SYMBOLIC POWERS OF RADICAL IDEALS

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Abstract. Hochster proved several criteria for when for a prime ideal P in a commutative Noetherian ring with identity, $P^n = P^{(n)}$ for all n. We generalize the criteria to radical ideals.

1. Introduction.

In [1], M. Hochster established several criteria for when for a prime ideal P in a Noetherian ring R, the n^{th} power P^n of P equals the n^{th} symbolic power $P^{(n)}$ of P for every positive integer n. He used a so-called test sequence of ideals in a polynomial ring over R to determine whether $P^n = P^{(n)}$ for all n. We extend Hochster's criteria to radical ideals.

Here is the set-up: let R be a Noetherian domain and P an ideal of R. Suppose that $\{a_1, a_2, \ldots, a_m\}$ is a generating set for P. Write the m-tuple as $\underline{\mathbf{p}} = (a_1, a_2, \ldots, a_m)$. Let $S = R[x_1, x_2, \ldots, x_m]$, where $x_1, x_2, \ldots, x_m$ are indeterminates over R.

Definition 1.1. *For an ideal $P = (a_1, \ldots, a_m)R$ of R, define recursively ideals of $S = R[x_1, \ldots, x_m]$:*

$$J_0(\underline{\mathbf{p}}) = 0 \quad and \quad J_{n+1}(\underline{\mathbf{p}}) = (\{\Sigma_{i=1}^m s_i x_i \;\big|\; s_i \in S \quad and \quad \Sigma_{i=1}^m s_i a_i \in J_n(\underline{\mathbf{p}})\})S$$

for $n \geq 0$. We write J_n for $J_n(\underline{\mathbf{p}})$ and denote $J = \cup_{n=1}^\infty J_n$. We call the sequence of ideals

$$PS + J_0,\ PS + J_1,\ \ldots,\ PS + J_n,\ \ldots$$

the test sequence of the m-tuple $\underline{\mathbf{p}}$.

Note that for each n, $J_n \subseteq J_{n+1}$. Since R is Noetherian, $J = J_n$ for all large n.

Hochster proved:

Theorem 1.2. [1, Theorem 1] *With the above notation, the following are equivalent for a prime ideal P in a Noetherian domain R:*

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A. The associated graded ring of R_P is a domain, and for every positive integer n, the n^{th} symbolic and ordinary powers of P agree.

B. The ideal $PS + J$ is prime.

C. For some integer n, $PS + J_n$ is a prime ideal of height m.

D. There is a height-m prime ideal Q of S such that $Q \subseteq PS + J$.

E. Let z be an indeterminate over R. Then z is a prime element in the subring $R[z, a_1/z, \ldots, a_m/z]$ of $R[z, 1/z]$.

As a generalization, we analyze the situation in which P is a radical ideal of a reduced Noetherian ring. We first define generalized symbolic powers of ideals. We then give some criteria regarding the equality of P^n and $P^{(n)}$.

2. Some basic results about test sequences

We start with some useful examples of test sequences:

Lemma 2.1. *Let R be a Noetherian ring and P an ideal generated by a regular sequence $a_1, a_2, \ldots, a_m$. For the m-tuple $\underline{\mathbf{p}} = (a_1, a_2, \ldots, a_m)$, denote $J_k = J_k(\underline{\mathbf{p}})$. Then*

$$J_1 = (x_j a_k - x_k a_j \;\big|\; 1 \le j < k \le m)S = J_2 = J_3 = \cdots = J.$$

Proof. The generators of J_1 are of the form $\sum_i s_i x_i$ such that $\sum_i s_i a_i = 0$. As $a_1, a_2, \ldots, a_m$ is a regular sequence, this means that the element $(s_1, \ldots, s_m) \in S^m$ is in the S-module generated by the Koszul relations $(0, \ldots, a_j, \ldots, -a_k, \ldots, 0)$, with $k < j$ and at most the kth and jth entries non-zero. Thus J_1 is generated by elements of the form $x_j a_k - x_k a_j$. It remains to prove that $J_1 = J_2$.

