ON THE IDEAL OF MINORS OF MATRICES OF LINEAR FORMS

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1. Introduction

The ideals generated by the minors of matrices whose entries are linear forms are not yet well-understood, unless the forms themselves satisfy some strong condition. One has a wealth of information if the matrix is *generic*, *symmetric generic* or *Hankel*; here we tackle 1-*generic matrices*. We recall the definition of 1-genericity introduced in [E2] by Eisenbud: Let F be a field and $X_1, \ldots, X_s$ be indeterminates over F. Let M be an $m \times n$ matrix of linear forms in $F[X_1, \ldots, X_s]$, with $m \le n$ and $s \ge m + n - 1$. By a *generalized row* of M one means a non-trivial F-linear combination of the rows of M. By a *generalized entry* of M one means a non-trivial F-linear combination of the entries of a generalized row of M. M is said to be 1-*generic* if every generalized entry is non-zero.

Generic, symmetric generic, Hankel matrices, as well as many others are all 1-generic. In this wider context, however, the only case of determinantal ideals fully understood is that of the ideals generated by the maximal minors. In fact, in [E2] it is proved that these ideals are prime. However, when one considers ideals generated by non-maximal minors, patterns get complicated by the fact that often these ideals are not prime.

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Recently, the authors have applied the theory of 1-generic matrices to problems arising from the theory of hyperdeterminants, see [GS]. In that context specifically emerged the interest of investigating the structure of ideals generated by non-maximal minors of 1-generic matrices. The possible lack of a combinatorial structure is one of the difficulties one encounters in this type of task. In the present work we concentrate on a specific class of 1-generic matrices that preserve some combinatorial aspects. This also allows to determine the primary decomposition of ideals obtained by taking the first partial derivatives of a class of trilinear forms studied in [BBG].

Let F be a field, $2 \leq m \leq n$ integers, and consider the matrix

$$M = \begin{bmatrix} r_{11}X_1 & r_{12}X_2 & r_{13}X_3 & \cdots & r_{1n}X_n \\ r_{22}X_2 & r_{23}X_3 & r_{24}X_4 & \cdots & r_{2,n+1}X_{n+1} \\ & & & \vdots & \\ r_{mm}X_m & r_{m,m+1}X_{m+1} & r_{m,m+2}X_{m+2} & \cdots & r_{m,m+n-1}X_{m+n-1} \end{bmatrix},$$

where $X_1, X_2, \ldots, X_{m+n-1}$ are indeterminates and all the coefficients r_{ij} are units in F. We call such matrices *generalized Hankel* matrices. The usual Hankel matrices have all coefficients r_{ij} equal to 1. Note that the indices on the r_{ij} are as follows: i denotes the row and j the index of the variable that this coefficient multiplies. Thus $j - i + 1$ is the column number in which the term $r_{ij}X_j$ appears.

The generalized Hankel matrices are examples of 1-generic matrices whose ideals of non-maximal minors might not be prime. They appear in [BBG], in connection with *diagonal non-degenerate trilinear forms of boundary format.* Let $h \geq n \geq m \geq 2$, a trilinear form $A = \sum_{\substack{1\leq i\leq h \\ 1\leq j\leq n \\ 1\leq k\leq m}} a_{ijk}X_iY_jZ_k$ has a *boundary format* if $h = n + m - 1$. In this case, it is said to be *diagonal* and *non-degenerate* if $a_{ijk} \neq 0$ if and only if $i = j + k - 1$. When this happens, the $m \times n$ matrix given by the second partial derivatives by the Z_k and the Y_j is generalized Hankel.

Hankel matrices play an important role in the theory of 1-generic matrices. In fact, by using linear changes of the variables, elementary row operations and elementary column operations, Eisenbud [E2] showed that a 1-generic $2 \times n$ matrix can be transformed into a scrollar space format, in other words a juxtaposition of Hankel matrices, with no overlaps among the variables in the submatrices.

The structure of 1-generic $m \times n$ matrices, with $m \geq 3$, is much less understood. In the present work we analyze the structure of $m \times n$ generalized Hankel matrices, with $m \geq 3$. In particular we determine the minimal primary decomposition of the ideals generated by the 2×2 minors of such matrices. An analysis of what can possibly happen in the case of the 3×3 minors when M is a generalized Hankel matrix of size 4×4 and 5×5 is given in [BCS].

By $I_2(M)$ we denote the ideal in the polynomial ring $F[X_1, \ldots, X_{m+n-1}]$ which is generated by the 2×2 minors of M.

In Section 2 we identify two integers intrinsic to M either of which allows to decide whether $I_2(M)$ is prime. In Section 3 we prove that $I_2(M)$ is either prime or else it has exactly two minimal components; when it is not a prime ideal, it sometimes also has one embedded component (see Theorems 3.2 and 3.4). We give all these components explicitly and we also give a numerical criterion for when each case occurs. In Section 4, we use this information to describe the set of the associated primes of the Jacobian ideal generated by the first partial derivatives of a diagonal non-degenerate trilinear forms of boundary format; see Theorem 4.6.

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2. When is $I_2(M)$ prime?

Assume the notation described. We first recall what is known in the special case:

Theorem 2.1: *Assume $r_{ij} = 1$ for all i, j. Then $I_2(M)$ is a prime ideal of height $m + n - 3$.*

Proof: Here, M is the usual Hankel matrix, and by the work of Gruson and Peskine [GP],

$$I_2(M) = I_2\left(\begin{bmatrix} X_1 & X_2 & X_3 & X_4 & \cdots & X_{m+n-2} \\ X_2 & X_3 & X_4 & X_5 & \cdots & X_{m+n-1} \end{bmatrix}\right),$$

the ideal of maximal minors of a 1-generic matrix. The latter is a prime ideal of height $m + n - 2 - 2 + 1 = m + n - 3$ by [E2]. ∎

By work of [GP], actually for every $t = 1, \ldots, m$, the ideal generated by the $t \times t$ minors of M is prime in this special case when all the r_{ij} equal 1. More information on the ideal of minors (and their powers) of classical Hankel matrices can be found in [C] and [W].

Without the assumption that the r_{ij} be equal to 1 for all indices i and j, however, the primeness property no longer holds.

To distinguish all the cases for which $I_2(M)$ is prime, we first need to define two integers s and t intrinsic to the matrix. The definition of these integers involves first transforming M into a special form by scaling the variables. Note that the scaling of variables does not change the number of primary components, the primeness, or the primariness properties! So, we start defining s.

