Erratum to

INTEGRAL CLOSURE OF IDEALS IN EXCELLENT LOCAL RINGS

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This paper was published in *J. Algebra*, **187** (1997), 422-445. We are grateful to Ray Heitmann for pointing out that Theorem 2.7 in the published version is wrong. The proof of the main theorem of the published paper used Theorem 2.7. Here we give new proofs of the main theorem as well as of some intermediate results. We point out that the main results still prove special cases of the Linear Artin Approximation Theorem.

The main theorem. Let (R, m) be an excellent local ring. Let I be an ideal of R. Then there exists a positive integer c such that

$$\overline{I+m^n} \subseteq \overline{I} + m^{\lfloor n/c \rfloor} \text{ for all } n.$$

As in Section 2 of [1], the proof of this theorem reduces to the case where (R, m) is a complete local normal integral domain and I is principal. However, contrary to the claims in [1], we may not assume that I is a radical ideal. In fact, Theorems 2.7 and 2.8 should be cut out of [1].

The following is a slight (and needed) generalization of [1, Theorem 3.9]. The proof here is essentially the same as the one in [1], only more direct.

Theorem 3.9. *Let (R, m) be a complete normal local domain and fR a non-zero principal ideal. In the case when R does not contain a field, we let p be a generator of the maximal ideal in a coefficient ring for R, and we assume that f satisfies one of the following properties: (i) f, p is a part of a system of parameters, or (ii) $f = ap^c$ for some positive integer c and some element a of R not contained in any minimal prime ideal over pR.*

Then there exist integers d and l such that for each n, every element in $\overline{fR+m^n}$ satisfies an integral equation of degree d over $fR + m^{\lfloor n/l \rfloor}$.

Proof: It is sufficient to prove that if JR is m-primary, then there exists an integer d such that for each n, every element in $\overline{fR+J^nR}$ satisfies an integral equation of degree d over $fR + J^{\lfloor n/d \rfloor}R$. (Note, however, that d depends on J!)

We use the Cohen Structure Theorem. Let $f_1, \ldots, f_l$ be a system of parameters in R. When R contains a coefficient field k, we may assume that $f_1 = f$, and we define $A = k[[f_1, \ldots, f_l]]$. When R contains a coefficient ring $(V, (p))$ of dimension 1, we may assume that p is f_1. In case (i) we may also assume that $f = f_2$ and in case (ii) we may assume that f_2 is a if a is not a unit. In case (ii) if a is a unit, as $fR = p^cR$, without loss of generality $a = 1$. We then define $A = V[[f_2, \ldots, f_l]]$. In either case, set $J = (f_1, \ldots, f_l)A$.

By the Cohen Structure Theorem, A is a regular local ring contained in R, R is module-finite over A, and JR is m-primary. We will prove the theorem for this JR. Furthermore, we will prove that the integral equation of degree d will have coefficients in A.

Let K be the fraction field of A and L the fraction field of R. By elementary field theory there exist fields L' and F such that all the inclusions $K \subseteq F \subseteq L'$ and $L \subseteq L'$ are finite, such that L' is Galois over F and such that F is purely inseparable over K. To simplify notation, as the coefficients of the integral equation will actually lie in A, we may replace R by the integral closure of R in L' and so we may assume that $L = L'$. Let $c = [L : F]$ and $e = [F : K]$. Let S be the integral closure of A in F. Then S is a complete normal local domain between A and R and the extension from S to R is Galois.

Let $u \in \overline{fR + (JR)^n}$. Consider the (at most) c conjugates of u over S, say $u = u_1, u_2, \ldots, u_c$. Write an integral equation for u over $fR + (JR)^n$:

$$u^k + \alpha_1 u^{k-1} + \alpha_2 u^{k-2} + \cdots + \alpha_k = 0$$

with $\alpha_i \in (fR + (JR)^n)^i$. By applying field automorphisms to this equation and by using the fact that (f) and J are ideals of A (and thus of S), we obtain that each u_i is integral over $fR + J^n R$. Let s_h be the sum of the products of the u_i, taken h at a time (hth symmetric function in the u_i). Then

$$u^c - s_1 u^{c-1} + \cdots + (-1)^c s_c = 0,$$

and $s_h \in \overline{(fR + (JR)^n)^h} \cap S$. We raise all this to the eth power. As e is either 1 or a power of the characteristic p of the given fields, we obtain

