideals.

3. Switchable sets and connectedness

In this section we set up the combinatorial structures used in the main results. We indicate in Remark 4.17 how these structures relate to those used by Fink [5].

Definition 3.1. *Let $t \in [n]$. A subset S of N is **switchable in the first t components** (**t-switchable for short**) if for all $a, b \in S$ with $d(a, b) = 2$, if $i \in [t]$, then $s(i, a, b) \in S$.*

Certainly the empty set and the full set N are t -switchable sets. Note that the notion of $(n - 1)$ -switchable is identical to the notion of n -switchable. Also, t -switchable is equivalent to the following condition: For any $a, b \in N$ and any distinct $i \in [t]$ and $j \in [n]$, then $a, s(\{i, j\}, a, b) \in S$ if and only if $s(i, a, b), s(j, a, b) \in S$.

Lemma 3.2. *Let S be a t -switchable subset of N , let $a \in S$, and let $b \in N$. Let $L \subseteq [t]$ be such that for all $l \in L$, $s(l, a, b) \in S$. Then $s(L, a, b) \in S$.*

Proof. We prove this by induction on $|L|$. If $|L| \leq 1$, this is the assumption. If $|L| \geq 2$, let i, j be distinct elements in L . By induction, $s(L', a, b) \in S$ for all $L' \subseteq L$ with $|L'| < |L|$. In particular, if $i, j \in L$ and $L_0 = L \setminus \{i, j\}$, then $c = s(L_0, a, b)$, $d = s(L_0 \cup \{i\}, a, b)$, $e = s(L_0 \cup \{j\}, a, b) \in S$. Also, $s(i, c, b) = d$ and $s(j, c, b) = e$ are in S , and therefore, since S is t -switchable, $s(L, a, b) = s(\{i, j\}, c, b) \in S$. \square

Definition 3.3. *Let S be a subset of N . We say that $a, b \in S$ are **connected in S** if there exist $a_0 = a, a_1, a_2, \dots, a_{k-1}, a_k = b \in S$ such that for all $j = 1, \dots, k$, a_{j-1} and a_j differ only in one component. We refer to a_0, \dots, a_k loosely as a **path** from a to b , and we refer to a_1, \dots, a_{k-1} as an **intermediate subpath** from a to b . Clearly any elements on the path from a to b are mutually connected. Also, connectedness is naturally an equivalence relation on S , and we refer to the equivalence classes as **connected components** of S .*

The following is immediate from the definition:

Lemma 3.4. *Let S be a t -switchable subset of N . Let $a, b \in S$. If $d(a, b) \leq 1$, then a and b are connected. If $d(a, b) = 2$ and $a_i \neq b_i$ for some $i \in [t]$, then $a, b, s(i, a, b), s(i, b, a)$ are pairwise connected in S , and both $s(i, a, b)$ and $s(i, b, a)$ form an intermediate subpath from a to b . \square*

Lemma 3.5. *Suppose that a and b are connected in a t -switchable set S . Let $a_0 = a, a_1, a_2, \dots, a_{k-1}, a_k = b$ be a path from a to b . Let $L \subseteq [t]$ and $i, j \in \{0, \dots, k\}$. Then $s(L, a_i, a_j) \in S$ and is connected to a in S .*

Proof. First suppose that $L = \{l\}$. If $|j - i| \leq 2$, then $s(l, a_i, a_j)$ is either $a_i \in S$ or it is in S by Lemma 3.4. So we may assume that $|j - i| \geq 3$. Without loss of generality, assume $i < j$. By induction on $|j - i|$ we have that $s(l, a_{i+1}, a_j) \in S$. Then $a_i, a_{i+1}, s(l, a_{i+1}, a_j)$ is a path in S , and therefore by induction, $s(l, a_i, a_j) = s(l, a_i, s(l, a_{i+1}, a_j)) \in S$. Hence by Lemma 3.2, $s(L, a_i, a_j) \in S$ for all $L \subseteq [t]$. Furthermore, if $L = \{l_1, \dots, l_k\}$, then $a_i, s(\{l_1\}, a_i, a_j), s(\{l_1, l_2\}, a_i, a_j), \dots, s(\{l_1, l_2, \dots, l_k\}, a_i, a_j)$ is a path in S , so that $s(L, a_i, a_j)$ is connected to a_i and hence to a . \square

In the definition of connectedness, the sets of indices where the consecutive a_i differ may not all be distinct. Furthermore, if $L \not\subseteq [t]$, then $s(L, a_i, a_j)$ need not be connected to a , as we show by the next example.

Example 3.6. Set

$$\begin{aligned}
S &= \{1, 2\} \times \{1\} \times \{1\} \times \{1, 2\} \\
&\cup \{1, 2\} \times \{1\} \times \{1, 2\} \times \{2\} \\
&\cup \{1, 2\} \times \{1, 2\} \times \{2\} \times \{2, 3\} \\
&\cup \{1, 2\} \times \{2\} \times \{2, 3\} \times \{2, 3\}.
\end{aligned}$$

Note that S is 1-switchable and consisting of a single connected component. The elements $(1, 1, 1, 1)$ and $(2, 2, 3, 3)$ are connected in S , but there is no path between them of length 4. Also, $s(2, (1, 1, 1, 1), (2, 2, 3, 3)) \notin S$. (See the comment after Proposition 5.1 for another point of view.)

4. Prime ideals minimal over $I^{(t)}$

This section has two main goals. One is to prove (in Theorem 4.4) that $\tilde{I}_S^{(t)}$ (see Definition 4.1) are prime ideals. The second is to prove that for any n and any $t \in [n]$, the prime ideals minimal over $I^{(t)}$ are of the form $P_S^{(t)}$ as S varies over maximal t -switchable subsets of N (see Definition 4.5). At the end of the section we look at $I^{(1)}$ more closely, especially when $n = 3$. We examine the minimal components of $I^{(t)}$ when $t = n$ in Section 5.

The following definition gives the notation for the variable and pure binomial parts of the minimal prime ideals for $I^{(t)}$.

Definition 4.1. Let S be t -switchable. Define

$$\tilde{I}_S^{(t)} = (f_{i,a,b} \mid i \in [t], a, b \text{ connected in } S), \quad \text{Var}_S^{(t)} = (x_a : a \notin S), \quad \text{and } P_S^{(t)} = \text{Var}_S^{(t)} + \tilde{I}_S^{(t)}.$$

Proposition 4.2. If S is t -switchable, then $P_S^{(t)}$ contains $I^{(t)}$.

Proof. We need to prove that $f_{i,a,b} \in P_S^{(t)}$ for any $a, b \in N$ differing exactly in components $i \in [t]$ and $j \in [n] \setminus \{i\}$. First suppose that $a \notin S$. Note that $s(i, a, b)$ and $s(i, b, a)$ differ exactly in components i and j , so that S being t -switchable implies either $s(i, a, b)$ or $s(i, b, a)$ is not in S . Thus either $x_{s(i,a,b)}$ or $x_{s(i,b,a)}$ is in $P_S^{(t)}$, so that $f_{i,a,b} \in P_S^{(t)}$. Thus we may assume that $a \in S$, and similarly that $b \in S$. But then by Lemma 3.4, a and b are connected in S , so that $f_{i,a,b} \in P_S^{(t)}$. \square

Lemma 4.3. Let $1 \leq t \leq n$. If S is a connected component in some t -switchable set, then $\tilde{I}_S^{(t)}$ is a prime ideal.