Discussion: By scaling the variables, we assume that for all $j = 1, \ldots, n$, $r_{1j} = 1$. We next show how to rescale the variables in such a way so that all the coefficients in the second row are 1 as well. For this, let j be the largest index such that $r_{2j} \neq 1$. Since we are trying to determine the primary decomposition of $I_2(M)$, without loss of generality we may divide the $(j-1)^{\text{th}}$ column by r_{2j}. At this point the coefficient in the first row and $(j-1)^{\text{th}}$ column is not 1 anymore, but after rescaling the variable X_{j-1} appropriately, all the coefficients in the first row are again 1. With this now all the coefficients r_{1i} are 1 and also $r_{2j} = r_{2,j+1} = \cdots = r_{2,n+1} = 1$. Coefficients different from 1 in row 2 may now only appear on the left of the spot $(2, j)$. We repeat this procedure until all the r_{1i} and all r_{2i} are 1. Thus without loss of generality the matrix M for which we want to study $I_2(M)$ is of the form

$$\begin{bmatrix} X_1 & X_2 & X_3 & \cdots & X_n \\ X_2 & X_3 & X_4 & \cdots & X_{n+1} \\ r_{33}X_3 & r_{34}X_4 & r_{35}X_5 & \cdots & r_{3,n+2}X_{n+2} \\ r_{44}X_4 & r_{45}X_5 & r_{46}X_6 & \cdots & r_{4,n+3}X_{n+3} \\ & & & \vdots & \\ r_{mm}X_m & r_{m,m+1}X_{m+1} & r_{m,m+2}X_{m+2} & \cdots & r_{m,m+n-1}X_{m+n-1} \end{bmatrix},$$

with all r_{ij} units in F. As before, without loss of generality we may multiply a row by a unit, so we divide the third row by r_{33}. Then by rescaling the last variable in the third row, namely X_{n+2}, we may assume that $r_{3,n+2} = 1$. Similarly, by continuing this process for each of the subsequent rows, we assume that all the coefficients in the first column and last column are 1. Eventually, without loss of generality, the matrix M for which we want to study $I_2(M)$ is of the form

$$\begin{bmatrix} X_1 & X_2 & X_3 & \cdots & X_n \\ X_2 & X_3 & X_4 & \cdots & X_{n+1} \\ X_3 & r_{34}X_4 & r_{35}X_5 & \cdots & X_{n+2} \\ X_4 & r_{45}X_5 & r_{46}X_6 & \cdots & X_{n+3} \\ & & & \vdots & \\ X_m & r_{m,m+1}X_{m+1} & r_{m,m+2}X_{m+2} & \cdots & X_{m+n-1} \end{bmatrix},$$

with all r_{ij} units in F.

Now we can finally define s:

Definition 2.2: *After converting M to the form as above, if some new r_{ij} is different from* 1*, define*

$$s = s(M) = min\{j : \text{ there exists } i \geq 3 \text{ such that } r_{ij} \neq 1\}.$$

Note that $s \geq 4$ and that the column in which X_s appears with the non-one coefficient is never the last nor the first one. Clearly s does not exist if and only if all the coefficients r_{ij} are equal to 1.

The assumption that s exists implies that $I_2(M)$ contains many monomials, and hence that $I_2(M)$ is not a prime ideal:

Lemma 2.3: *Whenever s is defined, for all $i = 1, \ldots, s-1$, $X_i X_s \in I_2(M)$.*

Proof: Without loss of generality M is in the special form. We will use the fact that the appearance of $r_{is} X_s$ does not appear in the last column nor in the first two rows of M.

In the case that $i + 2 < s$, the following is a submatrix of M:

$$\begin{bmatrix} X_i & X_{s-2} & X_{s-1} \\ X_{i+1} & X_{s-1} & X_s \\ X_{i+2} & r_{is} X_s & * \end{bmatrix}.$$

If $i + 1 < s - 2$, then also

$$\begin{bmatrix} X_{i+1} & X_{s-2} \\ X_{i+2} & X_{s-1} \end{bmatrix}$$

is a submatrix of M, so that

$$(r_{is} - 1) X_i X_s = (r_{is} X_i X_s - X_{i+2} X_{s-2}) - (X_i X_s - X_{i+1} X_{s-1}) - (X_{i+1} X_{s-1} - X_{i+2} X_{s-2})$$

is an element of $I_2(M)$, forcing $X_i X_s$ to be in $I_2(M)$ as well.

In the case that $i = s - 2$, M has the submatrix

$$\begin{bmatrix} * & X_{s-2} & X_{s-1} \\ X_{s-2} & X_{s-1} & X_s \\ X_{s-1} & r_{is} X_s & * \end{bmatrix}.$$

Then by taking appropriate 2×2 minors,

$$(r_{is} - 1) X_{s-2} X_s = (r_{is} X_{s-2} X_2 - X_{s-1}^2) - (X_{s-2} X_2 - X_{s-1}^2) \in I_2(M),$$

so that $X_{s-2}X_s \in I_2(M)$. Finally, in the case that $i = s-1$, the submatrix

$$\begin{bmatrix} * & X_{s-2} & X_{s-1} \\ X_{s-2} & X_{s-1} & X_s \\ X_{s-1} & r_{is}X_s & r_{i,s+1}X_{s+1} \end{bmatrix}$$

of M yields that $r_{is}X_{s-1}X_s - r_{i,s+1}X_{s-2}X_{s+1}$ and $X_{s-1}X_s - r_{i,s+1}X_{s-2}X_{s+1}$ both belong to $I_2(M)$. Therefore $(1-r_{is})X_{s-1}X_s \in I_2(M)$, so that $X_{s-1}X_s \in I_2(M)$. ■

Corollary 2.4: *For a given generalized Hankel matrix M, s exists if and only if $I_2(M)$ is not prime.* ■

We now define an integer t for the matrix M in a way similar to that used to define s. The integer t plays for the matrix obtained from M by rotating it by 180^o the same role as s does for M: by rescaling the variables and dividing the rows and columns by non-zero scalars, in a way so that all the coefficients in the last two rows and the first and last columns are 1, the matrix M can be converted to the form:

$$\begin{bmatrix} X_1 & r_{12}X_2 & r_{13}X_3 & \cdots & r_{1,n-1}X_{n-1} & X_n \\ X_2 & r_{23}X_3 & r_{24}X_4 & \cdots & r_{2n}X_n & X_{n+1} \\ & & & \vdots & & \\ X_{m-2} & r_{m-2,m-1}X_{m-1} & r_{m-2,m}X_m & \cdots & r_{m-2,m+n-4}X_{m+n-4} & X_{m+n-3} \\ X_{m-1} & X_m & X_{m+1} & \cdots & X_{m+n-3} & X_{m+n-2} \\ X_m & X_{m+1} & X_{m+2} & \cdots & X_{m+n-2} & X_{m+n-1} \end{bmatrix},$$

with all r_{ij} units in F.