$$\begin{aligned} u^{ce} &- s_1^e u^{e(c-1)} + \cdots + (-1)^{ce} s_c^e = 0, \quad \text{and} \\ s_h^e &\in \overline{(fR + (JR)^n)^h}^e \cap A \\ &\subseteq \overline{(f^{he}R + (JR)^{nhe})} \cap A \\ &\subseteq \overline{(f^{he}A + (JA)^{nhe})}, \end{aligned}$$

as $A \subseteq S$ is a module-finite extension. By Propositions 3.2 and 3.8, and by Corollary 3.4 of [1], then there exists an integer l such that

$$s_h^e \in f^{he}A + (JA)^{\lfloor n/l \rfloor} \subseteq (fA + (JA)^{\lfloor n/lhe \rfloor})^{he} \subseteq (fA + (JA)^{\lfloor n/lce \rfloor})^{he}.$$

Thus u satisfies an equation of integral dependence of degree ce over $fR + (JR)^{\lfloor n/lce \rfloor}$, all of whose coefficients are in A. ∎

The following proposition has no analogue in [1]. It is crucial for the inductive argument in the new proof of the main theorem:

Proposition A. *Let (R, m) be a Noetherian local integrally closed integral domain, and $f \in R$ satisfying the following:*

1. *There exists a positive integer c such that for all $n \geq 1$, $\overline{(f) + m^n} \subseteq (f) + m^{\lfloor n/c \rfloor}$.*

2. *For every $k = 1, \ldots, N$ there exist positive integers d and l such that for all n, every element of $\overline{(f^k) + m^n}$ satisfies an equation of integral dependence of degree d over $(f) + m^{\lfloor n/l \rfloor}$.*

Then for every $k = 1, \ldots, N$, there exists a positive integer c such that $\overline{(f^k) + m^n} \subseteq (f^k) + m^{\lfloor n/c \rfloor}$.

Proof: We prove this by induction on k. The case $k = 1$ is assumed. So assume $k > 1$. By induction, $\overline{(f^k) + m^n} \subseteq (f^{k-1}) + m^{\lfloor n/c' \rfloor}$ for some constant c' independent of n. We pick an element u in $\overline{(f^k) + m^n}$. Write $u = rf^{k-1} + s$ for some $r \in R$ and $s \in m^{\lfloor n/c' \rfloor}$. It suffices to prove that rf^{k-1} lies in $(f^k) + m^{\lfloor n/c \rfloor}$ for some c independent of n and u. Note that rf^{k-1} is integral over $(f^k) + m^{\lfloor n/c' \rfloor}$. Hence it suffices to prove that $(f^{k-1}) \cap \overline{(f^k) + m^{\lfloor n/c' \rfloor}}$ is contained in $(f^k) + m^{\lfloor n/c \rfloor}$ for some c independent of n, or even that $(f^{k-1}) \cap \overline{(f^k) + m^n}$ is contained in $(f^k) + m^{\lfloor n/c \rfloor}$ for some c independent of n. Thus without loss of generality we may assume that $u = rf^{k-1}$. Our goal is to prove that $r \in \overline{(f) + m^{\lfloor n/c'' \rfloor}}$ for some integer c'' independent of n and r, for then we know that $r \in (f) + m^{\lfloor n/c''' \rfloor}$ for some c''' independent of n and r, which proves that u lies in the desired ideal.

We first prove that a power of r lies in a good ideal, and for that we need the following detour:

Claim: $r^d \in (f) + m^{\lfloor n/l \rfloor - e}$ for some constant e independent of n.

Proof of the claim: By assumption there exists an integer d independent of n and r such that rf^{k-1} satisfies an integral equation of degree d over $(f^k) + m^{\lfloor n/l \rfloor}$, say: $(rf^{k-1})^d + \alpha_1 (rf^{k-1})^{d-1} + \cdots + \alpha_d = 0$, where $\alpha_i \in ((f^k) + m^{\lfloor n/l \rfloor})^i$.

We will recursively define $\beta_{d-i+1} \in ((f^k) + m^{\lfloor n/l \rfloor})^{d-i}$ for each $i \in \{0, \ldots, d-1\}$ such that

$$r^d (f^{k-1})^{d-i} + \alpha_1 r^{d-1} (f^{k-1})^{d-i-1} + \cdots + \alpha_{d-i} r^i + \beta_{d-i+1} = 0. \quad (\#)$$