Proof. Since S is a connected component in some t -switchable set, for any $a, b \in S$ and any $i \leq t$, $s(i, a, b)$ and $s(i, b, a)$ are in S . Therefore if we fix $i \leq t$, the set $\{x_a \mid a \in S\}$ naturally form a matrix M_i with rows indexed by the i th components and columns indexed by the remaining $n - 1$ components. Furthermore, each generator $f_{i,a,b}$ of $\tilde{I}_S^{(t)}$ is a 2×2 minor in M_i , and $\tilde{I}_S^{(t)}$ is generated by all the 2×2 minors of the matrices M_1, \dots, M_t . For each $i = 1, \dots, t$, let s_i be the number of rows of M_i , and let s_{t+1} be the number of tuples that

occur as the last $n - t$ entries in elements in S . Let V_i be a vector space of dimension s_i for $1 \leq i \leq t + 1$. Then the 2×2 minors of M_i naturally cut out the rank-1 tensors in the flattenings $V_i \otimes (\otimes_{j \neq i} V_j)$ of the tensor $V_1 \otimes \cdots \otimes V_t \otimes V_{t+1}$. Hence by Discussion 2.3, the ideal $\tilde{I}_S^{(t)}$ is a prime ideal. \square

Theorem 4.4. *If S is a t -switchable set, then the ideals $\tilde{I}_S^{(t)}$ and $P_S^{(t)}$ are prime.*

Proof. Partition $S = S_1 \cup \cdots \cup S_m$ into its connected components. Therefore, the S_i are pairwise disjoint. Then $\tilde{I}_S^{(t)} = \sum_{i=1}^m \tilde{I}_{S_i}^{(t)}$. By Lemma 4.3, each $\tilde{I}_{S_i}^{(t)}$ is a prime ideal. Let \overline{R} be the polynomial ring in the same variables as R but over the algebraic closure of the underlying field. Lemma 4.3 shows that each $\tilde{I}_{S_i}^{(t)} \overline{R}$ is a prime ideal as well. It is well known that in the polynomial ring over an algebraically closed field, if the generators of two prime ideals are polynomials in disjoint sets of variables, then the sum of the two prime ideals is also prime. Thus $\tilde{I}_S^{(t)} \overline{R}$ and $P_S^{(t)} \overline{R}$ are prime ideals. But the generators of these two prime ideals are in R , so since \overline{R} is a faithfully flat extension of R , $\tilde{I}_S^{(t)}$ and $P_S^{(t)}$ are contractions of the prime ideals $\tilde{I}_S^{(t)} \overline{R}$ and $P_S^{(t)} \overline{R}$ respectively, and hence are prime themselves. \square

Having established that $P_S^{(t)}$ is prime, we set up those t -switchable sets that correspond to minimal prime ideals for $I^{(t)}$.

Definition 4.5. *Let S be t -switchable. We say that S is **maximal t -switchable** if for all t -switchable subsets T of N properly containing S , $P_S^{(t)}$ and $P_T^{(t)}$ are incomparable.*

Remark 4.6. For brevity we state a few facts but omit the straightforward proofs. A maximal t -switchable set S is not empty. For all $a \in N \setminus S$, $S \cup \{a\}$ is not t -switchable. For every $i \in \{1, \dots, n\}$ and every $u \in [r_i]$ there exists $b \in S$ such that $b_i = u$.

Proposition 4.7. *Let S be a maximal t -switchable set in which all elements of S are pairwise connected. Then $S = N$.*

Proof. Certainly $S \subseteq N$, and both S and N are t -switchable. We first prove that $P_N^{(t)} \subseteq P_S^{(t)}$. Let $f_{i,a,b} \in P_N^{(t)}$, with $i \leq t$. If $a, b \in S$, then using that S is t -switchable and that all of its elements are connected, $f_{i,a,b} \in P_S^{(t)}$. So we may assume that either $a \notin S$ or $b \notin S$ and similarly that either $s(i, a, b) \notin S$ or $s(i, b, a) \notin S$. Hence $f_{i,a,b} \in \text{Var}_S^{(t)} \subseteq P_S^{(t)}$. Thus $P_N^{(t)} \subseteq P_S^{(t)}$, and by maximality of S , $S = N$. \square

Example 4.8. The 1-switchable set given in Example 3.6 is not maximal as it is a single connected component, but is not all of N .

Lemma 4.9. *For $a_0, a_1, c \in N$ and $i \in [n]$,*

$$x_{a_1} f_{i,a_0,c} - x_c f_{i,a_0,a_1} = x_{s(i,a_0,a_1)} f_{i,s(i,a_1,a_0),c} - x_{s(i,c,a_0)} f_{i,a_1,s(i,a_0,c)}.$$

In particular, if a_0 and a_1 differ at most in the i th component and one extra component, then

$$x_{a_1} f_{i,a_0,c} - x_{s(i,a_0,a_1)} f_{i,s(i,a_1,a_0),c} = -x_{s(i,c,a_0)} f_{i,a_1,s(i,a_0,c)} \in (f_{i,a,b} : a, b \in N, d(a,b) = 2).$$

More generally, let $a_0, a_1, \dots, a_k, c \in N$, and assume that for all $j = 1, \dots, k$, a_{j-1} and a_j differ at most in components l_j and i . Then

$$x_{a_1} x_{a_2} \cdots x_{a_k} f_{i, a_0, c} - x_{s(i, a_0, a_1)} x_{s(i, a_1, a_2)} \cdots x_{s(i, a_{k-1}, a_k)} f_{i, s(i, a_k, a_0), c}$$

is in $\sum_{j=1}^k (f_{i, a, b} : a, b \in N, d(a, b) = 2, a_{l_j} \neq b_{l_j})$. In particular, if a_k and c differ at most in components i and l_0 , then

$$x_{a_1} x_{a_2} \cdots x_{a_k} f_{i, a_0, c} \in \sum_{j=0}^k (f_{i, a, b} : a, b \in N, d(a, b) = 2, a_{l_j} \neq b_{l_j}).$$

Proof. The first statement is straightforward rewriting:

$$\begin{aligned} & x_{a_1} f_{i, a_0, c} - x_c f_{i, a_0, a_1} - x_{s(i, a_0, a_1)} f_{i, s(i, a_1, a_0), c} \\ &= -x_{a_1} x_{s(i, a_0, c)} x_{s(i, c, a_0)} + x_{s(i, a_0, a_1)} x_{s(i, a_1, a_0)} x_c \\ &\quad - x_{s(i, a_0, a_1)} x_c x_{s(i, a_1, a_0)} + x_{s(i, a_0, a_1)} x_{s(i, c, s(i, a_1, a_0))} x_{s(i, s(i, a_1, a_0), c)} \\ &= x_{s(i, a_0, a_1)} x_{s(i, c, s(i, a_1, a_0))} x_{s(i, s(i, a_1, a_0), c)} - x_{a_1} x_{s(i, a_0, c)} x_{s(i, c, a_0)} \\ &= x_{s(i, a_0, a_1)} x_{s(i, c, a_0)} x_{s(i, a_1, c)} - x_{a_1} x_{s(i, a_0, c)} x_{s(i, c, a_0)} \\ &= (x_{s(i, a_0, a_1)} x_{s(i, a_1, c)} - x_{a_1} x_{s(i, a_0, c)}) x_{s(i, c, a_0)} \\ &= x_{s(i, c, a_0)} (-f_{i, a_1, s(i, a_0, c)}). \end{aligned}$$

If $d(a_0, a_1) \leq 2$, then $f_{i, a_0, a_1} \in (f_{i, a, b} : a, b \in N, d(a, b) = 2)$ and a_1 and $s(i, a_0, c)$ still differ at most in the two components i and l , so that $f_{i, a_1, s(i, a_0, b)} \in (f_{i, a, b} : a, b \in N, d(a, b) = 2)$.