Definition 2.5: *After converting M to the form as above, define*

$$t = t(M) = max\{j : \text{ there exists } i \le m-2 \text{ such that } j-i+1 < n \text{ and } r_{ij} \neq r_{i,j+1}\}.$$

The requirement $j-i+1 < n$ means only that the column number is strictly less than n. Note that $t \le m+n-4$.

By symmetry:

Lemma 2.6: *Whenever t is defined, for all $j \ge t+1$, $X_tX_j \in I_2(M)$.* ■

Thus also t exists if and only if $I_2(M)$ is not prime, hence s exists if and only if t does. In the following assume that s (and t) exists.

Corollary 2.7: *If $s > t$, then $(X_1, \ldots, X_t)(X_s, \ldots, X_{m+n-1}) \subseteq I_2(M)$.*

Proof: Let $i \leq t$ and $j \geq s$. We will prove that $X_i X_j \in I_2(M)$. By the lemmas, this is already known to hold if either $i = t$ or $j = s$. Suppose that $i < t < s < j$. Then modulo one of the 2×2 minors of M, this monomial is in $I_2(M)$ if and only if $X_{i+1} X_{j-1}$ is in $I_2(M)$. In this way we raise the index of the first variable and simultaneously lower the index of the second variable until either i becomes t or j becomes s, at which point the previous lemmas prove the corollary. ■

There are even more monomials in $I_2(M)$:

Proposition 2.8: *For $i = 0, 1, \ldots, m+n-1-s$, $X_1^{i+1} X_{s+i} \in I_2(M)$, and analogously, for $j = 0, \ldots, t-1$, $X_{m+n-1}^{j+1} X_{t-j} \in I_2(M)$.*

Proof: We only prove the first part. The proof is by induction on i. The case $i = 0$ holds by Lemma 2.3. If $i > 0$,

$$X_1^{i+1} X_{s+i} = X_1^i X_1 X_{s+i} \equiv a X_1^i X_2 X_{s+i-1} \text{ modulo } I_2(M),$$

where a is a unit in F, but the latter is in $I_2(M)$ by induction on i. ■

It follows immediately that

Corollary 2.9: $X_1^{m+n-4}(X_s, \ldots, X_{m+n-1}) \subseteq I_2(M)$ *and* $X_{m+n-1}^{m+n-4}(X_1, \ldots, X_t) \subseteq I_2(M)$.

Proof: From Proposition 2.8 the elements $X_1 X_s, X_1^2 X_{s+1}, \ldots, X_1^{m+n-s} X_{m+n-1}$ are in $I_2(M)$. Since $s \geq 4$, one has $m+n-s \geq m+n-4$, thus $X_1^{m+n-4} X_j \in I_2(M)$ for all $j \geq s$. Analogously, using Proposition 2.8 and $t \leq m+n-4$, one can prove the statement about $X_{m+n-1}^{m+n-4}(X_1, \ldots, X_t)$ being included in $I_2(M)$. ■

3. Primary decomposition of $I_2(M)$

Having illustrated the case in which al the coefficients r_{ij} are 1 in Thoerem 2.1 we assume in this section that s (and t) exists.

Define the following three ideals in the ring $F[X_1, \ldots, X_{m+n-1}]$:

$$Q_1 = I_2(M) + (X_s, \ldots, X_{m+n-1}),$$
$$Q_2 = I_2(M) + (X_1, \ldots, X_t),$$
$$Q_3 = I_2(M) + (X_1^{m+n-4}, X_{m+n-1}^{m+n-4}).$$

Theorem 3.1: *Q_1, Q_2 and Q_3 are primary to the prime ideals $(X_2, \ldots, X_{m+n-1})$, $(X_1, \ldots, X_{m+n-2})$, and $(X_1, \ldots, X_{m+n-1})$, respectively.*

Proof: Clearly Q_3 is primary to the maximal homogeneous ideal because any minimal prime over Q_3 contains X_1 and X_{m+n-1} and therefore, by the shape of M, also all the other variables.

Now let N be the matrix obtained from M by setting $X_s, \ldots, X_{m+n-1}$ equal to zero. Then N is, after rescaling the variables $X_1, \ldots, X_{s-1}$, equal to a standard Hankel matrix with $X_s, \ldots, X_{m+n-1}$ equal to 0. But then $I_2(N) + (X_s, \ldots, X_{m+n-1})$ is primary to $(X_2, \ldots, X_{m+n-1})$ by [GP]. As Q_1 equals this ideal, it follows that Q_1 is primary to $(X_2, \ldots, X_{m+n-1})$.

The analogous proof works also for Q_2. ■

Theorem 3.2: *Assume that s and t exist. If $s > t$, then $I_2(M) = Q_1 \cap Q_2$ is a primary decomposition.*

Proof: It suffices to prove that $I_2(M) = Q_1 \cap Q_2$. For this:

$$\begin{aligned} I_2(M) &\subseteq Q_1 \cap Q_2 \\ &= (I_2(M) + (X_s, \ldots, X_{m+n-1})) \cap (I_2(M) + (X_1, \ldots, X_t)) \\ &= I_2(M) + (X_s, \ldots, X_{m+n-1}) \cap (I_2(M) + (X_1, \ldots, X_t)) \,. \end{aligned}$$

