Errata and Minor Additions
for
Integral Closure of Ideals, Rings, and Modules
by
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p. 8, lines 6–7 of proof of 1.3.3: replace "The lowest degree component" by "The lowest degree component monomial"; replace "components" by "component monomials".

p. 12, in Definition 1.4.7, now allow the Newton polyhedron to be the convex hull to be either in $\mathbb{R}^d$ or in $\mathbb{Q}^d$. This harmonizes with the subsequent general definition of the Newton polyhedron in 18.4.1.

p. 13, Theorem 1.4.10: The following proof is clearer:

Proof: Let $n \geq d$. It suffices to prove that I^n is integrally closed under the assumption that $I, I^2, \ldots, I^{n-1}$ are integrally closed. For this it suffices to prove that every monomial $X_1^{c_1} \cdots X_d^{c_d}$ in the integral closure of I^n lies in I^n. Let $\{\underline{X}^{\underline{v}_1}, \ldots, \underline{X}^{\underline{v}_t}\}$ be a monomial generating set of I. By the form of the integral equation of a monomial over a monomial ideal there exist non-negative rational numbers a_i such that $\sum a_i = n$ and the vector $(c_1, \ldots, c_d)$ is componentwise greater than or equal to $\sum a_i v_i$. By Carathéodory's Theorem A.2.1 (new version in errata!), by possibly reindexing the generators of I, there exist non-negative rational numbers $b_1, \ldots, b_d$ such that $\sum_{i=1}^d b_i \geq n$ and $(c_1, \ldots, c_d) \geq \sum_{i=1}^d b_i v_i$ (componentwise). As $n \geq d$, there exists $j \in \{1, \ldots, d\}$ such that $b_j \geq 1$. Then $(c_1, \ldots, c_d) - v_j \geq \sum_i (b_i - \delta_{ij}) v_i$ says that the monomial corresponding to the exponent vector $(c_1, \ldots, c_d) - v_j$ is integral over I^{n-1}. Since by assumption I^{n-1} is integrally closed, the monomial corresponding to $(c_1, \ldots, c_d) - v_j$ is in I^{n-1}. Thus $X_1^{c_1} \cdots X_d^{c_d} \in I^{n-1} \underline{X}^{\underline{v}_j} \subseteq I^n$. □

p. 15, Proposition 1.6.2: improved the proof (first need to reduce to S being finitely generated as a module over R).

p. 18, Theorem 1.7.3: A reference is added where the result appeared first: J. T. Arnold and R. Gilmer, On the contents of polynomials, *Proc. Amer. Math. Soc.* **24** 1970, 556–562.

p. 25, Lemma 2.1.7: The "Thus" part does not need the domain assumption.

p. 25, Lemma 2.1.8: M has to be finitely generated. In the second part, M has to be faithful over $R[x]$.

p. 26, Lemma 2.1.9 (3): M has to faithful over $R[x_1, \ldots, x_n]$. Modify the parts of the proof involving (3).

p. 28, Lemma 2.1.15 (2): The ring L contains all R_i so that the intersection makes sense.

pp. 31–32: The last two lines of page 31 and the first line of page 32 are redundant: P can be an arbitrary ideal and Q is not needed at all.

p. 32, line -16: z_i should be z_{i0} in the definition of $\mathfrak{n}$.

p. 34, line 2 of Section 2.3: "noni-negative" should be "non-negative"

p. 34, 2.3.1: clarify that each R_g is an additive subgroup of R. Add that elements of R_g and M_g are said to be homogeneous of degree g.

p. 35, line 5: variables should be over S rather than R. I added further clarifications there and fixed the double usage of the integer n.

p. 36, line 3 of 2.3.5: add "of" before "R".

p. 36, 2.3.5 (2) should say that $\overline{R}$ is a G-graded subring of S, inheriting the G-grading from S; p. 36, 2.3.5 (3) should say that the idempotents of $\overline{R}$ are of the form f/g, where f, g are homogeneous elements of R and g is a non-zerodivisor.

p. 37, line 7: r_i should be in P_i, $p_i \in \cap_{j \neq i} P_j \setminus P_i$. The P_i at the end of the second paragraph and at the beginning of the third should be $P_i S$.

p. 37, in the statement of 2.3.6, K is not defined (and is not needed): it is implicitly the total ring of fractions.

p. 38, line 4 of Corollary 2.3.7: $E \in \mathbb{N}^d$ should be $E \subseteq \mathbb{N}^d$.

p. 39, line 3: $R[S_0]$ should be $S_0[R_n]$.

p. 39, line -9: "torsion-free ideal" should be "ideal containing a non-zerodivisor or a unit". Similarly correct line 8 on p. 40.

p. 40, Lemma 2.4.3: We need to assume that $yJ \subseteq R$ for some non-zerodivisor y on R, and the conclusion is then that $\operatorname{Hom}_R(I, J)$ is identified with $\frac{1}{xy}(xyJ :_R I)$.

p. 41, example after Discussion 2.4.7: I is mistyped, it should be instead $I = (t^3, t^5)$, the rest is as was. The current principal I couldn't possibly be a counterexample.

p. 42, Exercise 2.3 (i): Change "A" to "R".

p. 43, line 4 of Exercise 2.10: "a R-submodule" should be "an R-submodule".

p. 43, Exercise 2.12 (ii): Also need to assume that the number of minimal prime ideals of R and S are the same. Otherwise $k[x] \to k[x] \times k$ is a counterexample. The new phrasing: Prove that $R :_R S$ is not contained in any minimal prime ideal of R and that R and S have the same number of minimal prime ideals if and only if R and S have the same total ring of fractions.