We prove the next display by induction. There is nothing to do if $k = 0$, and the previous part is the base case $k = 1$. Now suppose that $k > 1$. Then

$$\begin{aligned} & x_{a_1} x_{a_2} \cdots x_{a_k} f_{i, a_0, c} - x_{s(i, a_0, a_1)} x_{s(i, a_1, a_2)} \cdots x_{s(i, a_{k-1}, a_k)} f_{i, s(i, a_k, a_0), c} \\ &= x_{a_k} (x_{a_1} x_{a_2} \cdots x_{a_{k-1}} f_{i, a_0, c} - x_{s(i, a_0, a_1)} \cdots x_{s(i, a_{k-2}, a_{k-1})} f_{i, s(i, a_{k-1}, a_0), c}) \\ &\quad + x_{s(i, a_0, a_1)} \cdots x_{s(i, a_{k-2}, a_{k-1})} (x_{a_k} f_{i, s(i, a_{k-1}, a_0), c} - x_{s(i, a_{k-1}, a_k)} f_{i, s(i, a_k, a_0), c}). \end{aligned}$$

By induction on k the two parenthesized binomials are in $\sum_{j=1}^k (f_{i, a, b} : a, b \in N, d(a, b) = 2, a_{l_j} \neq b_{l_j})$, which finishes the proof for k . The last expression follows from the fact that $d(s(i, a_k, a_0), c) = d(a_k, c) = 2$. \square

Remark 4.10. Let S be a t -switchable set and $\mathbf{x}_S = \prod_{a \in S} x_a$. Define

$$I_S^{(t)} = (f_{i, a, b} \mid i \in [t], a, b \text{ connected in } S, d(a, b) = 2).$$

Discussion 2.3 and Lemma 4.3 imply that $\tilde{I}_S^{(t)} = I_S^{(t)} : \mathbf{x}_S^\infty$. (This was previously known only for $S = N$.) Thus the unique smallest binomial prime ideal that contains $I_S^{(t)}$ and that contains no monomials equals $\tilde{I}_S^{(t)}$. For results that follow we need the stronger fact that $\tilde{I}_S^{(t)} = I_S^{(t)} : \mathbf{x}_S$. This follows easily from Lemma 4.9. If $f_{i, a, b} \in \tilde{I}_S^{(t)}$, then $a, b \in S$ are t -connected and $i \in [t]$. By the definition of connectedness, there exist elements $a = a_0, a_1, \dots, a_k \in S$

such that for all $j = 1, \dots, k$, a_{j-1} and a_j differ only in one position, and $d(a_k, b) = 2$. By possibly modifying the i th components and the a_j , we may assume that a_k and b differ in the i th component and a_1, \dots, a_k are distinct. By Lemma 4.9, $x_{a_1} \cdots x_{a_k} f_{i,a,b} \in I^{(t)}$, and $x_{a_1} \cdots x_{a_k} \notin P_S^{(t)}$, since $a_j \in S$ for $1 \leq j \leq k$. Therefore $\mathbf{x}_S f_{i,a,b} \in I_S^{(t)}$ for all $f_{i,a,b} \in \tilde{I}_S^{(t)}$.

Lemma 4.11. *If P is a prime ideal minimal over $I^{(t)}$, then $P = P_S^{(t)}$ for some maximal t -switchable set S .*

Proof. We first note that $\tilde{I}^{(t)} = P_N^{(t)}$, and this is a prime ideal by Theorem 4.4.

Now let P be an arbitrary prime ideal minimal over $I^{(t)}$. Let S be the set of all $a \in N$ such that $x_a \notin P$. We know that S is not empty, for otherwise P is the ideal generated by all the variables, which properly contains the already established minimal prime ideal $\tilde{I}^{(t)}$.

Let $a, b \in S$ have $d(a, b) = 2$ and $a_i \neq b_i$ for some $i \in [t]$. Since P contains $I^{(t)}$ and $i \in [t]$, P contains $f_{i,a,b} = x_a x_b - x_{s(i,b,a)} x_{s(i,a,b)}$. Since $a, b \in S$, then $x_a x_b \notin P$, so that necessarily $x_{s(i,b,a)} x_{s(i,a,b)} \notin P$, and hence $s(i, b, a), s(i, a, b) \in S$. This proves that S is t -switchable, and so by Proposition 4.2, $I^{(t)} \subseteq P_S^{(t)}$, and by Theorem 4.4, $P_S^{(t)}$ is a prime ideal.

We next prove that $P_S^{(t)} \subseteq P$. By the construction of S , $\text{Var}_S^{(t)} \subseteq P$. Let $f_{i,a,b} \in \tilde{I}_S^{(t)}$, with $i \in [t]$ and a and b connected in S . By the definition of connectedness, there exist elements $a_0 = a, a_1, \dots, a_k \in S$ such that for all $j = 1, \dots, k$, a_{j-1} and a_j differ only in one component, and $d(a_k, b) = 2$. By Lemma 3.5, we may choose such a path so that a_k and b differ in the i th component. Then by Lemma 4.9, $x_{a_1} \cdots x_{a_k} f_{i,a,b} \in \tilde{I}_S^{(t)} \subseteq P$, and since $x_{a_j} \notin P$, it follows that $f_{i,a,b} \in P$, as desired. Thus $I^{(t)} \subseteq P_S^{(t)} \subseteq P$. Since $P_S^{(t)}$ is a prime ideal, by minimality of P , $P_S^{(t)} = P$.

Finally, let T be t -switchable and properly containing S . Then $\text{Var}_T^{(t)} \subsetneq \text{Var}_S^{(t)}$, so that $P_S^{(t)} \not\subseteq P_T^{(t)}$. By Proposition 4.2, $P_T^{(t)}$ contains $I^{(t)}$, by Lemma 4.3, $P_T^{(t)}$ is a prime ideal, and this combined with the fact that $P = P_S^{(t)}$ is minimal over $I^{(t)}$, implies that $P_T^{(t)} \not\subseteq P_S^{(t)}$. Therefore $P_S^{(t)}$ and $P_T^{(t)}$ are incomparable. Thus S is a maximal t -switchable set. \square

Lemma 4.12. *Let S be a maximal t -switchable set. Then $P_S^{(t)}$ is a minimal associated prime ideal of $I^{(t)}$.*

Proof. By Proposition 4.2, $P_S^{(t)}$ is a prime ideal containing $I^{(t)}$. Let P be a prime ideal contained in $P_S^{(t)}$ and minimal over $I^{(t)}$. By Lemma 4.11, $P = P_T^{(t)}$ for some maximal t -switchable set T . Since $P_T^{(t)} \subseteq P_S^{(t)}$, necessarily $\text{Var}_T^{(t)} \subseteq \text{Var}_S^{(t)}$, so that $S \subseteq T$. But then comparability of $P_S^{(t)}$ and $P_T^{(t)}$ and maximality of S force $S = T$. \square

These two lemmas prove:

Theorem 4.13. *The set of prime ideals minimal over $I^{(t)}$ equals the set of ideals of the form $P_S^{(t)}$ as S varies over the maximal t -switchable sets. \square*

Corollary 4.14. *Let S be a t -switchable set (maximal or not) such that $P_S^{(t)}$ is associated to $I^{(t)}$. Then $\tilde{I}_S^{(t)}$ is contained in the $P_S^{(t)}$ -primary component of $I^{(t)}$.*

Proof. Let $f_{i,a,b} \in \widetilde{I}_S^{(t)}$. So $a, b \in S$ are connected and $i \in [t]$. By Remark 4.10, $(\prod_{c \in S} x_c) f_{i,a,b} \in I^{(t)}$. By construction, $\prod_{c \in S} x_c^m \notin P_S^{(t)}$ for any m and hence $f_{i,a,b}$ is in the $P_S^{(t)}$ -primary component of $I^{(t)}$. \square

We note that Corollary 4.14 holds for binomial ideals in characteristic 0 in general by Eisenbud–Sturmfels [4, Theorem 7.1]. Our results for the specific binomial ideals $I^{(t)}$ are independent of the characteristic.