Let $\sum_i s_i x_i \in J_2$ with $\sum_i s_i a_i \in J_1$. Write $\sum_i s_i a_i = \sum_{j<k} l_{jk}(x_j a_k - x_k a_j)$ for some $l_{jk} \in S$. Then

$$\sum_{i=1}^{m} \left(s_i - \sum_{j=1}^{i-1} l_{ji} x_j + \sum_{k=i+1}^{m} l_{ik} x_k \right) a_i = 0,$$

so that

$$\sum_{i=1}^{m} s_i x_i = \sum_{i=1}^{m} \left(s_i - \sum_{j=1}^{i-1} l_{ji} x_j + \sum_{k=i+1}^{m} l_{ik} x_k \right) x_i \in J_1. \quad \square$$

In general, when the generating sequence does not form an R-sequence, the ideal J_2 may be bigger than J_1. One such example is given below:

Example 2.2. *Let $R = k[y_1, y_2]$ be a polynomial in two variables over a field k. Let $P = (a_1, a_2, a_3)R$, where $a_1 = y_1^2$, $a_2 = y_1 y_2$, and $a_3 = y_2^2$. The generating sequence (a_1, a_2, a_3) is not a regular sequence of R. In addition, $J_2 \neq J_1$.*

Proof. The module of relations on a_1, a_2, a_3 in $S = R[x_1, x_2, x_3]$ is generated by $(y_2, -y_1, 0)$ and $(0, y_2, -y_1)$, so that $J_1 = (y_2 x_1 - y_1 x_2, y_2 x_2 - y_1 x_3)S \subseteq (y_1, y_2)S$.

The element $x_1x_3-x_2^2$ is therefore not in J_1. But $x_1x_3-x_2^2 \in J_2$ as $x_1y_2^2-x_2y_1y_2 = y_2(x_1y_2-y_1x_2) \in J_1$. $\square$

Now let $S_r = R[x_1,\ldots,x_r]$ and consider an r-tuple $\underline{\mathbf{p}_r} = (a_1,\ldots,a_r)$, where $a_1,\ldots,a_r \in R$. Similar as in Definition 1.1, we denote

$$J_{k+1}(\underline{\mathbf{p}_r}) = (\{\Sigma_{i=1}^r s_ix_i \;\big|\; s_i \in S_r \quad \text{and} \quad \Sigma_{i=1}^r s_ia_i \in J_k(\underline{\mathbf{p}_r})\})S_r.$$

Lemma 2.3. *Let R be a Noetherian ring and $S = R[x_1,\ldots,x_m]$. Let $P = (a_1,a_2,\ldots,a_m)R$ be an ideal of R and $\underline{\mathbf{p}_m} = (a_1,a_2,\ldots,a_m)$. If $\sum_{i=r+1}^k g_ix_i = 0$, where $g_{r+1},\ldots,g_k \in S$ and $r+1 \le k \le m$, then $\sum_{i=r+1}^k g_ia_i \in J_1(\underline{\mathbf{p}_m})$.*

Proof. It is trivial when $k = r+1$. For $k > r+1$, $\sum_{i=r+1}^k g_ix_i = 0$ implies $g_k = \sum_{i=r+1}^{k-1} h_ix_i$ for some $h_i \in S$ since x_k is a regular element of S. Thus $\sum_{i=r+1}^{k-1}(g_i+h_ix_k)x_i = 0$. By induction hypothesis, $\sum_{i=r+1}^{k-1}(g_i+h_ix_k)a_i \in J_1(\underline{\mathbf{p}_m})$. On the other hand,

$$\begin{aligned}\sum_{i=r+1}^{k-1}(g_i+h_ix_k)a_i &= \sum_{i=r+1}^{k-1} g_ia_i + \sum_{i=r+1}^{k-1} h_i(x_ka_i - x_ia_k) + \sum_{i=r+1}^{k-1} h_ix_ia_k \\ &= \sum_{i=r+1}^{k} g_ia_i + \sum_{i=r+1}^{k-1} h_i(x_ka_i - x_ia_k) \in J_1(\underline{\mathbf{p}_m}).\end{aligned}$$

Since each $x_ka_i - x_ia_k$ is an element of $J_1(\underline{\mathbf{p}_m})$, $\sum_{i=r+1}^k g_ia_i \in J_1(\underline{\mathbf{p}_m})$. $\square$

Lemma 2.4. *Let R be a Noetherian ring and $P = (a_1,a_2,\ldots,a_m)R$, an ideal of R. Assume $a_m = \sum_{i=1}^{m-1} b_ia_i$, where each $b_i \in R$. For the m-tuple $\underline{\mathbf{p}_m} = (a_1,a_2,\ldots,a_m)$ and the $(m-1)$-tuple $\underline{\mathbf{p}_{m-1}} = (a_1,a_2,\ldots,a_{m-1})$,*

$$J_k(\underline{\mathbf{p}_m}) = \left(J_k(\underline{\mathbf{p}_{m-1}}) + \left(x_m - \sum_{i=1}^{m-1} b_ix_i\right)\right) R[x_1,\ldots,x_m]$$

and

$$J_k(\underline{\mathbf{p}_m}) \cap R[x_1,\ldots,x_{m-1}] = J_k(\underline{\mathbf{p}_{m-1}})$$

for all $k \ge 1$.