p. 43, Exercise 2.15 is wrong as stated. Replace by another exercise. Also, 2.14 and 2.15 are redundant, so both replaced.

p. 43, Exercise 2.16: the (i), (ii), (iii) should be replaced by: (i) $w = \frac{1+\sqrt{D_0}}{2}$ if $D_0 \equiv 1 \bmod 4$, (ii) $w = \sqrt{D_0}$ if $D_0 \not\equiv 1 \bmod 4$.

p. 44, Exercise 2.18: F should be declared as $(\mathbb{Q}_{\geq 0} E) \cap \mathbb{N}^d$. One exponent $d + n$ in (iv) should be $m + d$.

p. 44, Exercise 2.21: We could grade trivially. Find a non-trivial monoid.

p. 44: Exercise 2.23 uses almost integrality, which is defined in Exercise 2.26. Thus now 2.23 is moved to behind 2.26, which regretfully messes up the numbering.

p. 45: Exercise 2.26 (ix): Replace "X^n, Y^{n^2}" with "$X^n \cdot Y^{n^2}$".

p. 46, line 4 below the exercises: Change "A_P" to "R_P".

p. 47, Discussion 3.1.2: two lines below the display is the definition of y_j (and not of y_i).

p. 57, the fourth paragraph is wrong. Remove it (or replace R by $k[X,Y]/(X^3,XY)$). Reorganized Section 4.1 on page 57, and changed the old Proposition 4.1.2 to a stronger Proposition 4.1.3 (as needed in the new proof of the Mori–Nagata Theorem):

Proposition 4.1.3 Let $(R,\mathfrak{m})$ be a Noetherian local ring. Then $\mathfrak{m}$ is principal generated by a non-zerodivisor if and only if R is a Dedekind domain, and that holds if and only if every non-zero ideal in R is principal and generated by a non-zerodivisor. Furthermore, for such a ring R, every ideal is a power of $\mathfrak{m}$, and there are no rings strictly between R and its field of fractions. □

p. 57: the proof shows that part (1) can be phrased more strongly.

p. 59, proof of 4.2.3: If k is a finite field, then $k' = k$ and the paragraph three is wrong. Here is a correction:

Now assume that k is an arbitrary field, possibly finite. By repeated use of Corollary A.3.2, which says that all associated prime ideals in R are homogeneous, and by repeated use of the homogeneous Prime Avoidance Theorem (A.1.3), there exists a sequence $x_1,\ldots,x_m$ of homogeneous elements in R such that for all $i = 1,\ldots,m$, x_i is not in any prime ideal minimal over $(x_1,\ldots,x_{i-1})$. Let m be the maximal integer for which this is possible. By possibly lifting each x_i to a power, we may assume that all x_i have the same degree, say degree ed. By construction the only prime ideal in R containing $x_1,\ldots,x_m$ is the maximal homogeneous ideal $\mathfrak{M}$. Then $\mathfrak{M}^c \subseteq (x_1,\ldots,x_m)$ for some integer c, and in particular if $\tilde{R}$ is the subring of R generated over k by all homogeneous elements of degree ed, then the maximal ideal of $\tilde{R}$ raised to the cth power is contained in $(x_1,\ldots,x_m)\tilde{R}$. Thus as in the first paragraph we may assume that R is generated by elements of degree ed. By homogeneity and degree count $\mathfrak{M}^c = \mathfrak{M}^{c-1}(x_1,\ldots,x_m)$, so that $\mathfrak{M}$ is integral over $(x_1,\ldots,x_m)$. By Proposition 2.3.8, R is integral over its subring $A = k[x_1,\ldots,x_m]$. Since $\mathfrak{M}$ has height m by the construction of the x_i, there exists a minimal prime ideal P in R such that the height of $\mathfrak{M}/P$ is m and hence by the Dimension Formula (Theorem B.5.1) the height of $(x_1,\ldots,x_m)A = \mathfrak{M}\cap A$ modulo $A/(P\cap A)$ is m. This means that $\mathrm{ht}((x_1,\ldots,x_m)A) = m$ and so that $x_1,\ldots,x_m$ are algebraically independent over k. □

p. 64, line 3 of proof of 4.4.4: W should be W'.

p. 65, line -2: change p_0 to p

p. 71, add "is" in the definition of singular locus (4.5.4).

p. 78, line -19: $T\widehat{R}/Q_i$ should be $\widehat{R}/Q_i$; L_i+ should be $\overline{L}_i$.

p. 78, the last two paragraphs of the proof of 4.7.3: it is not clear how x/y could satisfy an equation of integral dependence over $\widehat{R}$. Here is a correction, by first changing the assumptions of the proposition: R is not assumed to be local but only semilocal, and then $\widehat{R}$ is the completion of R in the topology determined by its Jacobson radical:

If $\frac{x}{y} \in \overline{T} \cap K$, with x, y non-zero elements of R and y not in any minimal prime ideal of R, then $x \in y\overline{T}$. Since $x, y \in R$, by Proposition 1.6.1 the image of x in T is in the integral closure of yT, whence by Proposition 1.1.5 the image of x in $\widehat{R}$ is in the integral closure of $y\widehat{R}$, and by Proposition 1.6.2 $x \in \overline{yR}$, whence by Propositions 1.6.1 and 1.5.2, $x \in y\overline{R}$. Thus $\frac{x}{y} \in \overline{R}$, which proves that $\overline{R} = \overline{T} \cap K$.