Let f be an element of the intersection $(X_s, \ldots, X_{m+n-1}) \cap (I_2(M) + (X_1, \ldots, X_t))$. Without loss of generality, f is homogeneous and necessarily of degree at least 2. Write $f = \sum_{i \geq s} a_i X_i$, where the a_i are homogeneous elements of degree at least one. We first prove that we only need to consider the f of the form $f_0 X_{m+n-1}$. Namely, consider the term $a_i X_i$ with $s \leq i < m+n-1$. As $s > t$, by Corollary 2.7, we may assume that a_i involves only the variables $X_{t+1}, \ldots, X_{m+n-1}$. Now, for $j > t$, by going modulo appropriate 2×2 minors of M, the monomial $X_j X_i$ appearing in $a_i X_i$ may be reduced to a scalar multiple of the monomial $X_{j-1} X_{i+1}$. By repeating this and by using Corollary 2.7, we see that we only need to consider the cases when $a_s = \cdots = a_{m+n-2} = 0$. Thus we assume that $f = f_0 X_{m+n-1}$. As f is an element of $I_2(M) + (X_1, \ldots, X_t) = Q_2$, and Q_2 is primary to $(X_1, \ldots, X_{m+n-2})$, it follows that $f_0 \in I_2(M) + (X_1, \ldots, X_t)$. It remains to show that $X_{m+n-1}(X_1, \ldots, X_t)$ is contained in $I_2(M)$. But this follows by Corollary 2.7. ■

The following Lemma shows that, when $s \leq t$, another primary component is needed to get a primary decomposition of $I_2(M)$.

Lemma 3.3: *When $s \leq t$, the irredundant primary decomposition of $I_2(M)+(X_s,\dots,X_t)$ is*

$$(I_2(M)+(X_s,\dots,X_{m+n-1}))\cap(I_2(M)+(X_1,\dots,X_t)).$$

Proof: That the two ideals in the intersection are primary follows by Theorem 3.1. It remains to prove that their intersection equals $I_2(M)+(X_s,\dots,X_t)$.

Let N be the matrix obtained from M by setting $X_s,\dots,X_{m+n-1}$ to zero. Similarly, let N' be the matrix obtained from M by setting $X_1,\dots,X_t$ to zero. Then the two ideals above are $I_2(N)+(X_s,\dots,X_{m+n-1})$ and $I_2(N')+(X_1,\dots,X_t)$, respectively. As $s \leq t$, the intersection equals

$$\begin{aligned}
&= I_2(N)+I_2(N')+(X_s,\dots,X_{m+n-1})\cap(X_1,\dots,X_t)\\
&= I_2(N)+I_2(N')+(X_s,\dots,X_t)+(X_{t+1},\dots,X_{m+n-1})(X_1,\dots,X_t)\\
&\subseteq I_2(M)+(X_s,\dots,X_t)+(X_{t+1},\dots,X_{m+n-1})(X_1,\dots,X_t).
\end{aligned}$$

Modulo 2×2 minors of M, each monomial in $(X_{t+1},\dots,X_{m+n-1})(X_1,\dots,X_t)$ reduces to an element in $I_2(M)+(X_s,\dots,X_t)$, which proves the lemma. ∎

Theorem 3.4: *If $s \leq t$, then $I_2(M) = Q_1\cap Q_2\cap Q_3$ is an irredundant primary decomposition.*

Proof: The intersection $Q_1\cap Q_2\cap Q_3$ equals

$$\begin{aligned}
&= (I_2(M)+(X_s,\dots,X_{m+n-1}))\cap(I_2(M)+(X_1,\dots,X_t))\cap\left(I_2(M)+(X_1^{m+n-4},X_{m+n-1}^{m+n-4})\right)\\
&= I_2(M)+(X_1^{m+n-4},X_{m+n-1}^{m+n-4})\cap(I_2(M)+(X_s,\dots,X_{m+n-1}))\cap(I_2(M)+(X_1,\dots,X_t))\\
&= I_2(M)+\left((X_{m+n-1}^{m+n-4})+(X_1^{m+n-4})\cap(I_2(M)+(X_s,\dots,X_{m+n-1}))\right)\\
&\qquad\cap\left((X_1^{m+n-4})+(X_{m+n-1}^{m+n-4})\cap(I_2(M)+(X_1,\dots,X_t))\right)\\
&= I_2(M)+\left((X_{m+n-1}^{m+n-4})+(X_1^{m+n-4})\,(I_2(M)+(X_s,\dots,X_{m+n-1}))\right)\\
&\qquad\cap\left((X_1^{m+n-4})+(X_{m+n-1}^{m+n-4})\,(I_2(M)+(X_1,\dots,X_t))\right).\\
&= I_2(M)+(X_1^{m+n-4})\,(I_2(M)+(X_s,\dots,X_{m+n-1}))\\
&\qquad+(X_{m+n-1}^{m+n-4})\,(I_2(M)+(X_1,\dots,X_t))+(X_{m+n-1}^{m+n-4})(X_1^{m+n-4})\\
&= I_2(M)+(X_1^{m+n-4})(X_s,\dots,X_{m+n-1})+(X_{m+n-1}^{m+n-4})(X_1,\dots,X_t)+(X_{m+n-1}^{m+n-4})(X_1^{m+n-4}).
\end{aligned}$$

But all the monomial products above lie in $I_2(M)$ by Corollary 2.9, so that the intersection of the three Q_i's is indeed $I_2(M)$, thus proving the theorem. ∎

Observe that the height of $I_2(M)$ is $m+n-3$ when $I_2(M)$ is prime (i.e. when M is Hankel), and it is $m+n-2$ otherwise.

4. Primary decomposition of the Jacobian ideal of a trilinear form

The minimal primes of the Jacobian ideal J_A, obtained by taking the first partial derivatives of a trilinear form

$$A = \sum_{\substack{1\le i\le n+m-1 \\ 1\le j\le n \\ 1\le k\le m}} a_{ijk} X_i Y_j Z_k$$

for which $a_{ijk} \neq 0$ if and only if $i = j+k-1$, are described in [BBG]. There, among other results, the authors prove that the maximal irrelevant ideal is an associated prime of J_A. As mentioned in the introduction, the trilinear form above is said to be non-degenerate diagonal of boundary format. The matrix M whose (k,j) entry is given by the second partial derivative $A_{Z_k Y_j}$, has the form

$$M = \begin{bmatrix} a_{111}X_1 & \cdots & a_{nn1}X_n \\ a_{212}X_2 & \cdots & a_{n+1,n2}X_{n+1} \\ & \vdots & \\ a_{m1m}X_m & \cdots & a_{m+n-1nm}X_{m+n-1} \end{bmatrix},$$

where all the coefficients a_{ijk} are different from zero. Clearly, M is a generalized Hankel matrix.

Experimental evidence unveiled a possible pattern for the whole set of the associated primes of J_A. In [BBG] it was conjectured that the embedded associated primes of J_A were in fact the associated primes of the ideals

$$(Y_1, \ldots, Y_n, Z_1, \ldots, Z_m, I_t(M)),$$

with $1 \le t \le m-1$.