Proof. Let $\sum_{i=1}^m s_ix_i \in J_k(\underline{\mathbf{p}_m})$ such that $\sum_{i=1}^m s_ia_i \in J_{k-1}(\underline{\mathbf{p}_m})$. We want to show that $\sum_{i=1}^m s_ix_i$ is contained in the ideal generated by $J_k(\underline{\mathbf{p}_{m-1}})$ and $x_m - \sum_{i=1}^{m-1} b_ix_i$ in $R[x_1,\ldots,x_m]$. We can write $\sum_{i=1}^m s_ix_i = \sum_{i=1}^{m-1} t_ix_i + (x_m - \sum_{i=1}^{m-1} b_ix_i)s$ for some $s \in S$ and $t_i \in R[x_1,\ldots,x_{m-1}]$. It suffices to prove that $\sum_{i=1}^{m-1} t_ix_i$ is in $J_k(\underline{\mathbf{p}_{m-1}})$, or more generally that $J_k(\underline{\mathbf{p}_m}) \cap R[x_1,\ldots,x_{m-1}] \subseteq J_k(\underline{\mathbf{p}_{m-1}})$.

Let $f \in J_k(\underline{\mathbf{p}_m}) \cap R[x_1, \ldots, x_{m-1}]$. We may write $f = \sum_{i=1}^m s_i x_i$ such that $\sum_{i=1}^m s_i a_i \in J_{k-1}(\underline{\mathbf{p}_m})$. For each $i = 1, \ldots, m-1$, we write $s_i = t_i + f_i x_m$, where $t_i \in R[x_1, \ldots, x_{m-1}]$ and $f_i \in S$. Then $\sum_{i=1}^m s_i x_i = \sum_{i=1}^{m-1} t_i x_i + x_m(s_m + \sum_{i=1}^{m-1} f_i x_i) \in R[x_1, \ldots, x_{m-1}]$ implies that $s_m + \sum_{i=1}^{m-1} f_i x_i = 0$ and $\sum_{i=1}^m s_i a_i = \sum_{i=1}^{m-1} t_i a_i \in J_{k-1}(\underline{\mathbf{p}_m}) \cap R[x_1, \ldots, x_{m-1}]$. If $k = 1$, this says that $\sum_{i=1}^{m-1} t_i a_i = 0 \in J_{k-1}(\underline{\mathbf{p}_{m-1}})$, and if $k > 1$, then by induction $\sum_{i=1}^{m-1} t_i a_i \in J_{k-1}(\underline{\mathbf{p}_{m-1}})$. Thus for all $k \geq 1$, $\sum_{i=1}^m s_i x_i = \sum_{i=1}^{m-1} t_i x_i \in J_k(\underline{\mathbf{p}_{m-1}})$. □

As a generalization of Lemma 2.1, we have

Theorem 2.5. *Let R be a Noetherian ring and $P = (a_1, \ldots, a_m)R$ an ideal of R which is also generated by $a_1, \ldots, a_r$, where $0 < r < m$. Let $\underline{\mathbf{p}_m}$ and $\underline{\mathbf{p}_r}$ be as before. If $a_1, a_2, \ldots, a_r$ forms a regular R-sequence, then*

$$J_k(\underline{\mathbf{p}_m}) = J_1(\underline{\mathbf{p}_m})$$

for all $k \geq 1$.

Proof. Since $\{a_1, a_2, \ldots, a_r\}$ is a generating set of P, for each $i = r+1, \ldots, m$, we can write $a_i = \sum_{j=1}^r b_{ji} a_j$ for some $b_{ji} \in R$. Let $S = R[x_1, \ldots, x_m]$. Set $c_i = x_i - \sum_{j=1}^r b_{ji} x_j \in J_1(\underline{\mathbf{p}_m})$ for each $i = r+1, \ldots, m$. By repeated application of Lemma 2.4, for all $k \geq 1$,

$$J_k(\underline{\mathbf{p}_m}) = \left(J_k(\underline{\mathbf{p}_r}) + (c_{r+1}, \ldots, c_m)\right) S.$$

By Lemma 2.1, $J_k(\underline{\mathbf{p}_r}) = J_1(\underline{\mathbf{p}_r})$ for all $k \geq 1$, which finishes the proof. □