Now let $\alpha \in T^+ \cap \overline{K}$. Since $\overline{K}$ is the total ring of fractions of R^+, there exist $x, y \in R^+$ such that y is not in any minimal prime ideal of R^+ and such that $\alpha = \frac{x}{y}$. Let R' be the subring of R^+ generated over R by x and y. Since R' is a module-finite extension of R, it is Noetherian. By Lemma 2.7.1, each maximal ideal $\mathfrak{m}'$ in R' contracts to a maximal ideal in R, and by the Incomparability Theorem (2.2.3), $\mathfrak{m}'$ is minimal over $(\mathfrak{m}' \cap R)R'$. Since R' is Noetherian, for every maximal ideal $\mathfrak{m}$ of R, there are only finitely many prime ideals in R' minimal over $\mathfrak{m}R'$, which shows that R' is semilocal. Let T' be obtained from R' in the same way that T is constructed from R. Since $\widehat{R} \subseteq \widehat{R'}$ is module-finite, necessarily $T \subseteq T'$ and $T^+ \subseteq (T')^+$. Since y is not in any minimal prime ideal of R^+, it is not in any minimal prime ideal of R', and so not in any minimal prime ideal of T'. Thus $\alpha \in Q(T')$ is integral over T and hence over T', so that $\alpha \in \overline{T'}$. Thus $\alpha \in Q(R') \cap \overline{T'}$, and by the previous paragraph, $\alpha \in \overline{R'}$. It follows that $\alpha \in R^+$. □

p. 85, line 3 of 4.9.4: discrete valuation rings have not yet been defined. Change "Then $\overline{R}_Q$ is a discrete valuation ring of rank one between R and K" to Change "Then $\overline{R}_Q$ is a one-dimensional Noetherian integrally closed domain". Add in the proof: "The rest is the Krull–Akuziki Theorem (Theorem 4.9.2)."

p. 85, line 3 of proof of 4.9.5: change "Then" to ", then". In line 5, $x \in J$ rather than $x \in R$. Change "the conductor" in the last line to "J".

p. 87, the proof of the Mori–Nagata Theorem has a big gap (in the last paragraph). The corrected new proof is written below completely, even though some parts are repetitions of the old proof. (A reader familiar with valuations can make shortcuts in this proof.)

Let $P_1, \ldots, P_r$ be the minimal prime ideals of R. Then the total ring of fractions of R is $K_1 \times \cdots \times K_r$, where K_i is the field of fractions of R/P_i. By Corollary 2.1.13, the integral closure of R is the direct product of the integral closures of the R/P_i in K_i. Thus it suffices to prove that each integral closure of R/P_i is a Krull domain. This reduces the proof to the case where R is a Noetherian domain.

By Lemma 4.9.4, $\overline{R}$ satisfies the first property of Krull domains.

For an arbitrary non-zero $x \in \overline{R}$, there exists a non-zero $y \in R$ such that $yx \in R$. Lemma 4.9.5 proves that the contractions of height-one primes minimal over $xy\overline{R}$ are among the associated primes of xyR. There are only finitely many such contracted ideals. By Proposition 4.8.2 there are then only finitely many prime ideals in $\overline{R}$ that lie over these contractions, so that xy and hence x are contained in only finitely many prime ideals of $\overline{R}$ of height one. This proves that $\overline{R}$ satisfies the third property.

It remains to prove that $\overline{R}$ satisfies the second property of Krull domains. Suppose that the second property holds for $\overline{R}$ if R is a local domain. Then $\overline{R_\mathfrak{m}} = \bigcap_P (\overline{R_\mathfrak{m}})_P$, where P varies over all the height-one prime ideals in $\overline{R_\mathfrak{m}}$. Since $\overline{R_\mathfrak{m}}$ is a localization of $\overline{R}$, each such $(\overline{R_\mathfrak{m}})_P$ is a localization of $\overline{R}$ at a unique corresponding prime ideal Q in $\overline{R}$ of height one. Also, every prime ideal Q in $\overline{R}$ of height one contracts to a prime ideal in R contained in some maximal ideal $\mathfrak{m}$ of R, whence $\overline{R}_Q$ is a localization of $\overline{R_\mathfrak{m}}$ at a height-one prime ideal. Thus $\bigcap_\mathfrak{m} \overline{R_\mathfrak{m}} = \bigcap_Q \overline{R}_Q$, where Q varies over all the height-one prime ideals in $\overline{R}$, and $\mathfrak{m}$ varies over all the maximal ideals in R. Clearly $\overline{R} \subseteq \overline{R_\mathfrak{m}}$ for each maximal ideal $\mathfrak{m}$. If $\alpha \in \bigcap_\mathfrak{m} \overline{R_\mathfrak{m}}$, then the ideal $(\overline{R} :_R \alpha)$ is not contained in any maximal ideal of R, hence it must contain 1, whence $\alpha \in \overline{R}$. This proves that $\overline{R} = \bigcap_\mathfrak{m} \overline{R_\mathfrak{m}}$ as $\mathfrak{m}$ varies over all the maximal ideals of R, whence by what we already proved, $\overline{R} = \bigcap_Q \overline{R}_Q$, where Q varies over all the height-one prime ideals in $\overline{R}$. Thus it remains to prove the second property of Krull domains for $\overline{R}$ under the assumption that R is a local domain.