In this section we analyze this problem when $m = 3$ and $n \ge 3$. From this moment on we operate in the polynomial ring $F[X_1, \ldots, X_{n+2}, Y_1, \ldots, Y_n, Z_1, Z_2, Z_3]$. In this case $A = \underline{Z} M \underline{Y}^t$, where $\underline{Z}$ is the row-vector given by the Z, $\underline{Y}^t$ is the transpose of the row-vector given by the Y, and M is

$$\begin{bmatrix} r_{11}X_1 & r_{12}X_2 & r_{13}X_3 & \cdots & r_{1n}X_n \\ r_{22}X_2 & r_{23}X_3 & r_{24}X_4 & \cdots & r_{2,n+1}X_{n+1} \\ r_{33}X_3 & r_{34}X_4 & r_{35}X_5 & \cdots & r_{3,n+2}X_{n+2} \end{bmatrix}.$$

Explicitly, the generators of J_A are as follows:

$$
\begin{aligned}
A_{Y_j} &= r_{1j}X_jZ_1 + r_{2,j+1}X_{j+1}Z_2 + r_{3,j+2}X_{j+1}Z_3, 1 \leq j \leq m;\\
A_{Z_k} &= rkkX_kY_1 + \cdots + r_{k,k+n-1}X_{k,k+n-1}Y_n, k = 1, 2, 3;\\
A_{X_1} &= r_{11}Y_1Z_1,\\
A_{X_2} &= r_{22}Y_1Z_2 + r_{12}Y_2Z_1,\\
A_{X_i} &= r_{3i}Y_{i-2}Z_3 + r_{2i}Y_{i-1}Z_2 + r_{1i}Y_iZ_1, 3 \leq i \leq n,\\
A_{X_{n+1}} &= r_{3,n+1}Y_{n-1}Z_3 + r_{2,n+1}Y_nZ_2,\\
A_{X_{n+2}} &= r_{3,n+2}Y_nZ_3.
\end{aligned}
$$

We first need a standard linear algebra result. We write the proof here for completeness. Let C be an arbitrary $c \times d$ matrix. Let $i_1, \ldots, i_r$ be distinct elements of $\{1, 2, \ldots, c\}$ and let $j_1, \ldots, j_r$ be distinct elements of $\{1, 2, \ldots, d\}$. By $[i_1, \ldots, i_r|j_1, \ldots, j_r]$ we denote the $r \times r$ minor of C when taking rows $i_1, \ldots, i_r$ and columns $j_1, \ldots, j_r$.

Lemma 4.1: *Let F be a field and $T_1, \ldots, T_c$ be indeterminates over F, then in the polynomial ring $F[T_1, \ldots, T_c]$,*

$$(-1)^{sgn(i_1,\ldots,i_r)}T_{i_1}[i_1, \ldots, i_r|j_1, \ldots, j_r] + \sum_{i \neq i_1}(-1)^{sgn(i,i_2,\ldots,i_r)}T_i[i, \ldots, i_r|j_1, \ldots, j_r]$$

is in $I_1\left(\begin{bmatrix} T_1 & \cdots & T_c \end{bmatrix} C\right)$.

Proof: Let D be the $d \times c$ matrix as follows: entries outside rows $j_1, \ldots, j_r$ and outside columns $i_1, \ldots, i_r$ are all zero, the rest is the adjoint of the submatrix of C consisting of rows $i_1, \ldots, i_r$ and columns $j_1, \ldots, j_r$. Then CD is a $c \times c$ matrix which is zero outside the columns $i_1, \ldots, i_r$. In column i_h, row l, the entry is plus or minus the r-minor of C consisting of columns $j_1, \ldots, j_r$ and rows $i_1, \ldots, i_r$ after omitting the row i_h and adding row l. In particular, at most $c - r + 1$ entries in the column are non-zero. As

$$I_1\left(\begin{bmatrix} T_1 & \ldots & T_c \end{bmatrix} C\right) \supseteq I_1\left(\begin{bmatrix} T_1 & \ldots & T_c \end{bmatrix} CD\right),$$

the lemma follows. ■

Note that when $C = M$ and the variables $T_1, \ldots T_c$ are the variables Z_1, Z_2, Z_3, then $I_1\left(\begin{bmatrix} Z_1 & Z_2 & Z_3 \end{bmatrix} M\right)$ equals $(A_{Y_1}, \ldots, A_{Y_n}) \subseteq J_A$.

Proposition 4.2: *If $m = 3$ and $n \geq 3$,*

$$I_2(M)^{n-2}(\underline{Y})(\underline{Z}) \subseteq J_A.$$

Proof: It suffices to prove that for any 2×2 minors $\Delta_1, \Delta_2, \ldots, \Delta_{n-2}$ of M,

$$Y_j Z_k \Delta_1 \cdot \Delta_2 \cdots \Delta_{n-2} \in J_A,$$

for all $j = 1, \ldots, n$, $k = 1, 2, 3$. Note that each Δ is of the form $[i_1, i_2 | j_1, j_2]$ for some admissible i_1, i_2, j_1, j_2. Actually, j_1 and j_2 are unimportant here, so throughout this proof we write the minors simply as $[i_1, i_2|.]$. Furthermore, by the shorthand notation $[i_1, i_2|.]^a$ we mean the product of a possibly distinct minors of the form $[i_1, i_2|.]$ rather than an a-fold product of one element $[i_1, i_2|.]$.

We will use the following notation: if $P(j)$ is a statement about the integer j, then $\delta_{P(j)}$ equals 1 if $P(j)$ is true, otherwise $\delta_{P(j)}$ is 0.

Note also that a direct consequence of Lemma 4.1 is that either $Z_{i_1}[i_1, i_2|.] + Z_{i_3}[i_3, i_2|.]$ or $Z_{i_1}[i_1, i_2|.] - Z_{i_3}[i_3, i_2|.]$ is in J_A.

We proceed in steps:

Step 1: We reduce to the case when $k = 2$ or $k = 3$, i.e., we eliminate the case $k = 1$. So suppose that $k = 1$. As $A_{X_1} = r_{11} Y_1 Z_1 \in J_A$, without loss of generality $j > 1$. Also, as $A_{X_2} = r_{22} Y_1 Z_2 + r_{12} Y_2 Z_1 \in J_A$, by reducing modulo this element without loss of generality $j > 2$. Using the notation introduced above we may say that, for $j = 2, \ldots, n$, $Y_j Z_1 \in J_A + (Y_{j-1} Z_2, \delta_{j>2} Y_{j-2} Z_3)$. Thus, without loss of generality, we just need to prove the assertion for $k > 1$.