This gives some information on the test sequence of prime ideals in a regular ring:

Theorem 2.6. *Let R be a regular ring and P a prime ideal in R. Then there exists a generating set $\{a_1, \ldots, a_m\}$ of P such that with $\underline{\mathbf{p}} = (a_1, \ldots, a_m)$, for all integers $k \geq 1$, $J_k(\underline{\mathbf{p}})R_P = J_1(\underline{\mathbf{p}})R_P$.*

More generally, whenever P is an ideal and U a multiplicatively closed subset such that $U^{-1}P$ is generated by a regular sequence, there exists a generating set $\{a_1, \ldots, a_m\}$ of P such that with $\underline{\mathbf{p}} = (a_1, \ldots, a_m)$, for all integers $k \geq 1$, $U^{-1}J_k(\underline{\mathbf{p}}) = U^{-1}J_1(\underline{\mathbf{p}})$.

Proof: As $U^{-1}P$ is generated by a regular sequence, there exists a generating set such that the first r generators form a maximal regular sequence after localization at U. Let $\overline{J}_k(\underline{\mathbf{p}})$ be the corresponding k^{th} test ideal of $U^{-1}R$ for $\underline{\mathbf{p}}$. Clearly $U^{-1}J_k(\underline{\mathbf{p}}) = \overline{J}_k(\underline{\mathbf{p}})$. By Theorem 2.5, $\overline{J}_k(\underline{\mathbf{p}}) = \overline{J}_1(\underline{\mathbf{p}})$. Thus $U^{-1}\overline{J}_k(\underline{\mathbf{p}}) = U^{-1}J_1(\underline{\mathbf{p}})$.

The first part follows as in a regular ring, PR_P is generated by a regular sequence. □

3. Criteria for Radical Ideals

In this section we generalize Hochster's criterion to radical ideals, see Theorem 3.6.

Recall that $S = R[x_1, \ldots, x_m]$ and that $J_k = J_k(\underline{\mathbf{p}})$ refers to the kth test ideal with respect to the m-tuple $\underline{\mathbf{p}} = (a_1, \ldots, a_m)$. Clearly if U is a multiplicatively closed subset of R, then $U^{-1}J_k(\underline{\mathbf{p}}) = J_k(U^{-1}(\underline{\mathbf{p}}))$.

Definition 3.1. *Let R be a reduced Noetherian ring, P an ideal of R and U a multiplicatively closed subset of R. We define the n^{th} generalized symbolic power of P with respect to U to be*

$$P^{(n)} = P^n U^{-1} R \cap R.$$

If P is a radical ideal with the minimal primes $p_1, p_2, \ldots, p_t$, then the n^{th} generalized symbolic power of P with respect to $U = R \setminus (p_1 \cup \cdots \cup p_t)$ is called the n^{th} symbolic power of P.

In the proofs we will use the extended Rees algebra of P:

$$R' = R\left[z, \frac{P}{z}\right] = R\left[z, \frac{a_1}{z}, \frac{a_2}{z}, \ldots, \frac{a_m}{z}\right],$$

where z is an indeterminate over R. Note that

$$\frac{R'}{zR'} \cong \frac{R}{P} \oplus \frac{P}{P^2} \oplus \frac{P^2}{P^3} \oplus \cdots,$$

the associated graded ring of P.

For a ring A, we denote by $\mathcal{Z}(A)$ the set of all zero divisors of A. The following is well-known:

Remark 3.2. *Let R be a reduced Noetherian ring, P an ideal of R, and R' as above. Let U be a multiplicatively closed set of R. Then*

1. *$\mathcal{Z}(A)$ is the union of all associated prime ideals of A.*
2. *For each $n \geq 0$, $P^n = z^n R' \cap R$, and $P^n U^{-1} R \cap R = z^n U^{-1} R' \cap R$.*
3. *For a fixed $n > 0$, $P^n = P^n U^{-1} R \cap R$ if and only if $(P^n :_R u) = P^n$ for all $u \in U$. In particular, $P = PU^{-1}R \cap R$ if $U \cap \mathcal{Z}(R/P) = \emptyset$.*
4. *If $U \cap \mathcal{Z}(R'/zR') = \emptyset$, then $zU^{-1}R' \cap R' = zR'$ and $\mathrm{Rad}(zU^{-1}R') \cap R' = \mathrm{Rad}(zR')$.*

Our goal is to give similar criteria as those in [1] for radical ideals. First we establish some lemmas.