Let $\widehat{R}$ be the completion of R in the topology determined by the maximal ideal. Let $Q_1, \ldots, Q_s$ be all the minimal prime ideals in $\widehat{R}$. Set $T_i = \widehat{R}/Q_i$. By Proposition 4.7.3, R embeds canonically in $T = T_1 \times \cdots \times T_s$, the field of fractions K of R is contained in the total ring of fractions L of T, and $\overline{R} = \overline{T} \cap K$. Certainly $\overline{R} \subseteq \cap_Q \overline{T}_Q$ as Q varies over all the height-one prime ideals in $\overline{T}$. Now let $\alpha \in \cap_Q \overline{T}_Q \cap K$. By the form of prime ideals in direct sums, for each $i = 1, \ldots, s$, $\alpha \in \cap_Q (\overline{T}_i)_Q$, where Q varies over all the height-one prime ideals in $\overline{T}_i$. By Theorem 4.3.4 and by Proposition 4.10.4, $\overline{T}_i$ is a Krull domain, so that $\alpha \in \overline{T}_i$ for all i. By Proposition 2.1.16, $\alpha \in \overline{T}$ (the role of R in that proposition is played by T here, and the role of S is played by L). Hence $\alpha \in \overline{T} \cap K = \overline{R}$. This proves that $\overline{R} = \bigcap_Q (\overline{T}_Q \cap K)$, where Q varies over height-one prime ideals in $\overline{T}$.

We next prove that for any non-zero element b in $\overline{R}$ and any prime ideal P in $\overline{R}$ containing b there exists a height-one prime ideal in $\overline{R}$ contained in P and containing b. Let S_0 consist of the height-one prime ideals in $\overline{T}$ that contain b. Then S_0 is a finite set, and $b\overline{R} = \bigcap_Q (b\overline{T}_Q \cap \overline{R}) = \bigcap_{Q \in S_0} (b\overline{T}_Q \cap \overline{R})$. Each $b\overline{T}_Q$ is primary, hence so is each $b\overline{T}_Q \cap \overline{R}$, and by possibly merging and omitting we get an irredundant primary decomposition $b\overline{R} = q_1 \cap \cdots \cap q_s$, with each $\sqrt{q_i}$ contracted from at least one prime ideal in $\overline{T}$. By Lemma 4.8.4 there exists a Noetherian ring A between R and $\overline{R}$ such that with $p = P \cap A$, $\overline{R}_P$ is the integral closure of A_p. Note that $A[b]$ is Noetherian and that the integral closure of $A[b]_{P \cap A[b]}$ is $\overline{R}_P$, so by possibly changing notation we may assume that $b \in A$. By Propositions 1.6.1 and 1.5.2, $\overline{bA_p} = \overline{b\overline{R}_P} \cap A_p = b\overline{R}_P \cap A_p$, which is the intersection of all the $q_i \overline{R}_P \cap A_p$. Since A is Noetherian, $\overline{bA_p}$ has height one, whence some $q_i \overline{R}_P \cap A_p$ has height one. It follows by Theorem B.2.5 (Dimension Inequality) that $\operatorname{ht} q_i \overline{R}_P \le 1$, and so $q_i \subseteq P$ and $\operatorname{ht} q_i \le 1$. Since $b \in q_i$, necessarily q_i and the prime ideal $\sqrt{q_i}$ have height one.

We next prove that for any non-zero b in $\overline{R}$, all primary components of $b\overline{R}$ have height 1. We use notation as in the previous paragraph. Suppose for contradiction that the height of $p_1 = \sqrt{q_1}$ is not 1. By Exercise 4.30, $\overline{R}_{p_1} = \bigcap (\overline{T}_Q \cap K)$, where Q varies over those height-one prime ideals of $\overline{T}$ for which $Q \cap \overline{R} \subseteq p_1$. First suppose that each such Q contains b. Then $(\overline{R}_{p_1})_b = \bigcap ((\overline{T}_Q)_b \cap K) = K$, so that b is contained in every non-zero prime ideal of $\overline{R}_{p_1}$. By the established third property of Krull domains, this means that $\overline{R}_{p_1}$ has only finitely many height-one prime ideals. By the Prime Avoidance Theorem (A.1.1), there is $x \in p_1$ not contained in any of these height-one prime ideals, contradicting the previous paragraph. Thus necessarily there is a height-one prime ideal Q_0 in $\overline{T}$ such that $Q_0 \cap \overline{R} \subseteq p_1$ and $b \notin Q_0$. Set $P = Q_0 \cap \overline{R}$. Since the given primary decomposition of $b\overline{R}$ is contracted from a primary decomposition in a Noetherian ring, $p_1^t \subseteq q_1$ for some positive integer t. By irredundancy there is a possibly smaller positive integer t such that $p_1^t \cap q_2 \cap \cdots \cap q_s \subseteq b\overline{R}$ and $p_1^{t-1} \cap q_2 \cap \cdots \cap q_s \not\subseteq b\overline{R}$. Let $c \in p_1^{t-1} \cap q_2 \cap \cdots \cap q_s \setminus b\overline{R}$. Then $cP \subseteq cp_1 \subseteq b\overline{R}$. As b is a unit in $\overline{T}_{Q_0}$, it follows that $\frac{c}{b}P \subseteq \overline{R} \cap Q_0 = P$. We will get a contradiction when we prove that $\frac{c}{b} \in \overline{R}$, thus establishing that all primary components of $b\overline{R}$ have height one. Namely, for any $d \in P$ and any positive integer n, $(\frac{c}{b})^n d \in P \subseteq \overline{R}$. By Proposition 4.1.3, for each height-one prime Q in $\overline{T}$, there is $y \in \overline{T}$ such that $y\overline{T}_Q = Q\overline{T}_Q$, and we can write $d = uy^i$ and $\frac{c}{b} = vy^j$ for some integers i, j and some units u, v in $\overline{T}_Q$. Since $(\frac{c}{b})^n d = uv^n y^{i+nj} \in \overline{T}_Q$, necessarily $i + nj \geq 0$ for all n, whence $j \geq 0$, which says that $\frac{c}{b} \in \overline{T}_Q$. Since this holds for all Q, we get that $\frac{c}{b} \in \overline{R}$, which is the needed contradiction.