In the rest of the section we present some atypical behavior for $t = 1$. The next lemma helps connect maximal 1-switchable sets to the admissible bipartite graphs in [5].

Lemma 4.15. *Let $o, o' \in N$ differ at most in the first component. Then $I^{(1)} : x_o = I^{(1)} : x_{o'}$.*

Proof. By symmetry it suffices to prove that $I^{(1)} : x_o \subseteq I^{(1)} : x_{o'}$. Since $x_o(I^{(1)} : x_o) = I^{(1)} \cap (x_o) = (I^{(1)} \cdot Y + (x_o(Y - 1)))R[Y] \cap R$ for a variable Y , it follows that $I^{(1)} : x_o$ is generated by binomials of the form $g = x_{a_1} \cdots x_{a_r} - x_{b_1} \cdots x_{b_r}$. Note that $x_o g$ being in $I^{(1)}$ is the same as saying that there exists a sequential rewriting of o, a_1, \dots, a_r into o, b_1, \dots, b_r (after possibly reindexing b_1, \dots, b_r), such that at each step, only two indices change by switching their first components. This means that for each $i = 1, \dots, r$, a_i and b_i differ at most in the first component. Thinking of o as being in the 0th place on the original list o, a_1, \dots, a_r , and a_i in the i th place, we record each step in the rewriting process as a transposition (i, j) when we switch the first components of the i th and j th indices.

Let w be the composition of all these transpositions. We proceed by induction on the number of transpositions in w .

If the last transposition in w does not include 0, then the composition of all but the last transposition of w takes o, a_1, \dots, a_r to o, c_1, \dots, c_r for some c_1, \dots, c_r , and the remaining transposition in w takes o, c_1, \dots, c_r to o, b_1, \dots, b_r . This says that $x_{a_1} \cdots x_{a_r} - x_{c_1} \cdots x_{c_r} \in I^{(1)} : x_o$ and $x_{c_1} \cdots x_{c_r} - x_{b_1} \cdots x_{b_r} \in I^{(1)} : x_o$, and by induction on the number of transpositions in w , $x_{a_1} \cdots x_{a_r} - x_{c_1} \cdots x_{c_r} \in I^{(1)} : x_{o'}$ and $x_{b_1} \cdots x_{b_r} - x_{c_1} \cdots x_{c_r} \in I^{(1)} : x_{o'}$. Hence $x_{a_1} \cdots x_{a_r} - x_{b_1} \cdots x_{b_r} \in I^{(1)} : x_{o'}$. Thus we may assume that the last transposition in w includes 0.

With this set-up, we now perform the transpositions in w on o', a_1, \dots, a_r . If we reduce in this way to o', b_1, \dots, b_r , we have that $g \in I^{(1)} : x_{o'}$ as desired. Considering that w takes o, a_1, \dots, a_r to o, b_1, \dots, b_r , if o', a_1, \dots, a_r does not become o', b_1, \dots, b_r , w must take it to $o, b_1, \dots, b_{k-1}, s(1, b_k, o'), b_{k+1}, \dots, b_r$ for some $k \in \{1, \dots, r\}$ where $b_{k1} = o_1$ and k is such that in the transpositions involving o, o_1 lands in the first entry of b_k .

Let z be the composition of precisely those first consecutive transpositions by which the first component o_1 of o arrives in its final place in the k th position. Since the last transposition in w must involve 0, and since b_k has the same first component as o , by minimality the last transposition does not involve k . Therefore $z \neq w$. Note that $z^{-1} \circ w$ takes o, a_1, \dots, a_r to o, c_1, \dots, c_r , whereby the first component o_1 of o in o, a_1, \dots, a_r remains in the 0th position in o, c_1, \dots, c_r . Therefore $z^{-1} \circ w$ takes o', a_1, \dots, a_r to o', c_1, \dots, c_r , so that $x_{a_1} \cdots x_{a_r} - x_{c_1} \cdots x_{c_r} \in I^{(1)} : x_{o'}$. Furthermore, z takes o, c_1, \dots, c_r to o, b_1, \dots, b_r and since z has strictly fewer steps than w , $x_{c_1} \cdots x_{c_r} - x_{b_1} \cdots x_{b_r} \in I^{(1)} : x_{o'}$ by induction. Therefore $x_{a_1} \cdots x_{a_r} - x_{b_1} \cdots x_{b_r} \in I^{(1)} : x_{o'}$, which proves that $I^{(1)} : x_o = I^{(1)} : x_{o'}$. \square

In an earlier version of our paper we included a similar (but more complicated) proof that $I^{(1)} : x_o x_{o'} = I^{(1)} : x_o$, which implies that $I^{(1)}$ is radical. For brevity we omit our complicated proof because recently Ay and Rauh [1] proved, in a more straightforward way, that $I^{(1)}$ is radical for all n via the square-free nature of the leading terms of a Gröbner basis.

Lemma 4.16. *Let S be a maximal 1-switchable set. Then for all $a \in S$ and all $b \in N$, $s(1, a, b) \in S$. In other words, any element in N that differs from an element in S in at most the first component is also in S .*

Proof. Let $a \in S$ and $b \in N$. Then $c = s(1, a, b)$ differs from a in at most the first component. By Lemma 4.15, $I^{(1)} : x_a = I^{(1)} : x_c$. Since $a \in S$, it follows that $x_a \notin P_S^{(t)}$, so that for all positive integers m , $I^{(1)} : x_c^m = I^{(1)} : x_a^m$ is contained in $P_S^{(t)}$. Hence $I^{(1)} : x_c^\infty = I^{(1)} : x_a^\infty \subseteq P_S^{(t)}$ and therefore, $x_c \notin P_S^{(t)}$. Hence $c \in S$. \square

Remark 4.17. (Connection with admissible graphs in Fink [5].) Let S be a maximal 1-switchable set. Lemma 4.16 shows that the first component is unrestricted in each connected component of S . Let $S = S_1 \cup \dots \cup S_l$ be a partition of S into connected component. We prove in the two paragraphs below that when $n = 3$, $S_i = [r_1] \times S_{i2} \times S_{i3}$ where $S_{ij} \cap S_{kj} = \emptyset$ for all $i \neq k$ and both $j = 2, 3$. Therefore, each connected component corresponds to the complete bipartite graph $S_{i2} \times S_{i3}$, which is exactly Fink's representation in [5].