Lemma 3.3. *Let R be a Noetherian ring and $P = (a_1, a_2, \ldots, a_m)R$ an ideal. Let R', S and J be as above. Then R'/zR' is isomorphic to $S/(J + PS)$.*

In particular, $PS + J$ is a radical ideal if and only if zR' is a radical ideal.

Proof. Consider the surjective R-homomorphism ϕ from S to R'/zR', shown as composition below:

$$\begin{array}{rccccl} \phi: & S & \xrightarrow{\phi'} R' \longrightarrow & \dfrac{R'}{zR'} \cong \dfrac{R}{P} \oplus \dfrac{P}{P^2} \oplus \dfrac{P^2}{P^3} \oplus \cdots \\ & x_i & \mapsto & \dfrac{a_i}{z} \mapsto \dfrac{a_i + P^2}{P^2}. \end{array}$$

It suffices to prove that $\ker(\phi) = PS + J$. Note that each a_i maps to 0 in R/P, so that $PS \subseteq \ker(\phi)$. Cleary $\phi'(J_1) = 0$. Suppose that $\phi'(J_n) = 0$. Let $\sum s_i x_i \in J_{n+1}$ be such that $\sum s_i a_i \in J_n$. As $z\phi'(\sum s_i x_i) = \phi'(\sum s_i a_i) = 0$, it follows that $\phi'(\sum s_i x_i) = 0$. Thus by induction, $J \subseteq \ker(\phi') \subseteq \ker(\phi)$. This proves that $PS + J \subseteq \ker(\phi)$. To prove the opposite inclusion, let $f \in \ker(\phi)$. As ϕ is a graded homomorphism and $PS + J$ is a homogeneous ideal, it suffices to assume that f is a homogeneous element of S of degree d. Write $f = \sum_{|\nu|=d} f_\nu x^\nu$ for some $f_\nu \in R$. As $f \in \ker(\phi)$, this means that $\sum_{|\nu|=d} f_\nu a^\nu \in P^{d+1}$. Write $\sum_{|\nu|=d} f_\nu a^\nu = \sum_{i=1}^m \sum_{|\mu|=d} r_{i\mu} a^\mu a_i$ for some $r_{i\mu} \in R$. By definition of test sequences then $\sum_{|\nu|=d} f_\nu x^\nu - \sum_{i=1}^m \sum_{|\mu|=d} r_{i\mu} x^\mu a_i \in J_d$, whence

$$f = \sum_{|\nu|=d} f_\nu x^\nu - \sum_{i=1}^m \sum_{|\mu|=d} r_{i\mu} x^\mu a_i + \sum_{i=1}^m \sum_{|\mu|=d} r_{i\mu} x^\mu a_i \in J_d + PS \subseteq PS + J. \quad \square$$

Lemma 3.4. *Let R be a Noetherian ring and P an ideal. Let U be an arbitrary multiplicatively closed subset of R. Then the following are equivalent:*

(1) *$P^n U^{-1} R \cap R = P^n$ for every positive integer n, and the associated graded ring $\operatorname{gr}_{U^{-1}P}(U^{-1}R)$ is reduced.*
(2) *zR' is a radical ideal and $U \cap \mathcal{Z}(R'/zR') = \emptyset$.*

Proof. Assume the first statement. We first show that $U \cap \mathcal{Z}(R'/zR') = \emptyset$. Let $u \in U$ and $b \in R'$ such that $ub \in zR'$. Without loss of generality b is a homogeneous element of R' under the grading determined by the variable z. Thus we may write $b = b_0 z^{-n}$ for some integer n and some $b_0 \in P^n$. If n is negative, this means that $b_0 \in R$, $ub_0 \in P$, so that by assumption, $b_0 \in P$, whence $b = zR'$. Now let $n \geq 0$. Then $ub_0 \in z^{n+1}R' \cap R = P^{n+1}$ by Remark 3.2 (2). This implies that $b_0 \in P^{n+1}U^{-1}R \cap R = P^{n+1} = z^{n+1}R' \cap R'$, so that $b_0 \in z^{n+1}R'$ and thus $b \in zR'$. Hence $U \cap \mathcal{Z}(R'/zR') = \emptyset$.

By the assumption that the associated graded ring of $U^{-1}P$ is reduced and as $\operatorname{gr}_P(R) = R'/zR'$, it follows that $zU^{-1}R'$ is a radical ideal. Thus by Remark 3.2 (4), $zR' = zU^{-1}R' \cap R' = \operatorname{Rad}(zU^{-1}R') \cap R' = \operatorname{Rad}(zR')$, so zR' is a radical ideal of R'.