It follows that $b\overline{R} = \cap_P b\overline{R}_P \cap \overline{R}$, as P varies over height-one primes containing b, or even as P varies over all the height-one prime ideals in $\overline{R}$.

If $\frac{a}{b} \in \cap_P \overline{R}_P$, where P varies over the height-one prime ideals in $\overline{R}$, then $a \in \cap_P b\overline{R}_P \cap \overline{R} = b\overline{R}$, so $\frac{a}{b} \in \overline{R}$. This finishes the proof of the theorem. □

p. 89, line 1 of Exercise 4.9: remove "positive prime"; line 3: remove "of elements"

p. 90, Exercises 4.11: the summation on line 3 should start with 1, not with 0.

p. 92 (in the latest version p. 93): added a new exercise, which is used in the new proof of the Mori–Nagata Theorem:

Exercise 4.30 Let R be a domain contained in a field L. Assume that $R = \bigcap_{V \in S} V$, where S is a collection of one-dimensional integrally closed Noetherian domains contained in L such that every non-zero element of R is a non-unit in at most finitely many elements of S. Prove that for any multiplicatively closed subset W of R, $W^{-1}R = \bigcap_{V \in S_0} V$, where S_0 consists of those elements V of S in which all elements of W are units.

p. 94, below 5.1.5: The first part of "Clearly" is redundant.

p. 94, 5.1.6: the maximal ideal M also needs to contain $\mathfrak{m}/I$, not just all homogeneous elements of positive degree.

p. 95, paragraph above Section 5.2: change discussion of "general fiber". New: consider setting t^{-1} to a unit of R.

p. 96, 5.2.3: change the superscript on $\mathbb{Z}$ from "d" to "e".

p. 96, proof of 5.2.3: no need to pass to $\overline{R}[t]$, passing to $R[t]$ suffices.

p. 97, line 1 of the proof of 5.2.5: R should be S.

p. 99, Proposition 5.3.4: Add "domain" in front of the first "R". All calls to 5.3.4 have the domain assumption. (And a touch-up in the proof.) In addition, in part (3), need to name the base ring over which R is finitely generated as an algebra as A.

p. 105, line 2 of proof of 5.5.4: Replace "G" with "$G(T_1, \dots, T_n)$". In line 6 of the same proof, replace "$F = G$" with "$G = F$", and in line 7, replace "weight of F" with "weight of F_1".

p. 116, line 1: remove the second "V"

p. 116, line 1 of proof of 6.2.5: specify that m, n are integers and that m is positive.

p. 117, line 2 of 6.3.1: x has to be in K^*, not in K.

p. 118, line 1: add that we switch notation and call P now $\mathfrak{m}$.

p. 118, line -6: "hence T_M is a Noetherian valuation domain" uses a subsequent 6.4.4. Add "by Proposition 6.4.4" and switch sections 6.3 and 6.4.

p. 119, Lemma 6.4.3: Add the assumption that I is an ideal in R; in (1), state that m is a positive integer; modify the proof accordingly; in (2), assume that $u_1, \dots, u_n$ are units in R and that their differences are not in any $\mathfrak{m}_i \cap R$.

p. 121, 6.4.5: change all R to V: so Γ_V will make sense.

p. 122, line -2: change period to comma.

p. 127, Theorem 6.6.7, (2): the exponent on $\mathbb{Z}$ should be the rational rank, not n.

p. 130, paragraph below Example 6.7.1, add "alter" after "could" to get "Notice that we could alter this valuation..."

p. 131, Example 6.7.5: the generalized power series $\sum_n a_n t^{e_n}$ with strictly increasing rational e_n also need the assumption $\lim e_n = \infty$. Otherwise, one could not add $\sum_n t^{1-1/n}$ to t.

p. 131, Example 6.7.6: Summation in the third line should have index n rather than i. The construction of f_n is wrong. Here is a correction: "Let $k = \mathbb{R}$, X and Y variables over $\mathbb{R}$, V and v as in the previous example, and the map $k[X, Y] \to V$ sends X to t and Y to $\sum_{n=2}^{\infty} t^{e_n}$, with $e_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$. For any subset S of $\mathbb{Q}$ and for any $n \in \mathbb{N}$, let $nS = \{s_1 + \cdots + s_n : s_i \in S\}$. Set $S_1 = T_1 = \{e_n : n \geq 2\}$, and for $n \geq 2$, set $T_n = nS_{n-1}$ and $S_n = T_n \setminus \{\min\{T_n\}\}$. The elements of S_n and T_n are (some) sums of $n!$ elements in S_1. By induction on n and $k = 2, \dots, n$, T_n excludes exactly those $n!$-sums for which $(i-1)((i+1)(i+2)\cdots n)$ summands are e_i for $i = 2, \dots, k-1$ and for which strictly more than $(k-1)((k+1)(k+2)\cdots n)$ summands are e_k. The minimum element of T_n is $\sum_{k=2}^{n-1}(k-1)((k+1)(k+2)\cdots n)e_k + ne_n$, and $\min(S_n) = \sum_{k=2}^{n-1}(k-1)((k+1)(k+2)\cdots n)e_k + (n-1)e_n + e_{n+1}$, which equals an integer c_n plus $\frac{1}{n+1}$. Now set $f_0 = X$, $f_1 = Y$. If $f_{n-1} \in k[X, Y]$ has the image in V a power series in t with positive coefficients and with exponents exactly the elements of S_{n-1}, then $v(f_{n-1}^n)$ is the integer $nc_{n-1} + 1$, and for some $c \in k$, $f_n = f_{n-1}^n - cX^{nc_{n-1}+1} \in k[X, Y]$ maps to a power series in t whose exponents are exactly the elements of S_n. Thus $v(f_n) = \min S_n = c_n + \frac{1}{n+1}$. Hence for all $n \in \mathbb{N}$, f_n/X^{c_n} has v-value $\frac{1}{n+1}$, whence the value group of v contains $\mathbb{Q}$, and thus equals $\mathbb{Q}$."