We first argue that each connected component S_i has the form $[r_1] \times S_{i2} \times S_{i3}$. Let $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3) \in S_i$. It suffices to prove that $s(2, b, a)$, $s(3, b, a)$, $s(2, a, b)$, and $s(3, a, b) \in S_i$. By Lemma 4.16 it suffices to consider the case where $a_1 = b_1$, so that $d(a, b) \leq 2$. If $d(a, b) = 1$ then each switch is either a or b , and the conclusion follows. Now assume that $d(a, b) = 2$. Suppose that $s(2, b, a) \notin S$. Let $T = S \cup ([r_1] \times \{a_2\} \times \{b_3\})$. We prove that T is 1-switchable. Let $e, e' \in T$ satisfy $d(e, e') = 2$ and $e_1 \neq e'_1$. By symmetry it suffices to prove that $s(1, e, e') \in T$. If $e \in S$, then $s(1, e, e') \in S \subseteq T$ by Lemma 4.16, and if $e \in T \setminus S$, then $s(1, e, e') \in T$ by the definition of T . Thus T is 1-switchable. Using Lemma 4.16 it is also easy to see that $P_T^{(t)} \subsetneq P_S^{(t)}$, which contradicts the maximality of S . This proves that $s(2, b, a) \in S$, and since it is connected to b , it is in S_i . Analogous proofs show that $s(3, a, b)$, $s(2, a, b)$, $s(3, b, a)$ are in S_i . This proves that each connected component can be written in a "block form" $S_i = [r_1] \times S_{i2} \times S_{i3}$. (By Example 3.6, arbitrary t -switchable sets need not have a block form.)

Now suppose that $S_{i2} \cap S_{j2} \neq \emptyset$ for some distinct i, j . Let $a_2 \in S_{i2} \cap S_{j2}$. Then by Lemma 4.16 there exist $a = (a_1, a_2, a_3) \in S_i$ and $b = (a_1, a_2, b_3) \in S_j$. However, since $d(a, b) = 1$, a and b are connected, which is a contradiction since they are in distinct connected components. Therefore $S_{i2} \cap S_{j2} = \emptyset$. Similarly, $S_{i3} \cap S_{j3} = \emptyset$.

5. Prime Components for $t = n$

In this section we prove that the minimal components of $I^{(n)}$ are all prime ideals. Ay and Rauh [1] prove that $I^{(1)}$ is radical, so that all the components of $I^{(1)}$ are minimal and prime ideals. In Section 7 we show by example that $I^{(n)}$ may have embedded, and thus non-prime,

components. We consider it an interesting question to determine if the minimal components of $I^{(t)}$ are prime for all t (for examples, see Section 7).

For all t , by Theorem 4.13, every prime ideal minimal over $I^{(t)}$ is of the form $P_S^{(t)} = \text{Var}_S^{(t)} + \widetilde{I}_S^{(t)}$ for some maximal t -switchable set S . By Corollary 4.14, the binomial portion of the $P_S^{(t)}$ -primary component of $I^{(t)}$ is $\widetilde{I}_S^{(t)}$. Thus to prove that the minimal components of $I^{(n)}$ are prime, it suffices to prove that $\text{Var}_S^{(n)}$ is contained in the $P_S^{(n)}$ -primary component.

We first prove that the connected components of n -switchable sets can be given in a block form. By Remark 4.17, when $t = 1$ and $n = 3$, we also have a block form, but an arbitrary t -switchable set need not have it (see Example 3.6).

Proposition 5.1. *Let S be an n -switchable subset of N . Let S_1, \dots, S_l be the connected components. Then the following hold:*

- (1) *Each S_i is of the form $S_i = S_{i1} \times \dots \times S_{in}$ for some $S_{ij} \subseteq [r_j]$, $j = 1, \dots, n$.*
- (2) *If $a \in S_i$, $b \in N$, $j \in [n]$, and $s(j, a, b) \in S$, then $s(j, a, b) \in S_i$.*
- (3) *If i, j are distinct in $[l]$, there exist distinct $p_1, \dots, p_m \in [n]$ with $m \geq 3$ such that for all $k = 1, \dots, m$, $S_{ip_k} \cap S_{jp_k} = \emptyset$.*

Proof. For (1) it suffices to prove that whenever a and b are connected in S , then for all $K \subseteq [n]$, $s(K, a, b)$ is in S and connected to a and b . But this is precisely Lemma 3.5.

For (2), if $a \in S_i$, $b \in S$ such that $s(j, a, b) \in S$, then $d(a, s(j, a, b)) = 1$ so a and $s(j, a, b)$ are connected and therefore $s(j, a, b) \in S_i$.

Suppose that condition (3) fails. By possibly reindexing, $S_{1i} \cap S_{2i} \neq \emptyset$ for $i \geq 3$. By the block decomposition from (1), there exist $a \in S_1$, $b \in S_2$ such that $a_i = b_i$ for $i \geq 3$. Thus $1 \leq d(a, b) \leq 2$. But then by Lemma 3.4, a and b are connected, so they are both in $S_1 \cap S_2$, giving a contradiction. This proves (3). \square

By Lemma 4.3, when we think of elements of N as $(t+1)$ -tuples rather than n -tuples by reindexing $[r_{t+1}] \times \dots \times [r_n]$ by $[r_{t+1} \dots r_n]$, since $I^{(n-1)} = I^{(n)}$, we have that the t -switchable sets in general have a block form. However, when the last $n-t$ components are spelled out explicitly, there is not necessarily a block form; see Example 3.6.

Set the **distance** between connected components S_i and S_j to be the number of indices k such that $S_{ik} \cap S_{jk} = \emptyset$. We denote this distance as $d(S_i, S_j)$, just as for elements of N . Thus, part (3) of the theorem above proves that $d(S_i, S_j) \geq 3$. We note that the decomposition does not require all (coordinate) components of two connected components to be disjoint or equal, but for each pair S_i, S_j at least three have to be disjoint.

Lemma 5.2. *If $S \subseteq N$ is a maximal n -switchable set with connected components S_1, \dots, S_k , then for any connected component S_j there exists another connected component S_l , such that $d(S_j, S_l) = 3$.*

Proof. Without loss of generality $j = 1$. By Proposition 5.1 (3), for all $l \neq 1$, $d(S_1, S_l) \geq 3$. Suppose for contradiction that for all $l \neq 1$, $d(S_1, S_l) \geq 4$. By possibly reindexing the

components, $S_{11} \neq [r_1]$. Let $u \in [r_1] \setminus S_{11}$. Set $S'_1 = (S_{11} \cup \{u\}) \times S_{12} \times \cdots \times S_{1n}$, and $S' = S'_1 \cup S_2 \cup \cdots \cup S_k$. We argue that S' is n -switchable. Let $a, b \in S'$ such that $d(a, b) = 2$. For all $l \neq 1$, $d(S_1, S_l) \geq 4$, so that $d(S'_1, S_l) \geq 3$. By Proposition 5.1 (3), $d(S_i, S_l) \geq 3$ for all distinct i, l . Hence a and b must either both be in S'_1 or they must both be in S_i for some $i \geq 1$. By the structure of the connected components from Proposition 5.1, and by the definition of S'_1 , all the appropriate switches are contained in S' . Hence S' is still switchable. Since $S \subset S'$, $\text{Var}_{S'}^{(n)} \subsetneq \text{Var}_S^{(n)}$. Let $f_{i,a,b} \in \tilde{I}_{S'}^{(n)}$. If $a, b \in S$, then $f_{i,a,b} \in \tilde{I}_S^{(n)}$. Without loss of generality, suppose $a \notin S$. Then $a_1 = u$ and $x_a \in \text{Var}_S^{(n)}$. Hence the first component of at least one of $s(i, a, b)$ or $s(i, b, a)$ is u , so that at least one of $s(i, a, b)$ or $s(i, b, a)$ is not in S . Thus at least one of $x_{s(i,a,b)}$ or $x_{s(i,b,a)}$ is in $\text{Var}_S^{(n)}$, which implies that $f_{i,a,b} \in \text{Var}_S^{(n)} \subseteq P_{S'}^{(n)}$. Therefore $P_S^{(n)}$ properly contains $P_{S'}^{(n)}$, which contradicts the maximality of S . \square