Next assume that the second statement holds. As zR' is a radical ideal, $\operatorname{gr}_P(R)$ is reduced, and so trivially $\operatorname{gr}_{U^{-1}P}(U^{-1}R)$ is reduced.

Let $b \in P^n U^{-1}R \cap R = z^n U^{-1}R' \cap R$. There exists $u \in U$ such that $ub \in z^n R'$. We have to prove that $b \in P^n$. If not, then there exists an integer $k < n$ such that $b \in P^k$ and $b \notin P^{k+1}$. Then $\frac{b}{z^k} \in R'$ and $u \cdot \frac{b}{z^k} = \frac{ub}{z^n} \cdot z^{n-k} \in zR'$. Since u is not a zero divisor of R'/zR', then $\frac{b}{z^k} \in zR'$, so that $b \in z^{k+1}R' \cap R = P^{k+1}$, a contradiction. Thus necessarily $k \geq n$ and $b \in P^k \subseteq P^n$. □

Lemma 3.5. *Let P, S, J be as in the set-up, with P presented with m generators. Then all of the minimal primes of $PS + J$ are of height m. In particular,* $\mathrm{ht}(PS + J) = m$.

Proof. Let ψ be the $R[z]$-homomorphism of $S[z]$ onto $R' = R[z, P/z]$ which takes x_i to $\frac{a_i}{z}$ for each i. Let $I = \ker(\psi)$ and $I_0 = (a_1 - x_1 z,\ a_2 - x_2 z,\ \ldots,\ a_m - x_m z)S[z]$, both ideals of $S[z]$. Obviously, $I_0 \subseteq I$. After inverting z, both I and I_0 are generated by the regular sequence $a_1 - x_1 z, \ldots, a_m - x_m z$, so that $I = \cup_{n \geq 0}(I_0 : z^n)$. This implies that z is not a zero divisor on $S[z]/I$. It is easy to check that $PS + J = (I + zS[z]) \cap S$.

We claim that every minimal prime of I is of height m. When going up to the localization $S[z, 1/z]$ of $S[z]$ localized at z, the minimal primes of I in $S[z]$ correspond to the minimal primes of $IS[z, 1/z]$ in $S[z, 1/z]$ and the heights do not change since z is not a zero divisor of $S[z]/I$. But $IS[z, 1/z] = I_0 S[z, 1/z] = (x_1 - a_1/z, x_2 - a_2/z, \ldots,\ x_m - a_m/z)S[z, 1/z]$, and obviously all of the minimal primes of $I_0 S[z, 1/z]$ are of height m. Thus all the minimal primes of I in $S[z]$ are of height m. In addition, all minimal primes of $(I + zS[z])S[z]$ are of height $m + 1$, again because z is not a zero divisor of $S[z]/I$.

Let q be a minimal prime of $PS + J$ in S. In the polynomial ring $S[z]$ over S, $qS[z] + zS[z]$ is a minimal prime of $(PS + J + zS[z])S[z] = (I + zS[z])S[z]$, and so $m + 1 = \mathrm{ht}(qS[z] + zS[z]) = \mathrm{ht}(qS) + 1$. Hence $\mathrm{ht}(qS) = m$. □

Now we give similar criteria as those in [1] for radical ideals:

Theorem 3.6. *Let R be a reduced Noetherian ring and $P = (a_1, \ldots, a_m)$, a radical ideal of R. Let $U = R \setminus (p_1 \cup \cdots \cup p_t)$ and S, z be as above. Recall that $R' = R[z, Pz^{-1}]$. The following statements are equivalent:*

- *A'. For every integer $n > 0$, $P^n = P^{(n)}$, and the associated graded ring* $\mathrm{gr}_{U^{-1}P}(U^{-1}R)$ *is reduced.*
- *B'. The ideal $PS + J$ is a radical ideal of S and $U \cap \mathcal{Z}(S/(PS + J)) = \emptyset$.*
- *C'. For some positive integer n, $PS + J_n$ is a radical ideal of height m which has the same number of minimal primes as $PS + J$ has, and $U \cap \mathcal{Z}(S/(PS + J_n)) = \emptyset$. In this case, $PS + J_n = PS + J$.*
- *D'. The ideal $PS + J$ contains a height-m radical ideal Q which has the same number of minimal primes as $PS + J$ has, and $U \cap \mathcal{Z}(S/Q) = \emptyset$. In this case, $Q = PS + J$.*
- *E'. The ideal zR' is a radical ideal of R' and $U \cap \mathcal{Z}(R'/zR') = \emptyset$.*

Proof. Lemma 3.4 gives the equivalence of A′ and E′ by setting $U = R \setminus (p_1 \cup \cdots \cup p_t)$. By the isomorphism in Lemma 3.3, B′ and E′ are equivalent.