p. 135, Corollary 6.8.6: don't need the Noetherian assumption; this was already proved in Remark 1.3.2 (4).

p. 135, Corollary 6.8.7: don't need the Noetherian assumption, and I should not be 0.

p. 136, line -2, and p. 137, lines 1 and 3: change $\dots$ to $\cdots$. After the first sentence on page 137, add: "(If $k = l$, take $e = 1$.)"
p. 138, proof of 6.9.1, line 4 from qed up: remove a stray $\rfloor$.
p. 138, bottom line: "Thus $u \geq N$" should read "Thus $\liminf \operatorname{ord}_I(x^n)/n \geq N$."
p. 142, (ii) of Exercise 6.24: Change the second sentence to: "Prove that there are infinitely many K-valuations v such that for each n, v has center $\mathfrak{m}_n$ on R_n, and that there is at least one such valuation for which the residue field has positive transcendence degree over each $R_n/\mathfrak{m}_n$." The reference for the exercise should be Corollary of Theorem 10 on p. 21 Zariski–Samuel Volume II.
p. 142, line 4 of Exercise 6.25: "Watanabe" should be hyphenated differently.
p. 151, definition of I: change X_{i+1} into X_i; many subscripts j in M_j should be changed to i; K_{M_i} should be localized at M_i also in the rewriting. Also, in the definition of I, $X_1 \cdots X_i Y_i Z_i^{i-1}$ rather than $X_1 \cdots X_{i+1} Y_i Z_i^{i-1}$; in the definition of K, end with $Y_{i-2} Z_{i-2}^{i-3} - Y_{i-2}^{i-2}$ rather than $Y_{i-3} Z_{i-3}^{i-4} - Y_{i-3}^{i-3}$.
p. 151, line 1: replace N_I by $\mathcal{F}_I(R)$.
p. 154, line 2 of Corollary 8.2.2: change the order of "homogeneous" and "minimal".
p. 157, Corollary 8.2.5: The minimal number of generators is **at least** $\ell(I)$.
p. 157, line 2: uses existence of minimal reductions, and this is only proved in subsequent 8.3.6. Therefore, move the block starting with the paragraph before 8.3.6 and ending with the paragraph after the proof of 8.3.6 to after Example 8.3.4.
p. 157, Proposition 8.3.7: the assumption on the infinite residue field appears twice
p. 163, line 6: $c \leq k < n$ should be $c \leq k < n - m$
p. 165, Corollary 8.6.2: x^* is the image of x in $I/I^2 \subseteq \operatorname{gr}_I(R)$.
p. 173, Exercise 8.3: Add the assumption that the ring is Noetherian local.
p. 184, line 4 of Lemma 9.3.3: "dormula" should be "formula"
p. 188, line 2 below second display: add parentheses to get $\cap_{i=1}^r (I^n V_i \cap R)$.
p. 190, in Theorem 10.1.6, the Noetherian assumption is not needed.
p. 193, in proof of Proposition 10.2.2: Need to clarify what happens to V that do contain $\overline{S}_1$. For that, need Lemma 10.2.1.
p. 193, in Proposition 10.2.3: the choice of m has to be more careful, and there is a typographical error on the exponent of I in the display: "Let v_0 be the integer-valued valuation corresponding to V_0. Since $t = nv_0(I) - v_0(r)$ is positive, there exists m such that $v_0(c) < tm$. But $cr^m \in \bigcap_{V \neq V_0} I^{mn} V \cap R$, so that $\overline{I^{mn}} = \bigcap_{V \in T}^r I^{mn} V \cap R$ is properly contained in $\bigcap_{V \neq V_0} I^{mn} V \cap R$. Hence V_0 is not redundant." This finishes the proof.
p. 194, Proposition 10.2.5: the u_i need to be units in R.
p. 195, proof of Corollary 10.2.7: delete "since I"
p. 203, line 5 of Example 10.4.6: "a monomial valuation" should be "the monomial valuation".
p. 209, Exercise 10.12: changed "an an", and corrected the definition of $\mathfrak{m}$-full when the residue field is not infinite by inserting a new exercise.
p. 210, lines 1 and 2 (exercise 10.18 (ii)): "that are primary to $(X_1, \dots, X_n)$ and" and "with" are redundant. Remove.
p. 210, Exercise 10.19 (ii): parenthetical "integer-valued" should be "normalized"

p. 210, Exercise 10.20: added reference to Hübl–Swanson [132]. i (ii): parenthetical "integer-valued" should be "normalized"

p. 211, added two exercises:

Exercise 10.34: Let R be a Noetherian ring and I an ideal in R that is not contained in any minimal prime ideal of R. Let I_α be any rational power of I.

(i) Prove that $\mathcal{RV}(I_\alpha) = \mathcal{RV}(I)$.

(ii) Prove that for large integers n, $\mathrm{Ass}(R/I_{n\alpha}) = \mathrm{Ass}(R/\overline{I^n})$.