Lemma 5.3. *Let S be a maximal n -switchable set and $a \in N \setminus S$.*

- (1) *There exists $b \in S$ such that $d(a, b) \leq 2$.*
- (2) *For and $b \in S$ satisfying 1., and any i such that $a_i \neq b_i$, there exists $c \in S$ such that b and c are not connected, and $c_i = a_i$.*
- (3) *For any b and c and i from 2., there exists some $d \in S$ connected to b , such that $d(c, d) = 3$, and $d_i = b_i \neq a_i = c_i$.*
- (4) *For any b, c, d , and i as in 3., let j be such that $j \neq i$ and $d_j \neq c_j$. There exists a path $d = e_0, e_1, \dots, e_k = b$ in S such that $x_a x_{e_1} \cdots x_{e_k} f_{i,b,c} f_{j,d,c} \in I^{(n)}$.*

Proof. Suppose that $d(a, b) \geq 3$ for all $b \in S$. Set $T = S \cup \{a\}$. The new element a in T is not connected in T with any other element of T , so that T is t -switchable, and if $c, d \in S$ are connected in T , they must be connected in S . But then $P_S^{(t)}$ properly contains $P_T^{(t)}$, which contradicts the maximality of S . Hence there must exist $b \in S$ such that $d(a, b) \leq 2$. This proves 1.

Choose $b \in S$ such that $d(a, b) \leq 2$. Let S_b be the connected component of S containing b . By Proposition 4.7, $S_b \neq S$. By Proposition 5.1, $S_b = S_{b1} \times \cdots \times S_{bn}$, and since $a \notin S$, there exists $i \in [n]$ such that $a_i \notin S_{bi}$. Thus $a_i \neq b_i$.

By Lemma 5.2 there exists a connected component T of S such that $d(T, S_b) = 3$. Suppose that for all such T , either $a_i \notin T_i$ or $T_i \cap S_{bi} \neq \emptyset$. Build S'_b from S_b by replacing S_{bi} with $S'_{bi} = S_{bi} \cup \{a_i\}$, and build S' by replacing S_b in S with S'_b . We prove next that S' is n -switchable and that its connected components are the connected components of S except with S'_b taking the place of S_b . Let U be any connected component of S with $U \neq S_b$. If $d(U, S_b) \geq 4$, then $d(U, S'_b) \geq 3$. If instead $d(U, S_b) < 4$, then by Proposition 5.1, $d(U, S_b) = 3$. Hence by assumption either $a_i \notin U_i$ or $U_i \cap S_{bi} \neq \emptyset$. If $a_i \notin U_i$, then we still have $d(U, S'_b) = 3$. If instead $U_i \cap S_{bi} \neq \emptyset$, then $d(U, S'_b) = d(U, S_b) = 3$. Therefore S' is still n -switchable. We next prove that $P_S^{(n)}$ properly contains $P_{S'}^{(n)}$. Since $S \subsetneq S'$, $\text{Var}_{S'}^{(n)} \subsetneq \text{Var}_S^{(n)}$. Let $f_{j,e,o} \in \tilde{I}_{S'}^{(n)}$. Thus $e, o, s(j, e, o), s(j, o, e) \in S'$. If $e, o \in S$, then by the block structure e, o are connected in S , and hence $f_{j,e,o} \in P_S^{(n)}$. Otherwise either e or o is not in S . Without loss of generality,

say $e \notin S$. Then $e \in S'_b \setminus S_b$ and $e_i = a_i$. Hence either $s(j, e, o)$ or $s(j, o, e)$ has the i th component equal to a_i , so that by the block structure of S , either $s(j, e, o)$ or $s(j, o, e)$ is not in S . But then $f_{j,e,o} \in \text{Var}_{S'}^{(n)} \subseteq P_S^{(n)}$. This proves that $P_{S'}^{(n)} \subsetneq P_S^{(n)}$, which contradicts the maximality of S .

Thus there exists a connected component T of S such that $d(T, S_b) = 3$, $a_i \in T_i$, and $T_i \cap S_{bi} = \emptyset$. Let $J = \{j \in [n] : T_j \cap S_{bj} \neq \emptyset\}$. Choose $c \in T$ such that $c_i = a_i$ and $c_j \in T_j \cap S_{bj}$ for all $j \in J$. Similarly, choose $d \in S_b$ such that $d_i = b_i$ and $d_j \in T_j \cap S_{bj}$ for all $j \in J$. Since $d(T, S_b) = 3$ and $T_i \cap S_{bi} = \emptyset$, we have that $d(c, d) = 3$. This proves 2. and 3.

Let $d = e_0, e_1, \dots, e_l = b$ be a path from d to b in S . Then by n -switchability of S , $d = e_0, s(i, e_1, b), s(i, e_2, b), \dots, s(i, e_{l-1}, b), s(i, e_l, b) = b$ is, after omitting repetitions, a path from d to b in S in which all i th components in the path are the same. We rename this new path $d = e_0, e_1, \dots, e_k = b$. By Lemma 4.9, $x_{e_1} \cdots x_{e_k} f_{i,a,d} \equiv x_{e_0} \cdots x_{e_{k-1}} f_{i,a,b} \pmod{I^{(n)}}$. Since $d(a, b) = 2$, $f_{i,a,b} \in I^{(n)}$, so that $x_{e_1} \cdots x_{e_k} f_{i,a,d} \in I^{(n)}$.

The first expression of Lemma 4.9 (with a_0, a_1, c replaced by c, a, d) says that $x_a f_{i,c,d} - x_d f_{i,c,a} = x_{s(i,c,a)} f_{i,s(i,a,c),d} - x_{s(i,d,c)} f_{i,a,s(i,c,d)}$. Since $a_i = c_i$, this simplifies to $x_a f_{i,c,d} = x_c f_{i,a,d} - x_{s(i,d,a)} f_{i,a,s(i,c,d)}$. By the previous paragraph, $x_{e_1} \cdots x_{e_k} f_{i,a,d} \in I^{(n)}$, so that $x_{e_1} \cdots x_{e_k} x_a f_{i,c,d} \equiv -x_{e_1} \cdots x_{e_k} x_{s(i,d,a)} f_{i,a,s(i,c,d)} \pmod{I^{(n)}}$. Now $d(s(i, d, c), c) = 2$, and $s(i, d, c)$ and c differ in component j , so by the last expression in Lemma 4.9, $x_{s(i,d,a)} f_{j,d,c} = x_{s(i,d,c)} f_{j,d,c} \in I^{(n)}$. Thus modulo $I^{(n)}$, $x_{e_1} \cdots x_{e_k} x_a f_{i,c,d} f_{j,d,c} \equiv -x_{e_1} \cdots x_{e_k} x_{s(i,d,a)} f_{i,a,s(i,c,d)} f_{j,d,c}$, but the latter is in $I^{(n)}$, which finishes the proof. \square

Remark 5.4. The proof that there exists $b \in S$ such that $d(a, b) \leq 2$ only requires that S be a maximal t -switchable set (any t will do).