Step 2: Now we reduce to one of the two cases: either $k = 3$, or if $k = 2$, then $\{i_1, i_2\} = \{1, 3\}$. So let $k = 2$. By Lemma 4.1 applied to $C = M$ and $r = 2$,

$$Z_2[1, 2|.] \in J_A + (Z_3[1, 3|.]),$$

so without loss of generality $\{i_1, i_2\} \neq \{1, 2\}$. If $\{i_1, i_2\} = \{2, 3\}$, then similarly

$$Y_j Z_2[2, 3|.] \in J_A + (Y_j Z_1[1, 3|.]).$$

If $j = 1$, the last term is in J_A (as in Step 1), and if $j > 1$, then as in Step 1, the last term $Y_j Z_1[1, 3|.]$ lies in

$$J_A + (Y_{j-1} Z_2[1, 3|.], \delta_{j>2} Y_{j-2} Z_3[1, 3|.]).$$

Thus we see that, without loss of generality, either $k = 3$ or if $k = 2$, then $\{i_1, i_2\} = \{1, 3\}$.

Step 3: We reduce to the case $k = 3$. By Step 2 we only need to consider the elements $Y_j Z_2[1, 3|.]$. Now,

$$\begin{aligned} Y_j Z_2[1, 3|.]^{n-2} &\in J_A + (\delta_{j>1} Y_{j-1} Z_3, \delta_{j<n} Y_{j+1} Z_1)\,[1, 3|.]^{n-2} \\ &= J_A + \left(\delta_{j>1} Y_{j-1} Z_3[1, 3|.]^{n-2}\right) + \left(\delta_{j<n} Y_{j+1} Z_1[1, 3|.]^{n-2}\right) \\ &\subseteq J_A + \left(\delta_{j>1} Y_{j-1} Z_3[1, 3|.]^{n-2}\right) + \left(\delta_{j<n} Y_{j+1} Z_2[2, 3|.][1, 3|.]^{n-3}\right). \end{aligned}$$

By repeating this step until the index on j increases to $n-1$, as $Y_nZ_2 \in J_A + (Y_{n-1}Z_3)$, we see that we have reduced to the case $k = 3$.

Step 4: Now let $k = 3$. We reduce to the case $\{i_1, i_2\} \neq \{1, 3\}$. As $Y_nZ_3 \in J_A$, without loss of generality $j < n$. As

$$\begin{aligned} Y_jZ_3[1,3|.] &\in J_A + (Y_jZ_2[1,2|.]) \\ &\subseteq J_A + (Y_{j+1}Z_1[1,2|.], \delta_{j>1}Y_{j-1}Z_3[1,2|.]) \\ &\subseteq J_A + (Y_{j+1}Z_3[2,3|.], \delta_{j>1}Y_{j-1}Z_3[1,2|.]), \end{aligned}$$

we see that without loss of generality $\{i_1, i_2\} \neq \{1, 3\}$.

Finally, in order to prove that $I_2(M)^{n-2}\underline{YZ} \subseteq J_A$, it suffices to prove that

$$Y_jZ_3[1,2|.]^r[2,3|.]^t \in J_A,$$

where $r, t \in \{0, 1, \ldots, n-2\}$ and $r + t = n-2$. As $Y_nZ_3 \in J_A$, without loss of generality $j < n$. But from

$$\begin{aligned} Y_jZ_3[1,2|.] &\in J_A + (Y_{j+1}Z_2[1,2|.], \delta_{j<n-1}Y_{j+2}Z_1[1,2|.]) \\ &\subseteq J_A + (Y_{j+1}Z_3[1,3|.], \delta_{j<n-1}Y_{j+2}Z_3[2,3|.]) \end{aligned}$$

we conclude that each of the r minors of the form $[1,2|.]$ raises the index of Y by at least 1. As $Y_nZ_3 \in J_A$ we may assume, without loss of generality, that $r \leq n-1-j$. Also,

$$Y_jZ_3[2,3|.] \in J_A + (Y_jZ_1[1,2|.]),$$

whence as $Y_1Z_1 \in J_A$, without loss of generality $j > 1$. Then

$$\begin{aligned} Y_jZ_3[2,3|.] &\in J_A + (Y_jZ_1[1,2|.]) \\ &\subseteq J_A + (Y_{j-1}Z_2[1,2|.], \delta_{j>2}Y_{j-2}Z_3[1,2|.]) \\ &\subseteq J_A + (Y_{j-1}Z_3[1,3|.], \delta_{j>2}Y_{j-2}Z_3[1,2|.]), \end{aligned}$$

so that each of the t minors of the form $[2,3|.]$ reduces the index on Y by at least 1. Thus as $Y_1Z_1 \in J_A$, without loss of generality $t \leq j-2$. But then

$$n-2 = r+t \leq n-1-j+j-2 = n-3,$$

which is a contradiction. $\blacksquare$

Proposition 4.3: *If $m = 3$ and $n \geq 3$,*

$$\underline{X}\left(\underline{Y}\right)^2 \underline{Z} + \underline{XY}\left(\underline{Z}\right)^{n-1} \subseteq J_A$$

Proof: We prove the harder fact that $\underline{XY}\left(\underline{Z}\right)^{n-1} \subseteq J_A$ and leave the similar but easier fact that $\underline{X}\left(\underline{Y}\right)^2 \underline{Z} \subseteq J_A$ to the reader. Let Z be the $(n+m-1) \times n$ matrix whose (i,j) entry is

$$\frac{\partial^2 A}{\partial X_i \partial Y_j} = \sum_k a_{ijk} Z_k = a_{ij,i-j+1} Z_{i-j+1} = r_{i-j+1,i} Z_{i-j+1},$$

which is interpreted as zero if $i-j+1 < 1$ or if $i-j+1 > 3$. Observe that $\left(\underline{Z}\right)^{n-1} = I_{n-1}(Z)$.

It suffices to prove that $X_i Y_j Z_1^a Z_2^b Z_3^c \in J_A$ whenever a, b, c are non-negative integers adding up to $n-1$.