Exercise 10.35: Let $(R, \mathfrak{m})$ be a Noetherian analytically irreducible local ring and I an $\mathfrak{m}$-primary ideal. Prove that $|\mathcal{RV}(I)| = |\mathcal{RV}(I\widehat{R})|$.

p. 214, after Definition 11.1.5: the Hilbert polynomial $P_{I,M}(n)$ as written is actually $P_{I,M}(n+1)$. For uniformity of notation, we therefore change all binomials of the form $\binom{t+i}{i}$ to form $\binom{t-1+i}{i}$: this affects the paragraph above Lemma 11.1.1, the statement and proof of Lemma 11.1.1, the proof of Lemma 11.1.2, Discussion 11.3.6, and Exercise 11.1.

p. 218, Theorem 11.2.4 and p. 342, Theorem 17.4.8: "Associativity Formula" is changed to "Additivity and Reduction Formula". This terminology agrees with Nagata's. (This change also forced changes in referring to these theorems; see pages 220, 222, 230, 315, 342, 343, 346, 347, 348, 349, and the Index pages.)

p. 222, last line: delete "both the rings ... and ... are finitely generated over R, and"

p. 232: In each of Exercises 11.3 and 11.4, an extra closing parenthesis is needed in the last line of the exercise.

p. 238, proof of (3) of Proposition 12.3.2: We cannot immediately conclude that the height of $(g_1, \ldots, g_n)$ is n: we could multiply all g_i be a non-unit not in P. We only need that locally at P the height of $(g_1, \ldots, g_n)$ is n, but that is true since $R_{P\cap R}$ is a regular ring.

p. 248: paragraph above 13.2.1: Originally it was stated that in a two-dimensional regular local ring an ideal with a two-generated reduction has reduction number two. The correct statement is that every integrally closed ideal with a two-generated reduction has reduction number one. (This affects two sentences.)

p. 252, proof of 13.3.3, second paragraph: the two instances of R should be B.

p. 260, Definition 14.1.5 of m-full does not conform to Junzo Watanabe's original definition. Correct definitions are in exercises 10.11–10.13.

p. 257, statement of 14.4.4: add "integrally" before the second "closed"

p. 266, first sentence of the last paragraph of the section: this only holds for divisorial valuations!

p. 268, third line of the statement of 14.3.7: $S = R[\frac{\mathfrak{m}}{x}]$ rather than $S = R[\frac{y}{x}]$ (it is the same thing, IF y is defined so that $\mathfrak{m} = (x, y)$).

p. 268, line 14 of the proof of Proposition 14.3.7: change K to J.

p. 269, second line of the proof of 14.4.3: We need to check that $I^S = \overline{I^S}$ (rather than $\overline{I}^S$). Also, the part involving J needs to state that J is $\mathfrak{m}$-primary, and when applied to I times a power of $\mathfrak{m}$, this holds.

p. 272, line 11 (on third to the last line in the first paragraph): the second r should be r_i, other r remain r.

p. 272, Lemma 14.4.7: weaken the "regular local ring" assumption to "Noetherian local domain"

p. 272, third line of the proof of Lemma 14.4.7: write "and zero on $R[X] \setminus mR[X]$" rather than "and zero on $R[X] \setminus \{0\}$".

p. 272: line 4 of the second paragraph of the proof of 14.4.7: move one closing parenthesis from end of the line to the beginning of the line

p. 273, the last four lines of the proof of Theorem 14.4.8 have a gap. Here is the corrected version. "It remains to prove that $I^nV \cap R \subseteq \mathfrak{m}^{n \operatorname{ord} I}$. If not, then $b = n \operatorname{ord} I - \operatorname{ord}(I^nV \cap R)$ is positive, and I^n and $\mathfrak{m}^b(I^nV \cap R)$ have the same orders. Also, $\mathfrak{m}^b(I^nV \cap R) \subseteq (I^nV \cap R) \cap \mathfrak{m}^{n \operatorname{ord} I} = I^n$, so that $\deg(c(I^n)) \le \deg(c(\mathfrak{m}^b(I^nV \cap R))) = \deg(c(I^nV \cap R)) \le \operatorname{ord}(I^nV \cap R) < \operatorname{ord}(I^n)$. But then by Proposition 14.1.12, $I^n = \mathfrak{m}J$ for some ideal J, whence by unique factorization of integrally closed ideals, $I = \mathfrak{m}J'$ for some ideal J'. But this contradicts the assumption that I is simple."

p. 274, Definition 14.5.1 and Theorem 14.5.2. The count $1, \ldots, n-1$ should be replaced by $1, \ldots, n$.

p. 277, introduction to Section 14.6: note that the restriction to infinite residue field is not a severe restriction

p. 278, last display in the proof of 14.6.1: replace tr.deg by λ.

p. 279, the last part of the proof of 14.6.2: starting with "whose integral closure is the valuation ring corresponding to v'." on line 9, replace the rest of the proof by: "Thus $\Delta(v') = \lambda_{\kappa(\mathfrak{m}_{C'})}(\kappa(\mathfrak{m}_{\overline{C'}})) = \lambda_{\kappa(\mathfrak{m}_C)}(\kappa(\mathfrak{m}_{\overline{C}})) = \Delta(\overline{v}_I)$. (Here, $\mathfrak{m}_A$ denotes the unique maximal ideal of the ring A.) Since $(c,d)R'$ is a reduction of JR', by Proposition 14.6.1, $v'(JR') = v'((c,d)R') = v'(cY - d)$, so by Proposition 6.8.1,

$$\begin{aligned} \overline{v}_I(J)\Delta(\overline{v}_I) &= v'(J)\Delta(v') = v'(cY-d)\Delta(v') = \lambda_{C'}(C'/(cY-d)C') \\ &= \lambda(R(X,Y)/(bX-a, cY-d)). \end{aligned} \qquad \square$$

p. 283, on the line with "(3)": add "and cR".

p. 284, line 4 of 15.1.2: "the one" should be "that one", insert "can" before "insert".