Theorem 5.5. *The minimal components of $I^{(n)}$ are prime ideals.*

Proof. Let Q be a minimal component for $I^{(n)}$. By Theorem 4.13, the corresponding associated prime P is of the form $P_S^{(n)}$ for some maximal n -switchable set S . By Corollary 4.14, $\tilde{I}_S^{(n)} \subseteq Q$. It remains to prove that $\text{Var}_S^{(n)} \subseteq Q$.

Let \mathbf{f}_S be the product of those elements $f_{i,a,b} \in \tilde{I}^{(n)}$ that are not in $P_S^{(n)}$, and let $\mathbf{x}_S = \prod_{a \in S} x_a$, and $\rho_S = \mathbf{f}_S \mathbf{x}_S$. Since $P_S^{(n)}$ is a prime ideal, it follows that $\rho_S \notin P_S^{(n)}$.

Let $x_a \in \text{Var}_S^{(n)}$. Then $a \notin S$. Choose b, c, d, e_1, \dots, e_k as in Lemma 5.3 with the product $x_a x_{e_1} \cdots x_{e_k} f_{i,b,c} f_{j,d,c} \in I^{(n)}$. Since $b, c, d, e_1, \dots, e_k \in S$ and c is not connected either to b or to d , it follows that $x_{e_1} \cdots x_{e_k} f_{i,b,c} f_{j,d,c} \notin P_S^{(n)}$. Hence $x_{e_1} \cdots x_{e_k} f_{i,b,c} f_{j,d,c}$ divides ρ_S , and therefore $x_a \in I^{(n)} : \rho_S \subseteq Q : \rho_S$. Since Q is $P_S^{(n)}$ -primary and $\rho_S \notin P_S^{(n)}$, it follows that $Q : \rho_S = Q$, so that $x_a \in Q$. This proves that $\text{Var}_S^{(n)} \subseteq Q$, so that $P_S^{(n)} = \tilde{I}_S^{(n)} + \text{Var}_S^{(n)} \subseteq Q$, which finishes the proof. \square

Remark 5.6. We note that ρ_S , from the previous proof, is a product of more factors than absolutely necessary. For example, when $S = N$ and $n = 3$, then $I^{(n)} : \prod_{k=0}^{r_3} x_{0,0,k} = \tilde{I}_N^{(n)} = P_N^{(n)}$, so the proper factor $\prod_{k=0}^{r_3} x_{0,0,k}$ of ρ_N suffices. Furthermore, we note that the proof of Theorem 5.5 shows which diagonal minor factors p of ρ_S are necessary to argue $I^{(n)} : p \mathbf{x}_S = P_S^{(n)}$. These p are not necessarily unique.

6. Gröbner Bases

In this section we give a Gröbner basis for $\tilde{I}_S^{(t)}$ in many monomial orders including the reverse lexicographic and the lexicographic order under appropriate orderings of the variables. This generalizes the work of [2] and [10].

Recall that Caniglia, Guccione and Guccione [2] proved that the $r \times r$ minors of a generic $m \times n$ matrix form a Gröbner basis with respect to any “diagonal order”, where diagonal orders on a two-dimensional matrix are monomial orders such that for any $r \times r$ submatrix, the product of the variables on the main diagonal is the leading term of the minor of that submatrix. We start with an example which illustrates some of the subtleties that arise in extending the notion of diagonal orders for matrices to hypermatrices. One 2×2 minor of the $2 \times 2 \times 2$ hypermatrix is $x_{1,1,2}x_{2,2,1} - x_{1,2,2}x_{2,1,1}$. We might order monomials by comparing the first two components in the indices and following the notion of diagonal orders from [2], so that $x_{1,1,2}x_{2,2,1} > x_{1,2,2}x_{2,1,1}$. However, we get a different order from comparing the last two components in the indices, as the inequality reverses. Therefore, a generalization of a diagonal order to an n -dimensional hypermatrix must come equipped with a further prioritization of the components.

Definition 6.1. *We treat the last $n - t$ components $[r_{t+1}] \times \cdots \times [r_n]$ of N as one component, so essentially we assume that $t = n - 1$. (This also covers the case $t = n$.) When we write $s(K, a, b)$ in this sense and $t + 1 \in K$, we actually mean $s([t] \setminus K, b, a)$ in the usual sense. Let $\{\delta_1, \dots, \delta_{t+1}\} = [t + 1]$. A **t-diagonal order** on R relative to the enumeration $\delta_1, \dots, \delta_{t+1}$ is any monomial order $<$ with the following property: for any $a, b \in N$, if i is the smallest index in $[t + 1]$ such that $a_{\delta_i} \neq b_{\delta_i}$, and if $j > i$ such that $a_{\delta_j} > b_{\delta_j}$, then $x_a x_b > x_{s(\delta_j, a, b)} x_{s(\delta_j, b, a)}$ if and only if $a_{\delta_i} > b_{\delta_i}$.*

For example, the lexicographic order in which the variables are ordered in the lexicographic order on their indices $[r_1] \times \cdots \times [r_t] \times [r_{t+1} \cdots r_n]$ with $\delta_i = i$ for $i = 1, \dots, t + 1$ is a t -diagonal order. The degree reverse lexicographic order $x_a > x_b$ if $\sum_{i < j} |a_i - a_j| < \sum_{i < j} |b_i - b_j|$ or if $\sum_{i < j} |a_i - a_j| = \sum_{i < j} |b_i - b_j|$ and $a > b$ in the reverse lexicographic order with $\delta_i = t - i + 2$ is a t -diagonal order. Obviously many other options are possible. When $n = 2$, then any t -diagonal order is a diagonal order for 2×2 minors as given in [2].

Lemma 6.2. *Let $1 \leq t \leq n$, let S be a connected component in some t -switchable set. Set*

$$G = \{f_{K, a, b} : K \subseteq [t], a, b \in S\}.$$

Suppose that $a_1, \dots, a_r, b_1, \dots, b_r \in S$ have the property that for all $i = 1, \dots, t$, up to order, the r -list $a_{1i}, a_{2i}, \dots, a_{ri}$ is the same as the r -list $b_{1i}, b_{2i}, \dots, b_{ri}$, and such that, up to order, the r -list $(a_{1, t+1}, a_{1, t+2}, \dots, a_{1, n}), (a_{2, t+1}, a_{2, t+2}, \dots, a_{2, n}), \dots, (a_{r, t+1}, a_{r, t+2}, \dots, a_{r, n})$ is the same as the r -list $(b_{1, t+1}, b_{1, t+2}, \dots, b_{1, n}), (b_{2, t+1}, b_{2, t+2}, \dots, b_{2, n}), \dots, (b_{r, t+1}, b_{r, t+2}, \dots, b_{r, n})$. Then in any t -diagonal monomial order, $p = x_{a_1} x_{a_2} \cdots x_{a_r} - x_{b_1} x_{b_2} \cdots x_{b_r}$ reduces with respect to G to 0.