Step 1: As $Y_j Z_1 \in J_A + (Y_{j-1}\delta_{j>1} Z_2, Y_{j-2}\delta_{j>2} Z_3)$, without loss of generality we may assume $a = 0$.

Step 2: We reduce to the case when $c + j \leq n$ and $j \leq b + 1$.

As $Y_j Z_3 \in J_A + (Y_{j+1}\delta_{j<n} Z_2, Y_{j+2}\delta_{j<n-1} Z_1)$, it follows by iteration that

$$Y_j Z_3^c \in J_A + (Y_{j+l}, \ldots, Y_n)\left(Z_1, Z_2\right)^l Z_3^{c-l},$$

for $1 \leq l \leq c$. Since $Y_n Z_3 \in J_A$, without loss of generality one has $c + j \leq n$. As $b + c = n - 1$, thus without loss of generality, one has $j \leq b + 1$.

Step 3: We next reduce to the case $c = n - 1$. By steps 1 and 2 we may assume that $a = 0$ and $j \leq b + 1$. We use Lemma 4.1 on the matrix $C = Z$ with $r = n - 1$:

$$Y_j[2, \ldots, n | 3, \ldots, b{+}2, b{+}4, \ldots, n{+}2] \in \left(Y_1[1, \ldots, \hat{j}, \ldots, n | 3, \ldots, b+2, b+4, \ldots, n+2]\right) + J_A.$$

But $[2, \ldots, n | 3, \ldots, b{+}2, b{+}4, \ldots, n{+}2]$ equals $Z_2^b Z_3^c +$ a homogeneous polynomial of degree $n{-}1$ of strictly higher Z_3 degree, and as $j \leq b{+}1$, $[1, \ldots, \hat{j}, \ldots, n | 3, \ldots, b{+}2, b{+}4, \ldots, n{+}2]$ is a homogeneous polynomial of degree $n-1$ which is a multiple of Z_3^{c+1}.

Thus by repeating these three steps we get that $c = n - 1$, $a = b = 0$ and $j = 1$. So we only have to consider the elements of the form $X_i Y_j Z_3^{n-1}$. If $i \geq 3$, then as

$$X_i Z_3 \in J_A + (X_{i-1} Z_2, X_{i-2} Z_1),$$

and by applying the previous steps (which do not change the index of X at all), we may reduce the index of X to either 1 or 2. Thus we only have to consider the elements $X_1 Y_1 Z_3^{n-1}$ and $X_2 Y_1 Z_3^{n-1}$. But then by reducing modulo the elements $X_1 Y_1 + X_2 Y_2 +$

$\cdots + X_nY_n, X_2Y_1 + X_3Y_2 + \cdots + X_{n+1}Y_n$ of J_A, we reduce to proving that $X_iY_jZ_3^{n-1}$ lies in J_A whenever $j > 1$. But this follows by Step 2. $\blacksquare$

Proposition 4.4: *Every embedded component of J_A contains*

$$J_A + I_2(M)^{n-2} + I_3(M) + (\underline{Y})^3 + (\underline{Z})^n + (\underline{X})(\underline{Y})^2 + (\underline{X})(\underline{Z})^{n-1}.$$

Proof: By [GS, Theorem 5.2], the intersection of all the minimal components (and all the minimal primes) is $J_A + (\underline{Y})(\underline{Z})$. It now suffices to prove that the intersection of the displayed ideal with the minimal components is J_A:

$$\begin{aligned}
&(J_A + (\underline{Y})(\underline{Z})) \cap \left(J_A + I_2(M)^{n-2} + I_3(M) + (\underline{Y})^3 + (\underline{Z})^n + (\underline{X})(\underline{Y})^2 + (\underline{X})(\underline{Z})^{n-1}\right)\\
&= J_A + (\underline{Y})(\underline{Z}) \cap \left(J_A + I_2(M)^{n-2} + I_3(M) + (\underline{Y})^3 + (\underline{Z})^n + (\underline{X})(\underline{Y})^2 + (\underline{X})(\underline{Z})^{n-1}\right)\\
&\subseteq J_A + A_X + \underline{Y}A_Y + \underline{Z}A_Z + (\underline{Y})(\underline{Z})I_2(M)^{n-2} + (\underline{Y})(\underline{Z})I_3(M) + (\underline{Z})(\underline{Y})^3 + (\underline{Y})(\underline{Z})^n\\
&\quad + (\underline{X})(\underline{Y})^2(\underline{Z}) + (\underline{X})(\underline{Y})(\underline{Z})^{n-1}\\
&\subseteq J_A,
\end{aligned}$$

where the second to the last inclusion holds by multihomogeneity and the last inclusion holds by Propositions 4.2, 4.3, and [GS, Theorem 5.2]. $\blacksquare$

Lemma 4.5: *For all $r \geq 1$, $X_{n+2}^rY_1Z_3$ is not an element of J_A.*

Proof: Without loss of generality we may assume that M is in the normalized form so that in the first two rows all the coefficients are 1, and also the coefficients in the first and the last columns are all 1. We then consider a partial Gröbner basis of J_A under the lexicographic order $Z_1 > Z_2 > Z_3 > Y_n > \cdots > Y_1 > X_1 > \cdots > X_{n+2}$. Note that we only need to consider those elements of the Gröbner basis whose Y and Z degrees are at most one. Since

$$M = \begin{bmatrix} X_1 & X_2 & X_3 & \cdots & X_n \\ X_2 & X_3 & X_4 & \cdots & X_{n+1} \\ X_3 & r_{34}X_4 & r_{35}X_5 & \cdots & X_{n+2} \end{bmatrix},$$

the generators of J_A may be written as follows, with leading terms written first:

$$\begin{array}{rl}
A_X: & Y_1Z_1, \\
& Y_2Z_1 + Y_1Z_2, \\
& Y_3Z_1 + Y_2Z_2 + Y_1Z_3, \\
& Y_iZ_1 + Y_{i-1}Z_2 + r_{3i}Y_{i-2}Z_3, i = 4, \dots, n \\
& Y_nZ_2 + r_{3,n+1}Y_{n-1}Z_3, \\
& Y_nZ_3; \\
A_Y: & X_1Z_1 + X_2Z_2 + X_3Z_3, \\
& X_jZ_1 + X_{j+1}Z_2 + r_{3,j+2}X_{j+2}Z_3, j = 2, \dots, n-1 \\
& X_nZ_1 + X_{n+1}Z_2 + X_{n+2}Z_3; \\
A_Z: & X_nY_n + \cdots + X_1Y_1, \\
& X_{n+1}Y_n + \cdots + X_2Y_1, \\
& X_{n+2}Y_n + r_{3,n+1}X_{n+1}Y_{n-1} + \cdots + r_{34}X_4Y_2 + X_3Y_1.
\end{array}$$

The element $X_{n+2}^rY_1Z_3$ can only be a multiple of the term Y_1Z_3 appearing in A_{X_3} or of the term $X_{n+2}Z_3$ appearing in A_{Y_n}. Neither of these two terms is a leading term. In order to make $X_{n+2}^rY_1Z_3$ a leading term of some element in a Gröbner basis, that term will either come from $X_{n+2}^rA_{X_3}$ or from $X_{n+2}^{r-1}Y_1A_{Y_n}$.