If $i = 0$, set $\beta_{d+1} = 0$. Now assume we have defined β_{d-i+1} for some $i < d-1$. By the Artin-Rees Lemma there exists a positive integer e such that $m^n \cap (f^{k-1}) \subseteq f^{k-1} m^{n-e}$ for all $n \geq e$. In the following we may and do assume that $n/l \geq e$. With this we construct the next β using the equation displayed above and the following:

$$\begin{aligned} \alpha_{d-i} r^i + \beta_{d-i+1} &\in (f^{k-1}) \cap ((f^k) + m^{\lfloor n/l \rfloor})^{d-i} \\ &= (f^{k-1}) \cap \left(f^k ((f^k) + m^{\lfloor n/l \rfloor})^{d-i-1} + m^{\lfloor n/l \rfloor (d-i)}\right) \\ &= f^k ((f^k) + m^{\lfloor n/l \rfloor})^{d-i-1} + (f^{k-1}) \cap m^{\lfloor n/l \rfloor (d-i)} \end{aligned}$$

$$\subseteq \; f^{k-1}((f^k)+m^{\lfloor n/l\rfloor})^{d-i-1}+(f^{k-1})m^{\lfloor n/l\rfloor(d-i)-e}$$
$$\subseteq \; f^{k-1}((f^k)+m^{\lfloor n/l\rfloor})^{d-i-1}$$

as $n/l \geq e$. Thus we may write $\alpha_{d-i}r^i + \beta_{d-i+1} = f^{k-1}\beta_{d-i}$ for some $\beta_{d-i} \in ((f^k)+m^{\lfloor n/l\rfloor})^{(d-i-1)}$. To finish the induction step we only have to divide the displayed equation (#) by the nonzerodivisor f^{k-1}.

In the final step $i = d-1$ we thus obtain $r^d f^{k-1} + \alpha_1 r^{d-1} + \beta_2 = 0$. Therefore

$$r^d f^{k-1} = -\alpha_1 r^{d-1} - \beta_2 \in (f^{k-1}) \cap ((f^k)+m^{\lfloor n/l\rfloor}) = (f^k)+(f^{k-1})\cap m^{\lfloor n/l\rfloor} \subseteq (f^k)+f^{k-1}m^{\lfloor n/l\rfloor-e}.$$

It follows that $r^d \in (f)+m^{\lfloor n/l\rfloor-e}$. This completes the proof of the claim.

Now we are ready to prove that r is integral over $(f)+m^{\lfloor n/dlk(e+1)\rfloor}$. Recall that $rf^{k-1} \in \overline{(f^k)+m^n}$. It suffices to prove that for any valuation v on the field of fractions of R, $v(r) \geq \min\{v(f), \lfloor n/dlk(e+1)\rfloor v(m)\}$.

Since $rf^{k-1} \in \overline{(f^k)+m^n}$, $v(r)+(k-1)v(f) = v(rf^{k-1}) \geq \min\{kv(f), nv(m)\}$, therefore $v(r) \geq \min\{v(f), nv(m)-(k-1)v(f)\}$. If $v(r) \geq v(f)$, there is nothing to show, so we may assume that

$$\lfloor n/(e+1)\rfloor v(m) - (k-1)v(f) \leq nv(m) - (k-1)v(f) \leq v(r) < v(f).$$

This implies that $\lfloor n/(e+1)\rfloor v(m) < kv(f)$. Now we use our detour: as r^d lies in $(f)+m^{\lfloor n/l\rfloor-e} \subseteq (f)+m^{\lfloor n/l(e+1)\rfloor}$, then

$$dv(r) \geq \min\{v(f), \lfloor n/l(e+1)\rfloor v(m)\} \geq \min\{v(f), \lfloor n/lk(e+1)\rfloor v(m)\}.$$

If $dv(r) \geq \lfloor n/lk(e+1)\rfloor v(m)$, we are done, so we may assume instead that

$$\lfloor n/lk(e+1)\rfloor v(m) > dv(r) \geq v(f).$$

Thus

$$\lfloor n/lk(e+1)\rfloor v(m) > dv(r) \geq v(f) > \frac{1}{k}\lfloor n/(e+1)\rfloor v(m),$$

which is a contradiction. This finishes the proposition. ∎

The following is Theorem 3.10 in [1], presented here with a new proof:

Theorem 3.10. *Let (R,m) be a complete local normal domain and let (f) be a principal radical ideal. In case R does not contain a field, let $(V,(p))$ be a general coefficient ring of R and we also assume that either $fR = pR$ or that f, p is part of a system of parameters in R. Then for all k, $\overline{(f^k)+m^n} \subseteq (f^k)+m^{\lfloor n/c\rfloor}$ for some constant c independent of n.*