Proof. We may assume that p is reduced with respect to G . If $r = 1$ or $n = 1$, necessarily $p = 0$. Now suppose that $r, n > 1$. If the δ_1 entries appearing in a_1, \dots, a_r are all the same,

the same holds for the δ_1 entries in b_1, \dots, b_r , and by induction on n the binomial p reduces to 0 with respect to $\{f_{K,c,d} \in G : c_{\delta_1} = d_{\delta_1}\}$. So we may assume that $a_{1\delta_1} = a_{2\delta_1} = \dots = a_{s\delta_1} > a_{s+1,\delta_1} \geq \dots \geq a_{r\delta_1}$ for some positive $s < r$. If for some $i \in \{2, 3, \dots, t+1\}$, $j \leq s$ and $l > s$, the δ_i entry of a_j is strictly bigger than the δ_i entry of a_l , then $x_{a_1} \cdots x_{a_r}$ is not reduced with respect to G as it can be reduced with respect to $f_{\delta_i, a_j, a_l} \in G$ (recall that in the context of diagonal orders, $f_{t+1, a, b}$ stands for $f_{[t], a, b}$ in the usual sense). But p is assumed reduced, which gives a contradiction. So necessarily for all $i = 2, \dots, t+1$, all the minimal possible δ_i entries of a_1, \dots, a_r appear with correct multiplicities as δ_i entries of a_1, \dots, a_s . Analogously, $b_{1\delta_1} = b_{2\delta_1} = \dots = b_{s\delta_1} = \max\{b_{j\delta_1} : j = 1, \dots, r\} > b_{s+1,\delta_1}, \dots, b_{r\delta_1}$ and for each $i = 2, \dots, t+1$, all the minimal possible δ_i entries of b_1, \dots, b_r appear with correct multiplicities as δ_i entries of b_1, \dots, b_s . Thus $a_1, \dots, a_s, b_1, \dots, b_s$ satisfy the conditions of the lemma, and by necessity $a_{s+1}, \dots, a_r, b_{s+1}, \dots, b_r$ satisfy the conditions of the lemma. By induction on r , up to reindexing, by the reduced assumption, $a_1 = b_1, \dots, a_s = b_s, \dots, a_r = b_r$. Hence $p = 0$. \square

Remark 6.3. *The proof of the theorem above shows that if $t \geq n - 1$, then p reduces to 0 with respect to $G = \{f_{i,a,b} : i \in [n], a, b \in S\}$.*

Theorem 6.4. *Let $1 \leq t \leq n$, and let S be a t -switchable set. Then the set*

$$G = \{f_{K,a,b} : K \subseteq [t], a, b \in S \text{ are connected}\}$$

is a (non-minimal) Gröbner basis for $\tilde{I}_S^{(t)}$ in any t -diagonal monomial order.

Proof. Write $S = S_1 \cup \dots \cup S_r$, where the S_i are the connected components. Then $G = G_1 \cup \dots \cup G_r$, where $G_i = \{f_{K,a,b} : K \subseteq [t], a, b \in S_i\}$.

If $a, b \in S_i$ and $K = \{k_1, \dots, k_l\}$, then $f_{K,a,b} = \sum_{i=1}^l f_{k_i, s(\{k_1, \dots, k_{i-1}\}, a, b), s(\{k_1, \dots, k_{i-1}\}, b, a)}$, so that $G_i \subseteq \tilde{I}_{S_i}^{(t)} \subseteq (G_i)$. It follows that $G \subseteq \tilde{I}_S^{(t)} \subseteq (G)$.

If $f \in G_i, g \in G_j$ and $i \neq j$, then the S-polynomial of f and g trivially reduces to 0 with respect to G because the variables appearing in f are disjoint from the variables appearing in g . Observe that elements of G and S-polynomials of two elements from the same G_i are either 0 or are binomials of the form as in Lemma 6.2, hence they reduce with respect to G to 0, proving that G forms a Gröbner basis of $\tilde{I}_S^{(t)}$. \square

By Remark 6.3, if $t \geq n - 1$, then the smaller set $\{f_{i,a,b} : i \in [n], a, b \in S \text{ are connected}\}$ is a Gröbner basis of $\tilde{I}_S^{(t)}$.

Ha [10] proved the theorem above in the reverse lexicographic order with the lexicographic order on the variables. The theorem above, together with the structure theory of binomial ideals, gives a proof that $\tilde{I}_S^{(t)}$ are prime ideals, but it is more complicated than the proof we gave using flattenings of tensors in Lemma 4.3.

7. Examples

In this section we give examples showing that the primary decomposition structure of the conditional independence ideals $I^{(t)}$ can have embedded components. All computations were performed using the package Binomials [13] in the program Macaulay2 [8].

The first two examples show that $I^{(t)}$ is not radical in general. Thus the ideals $I^{(t)}$ are different from the conditional independence ideals in Herzog et al. [11], Ohtani [15], and Ay and Rauh [1].

Example 7.1. Let $r_1 = r_2 = r_3 = r_4 = 2$ and $t = 3, 4$. This ideal has 26 components, of which 17 are minimal and 9 are embedded. In particular, the maximal ideal, which is $P_\emptyset^{(t)}$, is associated, and contains every $P_S^{(t)}$. However, there are other prime ideals in between the minimal primes and this maximal associated prime. For example,

$$P_S^{(t)} = (x_{(1,1,1,1)}, x_{(1,1,1,2)}, x_{(1,1,2,1)}, x_{(1,1,2,2)}, x_{(1,2,1,1)}, x_{(1,2,1,2)}, x_{(1,2,2,1)}, \\ x_{(2,1,1,2)}, x_{(2,1,2,1)}, x_{(2,1,2,2)}, x_{(2,2,1,1)}, x_{(2,2,1,2)}, x_{(2,2,2,1)}, x_{(2,2,2,2)})$$

contains

$$P_T^{(t)} = (x_{(1,1,1,1)}, x_{(1,1,1,2)}, x_{(1,1,2,1)}, x_{(1,1,2,2)}, x_{(1,2,1,1)}, x_{(1,2,1,2)}, \\ x_{(2,1,2,1)}, x_{(2,1,2,2)}, x_{(2,2,1,1)}, x_{(2,2,1,2)}, x_{(2,2,2,1)}, x_{(2,2,2,2)}),$$

where $S = \{(1, 2, 2, 2), (2, 1, 1, 1)\}$ and $T = \{(1, 2, 2, 1), (1, 2, 2, 2), (2, 1, 1, 1), (2, 1, 1, 2)\}$. By Theorem 5.5, the $P_T^{(t)}$ -minimal component is $P_T^{(t)}$, but the $P_S^{(t)}$ component is much more complicated. For example, Macaulay2 gives it 101 generators.

Keeping $r_1 = r_2 = r_3 = r_4 = 2$ and changing t to 2, the ideal $I^{(t)}$ has 31 minimal primes and 11 embedded primes including the maximal ideal. For $t = 1$ and the same r_1, \dots, r_4 , the ideal $I^{(1)}$ has 17 components, and by work in [1] these components are all primes and there are no other components.

The following example shows that $I^{(t)}$ is not even radical for $n = 3$. Again, if $t = 1$ the work in [1] (and [5] since $n = 3$) proves $I^{(1)}$ is radical. Therefore the counterexample uses $t = 2, 3$. Thomas Kahle brought the following example to our attention:

Example 7.2. The simplest example is when $r_1 = r_2 = 2$ and $r_3 = 4$. The ideal $\tilde{I}^{(t)}$ with $t = 2, 3$ has 29 minimal components and one embedded component associated to the maximal ideal. We note that the minimal components are all prime by Theorem 5.5.

We consider it an interesting open question as to whether the minimal components are prime when $t \neq 1, n$. While the $I^{(t)}$ are not lattice basis ideals (see Example 2.4), one might first attack this question by considering lattice basis ideals in general, or just those with square-free generators. However, the following example given to us by Thomas Kahle shows that a general lattice basis ideal with square-free generators does not have to have prime minimal components.

Example 7.3. The ideal $(x_4x_8 - x_1x_9, x_4x_6 - x_7x_9, x_2x_5 - x_3x_9, x_2x_3 - x_5x_6)$ in variables x_1, x_2, \dots, x_9 over a field is equidimensional and it has 6 components, all of which are minimal and one of which is not prime.

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