In chapter 15 on computation, removed two instances of "absolute integral closure"– that was non-standardly referring to the integral closure of the ring in its field of fractions.

p. 286, line 18: change "when" to "where".

p. 292, Lemma 15.3.1 needs the assumption that J be non-zero.

p. 292, Theorem 15.3.2 needs the assumption that J be integrally closed and non-zero.

p. 292, last line: remove "Then".

p. 300, Exercise 15.18: the field needs to have characteristic 0, otherwise there are straightforward counterexamples.

p. 300, Exercise 15.19: change comma at the end of the first line to period. More importantly, $I \cap k(S)[T]$ should be $Ik(S)[T] \cap k[\underline{X}]$.

p. 306, proof of Theorem 16.2.3: perhaps there is no well-known structure theorem of finitely generated submodules of free modules over valuation domains. In the new version we provide a detailed and simple argument for what is needed.

p. 307, line 2-3 of 16.2.4: it should be Eisenbud, Huneke, and ULRICH.

p. 310, line 2 of 16.4.5 (2): remove "and if".

p. 329, line 2 of Exercise 16.13: $L = \cap_g \ker(\operatorname{Sym} M \xrightarrow{\operatorname{Sym} g} \operatorname{Sym} F)$ (in the text, "ker" is missing).

p. 329, in Exercise 16.16, clarify that the definition of Rees algebras that depends on the embedding is 16.2.1.

p. 336, statement of Proposition 17.3.2: in line 2, the indices on the a should be $a_{i1}, \ldots, a_{il_i}$ (rather than $a_{11}, \ldots, a_{1l_i}$), and in the last line, the indices on the first l_1 u should be $u_{11}, \ldots, u_{1l_1}$ (rather than $u_{i1}, \ldots, u_{il_i}$).

p. 337, statement of Proposition 17.3.3: No need to introduce ideals I_i.

p. 354, statement (2) of Corollary 17.7.3: Replace $e(I^{[d-i]}, J^{[i]}; M)$ by $e(I^{[i]}, J^{[d-i]}; M)$.

p. 371, third display: remove "big"

p. 392: all F_+ should be F^+ (positive parts of the fields), and actually, a different version is needed in Chapter 1. Here is the new statement with proof:

Theorem A.2.1 (Carathéodory's theorem) Let n be a positive integer, and $v_1, \ldots, v_r \in (\mathbb{R}_{\geq 0})^n$. Suppose that for some $a_i \in \mathbb{R}_{\geq 0}$, $v = \sum_i a_i v_i$. Then there exists a linearly independent subset $\{v_{i_1}, \ldots, v_{i_s}\}$ such that

$$v = \sum_j b_j v_{i_j}, b_j \geq 0, \text{ and } \sum_j b_j \geq \sum_j a_j.$$

The same result also holds if $\mathbb{Q}$ is used instead of $\mathbb{R}$.

Proof: If $v_1, \ldots, v_r$ are linearly independent, there is nothing to show. So assume that there exist $c_i \in \mathbb{R}$, not all zero, such that $\sum_i c_i v_i = 0$. Necessarily some c_i are positive and some are negative. By possibly multiplying by -1 and by reindexing we may assume that $\sum c_i \leq 0$ and that $c_1 > 0$. Note that for any i such that $c_i > 0$, $v = \sum_{j \neq i}(a_j - a_i \frac{c_j}{c_i}) v_j$ and $\sum_{j \neq i}(a_j - a_i \frac{c_j}{c_i}) = \sum_j a_j - \frac{a_i}{c_i} \sum_j c_j \geq \sum_j a_j$. By induction on r it suffices to show that v can be written as a linear combination of $r-1$ of the v_i, with non-negative coefficients that add up to a number greater than or equal to $\sum a_j$. For contradiction we assume the contrary. Then $a_j > 0$ for all j and from the previous paragraph applied to the case $i = 1$ we deduce that for some $j > 1$, $a_j - a_1 \frac{c_j}{c_1} < 0$. After reindexing without loss of generality $j = 2$. This implies that $c_2 > 0$. Next we apply the previous paragraph with $i = 2$ and either the conclusion holds or there exists $j \neq 2$ such that $a_j - a_2 \frac{c_j}{c_2} < 0$. Necessarily $j > 2$, $c_j > 0$, and by reindexing $j = 3$. We continue this process, and by the assumption we can only stop at the rth step to get all c_i positive, which is a contradiction. □

p. 393, Propositions A.3.1, A.3.2: the monoids should be totally ordered, not well-ordered.

p. 393, Proposition A.3.1, line 2: ideal I is not needed at all.

p. 395, in the Hilbert-Burch theorem, the matrix should be $n \times (n-1)$ rather than $(n-1) \times n$. The ideal generated by $(d_1, \ldots, d_n)$ should contain a non-zerodivisor.

p. 396, in the three-line display in the proof of Lemma A.5.1, we do not need a comma in line 1.

p. 405: references 7 and 8 should be switched. There is also a new reference: J. T. Arnold and R. Gilmer, On the contents of polynomials, *Proc. Amer. Math. Soc.* **24** 1970, 556–562.

p. 407: references 40, 41, 42, should be 42, 40, 41; references 51 and 52 should be switched.

p. 31: reference [111] should be W. Heinzer and C. Huneke, The Dedekind-Mertens lemma and the contents of polynomials, *Proc. Amer. Math. Soc.* **126** (1998), 1305–1309.

Back cover: “multiplicty” should be “multiplicity”