Proof: The case $k = 1$ holds by [2, Theorem 1.4]. Thus condition 1. of the previous proposition is satisfied. Condition 2. of the previous proposition is satisfied by Theorem 3.9, so that the corollary follows by Proposition A. ∎

Before we prove the main theorem, we need one new lemma:

Lemma B. *Let R be an integral domain, x and y non-zero elements of R and d, l positive integers such that for every positive integer n, every element of $\overline{(xy) + m^n}$ satisfies an integral equation of degree d over $(xy) + m^{\lfloor n/l \rfloor}$. Then there exists a positive integer k such that for every positive integer n, every element of $\overline{(x) + m^n}$ satisfies an integral equation of degree d over $(x) + m^{\lfloor n/k \rfloor}$.*

Proof: Let $r \in \overline{(x) + m^n}$. Then $ry \in \overline{(xy) + m^n}$. Thus there exist elements $r_i \in ((xy) + m^{\lfloor n/l \rfloor})^i$ such that

$$(ry)^d + r_1 (ry)^{d-1} + \cdots + r_{d-1} ry + r_d = 0.$$

Write $r_i = s_i (xy)^i + t_i$ for some $s_i \in R$ and some $t_i \in m^{\lfloor n/l \rfloor}$. Then

$$(ry)^d + s_1 (xy)(ry)^{d-1} + \cdots + s_{d-1} (xy)^{d-1} ry + s_d (xy)^d + t_1 (ry)^{d-1} + \cdots + t_{d-1} ry + t_d = 0.$$

Thus $t_1 (ry)^{d-1} + \cdots + t_{d-1} ry + t_d \in (y^d) \cap m^{\lfloor n/l \rfloor}$. By the Artin-Rees Lemma there exists an integer c such that $t_1 (ry)^{d-1} + \cdots + t_{d-1} ry + t_d \in y^d m^{\lfloor n/c \rfloor}$. But then dividing the integral equation above by y^d shows that r satisfies an integral equation of degree d over $(x) + m^{\lfloor n/cd \rfloor}$. ∎

Finally we can prove the general result for principal ideals in complete normal local domains. The theorem below is Theorem 3.12 in [1], presented here with a new proof:

Theorem 3.12. *Let (R, m) be a complete normal local domain. Let f be an element in R. Then there exists a positive integer c such that*

$$\overline{(f) + m^n} \subseteq (f) + m^{\lfloor n/c \rfloor} \quad \text{for all} \quad n.$$

Proof: If $f = 0$, the theorem is known by [2, Theorem 1.4]. So we may assume that f is not zero.

As R is normal, all the associated prime ideals of the ideal (f) are minimal over (f). By Corollary 3.4 of [1] it suffices to prove the theorem for the primary components of (f) in place of (f). Let P be an associated prime ideal of (f). As R is normal, the localization R_P is a one-dimensional regular local ring, so $fR_P = P^k R_P$ for some integer k. Thus the P-primary component of fR equals the kth symbolic power $P^{(k)}$ of P and it suffices to prove the theorem for all $P^{(k)}$ in place of (f).

Let $P = (a_1, \ldots, a_l)$. Let $X_1, \ldots, X_l$ be indeterminates over R and let S be the faithfully flat extension $R[X_1, \ldots, X_l]_{mR[X_1, \ldots, X_l]}$ of R. Note that as all the associated primes of xS

have height one and as S localized at height one prime ideals is a principal ideal domain, the ideal generated by $x = a_1X_1 + \cdots + a_lX_l$ is radical.

Suppose that this x satisfies the conditions of Theorem 3.10. Namely, either R contains a field, or instead if $(V, (p))$ is a coefficient ring of R, then either $x = p$ or x, p is a part of a system of parameters. Then by Theorem 3.10, for every positive integer k there exists an integer c such that $\overline{x^kS + m^nS} \subseteq x^kS + m^{\lfloor n/c \rfloor}S$ for all n. Note also that PS is associated to xS and that the PS-primary component of x^kS is $P^{(k)}S$ (as S_{PS} is a principal ideal domain). Thus there exists an element y in S such that $x^kS : y = P^{(k)}S$. As R is normal, then so is S, so that $x^kS = \overline{x^kS}$. An application of Lemma 3.11 of [1] shows that there exists an integer c' such that $\overline{P^{(k)}S + m^nS} \subseteq P^{(k)}S + m^{\lfloor n/c' \rfloor}S$ for all n. Finally,

$$\begin{aligned} \overline{P^{(k)} + m^n} &\subseteq \overline{P^{(k)}S + m^nS} \cap R \\ &\subseteq \left(P^{(k)}S + m^{\lfloor n/c' \rfloor}S\right) \cap R \\ &= P^{(k)} + m^{\lfloor n/c' \rfloor} \end{aligned}$$

as S is faithfully flat over R.