By Lemma 3.5, all the minimal primes of $PS+J$ are of height m. If an ideal Q of height m is contained in $PS+J$ and has the same number of minimal primes as $PS+J$ does, then the minimal primes of $PS+J$ are exactly the minimal primes of Q. Thus $\mathrm{Rad}(Q) = \mathrm{Rad}(PS+J)$. Furthermore, if Q is radical, then $Q = \mathrm{Rad}(PS+J) \supseteq PS+J$, so that $Q = PS+J$. Whence the equivalences of B′, C′, and D′ follow trivially.

Now it is clear that the statements A′, B′, C′, D′, and E′ are all equivalent. □

Remark 3.7. *Let R be an integral domain, P a prime ideal, and $U = R \setminus P$. The statements A′ - E′ are equivalent to the statements A - E in Theorem 1.2, respectively.*

Proof. It is enough to show that the condition $U \cap \mathcal{Z}(R'/zR') = \emptyset$ in E′ can be dropped with this special setting. From the isomorphism $\frac{R'}{zR'} \cong \frac{R}{P} \oplus \frac{P}{P^2} \oplus \frac{P^2}{P^3} \oplus \cdots = \mathrm{gr}_P R$, it is sufficient to show that $U \cap \mathcal{Z}(\mathrm{gr}_P R) = \emptyset$. Let $b \in \mathrm{gr}_P(R)$ be a non-zero homogeneous element of degree n, and let $ub = 0$ in $\mathrm{gr}_P(R)$ for some $u \in U$. By assumption zR' is an integral domain, i.e., $\mathrm{gr}_P(R)$ is an integral domain. Since b is non-zero, necessarily u must be zero, i.e., $u \in P$, which contradicts its choice. □

We give two applications of Theorem 3.6.

Corollary 3.8. *Let R be a reduced Noetherian ring and P a radical ideal generated by an R-sequence. Then $P^n = P^{(n)}$ for every positive integer n.*

Proof. Assume that $P = (a_1, a_2, \ldots, a_m)R$, where $a_1, a_2, \ldots, a_m$ is an R-sequence. As in Theorem 3.6, we set $S = R[x_1, x_2, \ldots, x_m]$ and $U = R \setminus (p_1 \cup \cdots \cup p_t)$, where $p_1, p_2, \ldots, p_t$ are the minimal primes of P in R.

Then $PS = (a_1, a_2, \ldots, a_m)S$ is a radical ideal of S with the minimal primes $p_1S, p_2S, \ldots, p_tS$ in S. Furthermore, $(a_1, a_2, \ldots, a_m)$ is an S-sequence. For each i, p_iS is of height m because it is minimal over an ideal generated by an S-sequence of m elements.

By Lemma 2.1, $J \subseteq PS$. So $PS + J = PS$. Furthermore, the isomorphism $S/PS \cong (R/P)[x_1, x_2, \ldots, x_m]$ implies that $U \cap \mathcal{Z}(S/PS) = \emptyset$. So the condition B′ in Theorem 3.6 is satisfied. Therefore $P^n = P^{(n)}$ for every positive integer n. □

Proposition 3.9. *Let $\mathrm{Y} = (y_{ij})$ be a $(2 \times r)$ matrix of indeterminates ($r > 1$) and $R = k[\{y_{ij}\}]$ be the polynomial ring over a field k. Let P be the ideal generated by the 2×2 permanents of Y, i.e.,*

P is generated by elements of form $y_{1i}y_{2j} + y_{2i}y_{1j}$ ($i \neq j$). Then

(1) *If $r = 2$ or 3, $P^n = P^{(n)}$ for all $n \in \mathbb{N}$;*
(2) *If $r > 3$, there exists a positive integer n such that $P^n \neq P^{(n)}$.*

Proof. It is shown in [3, Theorem 4.1] that P is a radical ideal with $\mathrm{ht}(P) = \min\{r,\ 2r-3\}$ for $r \geq 3$, so that clearly $\mathrm{ht}(P) = \min\{r,\ 2r-3\}$ for $r \geq 2$. For case $r = 2$ and $r = 3$, the number of generators of P is equal to the height of P, so that the genenerating set of permanents forms a regular sequence. It follows from Corollary 3.8 that $P^n = P^{(n)}$ for all n.