Assume that $X_{n+2}^rY_1Z_3$ comes from $X_{n+2}^rA_{X_3}$. As the leading term of A_{X_3} is Y_3Z_1, we first of all need to find an element g of the Gröbner basis whose leading term divides $X_{n+2}^rY_3Z_1$. This is in order to keep the possibility of $X_{n+2}^rY_1Z_3$ dividing one of the terms in the resulting S-polynomial. As the S-polynomial of A_{X_3} with itself is 0, the leading term of g necessarily divides either $X_{n+2}^rY_3$ or $X_{n+2}^rZ_1$. In the former case, g must be an element of A_Z, and in the latter case, an element of A_Y. But the only leading terms of A_Z with Y-degree at most 1 come from the polynomials as in Lemma 4.1 when applied to the matrix $C = X^t$, $V_i = Y_i$:

$$Y_n[.|.], Y_n[., .|j, n], j < n, \text{ and } Y_{n-1}[., ., .|j, n-1, n], j < n-1,$$

and similarly the only leading terms of A_Y with Z-degree at most 1 come from

$$Z_1X_i, i \leq n, Z_2[2, 3|., .], \text{ and } Z_3[1, 2, 3|., ., .],$$

and none of these divides $X_{n+2}^rY_1Z_3$ as desired.

So necessarily $X_{n+2}^rY_1Z_3$ comes from $X_{n+2}^{r-1}A_{Y_n}$. As the leading term of A_{Y_n} is X_nZ_1, as before we need to find an element g of the Gröbner basis whose leading term divides $X_{n+2}^{r-1}X_nZ_1$. As the S-polynomial of A_{Y_n} with itself is 0, the leading term of g necessarily divides either $X_{n+2}^{r-1}X_n$ or $X_{n+2}^{r-1}Z_1$. Of course the former is impossible. In the latter case, g is an element of A_Y, and as above, this is impossible. ∎

The following is a partial confirmation of the pattern conjectured in [BBG] and described at the beginning of this Section.

Theorem 4.6: *If $I_2(M)$ is not prime, then the set of associated primes of J_A consists of the minimal primes of J_A, the minimal primes over $I_2(M) + (\underline{Y}, \underline{Z})$, and the minimal primes over $I_1(M) + (\underline{Y}, \underline{Z})$.*

Proof: By Proposition 4.4 we know that every embedded prime contains all the Y_j, all the Z_k and $I_2(M)$. When $I_2(M)$ is not prime, one knows that an ideal containing $I_2(M) + (\underline{Y}, \underline{Z})$ has dimension at most 1; therefore, the embedded primes may have dimension 1 and 0. By quasihomogeneity, the only one possibility for dimension 0 is the irrelevant maximal ideal and in [BBG] it is proved that it is an associated prime. When the dimension is 1 we have to deal with the minimal primes over $I_2(M) + (\underline{Y}, \underline{Z})$.

Let P be a minimal prime over $I_2(M) + (\underline{Y}, \underline{Z})$. Suppose that P is not associated to J_A. By possibly rotating M by 180 degrees and appropriately renaming the variables, by Theorems 3.1, 3.2, 3.4, we may assume that $P = (X_1, \ldots, X_{n+1}) + (\underline{Y}, \underline{Z})$. Then by Proposition 4.4, J_A is the intersection of the minimal components with a possible $(X_2, \ldots, X_{n+2}) + (\underline{Y}, \underline{Z})$-primary component and a possible $(\underline{X}, \underline{Y}, \underline{Z})$-primary component. As by [GS], the intersection of the minimal components is $J_A + (\underline{Y})(\underline{Z})$, this means that for some large r, $X_{n+2}^r Y_1 Z_3 \in J_A$. But this contradicts the previous Lemma. ∎

References

[BBG] G. Boffi, W. Bruns and A. Guerrieri, On the Jacobian ideal of a trilinear form, *J. Algebra* **197** (1997), 521-534.

[BCS] P. Brodhead, M. Cummings and C. Seidler, Primary decomposition of ideals arising from Hankel matrices, REU project at Summer Institute in Mathematics for Undergraduates (SIMU) at the University of Puerto Rico – Humacao, 2000, preprint.

[C] A. Conca, Straightening law and powers of determinantal ideals of Hankel matrices, *Adv. Math.* **138** (1998), **no. 2**, 263–292.

[E1] D. Eisenbud, *Commutative Algebra with a View toward Algebraic Geometry,* Springer-Verlag, 1994.

[E2] D. Eisenbud, Linear sections of determinantal varieties, *Amer. J. Math.* **110** (1988), 541–575.

[GS] A. Guerrieri and I. Swanson, Jacobian ideals of trilinear forms: an application of 1-genericity, *J. of Algebra* **226**, (2000), 410–435.

[GrS] D. Grayson and M. Stillman, Macaulay2. 1996. A system for computation in algebraic geometry and commutative algebra, available via anonymous `ftp` from `math.uiuc.edu`.

[GPS] G.-M. Greuel, G. Pfister and H. Schönemann, Singular. 1995. A system for computation in algebraic geometry and singularity theory. Available via anonymous `ftp` from `helios.mathematik.uni-kl.de`.

[GP] L. Gruson and C. Peskine, C. Courbes de l'espace projectif: variétés de sécantes, *Enumerative geometry and classical algebraic geometry* (Nice, 1981), pp. 1–31, Progr. Math., **24**, Birkhäuser, Boston, Mass., 1982.

[W] J. Watanabe, Hankel matrices and Hankel ideals, *Proc. School Sci. Tokai Univ.* **32** (1997), 11–21.