This finishes the theorem for rings containing fields.

Now assume that R contains a coefficient field $(V, (p))$. The above proves the theorem for all f which are not contained in any minimal prime ideal over pV. Thus by Lemma 3.11 of [1], for all height one prime ideals P of R not containing p and all positive integers k there exists an integer c such that $\overline{P^{(k)} + m^n} \subseteq P^{(k)} + m^{\lfloor n/c \rfloor}$.

Next we shall prove the theorem in the special case $f = p$. Let $P_1, \ldots, P_N$ be all the prime ideals in R minimal over pR. Let $W = R \backslash (P_1 \cup \cdots \cup P_N)$. As R is normal, $W^{-1}R$ is a one-dimensional semi-local regular ring, thus a principal ideal domain. Let $x_i \in R$ be such that $x_iW^{-1}R = P_iW^{-1}R$. Therefore we may write $p = u'x_1^{n_1} \cdots x_N^{n_N}$ for some unit $u' \in W^{-1}R$. But then there exist $u, v \in W$ such that in R, $up = vx_1^{n_1} \cdots x_N^{n_N}$. Note that either u is a unit in R or else p, u is a part of a system of parameters. Thus by Theorem 3.9, for each positive integer k there exist integers d and l such that every element of $\overline{(up)^k + m^n}$ satisfies an equation of integral dependence of degree d over $(up)^k + m^{\lfloor n/l \rfloor}$. Thus by Lemma B, for each positive integer k there exist integers d and l such that for all $i = 1, \ldots, N$, every element of $\overline{(x_i)^k + m^n}$ satisfies an equation of integral dependence of degree d over $(x_i)^k + m^{\lfloor n/l \rfloor}$. This means that condition 2. of Proposition A is satisfied for each x_i. But $x_iR = P_i \cap Q_i$, where Q_i is either the unit ideal or a height one ideal modulo which p is a non-zerodivisor. As P_i is a radical ideal (even prime), by [2, Theorem 1.4], there exists a positive integer c such that for all $n \geq 1$, $\overline{P_i + m^n} \subseteq P_i + m^{\lfloor n/c \rfloor}$. By what we have proved, there exists a positive integer c' such that for all $n \geq 1$, $\overline{Q_i + m^n} \subseteq Q_i + m^{\lfloor n/c' \rfloor}$. Thus by Lemma 3.3 of [1], the theorem holds for x_i. In particular, condition 1. of Proposition A is satisfied for x_i. Thus by Proposition A, the theorem holds for all x_i^k, as k varies over all positive integers. Then by Lemma 3.11 [1], there exists an integer c such that for all $i = 1, \dots, N$,

$$\overline{P_i^{(k)} + m^n} \subseteq P_i^{(k)} + m^{\lfloor n/c \rfloor}.$$

Thus by Lemma 3.3 of [1], the theorem holds for $f = p$.

Hence condition 1. of Proposition A is satisfied for p, and condition 2. is satisfied by Theorem 3.9. Thus by Proposition A, the theorem holds for each $f = p^k$.

It remains to examine the case when f and p do not form a system of parameters. In this case there exist an integer e and elements $u \in W$ and $h \in R$ such that $fh = up^k$. We know the theorem for uR and $p^k R$. Since uR and $p^k R$ are part of a system of parameters, by Lemma 3.3 of [1] we also know the theorem for $(u) \cap (p^e) = (up^e) = (fh)$. This means that there exists an integer c such that $\overline{fhR + m^n} \subseteq fhR + m^{\lfloor n/c \rfloor}$.

Now pick $u \in \overline{fR + m^n}$. Then $hu \in \overline{fhR + m^n} \subseteq fhR + m^{\lfloor n/c \rfloor}$, so $hu \in fhR + m^{\lfloor n/c \rfloor} \cap hR$. By the Artin-Rees Lemma there exists an integer k independent of u and n such that $m^{\lfloor n/c \rfloor} \cap hR \subseteq hm^{\lfloor n/c \rfloor - k}$. Thus $hu \in fhR + hm^{\lfloor n/c \rfloor - k}$, so $u \in fR + m^{\lfloor n/c \rfloor - k}$. Thus also in this last case, $\overline{fR + m^n} \subseteq fR + m^{\lfloor n/c(k+1) \rfloor}$.

This finishes the proof of the theorem. ∎

References

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