For (2), suppose that $P = (a_1, a_2, \ldots, a_{n(n-1)/2})$, where $a_1, a_2, \ldots, a_{n(n-1)/2}$ are the generating permanents and $a_1 = y_{11}y_{22} + y_{21}y_{12}$. In [3] it is shown that P contains all products of three indeterminates chosen from three different columns but not from the same row. For example, both $y_{11}y_{22}y_{23}$ and $y_{21}y_{13}y_{24}$ are elements of P. Let

$$\alpha = y_{13}y_{23}y_{24}a_1 = y_{13}y_{23}y_{24}(y_{11}y_{22} + y_{21}y_{12}).$$

Then $\alpha \in P$. In addition, $\alpha \notin P^2$. This can be easily checked by Macaulay2.

However, $\alpha^2 \in P^3$. This is because

$$\begin{aligned}\alpha^2 &= y_{23}(y_{11}y_{22}y_{13})(y_{11}y_{24}y_{23})(y_{13}y_{24}y_{22}) \\ &\quad +2y_{13}(y_{13}y_{22}y_{21})(y_{23}y_{24}y_{12})(y_{11}y_{24}y_{23}) \\ &\quad +y_{13}(y_{13}y_{21}y_{24})(y_{23}y_{12}y_{21})(y_{24}y_{12}y_{23})\end{aligned}$$

and by above each of the nine elements in parentheses is in P. So we can represent α^2 as $\alpha^2 = \sum_{i_1i_2i_3} l_{i_1i_2i_3}a_{i_1}a_{i_2}a_{i_3}$ with $l_{i_1i_2i_3} \in R$. Let $\beta = [(y_{13}y_{23}y_{24})^2x_1]x_1 - \sum_{i_1i_2i_3}(l_{i_1i_2i_3}a_{i_1}x_{i_2})x_{i_3} \in S$. Note that $[(y_{13}y_{23}y_{24})^2a_1]a_1 - \sum_{i_1i_2i_3}(l_{i_1i_2i_3}a_{i_1}a_{i_2})a_{i_3}$ $= \alpha^2 - \alpha^2 = 0$, so $[(y_{13}y_{23}y_{24})^2a_1]x_1 - \sum_{i_1i_2i_3}(l_{i_1i_2i_3}a_{i_1}a_{i_2})x_{i_3} \in J_1$, which implies that $\beta = [(y_{13}y_{23}y_{24})^2x_1]x_1 - \sum_{i_1i_2i_3}(l_{i_1i_2i_3}a_{i_1}x_{i_2})x_{i_3} \in J_2 \subseteq J$. This implies that $(y_{13}y_{23}y_{24}x_1)^2 = \beta + \sum_{i_1i_2i_3}(l_{i_1i_2i_3}a_{i_1}x_{i_2})x_{i_3} \in J + PS$, i.e., $y_{13}y_{23}y_{24}x_1 \in \sqrt{J + PS}$.

However, under the homomorphism from Lemma 3.3, $y_{13}y_{23}y_{24}x_1$ is sent to the element $(y_{13}y_{23}y_{24}a_1 + P^2)/P^2$ in the graded ring gr_PR, which is nonzero. So $y_{13}y_{23}y_{24}x_1$ is not in the kernel $J + PS$. Therefore, $J + PS$ is not a radical ideal of S. By Theorem 3.6, $P^n \neq P^{(n)}$ for some positive integer n. □

Example 3.10. *Let k be a field and $R = k[x, y, z]$, where x, y, z are indeterminates over k. Let $P = (x, y) \cap (x - 1, z) \cap (y, 1 - zx)$, a radical ideal. Then $P^n = P^{(n)}$ for all positive integers n.*

Proof. Obviously, the three prime ideals $p_1 = (x, y)$, $p_2 = (x - 1, z)$, and $p_3 = (y, 1 - zx)$ are comaxmal and each of them is generated by an R-sequence. By Corollary 3.8, $p_i^n = p_i^{(n)}$ for all positive integers n and for $i = 1, 2, 3$. Thus $P^n = P^{(n)}$ for all n.

An application of Corollary 3.8 shows also the following:

Example 3.11. *Let k be a field and $R = k[x, y, z, u, v]/(xv - uy)$, where x, y, z, u, v are indeterminates over k, and let $P = (xy - u, yz)$. Then $P^n = P^{(n)}$ for all positive integers n.*

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